The Quarterly Journal of Mathematics Advance Access published on July 18, 2006
Quart. J. Math. 58 (2007), 31–45; doi:10.1093/qmath/hal013

ON FREYD'S GENERATING HYPOTHESIS

by MARK HOVEY†

(*Department of Mathematics, Wesleyan University, Middletown, CT 06459, USA*)

[Received 7 February 2006. Revised 15 May 2006]

Abstract

Freyd's generating hypothesis in stable homotopy theory is revisited and new consequences and equivalent forms of it are derived. A surprising such consequence is that I, the Brown–Comenetz dual of the sphere and the source of many counterexamples in stable homotopy, is the cofibre of a self-map of a wedge of spheres. It is also shown that a consequence of the generating hypothesis, that the homotopy of a finite spectrum that is not a wedge of spheres can never be finitely generated as a module over $\pi_* S$, is in fact true for many finite torsion spectra.

1. Introduction

Freyd's generating hypothesis [**4**] is perhaps the most important question in stable homotopy theory. A precise statement of it follows.

CONJECTURE 1.1 (Freyd's generating hypothesis) *If X and Y are finite spectra, and S is the sphere spectrum, then the natural map*

$$[X, Y] \to \mathrm{Hom}_{\pi_* S}(\pi_* X, \pi_* Y)$$

is a monomorphism.

If Y is fixed (perhaps not finite) and X is allowed to vary, a special case of the generating hypothesis is obtained, which will be referred to as Freyd's generating hypothesis with target Y. Here $[X, Y]$ denotes maps from X to Y in the stable homotopy category, and $\pi_* X = [S, X]_*$ denotes the homotopy groups of X. In practice, the stable homotopy category is implicitly assumed to be localized at a fixed integer prime p.

Freyd proves that the generating hypothesis actually implies that the map

$$[X, Y] \to \mathrm{Hom}_{\pi_* S}(\pi_* X, \pi_* Y)$$

is an isomorphism for all finite spectra X, Y. Kahn derived other consequences of the truth or falsity of the generating hypothesis in a series of papers, including [**11–13**].

Devinatz [**3**] has a program for proving the generating hypothesis with target S using the chromatic technology. This approach generalizes the previous work of Devinatz in [**1**], where he proves that if $f\colon X \to S$ has $\pi_* f = 0$, and p is odd, then the composite $X \to S \to L_1 S$ is null. The program

†E-mail: hovey@member.ams.org

depends on the truth of either the telescope conjecture (currently believed to be likely false) or a weak form of the chromatic splitting conjecture and several other conjectures.

In this article, the following theorem is proved. Let X_p denote the p-completion of a spectrum X.

THEOREM 1.2 *Let Y be a finite spectrum. The following are equivalent:*

(1) *Freyd's generating hypothesis with target Y;*
(2) *$\pi_* Y_p$ is an injective $\pi_* S$-module;*
(3) *$\pi_* Y_p$ is an injective $\pi_* S_p$-module;*
(4) *The natural map*

$$[X, Y_p] \to \mathrm{Hom}_{\pi_* S}(\pi_* X, \pi_* Y_p)$$

is an isomorphism for all spectra X.

The following theorem is also proved. Recall that the Spanier–Whitehead dual DX of X is defined by $DX = F(X, S)$, the spectrum of maps from X to S.

THEOREM 1.3 *Suppose Freyd's generating hypothesis with target S holds. Let R be a finite associative ring spectrum that is the Spanier–Whitehead self-dual, in the sense that DR is a suspension of R. Then, $\pi_*(R_p)$ is injective as a left R_*-module and as a left module over itself. In particular, the natural map*

$$[X, R_p] \to \mathrm{Hom}_{R_*}(R_* X, \pi_* R_p)$$

is an isomorphism for all X.

For example, this theorem means that if the generating hypothesis with target S holds, then $\pi_* M(p^n)$ is a self-injective ring for $p > 3$ and n arbitrary or for $p = 2$ and $n > 1$.

Freyd [**4**, Theorem 9.9] proved that the generating hypothesis is equivalent to $\pi_* Y$ being an injective $\pi_* S$-module for all finite torsion spectra Y. Lin [**16**] showed that $\pi_* Y$ is not an injective $\pi_* S$-module if Y is not torsion, but did not realize that completion would solve this problem if the generating hypothesis is true. The author's approach is different from Freyd's and yields a more precise result. In addition, Freyd does not mention part (4) of Theorem 1.2, which focuses attention on maps from infinite spectra X to Y_p. Infinite spectra X that might be worth studying in this context include the rational Eilenberg–MacLane spectrum $H\mathbb{Q}$ and $\Sigma^\infty BG_+$, the classifying space of a finite group, where the Segal conjecture states $[X, S_p]$.

Of course, even if the generating hypothesis is false, $\pi_* Y_p$ has some injective hull J_Y as a $\pi_* S$-module, so one can attempt to study the map $\pi_* Y_p \to J_Y$. In this article, it is shown that $\pi_* S_p \to J_S$ is a split monomorphism of abelian groups in degree 0, for example.

The methods of this article may also be helpful in investigating the generating hypothesis in other stable homotopy categories. Lockridge [**17**] has investigated this question; he shows that the generating hypothesis holds in the unbounded derived category $\mathcal{D}(R)$ of a commutative ring R if and only if R is von Neumann regular, for example.

Theorem 1.2 has a number of corollaries. Perhaps the most surprising of them is the following. Let IY denote the Brown–Comenetz dual of Y, so that

$$[X, IY] = \mathrm{Hom}_{\mathbb{Z}_{(p)}}\left(\pi_*(X \wedge Y), \frac{\mathbb{Q}}{\mathbb{Z}_{(p)}}\right)$$

for all X.

Corollary 1.4 *Let Y be a finite spectrum. Freyd's generating hypothesis with target Y holds if and only if $\pi_*(IY)$ is a flat π_*S-module. In particular, this implies that the natural map*

$$\pi_*(IY) \otimes_{\pi_*S} \pi_*X \to \pi_*(IY \wedge X)$$

is an isomorphism for all X. Furthermore, in this case, $\pi_*(IY)$ has projective dimension 1 as a π_*S-module and, if Freyd's generating hypothesis with target S holds as well, then there is a cofibre sequence

$$\Sigma^{-1}IY \xrightarrow{\delta} W \to W \to IY$$

for which W is a coproduct of spheres of varying dimensions and δ is a phantom map.

Note that the map δ cannot be 0, for then IY would be a coproduct of spheres itself. As $IY = DY \wedge I$ is BP-acyclic, this is impossible unless $IY = 0$, which is false.

On the other hand, there are several reasons to think that δ should be 0, and so the generating hypothesis should be false. For example, δ is a map from a *BP*-acyclic spectrum to a coproduct of spheres, and one might expect that a coproduct of spheres would be *BP*-local. Any bounded-below coproduct of spheres is *BP*-local, as is any suspension spectrum [**6**], but arbitrary coproducts of spheres are not known to be *BP*-local. Similarly, one might think that there are no phantom maps to W_p, which should also lead to a disproof of the generating hypothesis. Again, however, the fact that the spheres in W occur in infinitely many dimensions makes one unable to prove this.

However, some weak evidences can also be given that the generating hypothesis might be true. One of the most straightforward corollaries of the generating hypothesis is that, if Y is a finite spectrum and π_*Y is a finitely generated π_*S-module, then Y is a finite coproduct of spheres.

Theorem 1.5 *Suppose Y is a finite spectrum of type n, for some $n > 0$, and suppose the map $\pi_*Y \to \pi_*L_nY$ is non-zero. This hypothesis holds, for example, if Y is a ring spectrum or a μ-spectrum in the sense of* [**8**, Definition 4.8]. *Then, π_*Y is not a finitely generated π_*S-module.*

This theorem applies in particular to the ring spectrum $DX \wedge X$ for any finite torsion spectrum X and to the generalized Moore spectra $M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ for large enough values of the exponents [**2**]. The telescope conjecture [**18**] (which is true if $n = 1$) would imply that every finite torsion spectrum satisfies the hypotheses of Theorem 1.5, but even if the telescope conjecture fails, the author would be astounded if there were any non-zero finite spectra of type n for which the map $\pi_*Y \to \pi_*L_nY$ is zero. It is just that current techniques do not seem to be sufficient to prove this.

2. Proof of Theorem 1.2

Here is a basic result about injective π_*S-modules.

Lemma 2.1 *Suppose E is a spectrum such that π_*E is an injective π_*S-module. Then the natural map*

$$[X, E] \to \operatorname{Hom}_{\pi_*S}(\pi_*X, \pi_*E)$$

is an isomorphism for all spectra X.

This lemma shows that condition (2) of Theorem 1.2 implies condition (4). Note that Lemma 2.1 holds, with the proof given subsequently, in any monogenic stable homotopy category in the sense of [**9**].

Proof. As $\pi_* E$ is injective, the functor $\mathrm{Hom}_{\pi_* S}(\pi_* X, \pi_* E)$ is exact. The Brown representability theorem then implies that there is a spectrum J and a natural isomorphism

$$[X, J] \cong \mathrm{Hom}_{\pi_* S}(\pi_* X, \pi_* E)$$

for all X. The evident natural transformation

$$[X, E] \to \mathrm{Hom}_{\pi_* S}(\pi_* X, \pi_* E)$$

then gives a map $E \to J$ that is an isomorphism on homotopy groups.

The following proposition is the heart of the argument proving Theorem 1.2.

PROPOSITION 2.2 *Suppose Y is a spectrum such that there are no non-zero phantom maps to Y. Then Freyd's generating hypothesis with target Y holds if and only if $\pi_* Y$ is an injective $\pi_* S$-module.*

This proposition will also hold in any monogenic stable homotopy category.

Proof. The 'if' half of this proposition follows immediately from Lemma 2.1 and does not require the assumption about phantom maps. For the 'only if' half, assume

$$[X, Y] \to \mathrm{Hom}_{\pi_* S}(\pi_* X, \pi_* Y)$$

is injective for all finite X. Let J denote the injective hull of $\pi_* Y$ as a $\pi_* S$-module. Then the Brown representability theorem and Lemma 2.1 imply that there is a spectrum I with $\pi_* I = J$ and such that the natural map

$$[X, I] \to \mathrm{Hom}_{\pi_* S}(\pi_* X, J)$$

is an isomorphism for all X. In particular, there is a map $Y \to I$ corresponding to the inclusion $\pi_* Y \to J$. Consider the following commutative diagram.

$$\begin{array}{ccc} [X,Y] & \longrightarrow & [X,I] \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\pi_* S}(\pi_* X, \pi_* Y) & \longrightarrow & \mathrm{Hom}_{\pi_* S}(\pi_* X, J) \end{array}$$

The left-hand vertical map is injective for all finite X and the bottom horizontal map is always injective. It follows that the top horizontal map is injective for all finite X and hence that the fibre $F \to Y$ of the map $Y \to I$ is a phantom map. As there are no non-zero phantom maps to Y, Y must be a retract of I. Hence, $\pi_* Y$ is a retract of J and, therefore, is an injective $\pi_* S$-module.

To apply Proposition 2.2, the relationship between the generating hypothesis with target Y and the generating hypothesis with target Y_p needs to be known. The following lemma, due to the referee, makes this relationship easy to see.

LEMMA 2.3 *Suppose X is a bounded below spectrum such that $\pi_n X$ is a finitely generated abelian group for all n and F is a torsion-free abelian group. Then the natural map*

$$\mathrm{Hom}_{\pi_* S}(\pi_* X, \pi_* Y) \otimes F \to \mathrm{Hom}_{\pi_* S}(\pi_* X, \pi_* Y \otimes F)$$

is an isomorphism.

Proof. The natural map in question takes $f \otimes a$ to the map that takes x to $f(x) \otimes a$. The collection of all abelian groups M for which the corresponding natural map

$$\mathrm{Hom}_{\mathbb{Z}_{(p)}}(M, N) \otimes F \to \mathrm{Hom}_{\mathbb{Z}_{(p)}}(M, N \otimes F)$$

is an isomorphism contains $\mathbb{Z}_{(p)}$ and is closed under finite direct sums and cokernels (as F is flat). It, therefore, contains all finitely generated abelian groups, including $\pi_n X$ and $(\pi_* S \otimes \pi_* X)_n$ for all n. However, the diagram

$$\mathrm{Hom}_{\pi_* S}(\pi_* X, Z) \to \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\pi_* X, Z) \rightrightarrows \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\pi_* S \otimes \pi_* X, Z)$$

is an equalizer diagram of abelian groups, where the left-hand map is the obvious forgetful map and the two right-hand maps take $f\colon \pi_* X \to Z$ to the two ways to get maps $\pi_* S \otimes \pi_* X \to Z$ using the $\pi_* S$-module structure on $\pi_* X$ and on Z, respectively. As F is flat, tensoring with F will preserve equalizer diagrams and the lemma follows.

This gives the following proposition.

PROPOSITION 2.4 *Suppose Y is a finite spectrum. Then, Freyd's generating hypothesis with target Y holds if and only if Freyd's generating hypothesis with target Y_p holds.*

Proof. For X finite,

$$[X, Y] \otimes \mathbb{Z}_p \xrightarrow{\cong} [X, Y_p].$$

This is not only well known but also easy to prove because the collection of all X for which this map is an isomorphism contains S and is thick. By Lemma 2.3,

$$\mathrm{Hom}_{\pi_* S}(\pi_* X, \pi_* Y) \otimes \mathbb{Z}_p \xrightarrow{\cong} \mathrm{Hom}_{\pi_* S}(\pi_* X, \pi_* Y_p).$$

Now the fact that $\mathbb{Z}_p$ is faithfully flat over $\mathbb{Z}_{(p)}$ completes the proof.

It is useful to note that rational vector spaces are always injective $\pi_* S$-modules.

PROPOSITION 2.5 *Suppose V is a graded rational vector space. Then there is a unique $\pi_* S$-module structure on V extending the abelian group structure, and V is an injective $\pi_* S$-module with this structure.*

This proposition will hold if $\pi_* S$ is replaced by any ring R such that $R/\mathfrak{p} \cong \mathbb{Z}/p$, where $\mathfrak{p}$ is the ideal of p-torsion elements.

Proof. Let $\mathfrak{p}$ denote the ideal of p-torsion elements in $\pi_* S$. Then, as V is torsion-free, the only way to make $\pi_* S$ act on V is through the homomorphism $\pi_* S \to \pi_* S/\mathfrak{p} \cong \mathbb{Z}/p$.

Now suppose $f\colon M \to V$ is a map of $\pi_* S$-modules and $i\colon M \to N$ is an inclusion of $\pi_* S$-modules. Let $\operatorname{Tor}(M)$ denote the p-torsion in M, which is a $\pi_* S$-submodule. Then f factors through $\overline{f}\colon M/\operatorname{Tor}(M) \to V$ and furthermore i induces an inclusion $\overline{i}\colon M/\operatorname{Tor}(M) \to N/\operatorname{Tor}(N)$. As V is an injective abelian group, it follows that there is a map $\bar{g}\colon N/\operatorname{Tor}(N) \to V$ of abelian groups extending $\overline{f}$. However, $\bar{g}$ is in fact a map of $\pi_* S$-modules, as $N/\operatorname{Tor}(N)$ and V are torsion-free, so $\mathfrak{p}$ acts trivially. Hence,

$$N \to \frac{N}{\operatorname{Tor}(N)} \xrightarrow{\bar{g}} V$$

gives the desired extension of f.

The work so far implies the following proposition, independent of the generating hypothesis.

PROPOSITION 2.6 *Suppose X and Y are finite spectra. Then, the natural map*

$$\operatorname{Ext}^n_{\pi_* S}(\pi_* X, \pi_* Y) \to \operatorname{Ext}^n_{\pi_* S}(\pi_* X, \pi_* Y_p)$$

is an isomorphism for all $n \geq 1$.

Proof. As $\pi_n Y$ is a finitely generated abelian group for all n, the sequence

$$0 \to \pi_* Y \to \pi_* Y \otimes \mathbb{Z}_p \to \pi_* Y \otimes \frac{\mathbb{Z}_p}{\mathbb{Z}_{(p)}} \to 0$$

is exact. As $\mathbb{Z}_p/\mathbb{Z}_{(p)}$ is rational, $\pi_* Y \otimes \mathbb{Z}_p/\mathbb{Z}_{(p)}$ is an injective $\pi_* S$-module by Proposition 2.5. It follows that

$$\operatorname{Ext}^n_{\pi_* S}(\pi_* X, \pi_* Y) \to \operatorname{Ext}^n_{\pi_* S}(\pi_* X, \pi_* Y_p)$$

is an isomorphism for $n > 1$ and a surjection for $n = 1$, for all X. On the other hand, Lemma 2.3 states that the map

$$\operatorname{Hom}_{\pi_* S}(\pi_* X, \pi_* Y_p) \to \operatorname{Hom}_{\pi_* S}\left(\pi_* X, \pi_* Y \otimes \frac{\mathbb{Z}_p}{\mathbb{Z}_{(p)}}\right)$$

is isomorphic to the surjection

$$\operatorname{Hom}_{\pi_* S}(\pi_* X, \pi_* Y) \otimes \mathbb{Z}_p \to \operatorname{Hom}_{\pi_* S}(\pi_* X, \pi_* Y) \otimes \frac{\mathbb{Z}_p}{\mathbb{Z}_{(p)}}.$$

This completes the proof.

The last ingredient needed for Theorem 1.2 is the simple proof that conditions (2) and (3) are equivalent.

LEMMA 2.7 *Let Y be a finite spectrum. Then $\pi_* Y_p$ is an injective $\pi_* S$-module if and only if it is an injective $\pi_* S_p$-module.*

Proof. As $\pi_* S_p$ is a flat $\pi_* S$-module, the forgetful functor from $\pi_* S_p$-modules to $\pi_* S$-modules preserves injectives. Conversely, assume $\pi_* Y_p$ is an injective $\pi_* S$-module. Applying Baer's criterion suppose given an ideal $\mathfrak{a}$ of $\pi_* S_p$ and a map $f\colon\ \mathfrak{a} \to \pi_* Y_p$. Let $\mathfrak{b} = \mathfrak{a} \cap \pi_* S$. Then $\mathfrak{b}$ is an ideal of $\pi_* S$, and so there is an extension $\pi_* S \to \pi_* Y_p$ of $\pi_* S$-modules. This gives a map $\pi_* S_p = \pi_* S \otimes \mathbb{Z}_p \to \pi_* Y_p$ of $\pi_* S_p$-modules. When restricted to $\mathfrak{b}$, this map extends f. However, as $\mathfrak{a} = \mathfrak{b} \otimes \mathbb{Z}_p$, it follows that it is an extension of f on $\mathfrak{a}$ as well.

Theorem 1.2 is now proved.

Proof of Theorem 1.2. Suppose Freyd's generating hypothesis with target Y holds. Then Proposition 2.4 implies that the generating hypothesis with target Y_p holds. According to Proposition 2.2, as there are no non-zero phantom maps to Y_p, $\pi_* Y_p$ is an injective $\pi_* S$-module. Thus, condition (1) implies condition (2).

Lemma 2.7 states that conditions (2) and (3) are equivalent, and Lemma 2.1 states that condition (2) implies condition (4). Condition (4) obviously implies that Freyd's generating hypothesis holds with target Y_p, and then Proposition 2.4 implies that it holds with target Y as well.

3. Brown–Comenetz duality

In this section, the consequences of the generating hypothesis for Brown–Comenetz duals of finite spectra are investigated, proving Corollary 1.4.

Proof of Corollary 1.4. Suppose Y is finite. Then $Y_p = I^2 Y$, as is well known. Hence, $\pi_* Y_p = \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\pi_*(IY), \mathbb{Q}/\mathbb{Z}_{(p)})$. Now apply Lambek's theorem [**15**, Theorem 4.9] to conclude that $\pi_* Y_p$ is injective if and only if $\pi_*(IY)$ is flat. Once $\pi_*(IY)$ is flat, the map

$$\pi_*(IY) \otimes_{\pi_* S} \pi_* X \to \pi_*(IY \wedge X)$$

is a natural transformation of homology theories that is an isomorphism when $X = S$, so is always an isomorphism.

Now [**10**, Lemma 2] implies that, over a countable ring like $\pi_* S$, any flat module has projective dimension at most 1. As $\pi_* S$ is a local ring, projectives are free [**14**] (the graded case of this result is actually needed, which has been recently written in [**5**, Appendix A]). Thus, if $\pi_*(IY)$ had projective dimension 0, it would be free. From that, it is easy to conclude that IY is a coproduct of spheres. However, as $IY = DY \wedge I$ is BP-acyclic [**9**, Lemma B.11], IY would have to be trivial. As this is false, the projective dimension of IY is 1.

Thus, there is a short exact sequence

$$0 \to F_1 \to F_0 \to \pi_* IY \to 0$$

of $\pi_* S$-modules, where F_1 and F_0 are free. In fact, by tensoring over $\mathbb{Z}_{(p)}$ with $\mathbb{Q}$, it is seen that F_1 and F_0 are isomorphic. By choosing generators, a coproduct of spheres W with $\pi_* W \cong F_1 \cong F_0$ can be found. By looking at the image of the generators in homotopy, maps can be found as

$$W \xrightarrow{f} W \xrightarrow{g} IY$$

such that the induced maps on homotopy give the original free resolution of $\pi_* IY$. In fact, this sequence is a cofibre sequence. Indeed, the composite gf is null, so there is an induced map from the cofibre of f to IY, which one can easily see induces an isomorphism on homotopy.

Now, given Freyd's generating hypothesis with target S, the map $\Sigma^{-1} IY \xrightarrow{\delta} W$, that is, the fibre of f is phantom. Indeed, if F is finite, and $h\colon F \to \Sigma^{-1} IY$ is a map, then δh must factor through some a map $h'\colon F \to W'$ for some finite subcoproduct of spheres h'. If δh is non-zero, then h' is non-zero, and so, by Freyd's generating hypothesis with target S, must induce a non-trivial map on homotopy. Then δh also induces a non-trivial map on homotopy, as does the trivial map $f\delta h$. This contradiction implies δ is phantom.

Corollary 1.4 has some interesting consequences. Suppose that Freyd's generating hypothesis with target S holds, so that there is a cofibre sequence

$$\Sigma^{-1} I \xrightarrow{\delta} W \xrightarrow{f} W \to I,$$

where W is a coproduct of spheres and δ is a phantom map. Then, $E_* f$ is a monomorphism for all E and is an isomorphism for those E for which $E_* I = 0$, such as all BP-module spectra and I itself. In fact, $E^* f$ is an isomorphism for all BP-module spectra and all harmonic spectra E, as I is BP-acyclic and dissonant.

On the other hand, suppose E is one of the many spectra for which $[E, S]_* = 0$, such as I, $H\mathbb{F}_p$, $K(n)$ for $n > 0$ or any dissonant spectrum. Then, any map $E \to W$ goes to 0 in the corresponding product P of spheres and hence factors through the fibre of $W \to P$. As this is a phantom map, $[E, W]_*$ consists entirely of phantom maps that necessarily go to 0 in $[E, I]_*$. Hence $[E, f]_*$ is in fact surjective in this case. One might think that this happens because $[E, W]_* = 0$, but, in fact, $[E, W]_* = 0$ if and only if $E = 0$ because if $[E, W]_* = 0$, then $[E, I]_* = 0$, and so $E = 0$.

Another corollary is the following.

COROLLARY 3.1 *Suppose Freyd's generating hypothesis with target S holds. Then there is a product J of suspensions of I such that* $S_p \vee J \cong J$.

Proof. Apply the functor $F(-, I)$ to the cofibre sequence

$$\Sigma^{-1} I \xrightarrow{\delta} W \to W \to I$$

to get a cofibre sequence

$$F(I, I) = S_p \to J = F(W, I) \to J = F(W, I) \to F(\Sigma^{-1} I, I) = \Sigma S_p.$$

On homotopy, the last map takes a map $\alpha\colon \Sigma^i W \to I$ into the composite $\delta \circ \alpha$, which is necessarily 0 because there are no phantom maps to I. On the other hand, because $\pi_* S_p$ is an injective $\pi_* S$-module, any map into S_p that is trivial on homotopy is in fact trivial. Hence, the cofibre sequence splits, giving the corollary.

4. Other consequences of the generating hypothesis

In this section, Theorem 1.2 is used to draw some further consequences of the generating hypothesis, including Theorem 1.3. To begin, a more precise version of Freyd's 'faithful implies full' result [**4**, Proposition 9.7] is proved.

COROLLARY 4.1 *Suppose Y is a finite spectrum for which Freyd's generating hypothesis with target Y holds. Then the natural map*

$$[X, Y] \to \mathrm{Hom}_{\pi_* S}(\pi_* X, \pi_* Y)$$

is an isomorphism for all finite X.

Proof. This follows from part 4 of Theorem 1.2 and the method of proof of Proposition 2.4.

Proposition 2.6 immediately gives the following corollary of Theorem 1.2.

COROLLARY 4.2 *Suppose Y is a finite spectrum for which Freyd's generating hypothesis with target Y holds. Then*

$$\mathrm{Ext}^n_{\pi_* S}(\pi_* X, \pi_* Y) = 0$$

for all finite X and all $n > 0$.

Using the fact that $\mathbb{Z}_p/\mathbb{Z}_{(p)}$ is a rational vector space, the following consequence of the generating hypothesis is obtained.

COROLLARY 4.3 *Suppose Y is a finite spectrum for which Freyd's generating hypothesis with target Y holds. Then*

$$0 \to \pi_* Y \to \pi_* Y_p \to \frac{\pi_* Y_p}{\pi_* Y} \to 0$$

is an injective resolution of $\pi_* Y$ in the category of $\pi_* S$-modules. In particular, if Y is a finite spectrum of type 0, then $\pi_* Y$ has injective dimension 1.

The following lemma is needed for the proof of Theorem 1.3.

LEMMA 4.4 *Suppose R is a ring spectrum and M is an R-module spectrum such that M_* is injective as a left R_*-module. Then the natural map*

$$M^* X \to \mathrm{Hom}_{R_*}(R_* X, M_*)$$

is an isomorphism for all X.

Proof. The natural map in question takes $f\colon\ X \to M$ to the map $\mu_* \circ R_* f$, where $\mu\colon\ R \wedge M \to M$ is the structure map of M. As M_* is injective, the functor $\mathrm{Hom}_{R_*}(R_*(-), M_*)$ is a cohomology theory. As the natural transformation in question is an isomorphism when $X = S$, it is an isomorphism for all X.

Proof of Theorem 1.3. Suppose the generating hypothesis with target S holds and suppose that R is a finite ring spectrum that is Spanier–Whitehead self-dual. By Corollary 3.1, S_p is a retract of a product J of suspensions of I. By smashing with R, which commutes with products because R is finite, it is found that R_p is a retract, as an R-module, of a product of suspensions of $I \wedge R = F(DR, I)$. As R is Spanier–Whitehead self-dual, R_p is a retract, as an R-module, of a product of suspensions of IR. However, $\pi_*(IR) = \mathrm{Hom}_{\mathbb{Z}_{(p)}}(R_*, \mathbb{Q}/\mathbb{Z}_{(p)})$ is an injective R_*-module [**15**, Corollary 3.6B]. It follows that $\pi_* R_p$, as a retract of a product of injective R_*-modules, is an injective R_*-module. The same proof used in Lemma 2.7 implies that $\pi_* R_p$ is also injective as a left module over itself. Lemma 4.4 completes the proof.

Let R be a finite Spanier–Whitehead self-dual ring spectrum as in Theorem 1.3, and suppose the generating hypothesis holds for both S and R. Then, there is the isomorphism

$$[X, R_p] \cong \mathrm{Hom}_{\pi_* S}(\pi_* X, \pi_* R_p) \cong \mathrm{Hom}_{R_*}(R_* \otimes_{\pi_* S} \pi_* X, \pi_* R_p)$$

and also the isomorphism

$$[X, R_p] \cong \mathrm{Hom}_{R_*}(R_* X, \pi_* R_p).$$

These isomorphisms are related by the map

$$\sigma_X : \ R_* \otimes_{\pi_* S} \pi_* X \to R_* X,$$

and one might be tempted to think that σ_X has to be an isomorphism, and so R_* has to be flat over $\pi_* S$. However, it is actually known, under the generating hypothesis for S and R, that $\mathrm{Hom}_{R_*}(\sigma_X, \pi_* R_p)$ is an isomorphism. Thus, it can only be concluded that

$$\mathrm{Hom}_{R_*}(K_X, \pi_* R_p) = \mathrm{Hom}_{R_*}(C_X, \pi_* R_p) = 0$$

for all X, where K_X and C_X are the kernel and cokernel of σ_X, respectively.

5. Injective $\pi_* S$-modules

This section is concerned with injective $\pi_* S$-modules. Without assuming Freyd's generating hypothesis, it is still known that $\pi_* Y$ has some injective hull J_Y. Only little can be said about J_Y.

PROPOSITION 5.1 *The map $\pi_* S \to \pi_* S_p$ is an essential extension of $\pi_* S$-modules.*

Hence, whatever J_S is, at least it contains $\pi_* S_p$.

Proof. The only elements in $\pi_* S_p$, not in $\pi_* S$, are elements in $\pi_0 S_p \cong \mathbb{Z}_p$. Choose a non-zero $x \in \mathbb{Z}_p$ and suppose p^n divides x but p^{n+1} does not, so that x is congruent to an integer of the form $kp^n \in \pi_0 S$ modulo p^{n+1}, where k is a unit. Now choose an element $\alpha \in \pi_* S$ of order p^{n+1}, which can be done in the image of the J homomorphism. Then, $\alpha x = kp^n \alpha$, which is a non-trivial element of $\pi_* S$. Therefore, $(x) \cap \pi_* S$ is non-zero, completing the proof.

In fact, a little more is known about J.

PROPOSITION 5.2 *Let J denote the injective hull of $\pi_* S_p$ as a $\pi_* S_p$-module. The inclusion $\mathbb{Z}_p \to J_0$ is a split monomorphism of abelian groups.*

Proof. It will be proved that $\mathbb{Z}_p \to J_0$ is a pure monomorphism, that is, it will be shown that if there is an equation $x = p^n y$ for $x \in \mathbb{Z}_p$ and $y \in J_0$, then in fact there is an equation $x = p^n z$ for some $z \in \mathbb{Z}_p$. Indeed, it may as well be assumed that $x = p^k$, so that $p^k \alpha = p^n \alpha y$ for all $\alpha \in \pi_* S_p$. But then, if $n > k$, α may be taken as element of exact order p^n and it may be concluded that $p^k\alpha = 0$. This contradiction shows that $\mathbb{Z}_p \to J_0$ is pure.

However, it is well known that every pure monomorphism $i\colon\ \mathbb{Z}_p \to A$ splits for any abelian group A (so that $\mathbb{Z}_p$ is pure injective). Indeed, the purity of i guarantees that the map

$$i \otimes \frac{\mathbb{Q}}{\mathbb{Z}_{(p)}}\colon\ \frac{\mathbb{Q}}{\mathbb{Z}_{(p)}} \to A \otimes \frac{\mathbb{Q}}{\mathbb{Z}_{(p)}}$$

is still a monomorphism, so, as $\mathbb{Q}/\mathbb{Z}_{(p)}$ is injective, there is a map

$$A \otimes \frac{\mathbb{Q}}{\mathbb{Z}_{(p)}} \to \frac{\mathbb{Q}}{\mathbb{Z}_{(p)}}$$

extending the identity on $\mathbb{Q}/\mathbb{Z}_{(p)}$. By adjointness, this gives a map

$$A \to \operatorname{Hom}\left(\frac{\mathbb{Q}}{\mathbb{Z}_{(p)}}, \frac{\mathbb{Q}}{\mathbb{Z}_{(p)}}\right) \cong \mathbb{Z}_p$$

splitting i.

As $\pi_* S_p \to J$ is an essential extension, for every element $y \in J$, there is an element $\alpha_y \in \pi_* S_p$ with $\alpha_y y \in \pi_* S_p$. Choose a non-zero element x in $\pi_* S_p$ of the lowest possible degree such that $x = \gamma y$ for some $y \in J \backslash \pi_* S_p$. Proposition 5.2 states that the degree of x must be positive. Current knowledge of $\pi_* S$ is sufficient to rule out some possibilities for the pair (x, γ), but insufficient, as far as the author knows, to say anything systematic.

There is one more injective $\pi_* S$-module known, besides the rational ones and, conjecturally, $\pi_* S_p$.

PROPOSITION 5.3 *Let I denote the Brown–Comenetz dual of S. Then $\pi_* I$ is the injective hull of $\mathbb{F}_p$ as a $\pi_* S$-module.*

The same argument as in the following proof shows that $\pi_* I L_n S$ is an injective $\pi_* L_n S$-module for any n.

Proof. Let $\pi_* I = \operatorname{Hom}_{\mathbb{Z}_{(p)}}(\pi_* S, \mathbb{Q}/\mathbb{Z}_{(p)})$. As $\mathbb{Q}/\mathbb{Z}_{(p)}$ is an injective abelian group, $\pi_* I$ is injective by a standard result about injective modules [**15**, Corollary 3.6B]. The obvious inclusion $\mathbb{F}_p \to \pi_0 I \to \pi_* I$ is obviously a map of $\pi_* S$-modules. The action of $\pi_* S$ on $\pi_* I$ is given by

$$\mu\colon\ \pi_k S \otimes \operatorname{Hom}_{\mathbb{Z}_{(p)}}\left(\pi_n S, \frac{\mathbb{Q}}{\mathbb{Z}_{(p)}}\right) \to \operatorname{Hom}\left(\pi_{n-k} S, \frac{\mathbb{Q}}{\mathbb{Z}_{(p)}}\right),$$

where $\mu(x \otimes f)(y) = f(xy)$. In particular, if f is a non-trivial element of $\pi_{-n} I = \operatorname{Hom}(\pi_n S, \mathbb{Q}/\mathbb{Z}_{(p)})$, then there is an $x \in \pi_n S$ such that $f(x)$ is a non-zero element of $\mathbb{F}_p \subseteq \mathbb{Q}/\mathbb{Z}_{(p)}$. It follows that $\mu(x \otimes f)$ is a non-zero element of $\mathbb{F}_p$ and, therefore, that $\mathbb{F}_p \to \pi_* I$ is an essential extension of $\pi_* S$-modules.

Note that it is tempting to conclude from Proposition 5.3 that $\mathbb{F}_p$ has injective dimension 1 as a $\pi_* S$-module, which is wrong, however. The cokernel of $\mathbb{F}_p \to \pi_* I$ is isomorphic as a graded abelian group to $\pi_* I$ but not as a $\pi_* S$-module.

6. Infinitely generated homotopy

Whitehead realized that the generating hypothesis implies that the homotopy of a finite complex Y is not finitely generated over $\pi_* S$ unless Y is a finite coproduct of spheres [**4**, Proposition 9.5]. The proof of this fact is so easy that it is recalled here. Suppose Y has finitely generated homotopy, so that there is a cofibre sequence

$$F \xrightarrow{f} W \xrightarrow{g} Y \xrightarrow{h} \Sigma F,$$

where W is a finite wedge of spheres and $\pi_*(g)$ is surjective. Then $\pi_* h = 0$, so, by the generating hypothesis, $h = 0$. Hence, Y is a retract of W, so $\pi_* Y$ is projective and hence free. Thus, Y is itself a wedge of spheres.

Kahn [**11**] has shown that, for any finite spectrum Y, it is possible to attach two cells (one if Y is not torsion) to Y to get a new complex Z with $\pi_* Z$ not finitely generated. Thus, there are many finite spectra whose homotopy is not finitely generated.

The existence of v_n self-maps can be used to prove Theorem 1.5, which says that if X is a finite type n spectrum with $n > 1$ such that the map $\pi_* X \to \pi_* L_n X$ is non-zero, then $\pi_* X$ is not finitely generated.

Proof of Theorem 1.5. By the nilpotence theorem [**7**], X has a non-nilpotent self-map v of positive degree. This map can be assumed to have the Adams–Novikov filtration 0 [**8**, Theorem 4.6]. Let $\mathrm{AN}(\alpha)$ denote the Adams–Novikov filtration of an element $\alpha \in \pi_* X$. Choose element $\beta \in \pi_* X$ such that $\lim_{k\to\infty} \mathrm{AN}(v^k \beta)$ is minimal. Unfortunately, to do this, it is essential to know that there exists a β such that $\lim_{k\to\infty} \mathrm{AN}(v^k \beta)$ is finite. To see this, note that if this limit is not finite, then the analogous limit for the $E(n)$-Adams filtration is also infinite, as $E(n)$ is a BP-module spectrum. However, the $E(n)$-based Adams–Novikov spectral sequence for $L_n X$ converges strongly and has a horizontal vanishing line at the E_∞ term by [**8**, Proposition 6.5]. Hence, the image of $v^k \beta$ in $\pi_* L_n X$ must be zero; as v acts as a unit on $L_n X$, some β must map to 0 in $\pi_* L_n X$. Therefore, if β that does not map to 0 in $\pi_* L_n X$ is chosen, then $\lim_{k\to\infty} \mathrm{AN}(v^k \beta)$ is finite.

So now β is chosen such that $\lim_{k\to\infty} \mathrm{AN}(v^k \beta)$ is minimal. Choose a generating set $\{x_i\}$ for $\pi_* X$ as a $\pi_* S$-module and write

$$\beta = x_1 \circ \alpha_1 + \cdots + x_r \circ \alpha_r$$

for some $\alpha_j \in \pi_* S$. Then, for large k,

$$v^k \circ \beta = v^k \circ x_i \circ \alpha_i + \cdots + v^k \circ x_r \circ \alpha_r,$$

and $v^k \beta$ will have the least Adams–Novikov filtration among all the $v^k x_i$. This implies that there must be an i with α_i non-zero such that the Adams–Novikov filtration of $v^k x_i$ is the same as that of $v^k \beta$. Hence, α_i has the Adams–Novikov filtration 0, so is in $\pi_0 S$. Therefore, the degree of x_i is the same as the degree of β. By repeating the argument on $v^j \beta$, it is seen that there must be a generator of $\pi_* X$ in the degree of $v^j \beta$ for all $j \geq 0$. Thus, $\pi_* X$ is not finitely generated.

Now, the statement of Theorem 1.5 included the claim that the theorem holds when X is a μ-spectrum. This follows because if X is a μ-spectrum, then there is a unit $\eta\colon S \to X$ and a multiplication $\mu\colon X \wedge X \to X$ such that $\mu \circ (\eta \wedge 1)$ is the identity. In particular, if η went to 0 in $\pi_* L_n X$, then $L_n X$ itself would be zero, which is false because X is type n.

7. Generating hypothesis and thick subcategories

One difficulty that the generating hypothesis has always posed is that knowing the generating hypothesis with target Y does not seem to say very much about the generating hypothesis with other targets. Freyd's work does imply, however, that if the generating hypothesis with target Y is true for all finite torsion spectra Y, then it is true for all finite Y (this can be obtained from the proof of [**4**, Theorem 9.9]). In this section, Freyd's result is extended to finite spectra of type at least n.

PROPOSITION 7.1 *Suppose X is a type n finite spectrum for some n, with v_n self-map v. Let X/v^k denote the cofibre of v^k and consider the cofibre sequence*

$$Z \xrightarrow{\delta} X \to \prod_{k\geq 1} \frac{X}{v^k}.$$

Then, δ is a phantom map.

Proof. Suppose first that $n = 0$, so that $v = p$. If F is a finite spectrum, the group $[F, X]$ is finitely generated abelian, and, therefore, any $f\colon F \to X$ is not divisible by p^k for large enough k. Hence, the image of f in $[F, X/p^k]$ is non-zero for large enough k. Thus,

$$[F, X] \to \left[F, \prod \frac{X}{p^k}\right]$$

is a monomorphism, and so δ is phantom.

Now suppose $n \geq 1$, so that the map v has some positive degree d [**7**]. Let F be a finite spectrum, and suppose $f\colon F \to X$ is a non-trivial map. Then, the composite $F \to X \to X/v^k$ is non-trivial for some k. Indeed, if not, then f factors through $\Sigma^{dk}X$ for all k. For k large enough, every cell of F will be in a lower degree than all the cells of $\Sigma^{dk}X$, and so $[F, \Sigma^{dk}X] = 0$ and $f = 0$. Thus,

$$[F, X] \to \left[F, \prod \frac{X}{v^k}\right]$$

is a monomorphism, and so δ is phantom.

COROLLARY 7.2 *Suppose X is a type n finite spectrum for some n, with v_n self-map v. Then X_p is a retract of $\prod_{k\geq 1} X/v^k$.*

Proof. Recall that the completion X_p is the same as the Bousfield localization $L_{M(p)}$. The space $\prod X/v^k$ is already $L_{M(p)}$-local because each X/v^k is so. Hence there is cofibre sequence

$$L_{M(p)}Z \xrightarrow{L_{M(p)}\delta} L_{M(p)}X \to \prod_{k\geq 1} \frac{X}{v^k}.$$

The map $L_{M(p)}\delta$ is determined by its restriction to Z, which is phantom by Proposition 7.1. As there are no phantom maps to X_p, it is concluded that $L_{M(p)}\delta = 0$, giving the desired splitting.

COROLLARY 7.3 *Fix $n \geq 0$. The generating hypothesis with target Y is true for all finite spectra Y if and only if it is true for all finite Y of type at least n.*

Corollary 7.3 is as close as the author can come to showing that the collection of all Y for which the generating hypothesis with target Y is true is a thick subcategory.

Proof. It is enough to show that if the generating hypothesis with target Y is true for all finite Y of type at least k, then the generating hypothesis with target Y is true for all finite Y of type at least $k-1$. Suppose X has type $k-1$. Choose a v_{k-1} self-map v of X. By Corollary 7.2, X_p is a summand in $\prod X/v^k$. Each X/v^k has type k, and so $\pi_* X/v^k$ is an injective $\pi_* S$-module, by Theorem 1.2. It follows that $\pi_* X_p$ is an injective $\pi_* S$-module, and so the generating hypothesis with target X_p is true. But then Proposition 2.4 implies that the generating hypothesis with target X is true.

Another interesting corollary of Proposition 7.1 is the following. Let $\mathcal{C}_n$ denote the thick subcategory of finite spectra whose type is at least n.

COROLLARY 7.4 *The subcategory $\mathcal{C}_n$ generates and cogenerates the category of finite spectra.*

This means that, given a non-zero map $f\colon X \to Y$ of finite spectra, there are maps $g\colon Z \to X$ and $h\colon Y \to W$ with $Z, W \in \mathcal{C}_n$ and $f \circ g$ and $h \circ f$ both non-zero. This corollary was proved by Freyd [**4**, Proposition 6.8] in the case $n = 1$.

Proof. Let $f\colon X \to Y$ be a non-zero map. Suppose Y is of type k. Then it follows from Proposition 7.1 that there is a Z of type $k+1$, namely Y/v^r for large r, and a map $h\colon Y \to Z$ such that hf is non-zero. We can then proceed by induction to see that $\mathcal{C}_n$ cogenerates the category of finite spectra.

Given this, consider the Spanier–Whitehead dual Df of f. There is a spectrum V of type at least n and a map $k\colon DX \to V$ such that $k \circ Df$ is non-zero. Dualizing, it is seen that $f \circ Dk$ is non-zero, and DV also has type at least n.

Acknowledgements

The author thanks Dan Christensen and Don Kahn for some useful education about the generating hypothesis. The author also thanks the referee for Lemma 2.3, which replaces the author's more complicated approach to some of the results of the article.

References

1. E. S. Devinatz, K-theory and the generating hypothesis, *Amer. J. Math.* **112** (1990), 787–804. MR1073009 (91i:55011).
2. E. S. Devinatz, Small ring spectra, *J. Pure Appl. Algebra* **81** (1992), 11–16. MR1173820 (93g:55012).
3. E. S. Devinatz, The generating hypothesis revisited, *Stable and Unstable Homotopy* (Toronto, ON, 1996), Fields Institute Communications 19, American Mathematical Society, Providence, 1998, pp. 73–92. MR1622339 (99f:55016).
4. P. Freyd, Stable homotopy, *Proceedings of Conference on Categorical Algebra* (La Jolla, California, 1965), Springer, New York, 1966, pp. 121–172. MR0211399 (35 no. 2280).

5. I. Gordon and J. T. Stafford, Rational Cherednik algebras and Hilbert schemes, *Adv. in Math.* **198** (2005), 222–274. MR2183255.

6. M. J. Hopkins and D. C. Ravenel, Suspension spectra are harmonic, *Bol. Soc. Mat. Mexicana* **37**, 1, 2 (1992), 271–279, Papers in honor of José Adem (Spanish). MR1317578 (95j:55024).

7. M. J. Hopkins and J. H. Smith, Nilpotence and stable homotopy theory. II, *Ann. of Math.* **148** (1998), 1–49. MR1652975 (99h:55009).

8. M. Hovey and N. P. Strickland, *Morava K-Theories and Localisation*, Memoir 139, 666, American Mathematical Society, Providence, 1999, viii + 100. MR1601906 (99b:55017).

9. M. Hovey, J. H. Palmieri and N. P. Strickland, *Axiomatic Stable Homotopy Theory*, Memoir 128, 610, American Mathematical Society, Providence, 1997, x + 114. MR1388895 (98a:55017).

10. C. U. Jensen, On homological dimensions of rings with countably generated ideals, *Math. Scand.* **18** (1966), 97–105. MR0207796 (34 no. 7611).

11. D. W. Kahn, On stable homotopy modules. II, *Invent. Math.* **7** (1969), 344–353. MR0246293 (39 no. 7597).

12. D. W. Kahn, Stable spectral sequences and their applications, *Amer. J. Math.* **94** (1972), 1131–1154. MR0310876 (46 no. 9974).

13. D. W. Kahn, Relations in stable homotopy modules, *Proc. Amer. Math. Soc.* **40** (1973), 253–259. MR0436139 (55 no. 9089).

14. I. Kaplansky, Projective modules, *Ann. of Math*. **68** (1958), 372–377. MR0100017 (20 no. 6453).

15. T. Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics 189, Springer, New York, 1999. MR1653294 (99i:16001).

16. T. Y. Lin, Adams type spectral sequences and stable homotopy modules, *Indiana Univ. Math. J.* **25** (1976), 135–158. MR0391086 (52 no. 11908).

17. K. Lockridge, *The generating hypothesis in the derived category of R-modules*, preprint, 2005.

18. D. C. Ravenel, Localization with respect to certain periodic homology theories, *Amer. J. Math.* **106** (1984), 351–414. MR737778 (85k:55009).