THINKING ABOUT THE FREYD CONJECTURE

ANNA MARIE BOHMANN AND J.P. MAY

Abstract. We give a generalized version of the Freyd conjecture and a way to think about a possible proof. There are no new results yet.

Contents

1. The generalized Freyd conjecture

There are several sensible contexts in which we can work. We minimize initial assumptions by starting with just a triangulated category $\mathscr{T}$. Write $[X,Y]$ for the Abelian group of maps $X \longrightarrow Y$ in $\mathscr{T}$. Let $\mathscr{B}$ be a (small) full subcategory of $\mathscr{T}$ closed under its translation functor Σ and let $\mathscr{C}$ be the thick full subcategory of $\mathscr{T}$ generated by $\mathscr{B}$; write $\iota\colon \mathscr{B} \longrightarrow \mathscr{C}$ for the inclusion. For emphasis, we often write $\mathscr{B}(X,Y) = [X,Y]$ when $X,Y \in \mathscr{B}$ and $\mathscr{C}(X,Y) = [X,Y]$ when $X,Y \in \mathscr{C}$. The category $\mathscr{B}$ is pre-additive (enriched over $\mathscr{Ab}$), and $\mathscr{C}$ is additive (has biproducts). Let $\mathscr{P}\mathscr{B}$ and $\mathscr{P}\mathscr{C}$ denote the categories of Abelian presheaves defined on $\mathscr{B}$ and $\mathscr{C}$. They consist of the additive functors from $\mathscr{B}^{op}$ or $\mathscr{C}^{op}$ to $\mathscr{Ab}$ and the additive natural transformations between them.

Definition 1.1. Define the Freyd functor $\mathbb{F}\colon \mathscr{T} \longrightarrow \mathscr{P}\mathscr{B}$ by sending an object X to the functor $\mathbb{F}X$ specified on objects and morphisms of $\mathscr{B}$ by $\mathbb{F}X(-) = [-,X]$ and sending a map $f\colon X \longrightarrow Y$ to the natural transformation $f_* = [-,f]$. Define $\mathbb{Y}\colon \mathscr{T} \longrightarrow \mathscr{P}\mathscr{C}$ similarly.

We are interested only in the restrictions of $\mathbb{F}$ and $\mathbb{Y}$ to $\mathscr{C}$. Then $\mathbb{Y}$ becomes the standard Yoneda embedding $\mathscr{C} \longrightarrow \mathscr{P}\mathscr{C}$.

Conjecture 1.2 (The generalized Freyd conjecture)**.** *The functor $\mathbb{F}\colon \mathscr{C} \longrightarrow \mathscr{P}\mathscr{B}$ is faithful. Equivalently, $\mathbb{F}f = 0$ if and only if $f = 0$. We then say that the Freyd conjecture holds for the pair $(\mathscr{C}, \mathscr{B})$.*

We hasten to add that the conjecture is definitely false in this generality. Some additional hypotheses are certainly needed, but none will be relevant to our formal analysis. For example, we might assume that a map $f\colon X \longrightarrow Y$ in $\mathscr{C}$ is a weak equivalence if and only $f_*\colon \mathbb{F}X \longrightarrow \mathbb{F}Y$ is an isomorphism, although that is still

Date: July 13, 2008.

not sufficient in general. This hypothesis holds in the motivating example of stable homotopy theory.

Example 1.3. Take $\mathscr{T} = \mathrm{Ho}\mathscr{S}$ to be the stable homotopy category. Let $\mathscr{B}$ consist of the sphere spectra S^n for integers n. Then $\mathscr{C}$ is the homotopy category of finite CW spectra. Freyd [1] conjectured that a map f in $\mathscr{C}$ is zero if and only if it induces the zero homomorphism $f_*\colon \pi_*(X) \longrightarrow \pi_*(Y)$. By the following observation, this is a special case of our Conjecture 1.2.

Lemma 1.4. *In Example 1.3, the category $\mathscr{P}\mathscr{B}$ is isomorphic to the category $\mathscr{M}$ of left modules over the ring π_* of stable homotopy groups of spheres. Under this isomorphism, the Freyd functor $\mathbb{F}$ coincides with the stable homotopy group functor $\pi_*\colon Ho\mathscr{S} \longrightarrow \mathscr{M}$.*

Proof. For $T \in \mathscr{P}\mathscr{B}$, let $M_n = T(S^n)$. For an element $x \in \pi_q$, thought of by suspension as a map $x\colon S^{n+q} \longrightarrow S^n$, and an element $y \in M_n$, define the module product xy to be $T(x)(y) \in M_{n+q}$. The contravariant functoriality of T ensures that M is a left π_*-module. Conversely, given a left π_*-module M, define $T(S^n) = M_n$ and define $T(x)(y) = xy$. The module axioms ensure that T is a functor. This specifies the isomorphism of categories, and the consistency of $\mathbb{F}$ and π_* is clear. □

The following example originally prompted us to take a presheaf perspective on the Freyd conjecture.

Example 1.5. Let G be a compact Lie group and take $\mathscr{T} = \mathrm{Ho}G\mathscr{S}$ to be the equivariant stable homotopy category. Let $G\mathscr{B}$ consist of the orbit G-spectra $S^n_H \equiv \Sigma^n\Sigma^\infty(G/H)_+$ for integers n and closed subgroups H of G. The thick subcategory $G\mathscr{C}$ generated by $G\mathscr{B}$ is the category of retracts of finite G-CW spectra. The equivariant version of Freyd's conjecture asserts that a map f in $G\mathscr{C}$ is zero if and only if it induces the zero homomorphism $f_*\colon \pi^H_*(X) \longrightarrow \pi^H_*(Y)$ for all H, where

$$\pi^H_n(X) \equiv \pi_n(X^H) \cong [S^n_H, X]_G.$$

Again, this is a special case of our Conjecture 1.2.

Remark 1.6. The full subcategory $\mathscr{B}_0$ of $\mathscr{B}$ whose objects are the $S_H \equiv S^0_H$ is called the Burnside category. A Mackey functor (or G-Mackey functor) M is precisely an object of $\mathscr{P}G\mathscr{B}_0$. When G is finite, this agrees with the more usual algebraic definition [3, V§9] or [5, IX§4, XIX§3].

Definition 1.7. Define the graded Burnside category $G\pi_*$ to have objects the S_H and morphisms of degree q from S_H to S_J the elements of $\pi^H_q(S_J) = [S^q_H, S_J]_G$. Composition is induced by suspension and composition in $G\mathscr{B}$ in the evident fashion. Define a left $G\pi_*$-module M to consist of Abelian groups M^H_n together with homomorphisms $\pi^H_q(S_J) \otimes M^J_n \longrightarrow M^H_{n+q}$ satisfying the evident module identities.

If the module identities are written diagrammatically, one must write composition in the form

$$[S^{q+r+n}_H, S^{q+n}_J]_G \otimes [S^{q+n}_J, S^n_K]_G \longrightarrow [S^{q+r+n}_H, S^n_K]_G,$$

which is opposite to the usual convention. This is already familiar in the nonequivariant situation, where it corresponds to the sign difference in the product on π_* when one compares the composition product to the smash product.

Lemma 1.8. *In Example 1.5, the category $\mathscr{P}G\mathscr{B}$ is isomorphic to the category $G\mathscr{M}$ of left $G\pi_*$-modules. Under this isomorphism, the Freyd functor $\mathbb{F}$ coincides with the equivariant stable homotopy group functor $\pi_*^{(-)}: HoG\mathscr{S} \longrightarrow G\mathscr{M}$.*

Thus, in this case, the presheaf formulation of the Freyd conjecture appears more natural than the equivalent homotopy group reformulation. In fact, the presheaf perspective suggests a method of attack on the generalized Freyd conjecture. We have the forgetful functor

$$\mathbb{U} = \iota^* : \mathscr{P}\mathscr{C} \longrightarrow \mathscr{P}\mathscr{B}$$

given by restricting presheaves defined on $\mathscr{C}^{op}$ to the full subcategory $\mathscr{B}^{op}$. The functor $\mathbb{U}$ has a left adjoint prolongation functor

$$\mathbb{P} = \iota_! : \mathscr{P}\mathscr{B} \longrightarrow \mathscr{P}\mathscr{C}.$$

For $T \in \mathscr{P}\mathscr{B}$ and $K \in \mathscr{C}$, $\mathbb{P}T(K)$ is the categorical tensor product

$$\mathbb{P}T(K) = T \otimes_{\mathscr{B}} \mathscr{C}(K, -).$$

See, for example, [**?**, I§3] or [4, I§2]. Since $\mathscr{B}$ is a full subcategory of $\mathscr{C}$, the unit $\mathrm{Id} \longrightarrow \mathbb{U}\mathbb{P}$ of the adjunction is a natural isomorphism [**?**, I.3.2]. We focus attention on the counit $\varepsilon: \mathbb{P}\mathbb{U} \longrightarrow \mathrm{Id}$.

Observe that the Freyd functor $\mathbb{F}: \mathscr{T} \longrightarrow \mathscr{P}\mathscr{B}$ is the composite $\mathbb{U}\mathbb{Y}$. This leads to the following observation.

Proposition 1.9. *The Freyd conjecture holds for $(\mathscr{C}, \mathscr{B})$ if $\varepsilon: \mathbb{P}\mathbb{F}X \longrightarrow \mathbb{Y}X$ is an isomorphism for all $X \in \mathscr{C}$.*

Proof. Since the unit of the adjunction $(\mathbb{P}, \mathbb{U})$ is an isomorphism, $\mathbb{U}\varepsilon$ is an isomorphism by one of the triangle identities. Thus $\mathbb{F} \cong \mathbb{U}\mathbb{P}\mathbb{F}$. Therefore, for a map $f: X \longrightarrow Y$ in $\mathscr{C}$, $\mathbb{F}f = 0$ if and only if $\mathbb{P}\mathbb{F}f = 0$. By the Yoneda lemma, $f = 0$ if and only if $\mathbb{Y}f = 0$. If ε is an isomorphism, then $\mathbb{P}\mathbb{F}f = 0$ if and only if $\mathbb{Y}f = 0$. □

In fact, less is needed. Consider $\mathbb{Y}X(X) = [X, X]$. We will shortly prove the following result.

Proposition 1.10. *The Freyd conjecture holds for $(\mathscr{C}, \mathscr{B})$ if the identity map of X is in the image of ε for all $X \in \mathscr{C}$.*

We have the following starting point towards verification of the hypothesis.

Lemma 1.11. *The map $\varepsilon: \mathbb{P}\mathbb{F}X(J) \longrightarrow \mathbb{Y}X(J)$ is an isomorphism for all $X \in \mathscr{T}$ and $J \in \mathscr{B}$.*

Proof. We have seen that $\mathbb{U}\varepsilon$ is an isomorphism, and by definition $\mathbb{U}T(J) = T(J)$ for any $J \in \mathscr{B}$ and any $T \in \mathscr{P}\mathscr{C}$. □

Now consider an exact triangle

$$K \longrightarrow L \longrightarrow M \longrightarrow \Sigma K \tag{1.12}$$

in $\mathscr{T}$, where K, L, and M are in $\mathscr{C}$. We have the commutative diagram

(1.13)

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathbb{P}\mathbb{F}X(\Sigma K) & \longrightarrow & \mathbb{P}\mathbb{F}X(M) & \longrightarrow & \mathbb{P}\mathbb{F}X(L) & \longrightarrow & \mathbb{P}\mathbb{F}X(K) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} & \\
\cdots \longrightarrow & \mathbb{Y}X(\Sigma K) & \longrightarrow & \mathbb{Y}X(M) & \longrightarrow & \mathbb{P}\mathbb{F}X(L) & \longrightarrow & \mathbb{Y}X(K) & \longrightarrow \cdots .
\end{array}$$

Since $\mathbb{Y}X(K) = [K, X]$, the lower row is exact. By definition, $\mathscr{C}$ is the smallest subcategory of $\mathscr{T}$ that contains $\mathscr{B}$ and has the properties that a retract of an object of $\mathscr{C}$ is in $\mathscr{C}$ and that if two terms of an exact triangle are in $\mathscr{C}$ then so is the third. By an obvious retract argument and the five lemma, this gives the following conclusion.

Proposition 1.14. *If the top row of (1.13) is exact for every exact triangle (1.12) and every $X \in \mathscr{C}$, then $\varepsilon\colon \mathbb{P}\mathbb{F}X \longrightarrow \mathbb{Y}X$ is an isomorphism for every $X \in \mathscr{C}$ and the Freyd conjecture holds for $(\mathscr{C}, \mathscr{B})$.*

There is a reinterpretation of the exactness hypothesis that makes it reminiscent of the standard result that the adjoint of an exact functor between triangulated categories is exact. For K and X in $\mathscr{C}$, the Abelian group $\mathbb{P}\mathbb{F}X(K)$ is the tensor product of functors displayed in the evident coequalizer diagram (in $\mathscr{A}b$)

$$\begin{array}{c} \sum_{I,J\in\mathscr{B}} \mathscr{C}(J,X)\otimes\mathscr{B}(I,J)\otimes\mathscr{C}(K,I) \\ \downarrow\downarrow \\ \sum_{I\in\mathscr{B}} \mathscr{C}(J,X)\otimes\mathscr{C}(K,J) \\ \downarrow \\ \mathscr{C}(-,X)\otimes_{\mathscr{B}}\mathscr{C}(K,-). \end{array} \tag{1.15}$$

We use this to interpolate the proof of Proposition 1.10. The composition maps

$$\circ\colon \mathscr{C}(X,Y)\otimes\mathscr{C}(J,X) \longrightarrow \mathscr{C}(J,Y)$$

induce a pairing

$$\circ\colon \mathscr{C}(X,Y)\otimes\mathbb{P}\mathbb{F}X(K) \longrightarrow \mathbb{P}\mathbb{F}Y(K)$$

such that $f\circ z = \mathbb{P}\mathbb{F}f(z)$ for $z \in \mathbb{P}\mathbb{F}X(K)$ and the following diagram commutes:

$$\begin{array}{ccc} \mathscr{C}(X,Y)\otimes\mathbb{P}\mathbb{F}X(K) & \xrightarrow{\mathrm{id}\otimes\varepsilon} & \mathscr{C}(X,Y)\otimes\mathbb{Y}X(K) \\ \circ\downarrow & & \downarrow\circ \\ \mathbb{P}\mathbb{F}Y(K) & \xrightarrow[\varepsilon]{} & \mathbb{Y}Y(K). \end{array}$$

Take $K = X$ and suppose that $\varepsilon(z) = \mathrm{id}_X$. If $\mathbb{F}f = 0$, then $\varepsilon(f\circ z) = \varepsilon\mathbb{P}\mathbb{F}f(z) = 0$. By the diagram, this equals $f\circ\varepsilon(z) = f$ and so $f = 0$.

Let us write $\mathscr{P}_d\mathscr{B}$ and $\mathscr{P}_d\mathscr{C}$ for the categories of covariant additive functors on $\mathscr{B}$ and $\mathscr{C}$, and similarly write $\mathbb{U}_d$, $\mathbb{P}_d$, $\mathbb{F}_d$, and $\mathbb{Y}_d$ for the corresponding functors. We are just interchanging $\mathscr{B}$ and $\mathscr{C}$ with their opposite categories. Visibly, we again have $\mathbb{U}_d\mathbb{Y}_d = \mathbb{F}_d$ and again have an adjunction $(\mathbb{P}_d, \mathbb{U}_d)$ with $\mathbb{U}_d\mathbb{P}_d \cong \mathrm{Id}$. By symmetry, we have

$$\mathbb{P}\mathbb{F}X(K) = \mathbb{P}_d\mathbb{F}_dK(X). \tag{1.16}$$

But in this dual reformulation, the exactness hypothesis on K for fixed X is now a levelwise exactness statement about the composite functor $\mathbb{P}_d\mathbb{F}_d\colon \mathscr{C} \longrightarrow \mathscr{P}_d\mathscr{C}$. Since $\mathbb{F}_dK(J) = \mathscr{C}(K,J)$ for $J \in \mathscr{B}$, $\mathbb{F}_d$ clearly takes cofiber sequences in the variable $K \in \mathscr{C}$ to exact sequences for each fixed J. Thus a more general question to ask is whether or not $\mathbb{P}_d\colon \mathscr{P}_d\mathscr{B} \longrightarrow \mathscr{P}_d\mathscr{C}$ preserves levelwise exactness. That is, is it true that if $T' \longrightarrow T \longrightarrow T''$ is a sequence of diagrams $\mathscr{B} \longrightarrow \mathscr{A}b$ such that the sequence $T'(J) \longrightarrow T(J) \longrightarrow T''(J)$ is exact for all $J \in \mathscr{B}$, then the sequence $\mathbb{P}T'(X) \longrightarrow \mathbb{P}T(X) \longrightarrow \mathbb{P}T''(X)$ is exact for all $X \in \mathscr{C}$?

Observe that we have not yet used any hypothesis on $\mathscr{B}$, other than that it generates the thick subcategory $\mathscr{C}$. Thus all we have done is to give a purely formal reduction of the general problem.

2. The Freyd embedding

I've just skimmed Freyd's paper for the first time in many, many years, and jotted down some notes as I went: we should modernize and recall this:

Recall the construction of the embedding $\iota\colon \mathscr{T} \longrightarrow \mathscr{F}$ into an Abelian category. A triangulated category has weak kernels and cokernels, by the fill-in axiom, so the construction applies. Restate Freyd's basic result (his Theorem 3.1, page 128). Probably also his description of kernels (Lemma 3.1.4, page 129).

Recall the definition of a homology functor on a triangulated category (from Hovey, Palmieri, Strickland, 1.1.3, but don't confuse issues by calling exact triangles cofiber sequences). Now a generalization of Freyd's Lemma 4.1, page 133, is relevant: its proof is a preservation of epimorphism result, as you assumed: should state that formally.

The other thing we need, Freyd conj for $\mathscr{C}$ implies Freyd conj for $\mathscr{F} = \mathscr{F}_{\mathscr{C}}$ is clear. For any X in $\mathscr{F}$, there is an epi $g\colon Z \longrightarrow X$ with $Z \in \mathscr{C}$. If $f\colon X \longrightarrow Y$ is a map in $\mathscr{F}$ with $X \in \mathscr{F}$ and $Y \in \mathscr{C}$, then $\mathbb{F}f = 0$ implies $\mathbb{F}(f \circ g) = 0$, which implies $f \circ g = 0$ which implies $f = 0$.

Should also see how Freyd's Corollary 4.7, page 135, should generalize to give that $\mathscr{C}(X,Y)$ is finitely generated for all $X, Y \in \mathscr{C}$ (this will actually fit better in the next section — I know how it should go). The Lemma and Corollary 4.10 might generalize, but that would need figuring out what the right generalized version of ordinary homology means. (In any case, as it stands, this can be used as Freyd used it to avoid the theory of rationalization of spectra in proving that full implies faithful).

Freyd's Corollary 5.2, page 138, generalizes, assuming Prop. 5.1, and it is worth stating. I think Thm 5.3, page 139, also generalizes and is interesting.

Prop. 5.4, page 140, is cryptic and perhaps worth remarking. I think A in it is (or can be) any Abelian category, and the point is just that the top and left bottom horizontal arrows are iso's, while the right bottom horizontal arrow is not, so T can't be an iso on both left and right.

Let $\iota\colon \mathscr{T} \longrightarrow \mathscr{F}$ be the Freyd embedding, and identify $\mathscr{B}$ and $\mathscr{C}$ with their images in $\mathscr{F}$. Something should be said about the relationship between $\mathscr{C}$ and $\mathscr{D}$, where $\mathscr{D} \subset \mathscr{F}$ is the thick subcategory of $\mathscr{F}$ generated by $\mathscr{B}$. Should be true that $\mathscr{C} \subset \mathscr{D}$, but even that much is not quite clear to me. Presumably $\mathscr{D}$ is larger.

3. When does faithful imply full?

Conjecture 1.2 is the weak form of the Freyd conjecture. The strong form reads as follows.

Conjecture 3.1. *The functor $\mathbb{F}\colon \mathscr{C} \longrightarrow \mathscr{P}\mathscr{B}$ is full and faithful.*

Freyd [1, 9.7] showed that faithful implies full in the case of the stable homotopy category. That implication is not always true, but we show that it is true for the equivariant stable homotopy category when G is finite. We do not know whether or not this implication holds more generally when G is a compact Lie group. We explain here how to start the proof in general and how to complete the proof in the cases just cited. We begin by restricting our general context to a form closer to that encountered in our homotopical examples.

Definition 3.2. We say that $\mathscr{B}$ is graded if its set of objects is the disjoint union of subsets B_n for integers n, where $B_n = \Sigma^n B_0$ is the set of objects $\Sigma^n B$ for $B \in \mathscr{B}_0$. We require that $[\Sigma^n A, B] = 0$ for all $n < 0$ and all $A, B \in \mathscr{B}_0$. We say that $\mathscr{B}$ is of finite type if $[\Sigma^n A, B]$ is a finitely generated Abelian group for all $A, B \in \mathscr{B}_0$. For an object $X \in \mathscr{T}$, we define a presheaf $\mathbb{F}_n X \in \mathscr{P}\mathscr{B}_0$ by $\mathbb{F}_n X(B) = [\Sigma^n B, X]$ for $B \in \mathscr{B}_0$. Then the presheaf $\mathbb{F}X$ can be thought of as the graded presheaf $\mathbb{F}_* X$ consisting of the set of presheaves $\mathbb{F}_n X$.

The $\mathbb{F}_n X$ are the appropriate analogues of homotopy groups, and we have a general analogue of Lemma 1.4 and Example 1.5. That is, we have a graded category whose objects are the $B \in \mathscr{B}_0$ and whose morphisms of degree q from A to B are the elements of $\mathscr{B}(\Sigma^q A, B)$. The category of left modules over this graded category is isomorphic to the category $\mathscr{P}\mathscr{B}$.

Lemma 3.3. *If $\mathscr{B}$ is graded of finite type and $X, Y \in \mathscr{C}$, then $\mathscr{C}(X, Y)$ is a finitely generated Abelian group. Moreover, $\mathscr{C}(\Sigma^j B, Y) = 0$ for all $B \in \mathscr{B}_0$ and all sufficiently small j.*

Proof. The usual inductive argument for thick categories applies. □

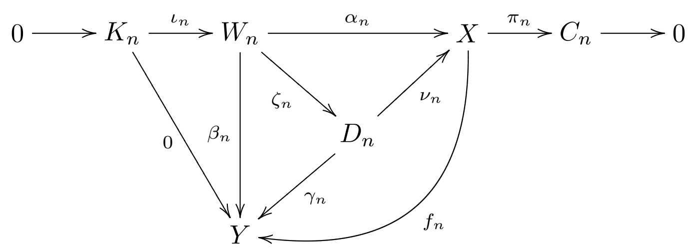
Assume that $\mathscr{B}$ is graded of finite type. Let $\phi\colon \mathbb{F}X \longrightarrow \mathbb{F}Y$ be a map of presheaves, where $X, Y \in \mathscr{C}$. To show that the functor $\mathbb{F}$ is full, we need to find a map $f\colon X \longrightarrow Y$ in $\mathscr{C}$ such that $\phi = \mathbb{F}f$. For each fixed j and each $B \in B_0$, choose a finite set $\{\alpha_{j,B}\}$ of generators for the group $\mathbb{F}_j X(B) = [\Sigma^j B, X]$. Let W_n be the coproduct of a copy of $\Sigma^j B$ for each $\alpha_{j,B}$, where $j \leq n$. This coproduct exists in $\mathscr{C}$ since there are finitely many summands.

Use the $\alpha_{j,B}$ to define a map $\alpha_n\colon W_n \longrightarrow X$. Then let $\beta_n\colon W_n \longrightarrow Y$ be the map in $\mathscr{C}$ obtained by adding up the composites $\phi(\alpha_{j,B})$. Let $\iota_n\colon K_n \longrightarrow W_n$ be a kernel of α_n in $\mathscr{F} = \mathscr{F}_{\mathscr{C}}$. By construction and the fact that $\alpha_n \circ \iota_n = 0$ in $\mathscr{F}$, the extension of $\mathbb{F}$ to $\mathscr{F}$ satisfies

$$\mathbb{F}(\beta_n \circ \iota_n) = \mathbb{F}(\beta_n) \circ \mathbb{F}(\iota_n) = \phi\mathbb{F}(\alpha_n)\mathbb{F}(\iota_n) = \phi\mathbb{F}(0) = 0.$$

There is an epimorphism $\xi\colon Z \longrightarrow K_n$ in $\mathscr{F}$ where Z is in $\mathscr{C}$, and then $\mathbb{F}(\beta_n \circ \iota_n \circ \xi)$ is zero. Since $\beta_n \circ \iota_n \circ \xi\colon X \longrightarrow Y$ is a map in the full subcategory $\mathscr{C}$ of $\mathscr{F}$ and we are assuming the Freyd conjecture, $\beta_n \circ \iota_n \circ \xi$ is zero in $\mathscr{F}$. Since ξ is an epimorphism, this implies that $\beta_n \circ \iota_n\colon K_n \longrightarrow Y$ is zero in $\mathscr{F}$.

Let $\zeta_n\colon W_n \longrightarrow D_n$ be a cokernel of ι_n. Since ι_n is a kernel of α_n, α_n factors through a monomorphism $\nu_n\colon D_n \longrightarrow X$. Since $\beta_n \circ \iota_n = 0$, β_n factors through a map $\gamma_n\colon D_n \longrightarrow Y$. Since Y is injective in $\mathscr{F}$, there is a map $f_n\colon X \longrightarrow Y$ such that $f_n \circ \nu_n = \gamma_n$. Also, let $\pi_n\colon X \longrightarrow C_n$ be a cokernel of α_n. We summarize these definitions in the following commutative diagram in $\mathscr{F}$.

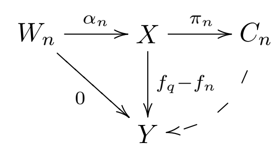


For $\Sigma^j B$, $B \in \mathscr{B}_0$ and $j \leq n$, we have constructed α_n so that

$$\mathbb{F}\alpha_n \colon [\Sigma^j B, W_n] \longrightarrow [\Sigma^j B, X]$$

is an epimorphism. Since $\mathbb{F}f_n \mathbb{F}\alpha_n = \mathbb{F}\beta_n = \phi \mathbb{F}\alpha_n$, it follows that $\mathbb{F}f_n = \phi$ when both are restricted to the union of the $\mathscr{B}_j$ for $j \leq n$. We want a map $f = f_\infty$ that works this way on all of $\mathscr{B}$.

By construction, we have an inclusion of summands $i_n \colon W_n \longrightarrow W_{n+1}$ such that $\alpha_{n+1} \circ i_n = \alpha_n \colon W_n \longrightarrow X$ and $\beta_{n+1} \circ i_n = \beta_n \colon W_n \longrightarrow Y$. Iterating, we have an inclusion of summands $i_{n,q} \colon W_n \longrightarrow W_q$ such that $\alpha_q \circ i_{n,q} = \alpha_n$ and $\beta_q \circ i_{n,q} = \beta_n$ for each $q > n$. Since $f_n \alpha_n = \beta_n$ for all n, $(f_q - f_n)\alpha_n = 0$. Therefore we obtain a factorization in the following diagram.



The following general conclusion is now immediate.

Proposition 3.4. *If there is an n such that $\mathscr{F}(C_n, Y)$ is finite, then the sequence of maps $f_q \colon X \longrightarrow Y$ has a constant cofinal sequence. Its common value $f \colon X \longrightarrow Y$ satisfies $\mathbb{F}f = \phi$.*

The hypothesis that $\mathscr{F}(C_n, Y)$ is finite for some n is quite special, and we now verify it in the case of Example 1.5 when G is finite.

Theorem 3.5. *If G is finite and the Freyd functor $\mathbb{F}$ for $G\mathscr{C} \subset HoG\mathscr{S}$ is faithful, then it is full and faithful.*

Proof. Using the notations above but letting $G\mathscr{F}$ denote the Freyd category of $G\mathscr{C}$, we must prove that $G\mathscr{F}(C_n, Y)$ is finite for some n. Since X is a retract of a finite G-CW spectrum, we may choose n to be the dimension of X and so ensure that $\underline{H}^G_r(X; M) = 0$ for $r > n$, where the homology in question is the $\mathbb{Z}$-graded Mackey functor valued Bredon homology with coefficients in a Mackey functor M. Explicitly, working in any version of the equivariant stable homotopy category Ho$G\mathscr{S}$ [3, 4],

$$\underline{H}^G_r(X; M)(G/H) = [S^r_H, X \wedge HM]_G,$$

where HM is the Eilenberg-Mac Lane spectrum of M. Since the functor $\underline{H}^G_r(-; M)$ is exact for each r, we can extend it to an exact functor on $G\mathscr{F}$ and obtain an exact sequence of Mackey functors

$$\underline{H}^G_r(W_n; M) \longrightarrow \underline{H}^G_r(X; A) \longrightarrow \underline{H}^G_r(C_n; M) \longrightarrow 0. \quad (3.6)$$

Since $\underline{H}_r^G(X;M) = 0$ for $r > n$, $\underline{H}_r^G(C_n;M) = 0$ for $r > n$. Now let $M = \underline{A}$ be the Burnside ring Mackey functor. It is proven in [2] that the rationalization of the sphere G-spectrum S_G is the Eilenberg-MacLane G-spectrum $H(\underline{A} \otimes \mathbb{Q})$. The rationalization of the sequence (3.6) can therefore be identified with the sequence

$$\underline{\pi}_r^G(W_n) \otimes \mathbb{Q} \xrightarrow{\alpha_n} \underline{\pi}_r^G(X) \otimes \mathbb{Q} \xrightarrow{\pi_n} \underline{\pi}_r^G(C_n) \otimes \mathbb{Q} \longrightarrow 0. \tag{3.7}$$

By construction, α_n induces an epimorphism on rational homotopy groups for $r \leq n$. Thus $\underline{\pi}_r^G(C_n) \otimes \mathbb{Q} = 0$ for $r \leq n$ and therefore for all integers r.

Write X_0 for the rationalization of a G-spectrum X. Then $\underline{\pi}_r^G(X) \otimes \mathbb{Q} \cong \underline{\pi}_r^G(X_0)$. In principle, we must deal with both $G\mathscr{F}$ and its rational variant $G\mathscr{F}_0$ defined using the stable homotopy category $G\mathscr{C}_0$ of rational finite G-spectra, since Y_0 is not in $G\mathscr{C}$. However, we have a rationalization functor $G\mathscr{F} \longrightarrow G\mathscr{F}_0$ that extends rationalization $\mathscr{C} \longrightarrow \mathscr{C}_0$ and becomes an equivalence on tensoring with $\mathbb{Q}$ (which doesn't change the target category).

It is proven in [2, App A] that the natural map

$$[X,Y]_G \longrightarrow \prod_r \mathrm{Hom}_{G\mathscr{M}_0}(\underline{\pi}_r^G(X), \underline{\pi}_r^G(Y)) \tag{3.8}$$

is an isomorphism for all rational G-spectra X and Y, where $G\mathscr{M}_0$ denotes the category of rational Mackey functors. This means that rational k-invariants are zero and is deduced from the fact that all rational Mackey functors are projective. When restricted to retracts of finite rational G-spectra, this result can be interpreted as saying that the strong form of the Freyd conjecture holds for the rational equivariant stable homotopy category. Of course, for fixed Y, both sides of (3.8) are cohomology theories in X, hence they are exact and extend to $G\mathscr{F}_0$.

Now return to our original situation and note that

$$[X,Y]_G \otimes \mathbb{Q} \cong [X, Y_0]_G \cong [X_0, Y_0]_G.$$

Reinterpreting everything just in terms of tensoring with $\mathbb{Q}$, we have the following commutative diagram. The left column is exact since C_n is the cokernel of α_n and $[X,Y]_G = G\mathscr{F}(X,Y)$ for $X, Y \in G\mathscr{C}$. The right column is also exact, but here the kernel is zero since $\pi_*^G(C_n) \otimes \mathbb{Q} = 0$. Therefore the rationalization of the finitely generated abelian group $G\mathscr{F}(C_n, Y)$ is zero, and $G\mathscr{F}(C_n, Y)$ is finite.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
G\mathscr{F}(C_n, Y) \otimes \mathbb{Q} & \longrightarrow & \prod_r \mathrm{Hom}_{G\mathscr{M}_0}(\underline{\pi}_r^G(C_n) \otimes \mathbb{Q}, \underline{\pi}_r^G(Y) \otimes \mathbb{Q}) \\
\downarrow{\scriptstyle \pi_n^*} & & \downarrow{\scriptstyle \pi_n^*} \\
[X,Y]_G \otimes \mathbb{Q} & \longrightarrow & \prod_r \mathrm{Hom}_{G\mathscr{M}_0}(\underline{\pi}_r^G(X) \otimes \mathbb{Q}, \underline{\pi}_r^G(Y) \otimes \mathbb{Q}) \\
\downarrow{\scriptstyle \alpha_n^*} & & \downarrow{\scriptstyle \alpha_n^*} \\
[W_n,Y]_G \otimes \mathbb{Q} & \longrightarrow & \prod_r \mathrm{Hom}_{G\mathscr{M}_0}(\underline{\pi}_r^G(W_n) \otimes \mathbb{Q}, \underline{\pi}_r^G(Y) \otimes \mathbb{Q})
\end{array}$$

□

References

[1] P. Freyd. Stable homotopy. 1966 Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965), pp. 121–172. Springer, New York.

[2] J.P.C. Greenlees and J.P. May. Generalized Tate Cohomology. Memoirs of the Amer. Math. Soc, No. 543. 1995.

[3] L. G. Lewis, J.P. May, and M. Steinberger and with contributions by J. E. McClure). Equivariant stable homotopy theory. Lecture Notes in Mathematics Vol. 1213. Springer-Verlag, 1986.

[4] M. A. Mandell and J.P. May. Equivariant orthogonal spectra and S-modules. Memoirs Amer. Math. Soc. Vol 159. 2002.

[5] J.P. May. Equivariant homotopy and cohomology theory. NSF-CBMS Regional Conference Series in Mathematics No. 91. Amer. Math. Soc. 1996.