THINKING ABOUT THE FREYD CONJECTURE

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We give a generalized version of the Freyd conjecture and a way to think about a possible proof. There are no new results yet.

There are several sensible contexts. We fix the following for definiteness. Let $\mathscr{V}$ be a stable model category with triangulated homotopy category $\mathscr{T}$. Write $[X, Y]$ for the Abelian group of maps $X \longrightarrow Y$ in $\mathscr{T}$. Let $\mathscr{B}$ be a (small) full subcategory of $\mathscr{T}$ closed under Σ and let $\mathscr{C}$ be the thick full subcategory of $\mathscr{T}$ generated by $\mathscr{B}$; write $\iota\colon \mathscr{B} \longrightarrow \mathscr{C}$ for the inclusion. Then $\mathscr{B}$ is pre-additive (enriched over $\mathscr{A}b$) and $\mathscr{C}$ is additive (has biproducts). Let $\mathscr{P}\mathscr{B}$ and $\mathscr{P}\mathscr{C}$ denote the categories of Abelian presheaves defined on $\mathscr{B}$ and $\mathscr{C}$. They consist of the additive functors from $\mathscr{B}^{op}$ or $\mathscr{C}^{op}$ to $\mathscr{A}b$ and the additive natural transformations between them.

Definition 0.1. Define the Freyd functor $\mathbb{F}\colon \mathscr{V} \longrightarrow \mathscr{P}\mathscr{B}$ by sending an object X to the functor $\mathbb{F}X$ specified on objects and morphisms of $\mathscr{B}$ by $\mathbb{F}X(-) = [-, X]$ and sending a map $f\colon X \longrightarrow Y$ to the natural transformation $f_* = [-, f]$. Define $\mathbb{Y}\colon \mathscr{V} \longrightarrow \mathscr{P}\mathscr{C}$ similarly.

We are interested only in the restrictions of $\mathbb{F}$ and $\mathbb{Y}$ to $\mathscr{C}$. Then $\mathbb{Y}$ becomes the standard Yoneda embedding $\mathscr{C} \longrightarrow \mathscr{P}\mathscr{C}$.

Conjecture 0.2 (The generalized Freyd conjecture)**.** *Assume, for example, that a map $f\colon X \longrightarrow Y$ in $\mathscr{C}$ is a weak equivalence if and only $f_*\colon \mathbb{F}X \longrightarrow \mathbb{F}Y$ is an isomorphism. Then $\mathbb{F}\colon \mathscr{C} \longrightarrow \mathscr{P}\mathscr{B}$ is a faithful functor. Equivalently, $\mathbb{F}f = 0$ if and only if $f = 0$. We then say that the Freyd conjecture holds for the pair $(\mathscr{C}, \mathscr{B})$.*

The hypothesis might admit other variants, but it holds in this form in the classical case. Certainly some hypothesis is required on $\mathscr{B}$, but none will be relevant to our formal analysis.

Example 0.3. Take $\mathscr{V} = \mathscr{S}$ to be a model for the stable homotopy category. Let $\mathscr{B}$ consist of the sphere spectra S^n for integers n (or cofibrant approximations if the S^n are not already cofibrant). Then $\mathscr{C}$ is the homotopy category of finite CW spectra. Freyd [1] conjectured that a map f in $\mathscr{C}$ is zero if and only if it induces the zero homomorphism $f_*\colon \pi_*(X) \longrightarrow \pi_*(Y)$. By the following observation, this is a special case of our Conjecture 0.2.

Lemma 0.4. *The category $\mathscr{P}\mathscr{B}$ is isomorphic to the category $\mathscr{M}$ of modules over the ring π_* of stable homotopy groups of spheres. Under this isomorphism, the Freyd functor $\mathbb{F}$ coincides with the stable homotopy group functor $\pi_*\colon \mathscr{S} \longrightarrow \mathscr{M}$.*

Proof. For $T \in \mathscr{P}\mathscr{B}$, let $M_n = T(S^n)$. For an element $x \in \pi_q$, thought of by suspension as a map $x\colon S^{n+q} \longrightarrow S^n$, and an element $y \in M_n$, define $xy = T(x)(y) \in M_{n+q}$. The functoriality of T ensures that M is a π_*-module. Conversely, given a π_*-module M, define $T(S^n) = M_n$ and define $T(x)(y) = xy$. The module axioms

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ensure that T is a functor. This specifies the claimed isomorphism of categories, and the consistency of $\mathbb{F}$ and π_* is clear. $\square$

The following example originally prompted us to take a presheaf perspective on the Freyd conjecture.

Example 0.5. Let G be a compact Lie group and take $\mathscr{V} = G\mathscr{S}$ to be a model for the equivariant stable homotopy category. Let $G\mathscr{B}$ consist of the orbit G-spectra $\Sigma^n(\Sigma^\infty(G/H)_+)$ for integers n and closed subgroups H of G (or cofibrant approximations if these are not already cofibrant). The thick subcategory, $G\mathscr{C}$, generated by $G\mathscr{B}$ is the homotopy category of retracts of finite G-CW spectra. The equivariant version of Freyd's conjecture asserts that a map f in $G\mathscr{C}$ is zero if and only if it induces the zero homomorphism $f_*\colon \pi_*^H(X) \longrightarrow \pi_*^H(Y)$ for all H, where

$$\pi_n^H(X) \equiv \pi_n(X^H) \cong [\Sigma^n(\Sigma^\infty(G/H)_+), X].$$

Again, this is a special case of our Conjecture 0.2.

Definition 0.6. By definition, a Mackey functor (or G-Mackey functor) M is precisely an object of $\mathscr{P}G\mathscr{B}$. When G is finite, this agrees with the more usual algebraic definition [2, V§9] or [5, IX§4, XIX§3]. Here $G\mathscr{B}$ is called the Burnside category. It is a Green functor, that being the appropriate version of a ring in the category of Mackey functors, and every Mackey functor is a module over $G\mathscr{B}$. This is the equivariant analogue of Lemma 0.4.

The presheaf perspective suggests a method of attack on the generalized Freyd conjecture. We have the forgetful functor

$$\mathbb{U} = \iota^*\colon \mathscr{P}\mathscr{C} \longrightarrow \mathscr{P}\mathscr{B}$$

given by restricting presheaves defined on $\mathscr{C}^{op}$ to the full subcategory $\mathscr{B}^{op}$. The functor $\mathbb{U}$ has a left adjoint prolongation functor

$$\mathbb{P} = \iota_!\colon \mathscr{P}\mathscr{B} \longrightarrow \mathscr{P}\mathscr{C}.$$

For $T \in \mathscr{P}\mathscr{B}$ and $K \in \mathscr{C}$, $\mathbb{P}T(K)$ is the categorical tensor product

$$\mathbb{P}T(K) = T \otimes_{\mathscr{B}} \mathscr{C}(K, -).$$

See, for example, [3, I§3] or [4, I§2]. Since $\mathscr{B}$ is a full subcategory of $\mathscr{C}$, the unit $\mathrm{Id} \longrightarrow \mathbb{UP}$ of the adjunction is a natural isomorphism [3, I.3.2]. We focus attention on the counit $\varepsilon\colon \mathbb{PU} \longrightarrow \mathrm{Id}$.

Observe that the Freyd functor $\mathbb{F}\colon \mathscr{V} \longrightarrow \mathscr{P}\mathscr{B}$ is the composite $\mathbb{UY}$. This leads to the following observation.

Proposition 0.7. *The Freyd conjecture holds for* $(\mathscr{C}, \mathscr{B})$ *if* $\varepsilon\colon \mathbb{PF}X \longrightarrow \mathbb{Y}X$ *is an isomorphism for all* $X \in \mathscr{C}$.

Proof. Since the unit of the adjunction $(\mathbb{P}, \mathbb{U})$ is an isomorphism, $\mathbb{U}\varepsilon$ is an isomorphism by one of the triangle identities. Thus $\mathbb{F} \cong \mathbb{UPF}$. Therefore, for a map $f\colon X \longrightarrow Y$ in $\mathscr{C}$, $\mathbb{F}f = 0$ if and only if $\mathbb{PF}f = 0$. By the Yoneda lemma, $f = 0$ if and only if $\mathbb{Y}f = 0$. If ε is an isomorphism, then $\mathbb{PF}f = 0$ if and only if $\mathbb{Y}f = 0$. $\square$

In fact, less is needed. Consider $\mathbb{Y}X(X) = [X, X]$. We will later prove the following result.

Proposition 0.8. *The Freyd conjecture holds for* $(\mathscr{C}, \mathscr{B})$ *if the identity map of* X *is in the image of* ε *for all* $X \in \mathscr{C}$.

We have the following starting point towards verification of the hypothesis.

Lemma 0.9. *For all* $X \in \mathscr{V}$, $\varepsilon\colon \mathbb{PF}X(J) \longrightarrow \mathbb{Y}X(J)$ *is an isomorphism for all* $J \in \mathscr{B}$.

Proof. We have seen that $\mathbb{U}\varepsilon$ is an isomorphism, and by definition $\mathbb{U}T(J) = T(J)$ for any $J \in \mathscr{B}$ and any $T \in \mathscr{P}\mathscr{C}$. $\square$

Now consider a cofiber sequence

$$K \longrightarrow L \longrightarrow M \longrightarrow \Sigma K \tag{0.10}$$

in $\mathscr{V}$, where K, L, and M are in $\mathscr{C}$. We have the commutative diagram
(0.11)

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{PF}X(\Sigma K) \arrow[r] \arrow[d, "\varepsilon"'] & \mathbb{PF}X(M) \arrow[r] \arrow[d, "\varepsilon"] & \mathbb{PF}X(L) \arrow[r] \arrow[d, "\varepsilon"] & \mathbb{PF}X(K) \arrow[r] \arrow[d, "\varepsilon"] & \cdots \\
\cdots \arrow[r] & \mathbb{Y}X(\Sigma K) \arrow[r] & \mathbb{Y}X(M) \arrow[r] & \mathbb{PF}X(L) \arrow[r] & \mathbb{Y}X(K) \arrow[r] & \cdots .
\end{tikzcd}$$

Since $\mathbb{Y}X(K) = [K, X]$, the lower row is exact. By the five lemma and the standing assumption that $\mathscr{C}$ is the thick subcategory generated by $\mathscr{B}$, this gives the following conclusion.

Proposition 0.12. *If the top row of (0.11) is exact for every cofiber sequence (0.10) and every* $X \in \mathscr{C}$, *then* $\varepsilon\colon \mathbb{PF}X \longrightarrow \mathbb{Y}X$ *is an isomorphism for every* $X \in \mathscr{C}$ *and the Freyd conjecture holds for* $(\mathscr{C}, \mathscr{B})$.

There is a reinterpretation of the exactness hypothesis that makes it reminiscent of the standard result that the adjoint of an exact functor between triangulated categories is exact. For K and X in $\mathscr{C}$, the Abelian group $\mathbb{PF}X(K)$ is the tensor product of functors displayed in the evident coequalizer diagram

$$\begin{tikzcd}
\sum_{I,J\in\mathscr{B}} \mathscr{C}(J,X) \otimes \mathscr{B}(I,J) \otimes \mathscr{C}(K,I) \arrow[d, shift left] \arrow[d, shift right] \\
\sum_{I\in\mathscr{B}} \mathscr{C}(J,X) \otimes \mathscr{C}(K,J) \arrow[d] \\
\mathscr{C}(-,X) \otimes_{\mathscr{B}} \mathscr{C}(K,-).
\end{tikzcd} \tag{0.13}$$

We use this to interpolate the proof of Proposition 0.8. The composition maps

$$\circ\colon \mathscr{C}(X,Y) \otimes \mathscr{C}(J,X) \longrightarrow \mathscr{C}(J,Y)$$

induce a pairing

$$\circ\colon \mathscr{C}(X,Y) \otimes \mathbb{PF}X(K) \longrightarrow \mathbb{PF}Y(K)$$

such that $f \circ z = \mathbb{PF}f(z)$ for $z \in \mathbb{PF}X(K)$ and the following diagram commutes:

$$\begin{tikzcd}
\mathscr{C}(X,Y) \otimes \mathbb{PF}X(K) \arrow[r, "\mathrm{id}\otimes\varepsilon"] \arrow[d, "\circ"'] & \mathscr{C}(X,Y) \otimes \mathbb{Y}X(K) \arrow[d, "\circ"] \\
\mathbb{PF}Y(K) \arrow[r, "\varepsilon"'] & \mathbb{Y}Y(K).
\end{tikzcd}$$

Take $K = X$ and suppose that $\varepsilon(z) = \mathrm{id}_X$. If $\mathbb{F}f = 0$, then $\varepsilon(f \circ z) = \varepsilon\mathbb{P}\mathbb{F}f(z) = 0$. By the diagram, this equals $f \circ \varepsilon(z) = f$ and so $f = 0$.

Let us write $\mathscr{P}_d\mathscr{B}$ and $\mathscr{P}_d\mathscr{C}$ for the categories of covariant additive functors on $\mathscr{B}$ and $\mathscr{C}$, and similarly write $\mathbb{U}_d$, $\mathbb{P}_d$, $\mathbb{F}_d$, and $\mathbb{Y}_d$ for the corresponding functors. We are just interchanging $\mathscr{B}$ and $\mathscr{C}$ with their opposite categories. Visibly, we again have $\mathbb{U}_d\mathbb{Y}_d = \mathbb{F}_d$ and again have an adjunction $(\mathbb{P}_d, \mathbb{U}_d)$ with $\mathbb{U}_d\mathbb{P}_d \cong \mathrm{Id}$. By symmetry, we have

$$\mathbb{P}\mathbb{F}X(K) = \mathbb{P}_d\mathbb{F}_dK(X). \tag{0.14}$$

But in this dual reformulation, the exactness hypothesis on K for fixed X is now a levelwise exactness statement about the composite functor $\mathbb{P}_d\mathbb{F}_d \colon \mathscr{C} \longrightarrow \mathscr{P}_d\mathscr{C}$. Since $\mathbb{F}_dK(J) = \mathscr{C}(K, J)$ for $J \in \mathscr{B}$, $\mathbb{F}_d$ clearly takes cofiber sequences in the variable $K \in \mathscr{C}$ to exact sequences for each fixed J. Thus a more general question to ask is whether or not $\mathbb{P}_d \colon \mathscr{P}_d\mathscr{B} \longrightarrow \mathscr{P}_d\mathscr{C}$ preserves levelwise exactness. That is, is it true that if $T' \longrightarrow T \longrightarrow T''$ is a sequence of diagrams $\mathscr{B} \longrightarrow \mathscr{A}b$ such that the sequence $T'(J) \longrightarrow T(J) \longrightarrow T''(J)$ is exact for all $J \in \mathscr{B}$, then the sequence $\mathbb{P}T'(X) \longrightarrow \mathbb{P}T(X) \longrightarrow \mathbb{P}T''(X)$ exact for all $X \in \mathscr{C}$?

Observe that we have not yet used any hypothesis on $\mathscr{B}$, other than that it generates the thick subcategory $\mathscr{C}$. Thus all we have done is to give a purely formal reduction of the general problem.

REFERENCES

[1] Peter Freyd. Stable homotopy. 1966 Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965), pp. 121–172. Springer, New York.

[2] L. G. Lewis, J.P. May, and M. Steinberger and with contributions by J. E. McClure). Equivariant stable homotopy theory. Lecture Notes in Mathematics Vol. 1213. Springer-Verlag, 1986.

[3] M. A. Mandell, J.P. May, S. Schwede, and B. Shipley. Model categories of diagram spectra. Proc. London Math. Soc. (3) 82(2001), 441–512.

[4] M. A. Mandell and J.P. May. Equivariant orthogonal spectra and S-modules. Memoirs Amer. Math. Soc. Vol 159. 2002.

[5] J.P. May. Equivariant homotopy and cohomology theory. NSF-CBMS Regional Conference Series in Mathematics No. 91. Amer. Math. Soc. 1996.