University of Chicago, REU 2006: K-Theory

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Contents

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LECTURE 1

Monday, June 19[1]

1. What is a Monoid?

DEFINITION 1.1. A **monoid** is a set M with

- an associative operation, i.e., a map $M \times M \to M$, written $(m, n) \mapsto mn$, satisfying

$$\forall m, n, p \in M,\ (mn)p = m(np),$$

- a two-sided identity, i.e., a distinguished element $e \in M$ satisfying

$$\forall m \in M,\ me = m = em.$$

DEFINITION 1.2. A monoid M is **commutative** or **abelian** if $mn = nm$ for all $m, n \in M$. We traditionally write commutative monoids additively instead of multiplicatively: i.e., we write 0 in place of e, and $m + n$ in place of mn.

DEFINITION 1.3. A **group** is a monoid with inverses; i.e.,

$$\forall m \in M,\ \exists m^{-1} \in M,\ mm^{-1} = e = m^{-1}m.$$

A group is **abelian** if the underlying monoid is abelian.

EXAMPLE 1.4. The nonnegative integers $\mathbb{N} = \mathbb{Z}^{\geq 0}$ under addition form an abelian monoid.

A first idea of K-theory is to replace a monoid with a group by formally throwing in inverses. The K stands for "class," which is spelled with a K in German.

2. Maps of Monoids

DEFINITION 1.5. Let M, N be monoids, with $e_M \in M$ and $e_N \in N$ their respective identity elements. A function $f : M \to N$ is a **homomorphism of monoids** or simply a **map of monoids** if $f(e_M) = e_N$ and $\forall m, m' \in M,\ f(mm') = f(m)f(m')$.

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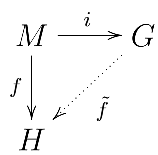
Definition 1.6. A homomorphism $f : M \to N$ is an **isomorphism** if there exists a homomorphism $f^{-1} : N \to M$ with $f \circ f^{-1} = f^{-1} \circ f = \text{identity}$. Two monoids M and N are **isomorphic** if there exists an isomorphism $f : M \to N$.

Remark 1.7. For monoids, a bijective homomorphism is in fact an isomorphism.

3. Universal Properties

How can we construct a group from a monoid? We will call the group that we build from a monoid M the "group completion" of M.

Definition 1.8. The **group completion** of M is a group G together with a map $i : M \to G$ so that for all groups H and maps of monoids $f : M \to H$, there exists a *unique* $\tilde{f} : G \to H$ making the following diagram commute:



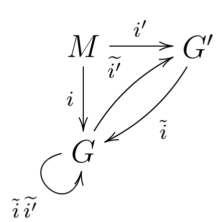
That is, $\tilde{f} \circ i = f$.

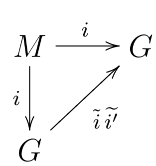
This definition is our first example of a **universal property**. It is critical that there exists a *unique* $\tilde{f}$.

Note that the group completion of M is more than just the group G; it is also the map $i : M \to G$, and the universal property says that if the group completion exists, then G is unique up to unique isomorphism.

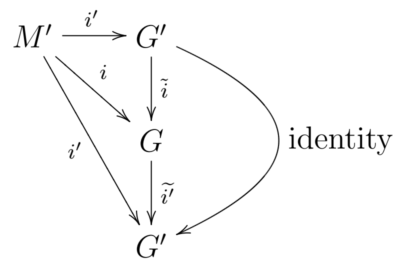
Remark 1.9. To say that a diagram commutes means that the composition of maps from one object to another object doesn't depend on the route you take.

3.1. Uniqueness in Universal Properties. Suppose we have $M \xrightarrow{i'} G'$ satisfying the same universal property as i and G. Then

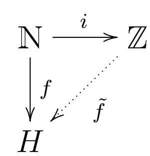




In the right-hand diagram, using the identity map $\text{id} : G \to G$ in place of $\tilde{i}\,\widetilde{i'}$ would also make the diagram commute. But the map produced by the universal property is unique, so it must be that $\tilde{i}\,\widetilde{i'} =$ id, and consequently, $\tilde{i}$ and $\widetilde{i'}$ are inverse isomorphisms, and G and G' are isomorphic. A better way to see this might be by examining the following diagram:



EXAMPLE 1.10. The group completion of $\mathbb{N}$ is $\mathbb{Z}$, because if H is a group and $f : \mathbb{N} \to H$ is any map of monoids, then defining $\tilde{f}(m-n) = f(m) - f(n)$ gives a well-defined map making the following diagram commute:



We have used additive notation for the multiplication in H, even though we did not assume that it is abelian. In fact, we can test the universal property for the group completion of an abelian monoid by mappings into abelian groups, because $\tilde{f}$ in the general case must land in an abelian subgroup of H.

In the previous example, we saw that $\mathbb{Z}$ was the group completion of $\mathbb{N}$ by explicitly checking that the universal property was satisfied. How can we know if we can do this in general? Does every monoid have a group completion? To explore questions like this, we need to provide a construction that will take a monoid, and produce a group G satisfying the universal property.

4. The First Construction

Let M be an abelian monoid, and consider the set of pairs $\{(m,n) : m,n \in M\}$, thinking intuitively that (m,n) should represent "$m-n$." We define an equivalence relation $(m,n) \sim (m',n')$ if $m+n' = m'+n$, which corresponds with our intuition, because if it were the case that $m-n = m'-n'$ then $m+n' = m'+n$.

Let $[m,n]$ be the equivalence class of the pair (m,n), and let G be the set of all such equivalence classes. Define an operation on G by $[m,n] + [p,q] = [m+p, n+q]$. You should check that this operation is well-defined and associative. Further define $0 = [0, 0]$, and observe that G is a monoid.

In fact, G is a group, because

$$[m, n] + [n, m] = [m + n, n + m] = [m + n, m + n] = [0, 0].$$

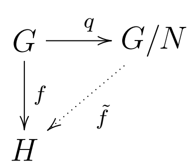
Thus we have constructed an abelian group G from an abelian monoid M.

It remains to verify that G is the group completion of M. Define $i : M \to G$ by setting $i(m) = [m, 0]$. Given a group H and a map $f : M \to H$, we define $\tilde{f}([m, n]) = f(m) - f(n)$ and we can check that this satisfies the appropriate universal property.

5. Review: Quotient Groups

Let G be a group, and $N \triangleleft G$, i.e., N is a normal subgroup in G, meaning that for any $g \in G$ and $n \in N$, the conjugate $gng^{-1} \in N$. We construct the **quotient group** G/N as the set of cosets gN, with the operation $gN \cdot hN = ghN$. There is a map $q : G \to G/N$ defined by $q(g) = gN$.

We can also define the quotient group by a universal property. The group G/N with $q : G \to G/N$ is the quotient group if the following is satisfied: for any group H and map $f : G \to H$ with $f(N) = e$, there exists a unique map $\tilde{f} : G/N \to H$ making the following diagram commute:



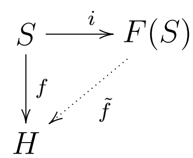
Definition 1.11. Let A, B be subgroups of a group G. The commutator $[A, B]$ subgroup is the subgroup generated by $aba^{-1}b^{-1}$ for all $a \in A$ and $b \in B$.

6. Other Universal Properties

We give two more examples of universal properties.

Definition 1.12. The **free group** generated by a set S is a group $F(S)$ and a map of sets $i : S \to F(S)$ such that for any group H and map of sets $f : S \to H$, there exists a unique map of groups

$\tilde{f} : F(S) \to H$ making the following diagram commute:

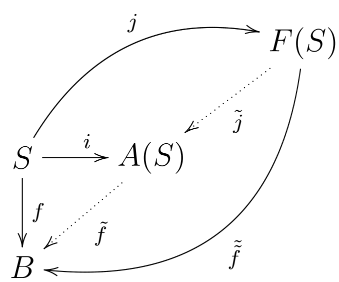


The set S need not be finite.

DEFINITION 1.13. The **free abelian group** generated by a set S is a group $A(S)$ and a map of sets $i : S \to A(S)$ such that for any *abelian* group H and map of sets $f : S \to H$, there exists a unique map of groups $\tilde{f} : A(S) \to H$ making the above diagram commute.

In some sense, these are the same universal property, but in different "worlds" of mathematics—the first in the "world" of groups, and the second in the "world" of abelian groups.

We will use $F(S)$ to construct $A(S)$. Consider the following diagram:



We claim $A(S) = F(S)/[F(S), F(S)]$. To see that this satisfies the universal property, first observe $\tilde{\tilde{f}}$ vanishes on $[F(S), F(S)]$ because for any $g, h \in F(S)$,

$$\begin{aligned} \tilde{\tilde{f}}(ghg^{-1}h^{-1}) &= f(ghg^{-1}h^{-1}) = f(g)f(h)f(g^{-1})f(h^{-1}) \\ &= f(g)f(h)f(g)^{-1}f(h)^{-1} = f(g)f(g)^{-1}f(h)f(h)^{-1} = e \end{aligned}$$

Second, the universal property for quotient groups says that a map vanishing on $[F(S), F(S)]$ factors through the quotient $A(S)$, which provides the required $\tilde{f}$.

7. Another Construction of the Group Completion

Using the notion of a free abelian group, we provide another construction of the group completion $G(M)$ of a monoid M. Let $A(M)$ be the free abelian group generated by M as a set, i.e.,

$$\begin{array}{ccc} M & \xrightarrow{j} & A(M) \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \tilde{f}} & \\ H & & \end{array}$$

We use $\oplus$ and $\ominus$ for the operations in the abelian group $A(M)$. Next, build the group completion $M \xrightarrow{i} G(M)$ by defining

$$G(M) = A(M)/\text{subgroup gen. by } m \oplus n \ominus (m+n) \text{ for all } m, n \in M.$$

The idea is that $A(M)$ doesn't know anything about the multiplication in M; after all, $A(M)$ is the free abelian group generated by the *set* M. In building $G(M)$, we have fixed this oversight by forcing $m+n$ to equal $m \oplus n$. We define $i : M \to G(M)$ to be the quotient map of $j : M \to A(M)$.

We need to check that $G(M)$ satisfies the universal property of group completions. That is, given any group H and map of monoids $f : M \to H$, can we construct a unique map $G(M) \to H$ making the appropriate diagram commute? Examine $\tilde{f} : A(M) \to H$, and observe

$$\begin{aligned} \tilde{f}\,(m \oplus n \ominus (m+n)) &= \tilde{f}(m) + \tilde{f}(n) - \tilde{f}(m+n) \\ &= f(m) + f(n) - f(m+n) \\ &= f(m) + f(n) - f(m) - f(n) = 0, \end{aligned}$$

so $\tilde{f}$ annihilates the subgroup generated by $m \oplus n \ominus (m+n)$, and hence by the universal property for quotient groups, the map $\tilde{f}$ factors through $G(M)$, providing the unique map $G(M) \to H$ making the appropriate diagram commute.

This construction looks very different from the first construction—but since we proved the uniqueness of the group completion up to canonical isomorphism, in fact these two constructions give the same group.

EXAMPLE 1.14. Let $q \in \mathbb{N}$, and let x, y be formal variables. Define a set

$$M_q = \{0\} \cup \{mx : m \in \mathbb{N}\} \cup \{ny : n \in \mathbb{N}\}$$

and an operation by the rules

$$\begin{aligned} 0 + c &= c \\ mx + px &= (m+p)x \\ my + py &= (m+p)y \\ mx + ny &= (qm+n)y \end{aligned}$$

In fact, $x + y = (q+1)y$ implies $mx + ny = (qm+n)y$. Then M_q is a monoid, and $G(M_q) = \mathbb{Z}$. To verify this last fact, define $i : M \to \mathbb{Z}$ by $i(0) = 0$, $i(ny) = n$ and $i(mx) = qm$. Then to make the diagram

$$\begin{array}{ccc} M & \xrightarrow{i} & \mathbb{Z} \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \tilde{f}} & \\ H & & \end{array}$$

commute we need only define $\tilde{f}(n) = f(ny)$. Alternatively, we can explicitly construct the group completion by using the first construction.

It is often easier to chase through universal properties than it is to chase through the details of a construction—and since verifying that the universal property holds is enough, relying on a universal property can often make for shorter, more concise proofs.

Example 1.15. Look at the surfaces handout. The set of homeomorphism classes of surfaces is a monoid under the operation of connected sum. Let P be the projective plane, S the sphere, and T be the torus. Then S is the identity element, and

$$\begin{aligned} mP \# nP &= (m+n)P \\ mT \# nT &= (m+n)T \\ mP \# nT &= (m+2n)P \end{aligned}$$

so the set of surfaces under connected sum forms a monoid, which is isomorphic to the monoid M_2 from the previous example, and which therefore has group completion equal to $\mathbb{Z}$.

8. Likewise for Rings

We can repeat much of the previous material about group completions, replacing monoids with semirings.

Definition 1.16. A **semiring** T is an abelian monoid (with operation $+$ and identity 0) such that T is also a monoid (with operation $\cdot$ and identity 1) and satisfies distributive laws:

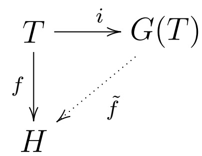
$$\begin{aligned} (s+s')t &= st + s't, \\ s(t+t') &= st + st'. \end{aligned}$$

We also require that 0 is a zero: $0 \cdot t = 0 = t \cdot 0$ for all t.

Thus, a semiring is nearly a ring, though we might be missing additive inverses.

We now define $G(T)$, the Grothendieck construction on the semiring.

Definition 1.17. The **Grothendieck construction** on a semiring T is a ring $G(T)$ together with a map $i : T \to G(T)$ such that for all rings H and maps of semirings $f : T \to H$, there exists a *unique* $\tilde{f} : G(T) \to H$ making the following diagram commute:



We can construct the ring completion of any semiring by repeating the first construction for T, thought of as an additive monoid, and observing that multiplication comes along for free. That is, because

$$(m - n) \cdot (p - q) = mp + nq - mq - np$$

we are led to define

$$[m, n] \cdot [p, q] = [mp + nq, mq + np].$$

You should check that this is well-defined.

9. Burnside Ring

Warning: algebraists might refer to the Burnside ring as $B(G)$, but topologists refer to it as $A(G)$, and I, being a topologist, will use this notation.

Definition 1.18. Let G be a group. A G-set S is a set S with an **action** of G, i.e., a map $G \times S \to$, written $(g, s) \mapsto g \cdot s$, such that

- $\forall s \in S$, $e \cdot s = s$,
- $\forall g, h \in G, s \in S$, $g \cdot (h \cdot s) = (gh) \cdot s$.

Definition 1.19. Let S and T be G-sets. Then a **G-map** is a map $f : S \to T$ such that $f(g \cdot s) = gf(s)$ for all $g \in G$ and $s \in S$. A G-map is a **G-isomorphism** if there exists an inverse G-map.

Let $[S]$ be the isomorphism class of the G-set S. We can turn the collection of isomorphism classes of finite G-sets into a semiring as follows:

- Define $[S] + [T] = [S \sqcup T]$, where the G-action on $S \sqcup T$ comes from the G-action on S and T.
- Define $0 = [\varnothing]$.
- Define $[S] \cdot [T] = [S \times T]$, where the G-action on $S \times T$ is the diagonal action $g(s, t) = (gs, gt)$.
- Define $1 = [\star]$, where $\star$ is a 1-point set with its unique G-action.

You should check that these definitions result in a semiring. Define $A(G)$, the **Burnside ring** of G, to be the Grothendieck construction on the semiring of isomorphism classes of finite G-sets.

Definition 1.20. Let G be a group, and H a subgroup of G. An **orbit** is the G-set $G/H = \{gH : g \in G\}$ of cosets, with the G-action $G \times G/H \to G/H$ given by $(k, gH) \mapsto kgH$.

Theorem 1.21. *Any finite G-set is isomorphic to the disjoint union of orbits.*

Corollary 1.22. *The Burnside ring $A(G)$ is the free abelian group generated by orbits.*

LECTURE 2

Wednesday, June 21[1]

We have two major goals for today. The first deals with idempotents.

DEFINITION 2.1. An **idempotent** in a ring R is an element $x \in R$ such that $x^2 = x$.

EXAMPLE 2.2. In $R = R_1 \times R_2$, idempotents include $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

Our first goal is to explain how this sort of decomposition relates to prime ideals.

Our second goal deals with simple groups.

DEFINITION 2.3. A **simple group** is a group G with no nontrivial normal subgroups.

It is now known how to classify all finite simple groups. The first step in this classification is the Feit-Thompson theorem—that an odd-order finite group is solvable, and hence not simple (where "simple" also means non-abelian).

DEFINITION 2.4. A finite group G is **solvable** if there exists subgroups $G_0, \ldots, G_s$ such that

$$G \rhd G_s \rhd G_{s-1} \rhd \cdots \rhd G_0 = \{e\}$$

and G_i/G_{i-1} is cyclic of prime order.

I will explain how to rephrase the Feit-Thompson theorem in the language of Burnside rings.

1. Spectrum of a Ring

Let R be a commutative ring.

DEFINITION 2.5. A ring R is **Noetherian** if every ascending chain of ideals is eventually constant. More precisely, if $I_0 \subset I_1 \subset I_2 \subset \cdots$ is an ascending chain of ideals, then there exists N so that for all $n > N$, $I_n = I_{n+1}$.

[1] TeXed by Jim Fowler.

Example 2.6. The ring $\mathbb{Z}$ is Noetherian.

Example 2.7. The ring $R[x_i|i \in \mathrm{N}]$, i.e., a ring adjoin infinitely many variables, is not Notherian, because

$$(x_0), (x_0, x_1), (x_0, x_1, x_2), \cdots$$

gives an infinite ascending chain of ideals I_i with $I_i \neq I_{i+1}$.

Theorem 2.8 (Hilbert Basis Theorem). *If the ring R is Noetherian, then $R[x]$ is Noetherian.*

Definition 2.9. An ideal P in a ring R is **prime** if $xy \in P$ implies $x \in P$ or $y \in P$.

Alternatively, an ideal P in a ring R is prime if and only if R/P is an integral domain.

Remark 2.10. Why are prime ideals important? Unique factorization into prime *numbers* fails; for instance, in $\mathbb{Z}[\sqrt{-5}]$, we have

$$2 \cdot 3 = (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5}).$$

Nonetheless, in this case there is still unique factorization into prime *ideals*.

Definition 2.11. The spectrum of a ring $\operatorname{Spec} R$, is the collection of all prime ideals.

On the surface, this seems uninteresting—but, surprisingly, $\operatorname{Spec} R$ is more than a set: it is a topological space. Usually we think of metric spaces (i.e., a set with a distance function), which are examples of topological spaces and provide plenty of intuition—but that intuition goes completely out the window for a topological space like $\operatorname{Spec} R$.

Definition 2.12. A **topological space** is a set X with a collection $\mathcal{U}$ of subsets of X. The sets in $\mathcal{U}$ are called the **open sets** of X, and must satisfy the following properties:

- The empty set $\varnothing$ and the whole space X are open sets, i.e., $\varnothing, X \in \mathcal{U}$.
- If $U_1, \ldots, U_n \in \mathcal{U}$, then $\bigcap_{i=1}^n U_i \in \mathcal{U}$.
- If $\{U_i\}_{i \in I}$ is an arbitrary set of sets in $\mathcal{U}$, then $\bigcup_{i \in I} U_i \in \mathcal{U}$.

The complement of an open set is an **closed set**. We call $\mathcal{U}$ a **topology** on the set X.

Remark 2.13. A set might be open, closed, both open and closed, or neither open nor closed.

DEFINITION 2.14 (Product of ideals). Let $\{I_1, \dots, I_n\}$ be a finite set of ideals. The product $I_1 \cdots I_n$ is the ideal whose elements are all sums of products $x_1 \cdots x_n$ with $x_i \in I_i$.

DEFINITION 2.15 (Sum of ideals). Let $\{I_i\}_{i \in I}$ be an arbitrary set of ideals. Then $\sum_i I_i$ is the ideal whose elements are all sums $\sum_i x_i$ with $x_i \in I_i$ and all but finitely many of the x_i equal to zero.

We now define a topology $\mathcal{U}$ on the set $\operatorname{Spec} R$. For I an ideal in R, define $V(I)$ to be the prime ideals containing I, i.e.,

$$V(I) = \{P \text{ a prime ideal in } R | P \supset I\}.$$

The V stands for "variety."

Set $\mathcal{U} = \{\operatorname{Spec} R - V(I) | I \text{ an ideal in } R\}$. We check that $\mathcal{U}$ gives a topology.

- Since $V(R) = \varnothing$, we have $\operatorname{Spec} R - \varnothing = \operatorname{Spec} R \in \mathcal{U}$.
- Since $V(0) = \operatorname{Spec} R$, we have $\operatorname{Spec} R - \operatorname{Spec} R = \varnothing \in \mathcal{U}$.
- Suppose $U_1, \dots U_n \in \mathcal{U}$, with $U_i = \operatorname{Spec} R - V(I_i)$ for an ideal I_i. Then,
$$V(I_1) \cup \cdots \cup V(I_n) = V(I_1 \cdots I_n),$$
and taking complements proves that $\bigcap_{i=1}^n U_i$ is $\operatorname{Spec} R - V(I_1 \cdots I_n) \in \mathcal{U}$.
- Let $\{U_i\}_{i \in I}$ be an arbitrary collection of open sets, with $U_i = \operatorname{Spec} R - V(I_i)$. Then,
$$\bigcap_i V(I_i) = V\left(\sum_i I_i\right).$$
Taking complements,
$$\begin{aligned} \bigcup_i U_i &= \bigcup_i (\operatorname{Spec} R - V(I_i)) \\ &= \operatorname{Spec} R - \bigcap_i V(I_i) \\ &= \operatorname{Spec} R - V\left(\sum_i I_i\right) \in \mathcal{U}, \end{aligned}$$
so the union of an arbitrary collection of open sets is in $\mathcal{U}$.

Our intuition often fails for general topological spaces because they may fail to satisfy the following very intuitive property:

DEFINITION 2.16. A topological space X is **Hausdorff** if for any two points $u, v \in X$ there exist open sets $U \ni u$ and $V \ni v$ with $U \cap V = \varnothing$.

Definition 2.17. The space $\operatorname{Spec} R$ is very far from being Hausdorff, as we already see by taking $R = \mathbb{Z}$ with its prime ideals 0 and (p) for prime numbers p. The proper closed sets $V((n))$ are the prime ideals given by the prime divisors of n, so the proper open subsets are the sets of all but finitely many non-zero primes.

Let X be a topological space with topology $\mathcal{U}$, and Y a subset of X. Then the **subspace topology** on Y is

$$\{U \cap Y | U \in \mathcal{U}\},$$

i.e., a set is open in the subspace topology if it is the intersection of an open set in X with Y.

Define

$$D(r) = \{P \text{ a prime ideal in } R : r \notin P\} = \operatorname{Spec} R - V((r)),$$

which is the open set associated with the ideal (r). Every prime ideal is in $D(r)$ for some $r \in R$. If $P \in D(r) \cap D(s)$, then $P \in D(rs) \subset D(r) \cap D(s)$. This is very similar to Euclidean space—every point is in some open ball, and the intersection of two open balls contains an open ball. Such a collection is called a **neighborhood basis**.

Precisely, in $\mathbb{R}$, define an open ball around x of radius ϵ to be

$$U(\epsilon, x) = \{y \in \mathbb{R} : |y - x| < \epsilon\}.$$

Every point in $\mathbb{R}$ is in some $U(\epsilon, x)$, and $U(\epsilon_1, x_1) \cap U(\epsilon_2, x_2)$ contains some $U(\epsilon, x)$.

A space X is **disconnected** if it has disjoint open subsets U and V with $U \cap V = \varnothing$ and $U \cup V = X$. We say that points $x, y \in X$ are in the same component, written $x \sim y$, if there exists a connected $C \subset X$ with $x, y \in C$. The equivalence classes are called **components**, and X is the disjoint union of its components.

2. Relating Connectedness and Idempotents

Our goal is to relate the topological notion of connectedness for $\operatorname{Spec} R$ to the algebraic notion of idempotents.

We say that e_1 and e_2 are **orthogonal** if $e_1 e_2 = 0$. An idempotent is **indecomposable** if it is not a sum of idempotents. If e is decomposable, i.e., $e = e_1 + e_2$, then

$$e = e^2 = (e_1 + e_2)^2 = {e_1}^2 + {e_2}^2 + 2e_1e_2 = e_1 + e_2 + 2e_1e_2,$$

therefore $2e_1e_2 = 0$. Thus, if 2 is a unit in R or R is an integral domain, then e_1 and e_2 are orthogonal.

Theorem 2.18. *Let R be a Noetherian ring. There is a bijective correspondence between:*

- *decompositions* $1 = e_1 + \cdots + e_n$ *with the* e_i*'s orthogonal idempotents,*
- *decompositions of* R *as the direct sum of* n *ideals,*
- *decompositions of* $\operatorname{Spec} R$ *as the disjoint union of* n *open and closed subsets.*

3. Burnside ring

In the first lecture we defined $A(G)$, the Grothendieck ring of the finite G-sets. We now deepen our understanding of its structure. We are headed towards the following theorem, which includes the promised reinterpretation of the Feit-Thompson theorem in terms of the Burnside ring: their theorem says that if G has odd order, then 0 and 1 are the only idempotents in $A(G)$.

Theorem 2.19. *The group* G *is solvable if and only if* 0 *and* 1 *are the only idempotents in* $A(G)$.

The ring $A(G)$ is difficult to understand directly; instead, for each subgroup H, we will define a homomorphism $\chi_H : A(G) \to \mathbb{Z}$. We define $\chi_H([S]) = |S^H|$, that is, χ_H sends a set S to the cardinality of the points fixed by H. We first check that this is well-defined; if S and S' are isomorphic as G-sets, then S^H is isomorphic to $(S')^H$, and therefore their cardinalities are equal. Note that $\chi_H([\varnothing]) = 0$ and $\chi_H([\star]) = 1$. Further, $(S \times T)^H = S^H \times T^H$, so

$$\begin{aligned} \chi_H([S]\cdot[T]) &= \chi_H([S\times T]) = |(S\times T)^H| \\ &= |S^H \times T^H| = |S^H|\cdot|T^H| = \chi_H([S])\cdot\chi_H([T]). \end{aligned}$$

Likewise, we check that $\chi_H([S]+[T]) = \chi_H([S]) + \chi_H([T])$. Thus, χ_H is a homomorphism from the semiring of isomorphism classes of finite G-sets to $\mathbb{Z}$ and extends by the universal property to a homomorphism of rings χ_H from $A(G)$ to $\mathbb{Z}$.

If $H' = gHg^{-1}$, so that H' and H are conjugate, then G/H' is isomorphic as a G-set to G/H. Define $C(G) = \prod_{(H)} \mathbb{Z}$, that is, the product of a copy of $\mathbb{Z}$ for each conjugacy class of subgroups of G. Define $\chi : A(G) \to C(G)$ by $\chi([S]) = (\chi_H([S]))$, so that the H-th coordinate of $\chi([S])$ is $\chi_H([S])$.

Proposition 2.20. *The homomorphism* $\chi : A(G) \to C(G)$ *is a monomorphism (i.e., injective).*

Example 2.21. Suppose S is a G-set such that every $t \in S$ is gs for some $g \in G$ and a fixed $s \in S$. In other words, S consists of a single orbit. Then $S \cong G/H$ by $gs \mapsto gH$, where $H = \{g | gs = s\} = G_s$ is the isotropy (or stabilizer) group of s. In general, a G-set looks like $\sum_{(H)} a_H[G/H]$, up to isomorphism.

Understanding the multiplication in $A(G)$ can be difficult. For example, $G/H \times G/K$ is $\coprod_i G/J_i$, but it takes some work to figure out how the J_i's relate to H and K. But since χ is a monomorphism, we can multiply in $C(G)$ instead of $A(G)$, and multiplication in $C(G)$ is easy—after all, $C(G)$ is a product of copies of $\mathbb{Z}$.

The proof of the above proposition requires a definition.

DEFINITION 2.22. Let H, K be subgroups of G. Then H is **subconjugate** to K, written $[G/H] < [G/K]$, if there exists $g \in G$ such that $gHg^{-1} \subset K$.

PROOF THAT χ IS A MONOMORPHISM. Assume not; if χ is not injective, then there is some nonzero $a = \sum_{(H)} a_H[G/H]$ with

$$\chi\left(\sum_{(H)} a_H[G/H]\right) = 0.$$

Choose J maximal (with respect to subconjugacy) such that $a_J \neq 0$. If J is not subconjugate to H, then $\chi_J([G/H]) = |(G/H)^J| = 0$. Consequently,

$$\begin{aligned}\chi_J\left(\sum_{(H)} a_H[G/H]\right) &= \sum_{(H)} a_H\chi_J([G/H]) \\ &= \sum_{(H) \text{ with } [G/J]<[G/H]} a_H\chi_J([G/H]) \\ &= a_J\chi_J([G/J]).\end{aligned}$$

But $\chi_J([G/J]) \neq 0$, so $\chi(a) \neq 0$, a contradiction. We conclude χ is a monomorphism. □

DEFINITION 2.23. A group G is **perfect** if $G = [G, G]$.

We consider perfect subgroups of a group G. For any subgroup H of G, there is a descending chain $H_s \lhd H_{s-1} \lhd \cdots \lhd H_0 = H$ with H_s perfect and H_i/H_{i+1} a cyclic group of prime order; H_s is the maximal perfect subgroup of H.

4. Prime ideals in the Burnside Ring

Let H be a chosen conjugacy class, and let p be a prime ideal in $\mathbb{Z}$ (possibly zero). Define

$$q(H, p) = \{x \in A(G) : \chi_H(x) \in p\}.$$

Then $q(H,p)$ is a prime ideal of $A(G)$, and every prime ideal in $A(G)$ is of this form. But there are some duplicates, namely $q(H,p) = q(H_p,p)$ where H_p is the maximal "p- perfect" subgroup of H. Here a group is p-perfect if it has no non-trivial quotient p-groups.

We know all the idempotents in $C(G)$—the only idempotents in $\mathbb{Z}$ are 0 and 1, so it is clear which elements in $\prod \mathbb{Z}$ are idempotent.

Let $\pi(\operatorname{Spec} A(G))$ be the set of components of the topological space $\operatorname{Spec} A(G)$, which, by theorem 2.18, correspond to the idempotents in $A(G)$. The claim is that the elements of $\pi(\operatorname{Spec} A(G))$ are in one-to-one correspondence with the conjugacy classes of perfect subgroups of G. That is, every prime ideal $q(H,p)$ is in the same component as one and only one $q(P,0)$, where P is perfect.

Being solvable means there are no non-zero perfect subgroups, so for a solvable group G, $\operatorname{Spec} A(G)$ has only one component.

LECTURE 3

Friday, June 23 [1]

1. Prime Spectra

For a ring R and an ideal I, we define

$$V(I) := \{P \mid P \supset I\}$$

where the P are prime ideals of R. The *Zariski topology* on the set of prime ideals of R is defined by declaring subsets $V(I)$ closed for all I. Denote this topological space by $\mathrm{Spec}(R)$.

PROPOSITION 3.1. *If R is an integral domain, then* $\mathrm{Spec}(R)$ *is connected.*

PROOF. Suppose that $\mathrm{Spec}(R) = C_1 \amalg C_2$ where the C_i are open and hence closed. Since R is an integral domain, (0) is prime. Say $(0) \in C_1$. However, every $P \supset (0)$, so $V(0) = \mathrm{Spec}(R) = C_1$. □

Consider $R = \prod_{j\in J} R_j$ where R_j is an integral domain. Let $I_j = (0, 0, \ldots, 0, R_j, 0, \ldots, 0)$ where R_j is in the jth slot and let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 is in the jth slot. Denoting by 1 the identity in R, that is $(1, 1, 1, \ldots, 1, 1)$, we see that $V((1 - e_j))$ and $V((1 - e_k))$ are disjoint for $j \neq k$. This follows by noting that the only primes in $V((1 - e_k))$ are uniquely associated to primes in R_k while primes in $V((1 - e_j))$ are uniquely associated to primes in R_j.

1.1. What are the primes in $\mathbb{Z} \times \mathbb{Z}$?

CLAIM 3.2. The only primes in $\mathbb{Z} \times \mathbb{Z}$ are

$$\begin{aligned} &((1, q)), \quad ((p, 1)) \ \ p, q \text{ prime} \\ &((1, 0)), \quad ((0, 1)). \end{aligned}$$

Now, take $\prod_{i=1}^{n} \mathbb{Z}_i$ with basis elements e_i and $\mathbb{Z}_i \cong \mathbb{Z}$. Then

$$\coprod_{i=1}^{n} V((1 - e_i)) = V\left(\prod_{i=1}^{n} \mathbb{Z}_i\right). \tag{1}$$

[1] TEXed by Emma Smith.

2. Simplicial digression

Consider a possibly infinite simplicial set, that is, an ordered set $\mathscr{S} = \{\text{vertices}\}$ and a set of simplices $S = \{\text{distinguished subsets of } \mathscr{S}\}$ with the rules that every vertex is in some simplex and any subset of a simplex is a simplex. Now, since our vertices are ordered, we can glue our abstract simplices together to construct a topological space. We identify each abstract simplex with a copy of the standard n-simplex Δ_n defined as

$$\Delta_n = \{(t_0, \ldots, t_n) \mid \sum_{i=0}^{n} t_i = 1\} \subset I^{n+1}$$

where $I = [0,1] \subset \mathbb{R}$. This space has $n+1$ vertices, the ith being 1 in the ith coordinate, $0 \leq i \leq n$. Let $S_n = \{\sigma \in S \mid \sigma \text{ has } n+1 \text{ elements}\}$. For each $\sigma \in S_n$ take a copy of Δ_n with vertices relabeled by the elements of σ, in order. Take the disjoint union of all of these simplices and identify any faces that are labeled by the same simplices of S.

A partially ordered set is a set A together with a transitive relation $\leq$ such that $a \leq b$ and $b \leq a$ implies $a = b$. A partially ordered set A gives a simplicial set S whose vertices are the points of A and whose n-simplices are the chains $a_0 < a_1 < \ldots < a_n$. Thus it gives a topological space. h This gives us another way of thinking about Spec(R). The set of prime ideals of R is a partially ordered set under the operation of inclusion. Consider chains of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n.$$

These determine simplices. Now Spec(R) may be drawn as in Figures 1 and 2. Note that Spec($\mathbb{Z} \times \mathbb{Z}$) is disconnected and that $(1,0)$ and $(0,1)$ are orthogonal idempotents. The intuitive content of our description of components of Spec(R) in terms of idempotents is that the space associated with the partially ordered set of primes always breaks up into components corresponding to idempotents in this fashion. The picture becomes particularly clear for Burnside rings, where we just see graphs with components corresponding to idempotents.

Definition 3.3. The **dimension** of a commutative ring is the maximal length n of a chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ in the ring.

Example 3.4. Let $R = k[x_1, \ldots, x_n]$. Then a maximal length chain is

$$(0) \subset (x_1) \subset (x_1, x_2) \subset \ldots \subset (x_1, \ldots, x_n),$$

so we say that $k[x_1, \ldots, x_n]$ is of dimension n.

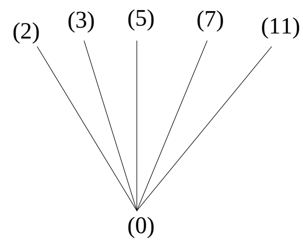


FIGURE 1. Spec($\mathbb{Z}$) for the lowest few primes.

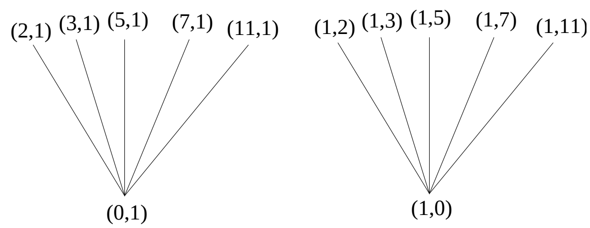


FIGURE 2. Spec($\mathbb{Z} \times \mathbb{Z}$) for the lowest few primes.

3. Feit-Thomson Theorem

Because it was hard going, we are going to repeat some things we did before. Take the Burnside ring $A(G)$, that is the free abelian with one basis element $[G/H]$ for each conjugacy class (H) of subgroups in G. Recall that $H \sim H'$ if there exists a $g \in G$ such that $gHg^{-1} = H'$.

Define $\chi_H : A(G) \to \mathbb{Z}$ by

$$[S] \to \left|S^H\right| := \left|\{s \mid hs = s \ \forall h \in H\}\right|.$$

Define $C(G) = \prod_{(H)} \mathbb{Z}_H$ and $e_J = 1$ in the Jth spot and 0's elsewhere. Then we can define

$$\begin{aligned} \chi : A(G) &\to C(G) \\ \sum_H a_H[G/H] &\mapsto \sum_J \left(\sum_H a_H \left|(G/H)^J\right| e_J \right) \end{aligned}$$

Just believe that this is a monomorphism. We proved it before.

3.1. Pulling back prime ideals. We wish to consider Spec$(A(G))$, but the only spectrum that we understand is that of a product of $\mathbb{Z}$'s. We shall deal with this by observing how primes work with homomorphisms.

Let $f : R \to S$ be a ring homomorphism. Let $Q \subset S$ be a prime ideal. Define $P \subset R$ by $P = \{x \mid f(x) \in \mathbb{Q}\} = f^{-1}(Q)$.

LEMMA 3.5. *P is prime in R.*

Proof. Let $xy \in P$ for $x, y \in R$. Then since f is a homomorphism, $f(xy) = f(x)f(y)$. Also, as $xy \in P$ we know that $f(x)f(y) = f(xy) \in Q$. Hence, since Q is prime, either $f(x) \in Q$ or $f(y) \in Q$. Thus, either $x \in P$ or $y \in P$ so P is also prime. $\square$

Now χ is the ring homomorphism under which we wish to pull back prime ideals. We know the primes in $C(G)$ by equation (1). We shall denote these pullbacks as follows:

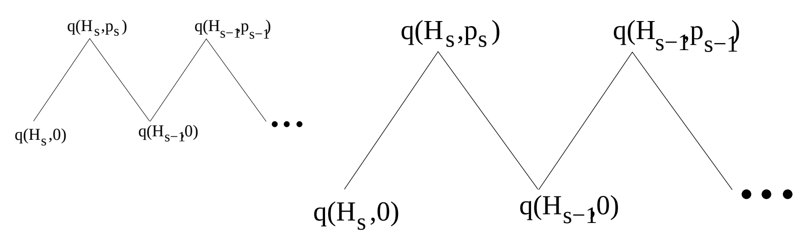
$$q(H,p) = \chi^{-1}\left((1,1,\dots,1,p,1,\dots,1)\right)$$

where p is the Hth coordinate and p is prime or 0 (thought of as prime ideals). Notice that $q(H,0) \subset q(H,p)$ for all primes p. These are the only inclusions, so $A(G)$ has dimension 1.

3.2. Now for some group theory. Suppose $H \lhd J$ and $J/H = \Pi_p$ (cyclic group of order p). Then $q(H,p) = q(J,p)$. To see this, look at a finite J-set S. Then $S^J = \left(S^H\right)^{J/H}$ since $js = s$ and $jhs = js$ if $hs = s$ for all $h \in H$. We claim that

$$\chi_J(S) \equiv \chi_H(S) \mod p.$$

This means that the number of elements that are fixed by H but are not fixed by all of J is divisible by p. Look at elements that are not fixed by J. This means that they are not fixed by the cyclic group J/H of order p, and the number of these is divisible by p because not being fixed they must break up into orbits with p elements each. Hence, we have graphs



Hence, we see that every chain of subgroups

$$H_s \lhd \cdots \lhd H_2 \lhd H_1 \lhd H_0 = H$$

where H_s is a maximal perfect subgroup of H and each quotient $H_i/H_{i+1} = \Pi_{p_i}$ belongs to precisely one connected component of our spectrum. Therefore, there is one conjugacy class for each perfect subgroup.

Remark 3.6. Just as we defined $\operatorname{Spec}(R)$, we can consider the maximal spectrum $m\operatorname{Spec}(R) \subset \operatorname{Spec}(R)$ which consists of the maximal ideals of R. We give it the subspace topology.

If $R = \mathbb{C}[x_1, \ldots, x_n]$, then $m\,\mathrm{Spec}(R) \cong \mathbb{C}^n$ with the Zariski topology. In fact, for any field $\mathbb{F}$, the ideals of $\mathbb{F}[x_1, \ldots, x_n]$ of the form $((x_1 - a_1, \ldots, x_n - a_n))$ for $a_1, \ldots, a_n \in \mathbb{F}$ are certainly maximal. Quotienting by such ideal is equivalent to evaluating polynomials at $(a_1, \ldots, a_n)$. Thus $\mathbb{F}$ injects isomorphically into the quotient $\mathbb{F}[x_1, \ldots, x_n]/(x_1 - a_1, \ldots, x_n - a_n)$. Hilbert's Nullstellensatz states that all maximal ideals are of this form when F is algebraically closed. We will look at that next.

LECTURE 4

Monday, June 26 [1]

1. Algebraic Geometry

1.1. Weak Nullstellensatz.

DEFINITION 4.1. A field F is said to be **algebraically closed** if every polynomial f in $F[x]$ of positive degree has a root in F. Since $F[x]$ is a PID, this is equivalent to saying that every maximal ideal of $F[x]$ is of the form $(x - a)$ for some $a \in F$.

EXAMPLE 4.2. By the fundamental theorem of algebra, the field of complex numbers $\mathbb{C}$ is algebraically closed.

Henceforth, we fix a field F. For each $n \geq 1$, we define **affine n-space** to be the set $\mathbb{A}^n = F^n$ and write $A = F[x_1, \ldots, x_n]$ for its corresponding polynomial ring.

THEOREM 4.3 (Weak Nullstellensatz). *If F is an algebraically closed field, then every maximal ideal of $F[x_1, \ldots, x_n]$ is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ with $a_1, \ldots, a_n \in F$.*

In other words, when F is algebraically closed, $\mathbb{A}^n$ is in one to one correspondence with $mSpec(F[x_1, \ldots, x_n])$ via

$$(a_1, \ldots, a_n) \mapsto (x_1 - a_1, \ldots, x_n - a_n).$$

The proof of this will be given later.

1.2. Zariski Topology.

DEFINITION 4.4. For any $T \subseteq A$, we can associate a **zero set**

$$\mathcal{Z}(T) = \{P \in \mathbb{A}^n \mid f(P) = 0, \forall f \in T\}$$

in $\mathbb{A}^n$. A subset Y of $\mathbb{A}^n$ is an **algebraic set** if there is a $T \subseteq A$ for which $Y = \mathcal{Z}(T)$.

REMARK 4.5. It is immediate that $\mathcal{Z}$ is inclusion reversing. By this we mean, if $T_1 \subseteq T_2 \subseteq A$ then $\mathcal{Z}(T_1) \supseteq \mathcal{Z}(T_2)$. Secondly, if I is the ideal generated by T in A, it is easy to see that $Z(I) = Z(T)$. Indeed

[1] TEXed by Alan Anders.

$T \subseteq I \Rightarrow \mathcal{Z}(T) \supseteq \mathcal{Z}(I)$. For the other inclusion, suppose $P \in \mathcal{Z}(T)$ so that $f(P) = 0$ for all $f \in T$. Since every element $g \in I$ is a sum $g = \sum_{i=1}^{q} h_i f_i$ with $f_i \in T$ and $h_i \in A$, we have $g(P) = 0$ as well.

PROPOSITION 4.6. *$\mathcal{Z}$ satisfies the following properties:*

(i) $\mathcal{Z}(T_1T_2) = \mathcal{Z}(T_1) \cup \mathcal{Z}(T_2)$ $\quad (T_1, T_2 \subseteq A)$;

(ii) $\mathcal{Z}(\cup_i T_i) = \cap_i \mathcal{Z}(T_i)$ $\quad (\forall i \in I, T_i \subseteq A)$;

(iii) $\mathcal{Z}(\emptyset) = \mathbb{A}^n$;

(iv) $\mathcal{Z}(A) = \emptyset$;

where in the above identities $T_1T_2 = \{f_1f_2 \mid f_i \in T_i\}$.

PROOF. With a little thought, the proofs of these identities become quite obvious. We provide (i) and leave the rest to the reader.

(i) If $P \in \mathcal{Z}(T_1T_2)$ and $P \notin \mathcal{Z}(T_1)$, then there is an $f_1 \in T_1$ for which $f_1(P) \neq 0$. So whenever $f_2 \in T_2$, we have $f_1(P)f_2(P) = 0$ implying $f_2(P) = 0$. Conversely, if P is in $\mathcal{Z}(T_1)$ or $\mathcal{Z}(T_2)$ then $f(P) = 0$ for all $f \in T_1$ or all $f \in T_2$. So clearly, $(f_1f_2)(P) = f_1(P)f_2(P) = 0$ for all $f_i \in T_i$. □

DEFINITION 4.7. It is apparent from the last proposition that we can give $\mathbb{A}^n$ a topology whose closed sets are precisely the algebraic subsets. Equivalently, define the open sets of $\mathbb{A}^n$ to be the complements of algebraic sets. This is called the **Zariski topology**.

1.3. Strong Nullstellensatz.

DEFINITION 4.8. To complete our dictionary of correspondences between subsets of affine space and ideals of our polynomial ring in n-variables, we form for each $Y \subseteq \mathbb{A}^n$, the **ideal of** Y in A by setting

$$\mathcal{I}(Y) = \{f \in A \mid f(P) = 0, \forall P \in Y\}.$$

To relate $\mathcal{I}$ and $\mathcal{Z}$, we introduce the notion of the **radical of** an ideal I in a ring R, which is defined to be

$$\sqrt{I} = \{f \in R \mid \exists q > 0, f^q \in I\}.$$

Also, we say that an ideal I is **radical** if $I = \sqrt{I}$.

EXAMPLE 4.9. In general, every prime ideal of any ring is always radical. To get an idea of the radical of an ideal, take p as a prime number in $\mathbb{Z}$. Then if $q > 0$, $\sqrt{(p^q)} = (p)$.

REMARK 4.10. Before we prove anything, we note that $\mathcal{I}$, like $\mathcal{Z}$ is inclusion reversing. Also, $I \subseteq \sqrt{I} \subseteq \mathcal{I}\mathcal{Z}(I)$ always holds. Indeed, if $f \in \sqrt{I}$ then there is a $q > 0$ for which $f^q \in I$. So if $P \in \mathcal{Z}(I)$, then $f(P) = 0$ provides $f^q(P) = 0$ as well.

The closure $\bar{Y}$ of a subspace Y of a topological space is the smallest closed subspace that contains it, namely the intersection of all closed subspaces that contains it.

PROPOSITION 4.11. *If* $Y \subseteq \mathbb{A}^n$, *then* $\mathcal{Z}\mathcal{I}(Y) = \bar{Y}$.

PROOF. If $P \in Y$, then $f(P) = 0$ for all $f \in \mathcal{I}(Y)$. Hence, $Y \subseteq \mathcal{Z}\mathcal{I}(Y)$. And since the image of $\mathcal{Z}$ consists of closed sets, $\bar{Y} \subseteq \mathcal{Z}\mathcal{I}(Y)$. Conversely, let $W = \mathcal{Z}(I)$ be a closed set containing Y where I is an ideal of A. Then $\mathcal{I}(Y) \supseteq \mathcal{I}(W) \Rightarrow \mathcal{Z}\mathcal{I}(Y) \subseteq \mathcal{Z}\mathcal{I}(W)$. By the previous remark, $I \subseteq \mathcal{I}\mathcal{Z}(I) \Rightarrow W = \mathcal{Z}(I) \supseteq \mathcal{Z}\mathcal{I}(W) \supseteq \mathcal{Z}\mathcal{I}(Y)$. This completes the proof. □

THEOREM 4.12 (Strong Nullstellensatz - Rabinowitsch). *If* F *is algebraically closed, then* $\mathcal{I}\mathcal{Z}(I) = \sqrt{I}$ *for any ideal* I *in* A.

PROOF. Let $f \in \mathcal{I}\mathcal{Z}(I)$ be nonzero. Then let J be the ideal generated by I and $fx_{n+1} + 1$ in $A' = F[x_1, \dots, x_{n+1}]$. Thus, $J = A'I + A'(fx_{n+1} + 1)$. We claim $\mathcal{Z}(J) = \emptyset$. Suppose otherwise and that $P' = (a_1, \dots, a_{n+1}) \in \mathcal{Z}(J)$. Then, writing $P = (a_1, \dots, a_n)$, we must have $P \in \mathcal{Z}(I)$. Hence, $f(P) = 0$. So

$$(fx_{n+1} - 1)(P') = f(P)a_{n+1} - 1 = -1 \neq 0$$

which contradicts our assumption that $P' \in \mathcal{Z}(J)$.

By the weak Nullstellensatz, $\mathcal{Z}(J) = \emptyset$ implies $J = A'$. This provides $g, k \in A'$ and an $h \in I$ for which

$$1 = gh + k \cdot (fx_{n+1} - 1).$$

Next, observe that $A' = A[x_{n+1}]$ and write $g = \sum_{i=0}^{q} g_i x_{n+1}^i$ where $g_i \in A$. Also, consider $1/f$ as an element in the quotient field of A. Then we can evaluate our above expression in x_{n+1} at $1/f$ which gives the equality

$$1 = \left(\sum_{i=0}^{q} g_i (1/f)^i \right) h$$

in A. Hence, if we multiply through by f^q and note that $h \in I$, we see $f^q \in I$ as needed. □

COROLLARY 4.13. *There is a one-to-one inclusion reversion correspondence between algebraic sets of* $\mathbb{A}^n$ *and radical ideals of the corresponding* F*-algebra* A*, given by* $Y \mapsto \mathcal{I}(Y)$ *and* $I \mapsto \mathcal{Z}(I)$.

1.4. On our way to the proof of the weak Nullstellensatz. The following notion will be helpful in developing our proof.

DEFINITION 4.14. Recall that when R is an integral domain we have its quotient field or field of fractions $Frac(R) = \{r/s \mid s \neq 0\}$. We may consider the subring $R[a^{-1}]$ of $Frac(R)$

$$R[a^{-1}] = \{r/a^n \mid n \geq 0\}$$

for each nonzero $a \in R$. An integral domain R is said to be a ***G*-domain** if $R\left[a^{-1}\right]$ is a field for some nonzero $a \in R$.

EXAMPLE 4.15. Consider the subring R of $\mathbb{Q}$ where

$$R = \{a/b \in \mathbb{Q} \mid a, b \in \mathbb{Z},\ 2 \not| \, b\}.$$

It is easy to check $R[2^{-1}] = \mathbb{Q}$ so that R is a G-domain.

PROPOSITION 4.16. *If R is an integral domain, then $R\left[x\right]$ is never a G-domain.*

PROOF. Suppose otherwise. Then there is an $f \in R\left[x\right]$ for which $R\left[x\right]\left[f^{-1}\right]$ is a field. This implies that $\deg f > 0$, since if $f \in R$, then $R[x][f^{-1}] = R[f^{-1}][x]$ is not a field. We must have $(1+f)^{-1} = g/f^n$ for some $g \in R\left[x\right]$. This implies $f^n = g(1+f)$. Hence, f^n is congruent to $0 \mod (1+f)$. Since f is clearly congruent to $-1 \mod (1+f)$, in the ring $R\left[x\right]/(1+f)$, we have $(-1)^n = 0$. This implies that $(1+f)$ is the unit ideal. Since $\deg f > 0$, this is impossible. ☐

LECTURE 5

Monday, June 26 [1]

1. Algebraic Geometry Continued

1.1. Irreducible algebraic sets.

DEFINITION 5.1. An algebraic set V is **irreducible** if whenever $V = V_1 \cup V_2$ where V_1, V_2 are algebraic sets, then $V = V_1$ or $V = V_2$.

To see what these correspond to under $\mathcal{I}$, we first prove the following lemma.

LEMMA 5.2. *If P is a prime ideal in a commutative ring R for which $P \supseteq I_1 \cap \cdots \cap I_n$ where $I_1, \ldots, I_n$ are ideals, then $P \supseteq I_i$ for some i.*

PROOF. If $P \not\supseteq I_i$ for all i, then there are $x_i \in I_i$ with $x_i \notin P$. Because P is a prime ideal, $x_1 \cdots x_n \notin P$. But $x_1 \cdots x_n \in I_1 \cap \cdots \cap I_n \subseteq P$. This is a contradiction. □

PROPOSITION 5.3. *An algebraic set V is irreducible iff $\mathcal{I}(V)$ is a prime ideal.*

PROOF. $\Rightarrow$: Suppose V is irreducible. Let $f_1 f_2 \in \mathcal{I}(V)$. Hence,

$$\mathcal{Z}(f_1) \cup \mathcal{Z}(f_2) = \mathcal{Z}(f_1 f_2) \supseteq \mathcal{Z}\mathcal{I}(V) = V$$

using proposition 4.5, 4.6, and 4.11 from last time. Intersecting both sides with V and using the irreducibility of V, we find $\mathcal{Z}(f_i) \cap V = V$ for some $i = 1, 2$. Therefore, $\mathcal{Z}(f_i) \supseteq V$ which means $f_i(P) = 0$ for all $P \in V$. In other words, $f_i \in \mathcal{I}(V)$.

$\Leftarrow$: Suppose $\mathcal{I}(V)$ is prime and $V = V_1 \cup V_2$. Then $\mathcal{I}(V) = \mathcal{I}(V_1) \cap \mathcal{I}(V_2)$. By the lemma, $\mathcal{I}(V_i) \subseteq \mathcal{I}(V)$ for some i. Taking $\mathcal{Z}$ on both sides, $V_i \supseteq V$. Since $V_i \subseteq V$, equality must hold. □

1.2. The proof of the weak Nullstellensatz.

DEFINITION 5.4. Suppose B is an integral domain and A a subring of B. An element $b \in B$ is **integral over** A if there is a monic polynomial $f \in A[x]$ for which $f(b) = 0$. And B is an **integral extension** of A if every element of B is integral over A.

[1] TEXed by Alan Anders.

REMARK 5.5. Denote the units of a ring A as $U(A)$. Recall a unit is an element of a ring which has a multiplicative inverse.

LEMMA 5.6. *If B is an integral extension of A then $A \cap U(B) = U(A)$.*

PROOF. Clearly, $U(A) \subseteq A \cap U(B)$. So suppose $a \in A \cap U(B)$. Hence, there is a $b \in B$ for which $ab = 1$. There is a polynomial $f = x^n + \sum_{i=0}^{n-1} a_i x^i$ for which $f(b) = 0$. Then $0 = a^{n-1} f(b) = b + \sum_{i=0}^{n-1} a_i a^{n-1-i}$. This implies that $b \in A$ and thus $a \in U(A)$. □

COROLLARY 5.7. *If a field is an integral extension of a ring R, then R must be a field as well.*

THEOREM 5.8. *If R is an integral domain and M is a maximal ideal of $R[x_1, \ldots, x_n]$ where $M \cap R = 0$, then there is a nonzero element $a \in R$ for which R_a is a field and $K = R[x_1, \ldots, x_n]/M$ is a finite field extension of R_a. In particular, if R is not a G-domain, then $R \cap M \neq 0$.*

For a proof of this, see the handout. This does indeed imply the weak Nullstellensatz. If F is an algebraically closed field and M is a maximal ideal of $F[x_1, \ldots, x_n]$, where $n \geq 2$, then take R in the theorem to be $F[x_1]$, and consider $M \cap R$. Since F is algebraically closed, f splits into linear factors and one of those factors must be in M, say $x_1 - a_1 \in M$. Similarly there are $x_i - a_i \in M$ for all i. Thus M contains and therefore equals the maximal ideal $(x_1 - a_1, \cdots, x_n - a_n)$.

LECTURE 6

Friday, June 30 [1]

REMARK 6.1. All information can be found in the first chapter of Peter May's "A Concise Course in Algebraic Topology"

REMARK 6.2. Wikipedia has vignettes about the history of the Fundamental Theorem of Algebra. Leibnitz thought he had a counterexample in the fourth degree polynomial $x^4 - a^4 = 0$.

1. Fundamental Group

Algebraic topology assigns discrete algebraic invariants to topological spaces and continuous maps. More narrowly, one wants the algebra to be invariant with respect to continuous deformations of the topology. Typically, one associates a group $A(X)$ to a space X and a homomorphism $A(p) : A(X) \to A(Y)$ to a map $p : X \to Y$. One usually writes $A(p) = p_*$.

REMARK 6.3. Here is the general plan of attack for algebraic invariants. Start with your construction A. Then compute A on spaces you know and love. Finally, take a problem of interest and manipulate it to a form your prior calculations allow you to solve. Voila!

The fundamental group is the first invariant studied in algebraic topology classes. Although at the moment we shall follow the plan for the fundamental group, later we shall follow this plan with topological K-theory, as well. On S^1 our construction will be called 0, though, and we shall compute values for all S^n.

1.1. Our construction. Let $A = \pi_1$, X be a space, and x a point of X. Thus, we are interested in objects $A(X, x) = \pi_1(X, x)$. Look at continuous maps $f : I \to X$ with the properties $f(0) = x$ and $f(1) = y$. We say that such a function defines a path from x to y, as depicted in Figure 1.

DEFINITION 6.4. Now let f, g be two paths from x to y in X. We say that f is **homotopic** to g, written $f \simeq g$, if there exists an

[1] TEXed by Emma Smith.

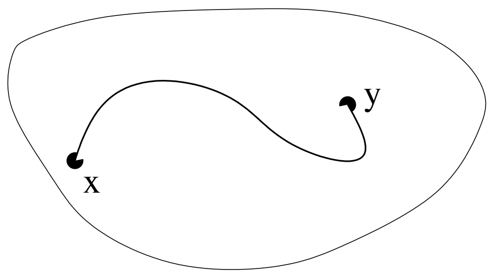


FIGURE 1. A basic path from x to y.

$h: I \times I \to X$ with

$$h(s,0) = f(s),\ h(s,1) = g(s),\ h(0,t) = x,\ \text{and}\ \ h(1,t) = y.$$

for all $s, t \in I$.

In other words, we are continuously deforming f into g while holding the endpoints fixed, as shown in Figure 2.

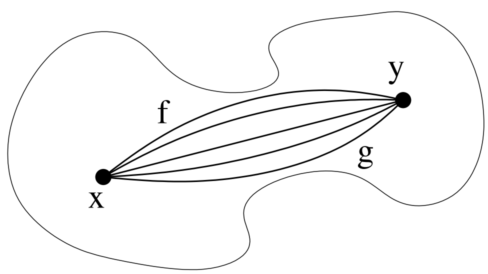


FIGURE 2. A continuous deformation of path f into path g from x to y.

We write the homotopy class of f as $[f]$. This is an equivalence relation. Let $f : x \to y$ and $g : y \to z$ be paths in X. Define

$$(g \cdot f)(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1. \end{cases}$$

This passes to equivalence classes by

$$[g][f] := [g \cdot f].$$

Check that this is a reasonable definition because

$$f \simeq f' \ \Rightarrow\ g \cdot f \simeq g \cdot f'.$$

This multiplication operation, after passing to equivalence classes, is associative and has left and right identities. Suppose $f : x \to y$,

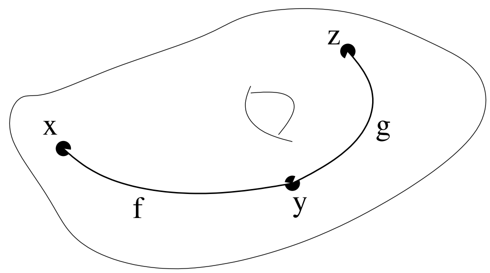


FIGURE 3. The composition of two maps $f : x \to y$ and $g : y \to z$.

$g : y \to z$, and $k : z \to w$ are paths in X. Then

$$\begin{aligned}(k \cdot g) \cdot f &\simeq k \cdot (g \cdot f) \\ ([k][g])[f] &= [k]([g][f]).\end{aligned}$$

See, for example, Figures 4, 5.

We also have inverses. Define $f^{-1}(s) = f(1 - s)$, i.e. f^{-1} traverses the path f in the opposite direction and hence is a path from y to x. Note that $f^{-1}f \simeq c_x$ where c_x is the constant function at x, as shown in Figure 6.

1.2. Loops. Now, look at paths f where $f(0) = x = f(1)$. We shall call such paths loops. What we have just shown, above, is that all the loops with a fixed basepoint x form a group. This group is called the fundamental group of X with basepoint x. Notice that there is an awkwardness because in general we must pick a base point, although as depicted in Figure 7, we can change basepoints along paths. For a path a from x to y define $\gamma[a] : \pi_1(X, x) \to \pi_1(X, y)$ by $(\gamma[a])([f]) := [a][f][a]^{-1}$. It is easy to check that $\gamma[a]$ depends ony on the equivalence class of a and is a homomorphism of groups. For a path b from y to z we see that $\gamma[b \cdot a] = \gamma[b] \circ \gamma[a]$. It follows that $\gamma[a]$ is an isomorphism with inverse $\gamma[a^{-1}]$. For a path b from y to x we have $\gamma[b \cdot a][f] = [b \cdot a][f][(b \cdot a)^{-1}]$. If the group $\pi_1(X, x)$ happens to be abelian, then this is just $[f]$. By taking $b = (a')^{-1}$ for another path a' from x to y we see that, when $\pi_1(X, x)$ happens to be abelian, $\gamma[a]$ is independent of the choice of the path class $[a]$. Thus, in this case, we have a canonical way to identify $\pi_1(X, x)$ with $\pi_1(X, y)$.

1.3. Higher homotopy groups. You may have noted that the fundamental group is called π_1. Higher homotopy groups are denoted π_n and are defined as homotopy classes of maps from $(S^n, 1) \to (X, x)$. By restricting to the first coordinate, a multiplication can be defined.

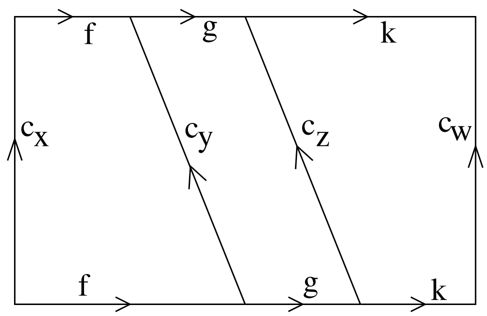


FIGURE 4. A picture of a homotopy between $(f \circ g) \circ k$ and $f \circ (g \circ k)$.

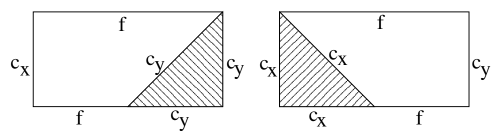


FIGURE 5. A picture of a homotopy between a path and the path either right or left composed with the constant path.

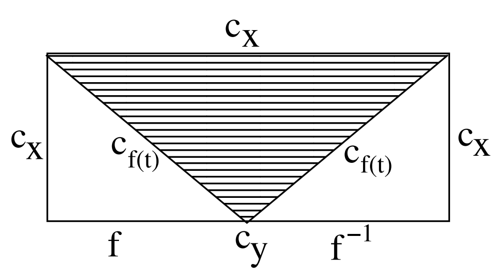


FIGURE 6. A picture of a homotopy between two inverse paths and the constant path.

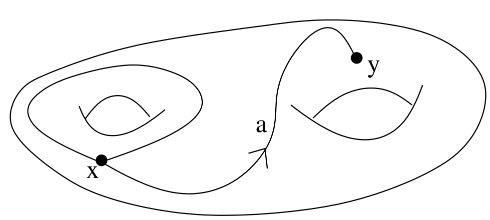


FIGURE 7. A picture of a base point change from x to y.

Say we have $p, q : X \to Y$. We say that $p \simeq q$ if there exists a continuous map $h : X \times I \to Y$ such that

$$h(s, 0) = p(s) \text{ and } h(s, 1) = q(s).$$

Say that p, q are homotopic if $h(1, t) = x$ for all $t \in I$.

REMARK 6.5. Notice that our definition for π_1 is consistent with this since $S^1 = I/\partial I$.

EXAMPLE 6.6. $\pi_n(S^n) = \mathbb{Z}$

A few facts:

- For simply connected compact spaces if there is one nontrivial higher homotopy group then there are infinitely many.
- There is no non-contractible simply connected compact manifold (or finite CW complex) for which all the homotopy groups are known.
- Amazingly, however, it has been proven that for $n \geq 2$, π_n is abelian.
- If X is a finitely generated simplicial complex which is simply connected, then $\pi_n(X)$ is finitely generated.

EXAMPLE 6.7. $\pi_2(S^1 \vee S^2)$ is not finitely generated.

THEOREM 6.8 (Serre, 1955). *All homotopy groups of spheres are finite with two exceptions. For $n \geq 1$,*

$$\pi_n(S^n) = \mathbb{Z} \quad \pi_{4n-1}(S^{2n}) = \mathbb{Z} \oplus \text{ finite group.}$$

1.4. Back to base points. Say we have a map $p : X \to Y$. Then we have $p_* : \pi_1(X, x) \to \pi_1(Y, p(x))$ via the rule $[f] \mapsto [p \circ f]$. Note that $[f] \in \pi_1(X, x)$ means that $f : I \to X$ based at x. Hence, we have $I \xrightarrow{f} X \xrightarrow{p} Y$. Let q be another map from X to Y which is homotopic to p. If we take a path a from $p(x)$ to $q(x)$, then taking $a(x)(s) = h(x, s)$ where h is the homotopy between p and q we have the following diagram.

$$\begin{array}{ccc} & \pi_1(X,x) & \\ \swarrow^{p_*} & & \searrow^{q_*} \\ \pi_1(Y,p(x)) & \xrightarrow[\cong]{\gamma[a]} & \pi_1(Y,q(x)) \end{array}$$

This diagram commutes. Check by noting that $f : I \to X$ and consider $h(f(x), t)$ as $t : 0 \to 1$. Then stare.

Homotopic maps <u>almost</u> determine the same maps, where almost means up to the bottom isomorphism.

1.4.1. *When is this isomorphism independent of the choice of paths?* If π_1 is commutative, then yes. Generally, no. Say $a : x \to y$ and $b : y \to x$. Then, $\gamma[ba] : x \to x$ and

$$(\gamma[b])(\gamma[a])([f]) = (\gamma[ba])([f]) = [ba][f][ba]^{-1} = [f]$$

when the group is commutative.

1.5. Calculations.

CLAIM 6.9. $\pi_1(\mathbb{R}, 0) = 0$ where 0 is the trivial group.

PROOF. Let $f : I \to \mathbb{R}$. Contract $\mathbb{R}$ to a point by $h(s,t) = f(s)t$. □

Note that the same proof works for $D^2 = \{(x + iy) \mid ||x + iy|| \leq 1\}$. Thus, $\pi_1(D^2, 0) = 0$.

CLAIM 6.10. $\pi_1(S^1, 1) = \mathbb{Z}$

PROOF.

$$\mathbb{R}/\mathbb{Z} = I/\partial I \to^{\alpha} S^1$$

where $\alpha : t \to e^{i2\pi t}$. Let p_n be a polynomial mapping S^1 to S^1 such that $p_n(z) = z^n$. Then let $f_n = p_n \circ \alpha$ and $j : \mathbb{Z} \to \pi_1(S^1, 1)$ with $j(n) = [f_n]$. Then $[f_n][f_m] = [f_{n+m}]$. This is a homomorphism of abelian groups. Consider a map

$$\pi : \pi_1(S^1, 1) \to \mathbb{Z} \quad \pi j = \text{ id}, \pi\ 1-1$$

then π is an isomorphism. Look at $\mathbb{R} \to S^1 \simeq I/\partial I$. Take a piece of the circle as shown in Figure 8.

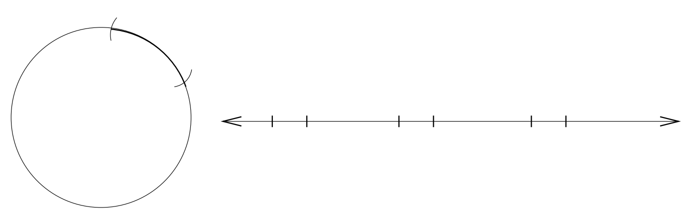
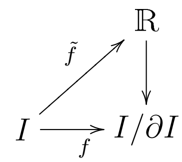


FIGURE 8. A picture of the lifting from a segment on the circle to $\mathbb{Z}$ segments on the real line.

Then in $\mathbb{R}$ this corresponds to $\mathbb{Z}$ copies of the piece as shown in figure 8. Thus, we can lift maps to $\mathbb{R}$ per the diagram

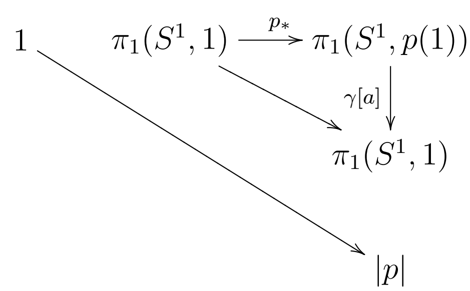


with $\tilde{f}(0) = 0$ and $\pi[f] = \tilde{f}(1)$ by subdividing the leftmost I into small enough divisions that they lift uniquely. $\square$

2. Fundamental Theorem of Algebra

THEOREM 6.11. *Every polynomial in one variable has a complex root.*

DEFINITION 6.12. Let $p : S^1 \to S^1$. Then $\mathbf{deg(}p\mathbf{)} = |p|$ with



PROOF. Say

$$p(x) = x^n + c_1 x^{n-1} + \cdots + c_{n-1}x + c_n.$$

Suppose for contradiction that there are no complex roots of p. In particular, therefore, there are no complex roots on the unit circle. Define

$$\hat{p}(x) = \frac{p(x)}{|p(x)|} \quad x \in S^1.$$

Assume $p(x) \neq 0$ for $|x| \geq 1$. Then define

$$j(x,t) = \frac{k(x,t)}{|k(x,t)|}$$

where

$$k(x,t) = x^n + t(c_1 x^{n-1} + t c_2 x^{n-2} + \cdots + t^{n-1} c_n)$$

and

$$k(x,0) = x^n \quad k(x,1) = p(x) \quad j(x,1) = \hat{p}(x).$$

Thus, $\deg(\hat{p}) = n$. On the other hand, assume $p(x) \neq 0$ for $|x| \leq 1$. Define

$$h(x,t) = \frac{p(tx)}{|p(tx)|}.$$

Then

$$\begin{cases} h(x,0) = \frac{p(0)}{|p(0)|} \\ h(x,1) = \hat{p}(x) \end{cases} \Rightarrow \deg(\hat{p}) = 0.$$

Hence, we have shown that the only time there are no zeros is if $n = 0$ or in other words, p is a constant function. Thus, if $n \geq 1$, p has a complex root. $\square$

2.1. Brouwer fixed point theorem.

THEOREM 6.13. *There does not exist a retraction $r : D^2 \to S^1$ such that $r(x) = x$ for $x \in S^1$.*

Intuition: in order to retract, we must rip the disk somewhere. Also, if $X \to^p Y \to^r Z \ \Rightarrow\ r_* \circ p_* = (rp)_*$.

PROOF. Suppose there is. Then

$$\begin{array}{ccc} \pi_1(S^1) & \xrightarrow{i_*} & \pi_1(D^2) \\ & \searrow^{id_*} & \downarrow r_* \\ & & \pi_1(S^1) \end{array}$$

Now, since $\pi_1(D^2) = 0$, this must be a contradiction since we cannot factor the identity map through 0. $\square$

COROLLARY 6.14. *Let $f : D^2 \to D^2$. Then there exists a y such that $f(y) = y$.*

Suppose not. Then for all y we have $f(y) \neq y$ so we can take the map to S^1 given by the intersection point of S^1 and the ray from $f(x)$ to x. If $y \in S^1$ this ray will return y so this would be a retraction as above.

LECTURE 7

Tuesday, July 17 [1]

1. Vector bundles

First, we explain vector bundles. The idea is that we all know what vector spaces are. That's linear algebra. We also know about topological spaces. We would like to blend the two together. So we start with a topological space (say a subset of Euclidean space), and for each point in that space, we will have "over it" a vector space.

Say the topological space is B. We would like a surjective continuous map

$$E \xrightarrow{p} B$$

(E is the total space, B is the base space) with the property that for each point, the fibre $p^{-1}(b)$ has a fixed isomorphism to $\mathbb{R}^n$ or $\mathbb{C}^n$. That's not enough, but here's a picture.

Think of the Moebius strip. The Moebius strip will be E. There is a central circle. The map p in this case, is simply projection to that central circle.

On the other hand, we could also have just the product

$$S^1 \times \mathbb{R}.$$

These are examples of two different bundles over the same space.

We always have an example of a "trivial bundle"

$$B \times \mathbb{R}^n \to B.$$

In a sense, this is the most important example since we require that every bundle be locally trivial:

Definition 7.1. A **local trivialization** for

$$E \xrightarrow{p} B$$

is an open cover of B, $\mathcal{O} = \{U\}$, and for each U, a homeomorphism

$$\phi_u : U \times \mathbb{R}^n \hookrightarrow p^{-1}(U)$$

[1]TeXed by Mohammed Abouzaid.

such that the diagram

$$\begin{array}{ccc} U \times \mathbb{R}^n & \xrightarrow{\phi_U} & p^{-1}(U) \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle p} & \\ & U & \end{array}$$

commutes.

Recall also, that we have a canonical identification of $p^{-1}(b)$ with $\mathbb{R}^n$ such that the the restriction of ϕ_U to $\{b\} \times \mathbb{R}^n$ is a linear map.

To simplify life, we will assume that B is compact.

Definition 7.2. A topological space B is **compact** if every open cover has a finite subcover.

We would like our choice of local trivialization to be compatible. Consider the composition

$$(U \cap V) \times \mathbb{R}^n \xrightarrow{\phi_U} p^{-1}(U \cap V) \xrightarrow{\phi_V^{-1}} (U \cap V) \times \mathbb{R}^n$$

Since all of these maps have to be isomorphisms of vector spaces on each fibre, we obtain a map

$$U \cap V \to GL_n(\mathbb{R}).$$

Definition 7.3. A **vector bundle** is a surjective continuous map

$$E \xrightarrow{p} B$$

which is locally trivial for a cover $\mathcal{O}$ such that the corresponding maps

$$U \cap V \to GL_n(\mathbb{R})$$

are continuous.

Note that $GL_n(\mathbb{R})$ is a subset of the set of $n \times n$ matrices, and hence is topologized as a subset of $\mathbb{R}^{n^2}$.

Later, we will see an important theorem stating that every vector bundle embeds in a trivial bundle.

2. Tangent bundles

Given a manifold, we can study embeddings

$$M \hookrightarrow \mathbb{R}^q.$$

The Whitney embedding theorem guarantees that if M is n dimensional, we can choose q to be $2n + 1$.

For example, we can take the n-dimensional sphere

$$S^n \to \mathbb{R}^{n+1}.$$

In this case, at every x, we can study the set of v which are orthogonal to x. This is an n-dimensional vector space at every point, called the tangent space.

So have a subset of $S^n \times \mathbb{R}^{n+1}$ consisting of

$$\{(x, v) | v \perp x\}$$

which we call the tangent space of S^n. We write this at $\tau(S^n)$, which is a vector bundle S^n.

We can also consider the set of points which are parallel to $x \in S^n$. We can do this for every manifold which embeds in $\mathbb{R}^n$ to obtain a tangent and normal bundle which both embed in the trivial bundle $M \times \mathbb{R}^q$. We can topologize them as subsets, and check that they satisfy the appropriate axioms.

One way of checking that these are bundles, is to use upper and lower hemispheres as the appropriate cover.

In this course, we're not going to study bundles one at a time. Rather, we will consider all bundles over a fixed base space, and study the appropriate structure.

First, we need

Definition 7.4. An **isomorphism** of vector bundles E and E' over the same base B is a map

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle g'} & \\ & B & \end{array}$$

which is a linear isomorphism on each fibre.

It turns out that the inverse function is automatically continuous, and induces the inverse isomorphism of vector spaces when restricted to each fibre. We will only be looking at isomorphism classes of vector bundles. We will use ξ for a vector bundle, and $[\xi]$ for its class.

If B is connected, the dimension of the vector bundle is constant. However, if B is not connected, there are two different notions. Either we study n-plane bundles, in which the dimension is kept fixed, or, more generally, we can study vector bundles, where the dimension is allowed to vary in the different components. For simplicity, we will always assume connectivity of B.

DEFINITION 7.5. $\mathcal{E}_n(B)$ is the set of equivalence classes of n-plane bundles over B.

One may complain that this may not be a set, but we will come to this set-theoretic point later.

DEFINITION 7.6. Vect(B) is the set of isomorphism classes of vector bundles over B.

If B is path connected, then

$$\mathrm{Vect}(B) = \coprod_{n \geq 0} \mathcal{E}_n(B),$$

since every vector bundle has a fixed dimension.

Question: What are the isomorphism classes of 0-dimensional bundles?

Answer: There is only one such bundle, consisting of

$$\mathrm{id} : B \to B,$$

we call this $[\epsilon_0]$.

This should be thought of as the analogue of the trivial vector space whose only element is 0. This is a general principle. Anything you can do to vector spaces, I can do to vector bundles.

For example, given ξ and χ, let us write $\xi_b = \xi^{-1}(b)$ for the fibre at a point b. This is standard notation where the projection is given the same name as the vector bundle. We can now define the Whitney sum of ξ and χ to be be the vector bundle $\xi \oplus \chi$ whose fibre is given by

$$(\xi \oplus \chi)_b = \xi_b \oplus \chi_b.$$

We will come back later to why this operation is well defined. But we can check that this operation is compatible with isomorphism classes. Further, the 0-dimensional bundle acts as the zero for this operation. The result is therefore an abelian monoid, modulo the details we haven't checked.

We can now apply the Grothendieck construction to this monoid, and obtain

$$K(B)$$

which is the Grothendieck group of B. When we write $K(B)$, we mean the Grothendieck group of complex vector bundles over B, which we sometimes also write $KU(B)$.

There is also a Grothendieck group of real vector bundles

$$KO(B).$$

But we need more than a group. Rather, we would like to have a ring. First, we need to discuss tensor products.

3. Tensor Products

Let R be a commutative ring, in particular, it could be a field, which is the case we're interested in. Suppose we have M, N two R modules (commutativity implies that we don't have to worry about left and right modules). In other words, we have abelian groups together with a multiplication

$$R \times M \to M$$

which is associative and bilinear. In general, a bilinear map

$$M \times N \xrightarrow{f} P$$

is a map which is linear in both variables separately.

$$\begin{aligned} f(m,n) + f(m',n) &= f(m+m',n) \\ f(rm,n) &= rf(m,n). \end{aligned}$$

We will now define the tensor product $M \otimes_R N$ via its universal property:

$$\begin{array}{ccc} M \times N & \xrightarrow{f} & P \\ \downarrow{\scriptstyle i} & \nearrow{\scriptstyle \tilde{f}} & \\ M \otimes_R N & & \end{array} .$$

In order to check that this makes sense, we must provide a construction.

Let F be the free R module on the set $M \times N$. Elements of F are formal sums $\sum r_i(m_i, n_i)$ where all but finitely many entries in the sum vanish. We define

$$M \otimes_R N = F/\sim$$

where the equivalence relation is generated by

$$\begin{aligned} (m,n) + (m',n) - (m+m',n) &= 0 \\ (rm,n) - r(m,n) &= 0 \\ (m,n) + (m,n') - (m,n+n') &= 0 \\ (m,rn) - r(m,n) &= 0. \end{aligned}$$

Note the fact that our relations are symmetric in M and N.

Example 7.7. Let V and W be finite dimensional vector spaces of $\mathbb{F}$. In order to be concrete, we choose bases $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$

of V and W respectively. Consider a vector space $V \otimes W$ spanned by a basis $\{v_i \otimes w_j\}_{1 \le i \le m}^{1 \le j \le n}$ and define a map

$$\begin{array}{c} (V \times W) \\ \Big\downarrow{\scriptstyle \iota} \\ V \otimes W \end{array}$$

where

$$\iota(v_i, w_j) = v_i \otimes w_j$$

on basis elements, and is extended by bilinearity to every element of $V \times W$. Given any bilinear function to P, we can construct a linear map $\tilde{f}$ such that the diagram

$$\begin{array}{ccc} (V, W) & \xrightarrow{f} & P \\ \Big\downarrow{\scriptstyle \iota} & \nearrow_{\tilde{f}} & \\ V \otimes W & & \end{array}$$

commutes.

This proves the existence (and uniqueness) of the tensor product of vector spaces.

We have a distributivity property

$$(V \oplus V') \otimes W \cong (V \otimes W) \oplus (V' \oplus W),$$

which can be proved either by the universal property, or by the construction.

Note that

$$0 \otimes W = 0$$

while

$$\mathbb{F} \otimes W \cong W.$$

So direct sum looks like addition, and tensor product looks like multiplication. Formally, we can say that the set of isomorphism classes of vector spaces over a fixed field is a semi ring under these two operations, however, this is a boring object, since isomorphism classes of vector spaces are determined by their dimension.

However, if we start with an arbitrary ring, and we restrict to good (projective) finitely generated modules, we obtain

$$K_0(R)$$

the K-theory group of the ring R. The subscript is referring to the existence of higher K-theory groups. This is in fact the beginning of the rich subject of algebraic K-theory.

4. Topological K-theory

We now repeat the same procedure with vector bundles. First, we define the tensor product of vector bundles. Given ξ and ξ', we define $\xi \otimes \xi'$ to be the new vector bundle whose fibre is

$$(\xi \otimes \xi')_b = \xi_b \otimes \xi'_b.$$

So $\mathrm{Vect}(B)$ is a semi ring under Whitney sum and tensor product with

$$0 = [\epsilon_0]$$
$$1 = [\epsilon_1].$$

where

$$\epsilon_1 : B \times \mathbb{R} \to B,$$

is just given by projection to the first factor.

Unfortunately, we have been keeping B fixed this whole time. We need an aside

Remark 7.8 (Aside on Categories). A category is a field of mathematics. It has objects, say, spaces or groups, and maps between a pair of objects. We write X, Y, $\cdots$ for the objects, and $\mathcal{C}(X,Y)$ for the set of maps between these objects.

We have composition maps

$$\mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$$

which satisfy associativity, and we have chosen identities

$$\mathrm{id}_X \in \mathcal{C}(X,X).$$

Now a functor acts on categories

$$F : \mathcal{C} \to \mathcal{D}$$

assigning an object FX of $\mathcal{D}$ to every object C of $\mathcal{C}$, and acts on morphisms in one of two different ways. A functor can either be covariant

$$F : \mathcal{C}(X,Y) \to \mathcal{D}(FX,FY),$$

or contravariant

$$F : \mathcal{C}(X,Y) \to \mathcal{D}(FY,FX).$$

We would like to say that K is a contravariant functor from topological spaces to groups.

Given a continuous map

$$f : B' \to B$$

we will construct a map of rings

$$K(f) : K(B) \to K(B').$$

Further, as in the proof of fundamental theorem of algebra, we would like for this construction be to homotopy invariant

$$f \cong g \Rightarrow K(f) = K(g).$$

Given a vector bundle over B, and a map $f : B' \to B$, we will construct the pullback of E. Roughly speaking,

$$E' = \{(b', e) | f(b') = p(e)\} \subset B' \times E.$$

We claim that this bundle fits in a commutative diagram

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B' & \xrightarrow{f} & B \end{array}.$$

LECTURE 8

Wednesday, July 19 [1]

Recall that we defined vector bundles to be surjective maps

$$E \to B$$

whose fibres

$$E_b \equiv p^{-1}(b)$$

are identified with either $\mathbb{R}^n$ or $\mathbb{C}^n$. Further, we require a local triviality condition relative to a cover $\mathcal{O}$ of B.

We saw examples such as the Moebius strip, and the tangent and normal bundles of S^n. However, our goal is to consider

$$\mathcal{E}_n^O(B),$$

and

$$\mathcal{E}_n^U(B),$$

the sets of equivalence classes of (respectively) real and complex vector bundles over B. We also introduced

$$\mathrm{Vect}(B),$$

the set of equivalence classes of bundles of arbitrary (finite) dimension. $\mathrm{Vect}(B)$ is closed under direct sums and tensor products, and is in fact a semi-ring. We defined

$$K(B) = KU(B)$$

to be the Grothendieck group of this semi-ring of complex vector bundles, and

$$KO(B)$$

to be the Grothendieck group of real vector bundles.

From the point of view of mathematics in general, KU is more important, but for an algebraic topologist, KO is a richer object.

We also introduced the notion of categories. One important convention is that we will write an element of $\mathcal{C}(X, Y)$, a map from X to Y, as an arrow

$$X \to Y.$$

[1] TEXed by Mohammed Abouzaid.

1. A Category of vector bundles

Consider the category Vect whose objects are vector bundles over arbitrary spaces. We do not take the quotient by equivalence.

A morphism between the vector bundles p and p' is a commutative diagram of continuous maps

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{f} & B'. \end{array}$$

There are two choices here. We can either require that g be a linear map on each fibre, or, more restrictively, that g induce an isomorphism on each fibre.

The composition of morphisms

$$\begin{array}{ccccc} E & \xrightarrow{g} & E' & \xrightarrow{g'} & E'' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p''} \\ B & \xrightarrow{f} & B' & \xrightarrow{f'} & B'', \end{array}$$

is simply given by composing g with g' and f with f'

$$\begin{array}{ccc} E & \xrightarrow{g'\circ g} & E'' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p''} \\ B & \xrightarrow{f'\circ f} & B''. \end{array}$$

In any category, we can formulate the notion of Cartesian product. Given X and Y objects, the Cartesian product is an object $X \times Y$ together with maps to X and Y satisfying the universal property

$$\begin{array}{ccccc} X & \longleftarrow & X \times Y & \longrightarrow & Y \\ & \nwarrow & \vdots\uparrow & \nearrow & \\ & & Z. & & \end{array}$$

Not every category has Cartesian products. However, since Cartesian products exist for sets, it is sometimes possible to construct a Cartesian product by first taking the Cartesian product of sets, then equipping it with the appropriate structure.

We claim that Vect is a category with Cartesian products. Indeed, we can define the Cartesian product of

$$\begin{array}{ccc} D & & E \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\ A & & B \end{array}$$

to be the map

$$\begin{array}{c} D \times E \\ \big\downarrow{\scriptstyle p \times q} \\ A \times B, \end{array}$$

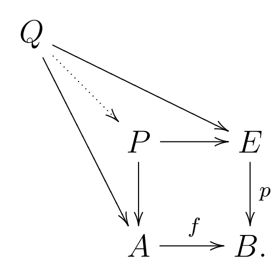
where the linear structure on the fibres is given by the product of the linear structures, and local triviality follows by taking the cover of $A \times B$ which consists of products of the open sets of A and B which are used to prove local triviality of p and q.

Note that in order for this object to be a Cartesian product, we must allow the maps of vector bundles

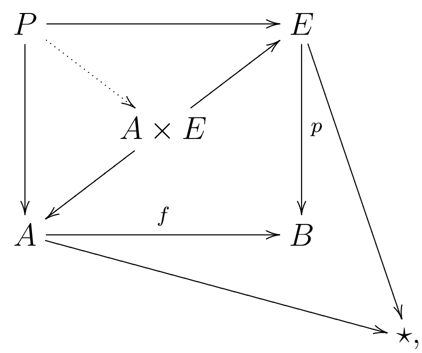
$$\begin{array}{ccccc} D & \longleftarrow & D \times E & \longrightarrow & E \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p \times q} & & \big\downarrow{\scriptstyle q} \\ A & \longleftarrow & A \times B & \longrightarrow & B \end{array}$$

which are NOT isomorphisms on each fibre. This is a general principle that requires us to enlarge our categories in order to be able to obtain a richer structure. Note that we must verify that the universal property is satisfied for this construction.

We now define pullback in an arbitrary category. Given two maps p and f in any category, we define their pullback P to be the object equipped with maps to E and A satisfying the a universal property



If we let B be a point, then P is the Cartesian product. In the category of sets, we can map B to a point, and consider the diagram



in particular, the universal property of the Cartesian product implies that P is equipped with a map to $A \times E$. In the category of sets, we can in fact define the pullback to be

$$\{(a,e)|f(a) = p(e)\} \subset (A, E).$$

We must now prove the existence of pullbacks in Vect. Given a vector bundle p and a map of topological spaces f, we define $f^*(E)$ to be the pullback of topological spaces

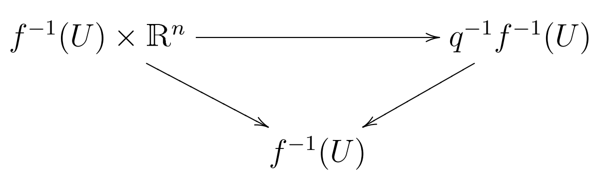
$$\begin{array}{ccc} f^*(E) & \xrightarrow{g} & E \\ \downarrow & & \downarrow p \\ A & \xrightarrow{f} & B \end{array} .$$

LEMMA 8.1. *f^*E is a vector bundle.*

PROOF. Note that

$$q^{-1}(a) = \{(a,e)|f(a) = p(e)\} = p^{-1}(f(a)).$$

So, in particular, the fibres of q are vector spaces. As to local triviality, we consider a cover $\mathcal{O} = \{U\}$ of open sets $U \subset B$. Now $f^{-1}(U)$ is an open set in A, and we can construct a trivialization



by using the trivialization of p. □

Note that this means that given a map f from A to B, we obtain a well-defined map

$$\mathcal{E}_n(B) \xrightarrow{f^*} \mathcal{E}_n(A)$$

once we check that an isomorphism of vector bundles over B gives an isomorphism of their pullbacks over A.

Let $\mathcal{U}$ be the category of topological spaces (secretly, compact and connected). And for each n, we have a functor $\mathcal{E}_n$

$$\mathcal{U} \xrightarrow{\mathcal{E}_n} \text{Sets} .$$

In particular, we are assigning not only a set $\mathcal{E}_n(B)$ to every topological spaces B, but also a map of sets

$$f^*\colon\ \mathcal{E}_n(B) \to \mathcal{E}_n(A)$$

to every map

$$f\colon\ A \to B.$$

Note that the order was reversed in this procedure. Further, we must prove that the functor preserves composition (although it reverses the order in which it's taken).

We can now make our construction of Whitney sums rigorous. Indeed, given two bundles p and q over A and B,

$$\begin{array}{ccc} D & & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ A & & B \end{array}$$

the Cartesian product

$$\begin{array}{c} D \times E \\ \downarrow{\scriptstyle p\times q} \\ A \times B, \end{array}$$

can be thought of as an **external** direct sum of vector bundles. If our bundles are bundles over the same base space we simply pull back this direct sum over the diagonal

$$\begin{array}{ccc} D \oplus E & \longrightarrow & D \times E \\ \downarrow & & \downarrow{\scriptstyle p\times q} \\ B & \xrightarrow{\Delta} & B \times B. \end{array}$$

This define the Whitney sum rigorously. We can define tensor products in the same way, by first defining an **external tensor product** of vector bundles, then pulling back this construction along the diagonal. In general, to check continuity, it is easiest to think of vector bundles as a set of continuous maps

$$U \cap V \xrightarrow{\phi_{U\cap V}} GL_n(R)$$

satisfying an appropriate co-cycle condition. Any continuous operations on $GL(R)$ yields a natural construction on vector bundles.

Recall that if B is connected,

$$\mathrm{Vect}(B) = \coprod_{n \geq 0} \mathcal{E}_n(B).$$

Note that we have a set valued functor $\mathcal{E}_n$ for each n. By taking their "disjoint union" we obtain a set valued functor

$$\mathcal{U} \xrightarrow{\mathrm{Vect}} \mathrm{Sets}\ .$$

Now, $\mathrm{Vect}(B)$ had the structure of a semi ring. In fact, given a map $f\colon\ A \to B$, its pullback

$$f^*\colon\ \mathrm{Vect}(B) \to \mathrm{Vect}(A)$$

is a map of semi-rings. This entails checking that

$$f^*(p \oplus q) \cong f^*(p) \oplus f^*(q)$$

which can be checked easily at the level of fibres. The same should be done for the tensor product.

We also have the Grothendieck construction which is a functor from semi-rings to rings. Indeed, given ϕ, a map of semi-rings, there exists a unique map $\tilde{\phi}$ making the diagram

$$\begin{array}{ccc} R & \xrightarrow{\phi} & S \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ K(R) & \xrightarrow{\tilde{\phi}} & K(S) \end{array}$$

commute. The existence (and uniqueness) of $\tilde{\phi}$ is simply a consequence of the universal property of $K(R)$. In particular, if A and B are spaces, and

$$f\colon\ A \to B$$

is a continuous map, we obtain a map of rings,

$$K(B) \xrightarrow{K(f)} K(A)$$

in particular, K is a functor from spaces to rings.

THEOREM 8.2. *If $f \simeq f'\colon\ A \to B$ then*

- $\mathcal{E}_n(f) = \mathcal{E}_n(f')$
- $\mathrm{Vect}(f) = Vect(f')$
- $K(f) = K(f')$.

This is a surprising (though not difficult to prove) fact. The conclusion is the existence of isomorphisms of vector bundles, which is a rigid notion, while the assumption, that of the existence of a homotopy, is a much weaker condition. As a consequence, we will be able to prove important results in topology. For example, Milnor's proof that there are different smooth structures on the n-sphere used cobordism theory, which is closely related to K-theory. Indeed, the classification of such structure, uses the homotopy groups of spheres, and relies on analogous constructions.

Recall that a homotopy from f to f' is simply a map

$$h\colon\ A \times I \to B$$

such that $h(a,0) = f$ and $h(a,1) = f'$.

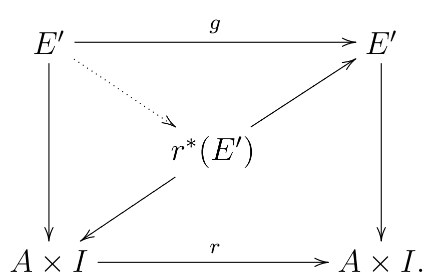
Proof of Theorem 8.2. The idea is to look at $h^*(E)$, which is a bundle over $B \times I$.

Claim 8.3. For any bundle E', there exists a map g of vector bundles

$$\begin{array}{ccc} E' & \xrightarrow{g} & E' \\ \downarrow & & \downarrow \\ A \times I & \xrightarrow{r} & A \times I, \end{array}$$

where $r(b,t) = (t,1)$.

Let us assume the claim holds. If we consider the pull-back, we obtain a diagram



Now, simply because these are maps of bundles, we can study the fibres, and conclude that the map

$$E' \to r^* E'$$

is an isomorphism.

Let us consider the case where $E' = h^*E$; i.e. it is pulled back from the homotopy. In this case,

$$h|_{A\times\{1\}} = f'$$

so

$$r^*E' = (f')^*E \times I \to A \times I.$$

In particular, the factor I does not enter in the construction of r^*E'. But the claim implies that h^*E' is isomorphic to this vector bundle. Since the same property hold for the restriction to $A \times \{0\}$, we conclude that

$$f^*(E) \cong f'^*(E).$$

□

Preview: On Friday, we will construct spaces $BO(n)$ and $BU(n)$, and we will show that for real vector bundles,

$$\mathcal{E}_n \cong [B, BO(n)]$$

where the right hand side refers to homotopy classes of maps from B to $BO(n)$. An analogous statement holds for complex vector bundles.

LECTURE 9

Friday, July 21 [1]

We will eventually prove the Bott Periodicity theorem

$$K(X \times S^2) \cong K(X) \otimes K(S^2).$$

Using the fact that a vector bundle over a point is just a vector space, and vector spaces are determined up to isomorphism by their dimension, we see that

$$K(\star) = \mathbb{Z}.$$

If we choose a point $\star \in X$, the inclusion induces

$$K(X) \xrightarrow{\epsilon} K(\star)$$

which is just recording the dimension of the fibre at $\star$. We call the kernel of ϵ the **reduced** K-theory $\tilde{K}(X)$.

It is a crucial fact that

$$\tilde{K}(S^2) \cong \mathbb{Z}.$$

This can be proved concretely as is done in Atiyah's book, by using the cover of S^2 by the northern and southern hemisphere. We can also use a more-homotopy theoretic approach.

Consider X and Y two topological spaces. Consider the set

$$\mathrm{Maps}(X, Y)$$

of continuous maps from X to Y. We can partition this set into equivalence classes of **homotopic maps**, where

$$f \simeq g$$

if there exists a map

$$h\colon X \times I \to Y$$

such that

$$h \circ i_0 = f$$
$$h \circ i_1 = g,$$

where $i_0(x) = (x, 0)$, and $i_1(x) = (x, 1)$. One should check that this is an equivalence relation by proving transitivity, reflexivity, etc.

[1] TEXed by Mohammed Abouzaid.

We write $[X, Y]$ for the quotient of the set of maps by this equivalence relation.

This allows us to define the homotopy category of spaces, whose

- Objects are topological spaces,
- Morphisms are homotopy classes of maps, i.e the set of morphisms from X to Y is $[X, Y]$.
- Composition is given by choosing representatives of our homotopy classes, then composing them as ordinary maps of spaces, and finally taking the homotopy class of the composite. One should check that this is independent of the choice of representatives.

We are about to prove that

$$\mathcal{E}_n^{\mathbb{R}}(X),$$

the set of isomorphism classes of real n-plane bundles over X, can be identified with

$$[X, BO(n)],$$

where $BO(n)$ is a space we are about to construct. A stronger fact is that the functors $\mathcal{E}_n$ and $[\_, BO(n)]$ are naturally isomorphic functors.

Given two contravariant functors

$$F, G\colon \mathcal{C} \to \mathcal{D}$$

a natural transformation

$$\eta : F \to G$$

is a " map between these two functors." More precisely, for every object X of $\mathcal{C}$, we have a map

$$\eta_X\colon F(X) \to G(X)$$

such that for every morphism $f\colon X \to Y$, the diagram

$$\begin{array}{ccc} F(Y) & \xrightarrow{F(f)} & F(X) \\ \downarrow{\scriptstyle \eta_Y} & & \downarrow{\scriptstyle \eta_X} \\ G(Y) & \xrightarrow{G(f)} & G(X) \end{array},$$

commutes.

So if we let $G(X) = \mathcal{E}_n^{\mathbb{R}}(X)$, and $F(X) = [X, BO(n)]$, we obtain contravariant functors from the homotopy category of spaces to sets. To say that $\mathcal{E}_n^{\mathbb{R}}$ is a functor means that given a continuous map

$$f\colon X \to Y,$$

then we can pull-back bundles from Y to X

$$\begin{array}{ccc} f^*(E) & \xrightarrow{g} & E \\ \downarrow & & \downarrow p \\ X & \xrightarrow{f} & Y \end{array} .$$

and hence define a map

$$\begin{aligned} \mathcal{E}_n(f)\colon\ \mathcal{E}_n(Y) &\to \mathcal{E}_n(X) \\ [E] &\mapsto [f^*E]. \end{aligned}$$

We are now ready to define $BO(n)$. Recall that an inner product on a real vector space V is a bilinear, symmetric, non-degenerate pairing. Recall that two vectors are orthogonal if their inner product vanishes. A subset $\{b_i\}$ of V is said to be orthonormal if

$$\langle b_i, b_j \rangle = \delta_{i,j}.$$

In other words, inner products among the vectors b_i and b_j vanish unless $i = j$, in which case the inner product is equal to one.

Let

$$V_n(\mathbb{R}^q) \subset \mathbb{R}^{nq}$$

denote the set of n-tuples of orthonormal vectors in $\mathbb{R}^q$. In particular, $V_n(\mathbb{R}^q)$ inherits a natural topology from the above inclusion. This is the Stiefel manifold (variety). Note that $V_n(\mathbb{R}^q)$ is empty if $n > q$.

Given an n-tuple of orthonormal vectors, we obtain a basis for an n dimensional subspace of $\mathbb{R}^q$ which is simply their span. In particular

$$V_2(\mathbb{R}^3)$$

is simply the set of possible orthonormal bases for planes in $\mathbb{R}^3$. Let us define

$$G_n(\mathbb{R}^q)$$

to be the set of all n-planes in $\mathbb{R}^q$. When $n = 1$, we simply recover the projective space $\mathbb{RP}^q$ of lines in $\mathbb{R}^q$. We have a map

$$V_n(\mathbb{R}^q) \xrightarrow{\pi} G_n(\mathbb{R}^q)$$

which takes every orthonormal set to the plane that it spans. We can specify the quotient topology on $G_n(\mathbb{R}^q)$ which makes this map continuous. With this topology, we call $G_n(\mathbb{R}^q)$ the Grassmannian, or the Grassmann manifold.

On $G_n(\mathbb{R}^q)$, we have a natural subbundle of the trivial bundle

$$\begin{array}{c} G_n(\mathbb{R}^q) \times \mathbb{R}^q \\ \downarrow \\ G_n(\mathbb{R}^q). \end{array}$$

which as a set is given by

$$E(\gamma_n^q) = \{(x, v) | v \in x\}.$$

Let γ_n^q denote the projection

$$\begin{aligned} E(\gamma_n^q) &\to G_n(\mathbb{R}^q) \\ (x, v) &\mapsto x. \end{aligned}$$

From the point of view of n-plane bundles, the choice of q is arbitrary. However, if we embed $\mathbb{R}^q$ into $\mathbb{R}^{q+1}$, we obtain a map

$$G_n(\mathbb{R}^q) \to G_n(\mathbb{R}^{q+1})$$

letting q go to infinity, we obtain a space

$$BO(n) \equiv G_n(\mathbb{R}^\infty).$$

The right hand side is topologized as an increasing union. This is not the topology that an analyst would give to $\mathbb{R}^\infty$. Note that in fact, we only care about the **homotopy type** of the space $BO(n)$.

Let us now explain why $F(X) = [X, BO(n)]$ is a functor. This is in fact a general fact.

Lemma 9.1. *If $\mathcal{C}$ is any category, and Y is an object of $\mathcal{C}$, then*

$$F(X) \equiv \mathcal{C}(X, Y)$$

is a functor from $\mathcal{C}$ to Sets*.*

Proof. Given

$$f\colon\ X \to X',$$

precomposition with f yields a map

$$\mathcal{C}(X', Y) \to \mathcal{C}(X, Y).$$

Associativity of composition of morphisms in $\mathcal{C}$ yields the desired properties of F. □

In general, functors of this type are called **representable**. The idea is that Y represents the functor F. In our case, Y will be $BO(n)$.

Note that the bundles $E(\gamma_n^q)$ glue together to give a bundle

$$EO(n) \xrightarrow{\gamma_n} BO(n)$$

such that the fibre over every point of $BO(n)$ (which represents an n-plane in $\mathbb{R}^\infty$), is simply that n-dimensional vector space.

We define a natural transformation

$$\begin{aligned}\Phi\colon\ [X, BO(n)] &\to \mathcal{E}_n(X)\\ [g\colon\ X \to BO(n)] &\mapsto [g^*(\gamma_n)].\end{aligned}$$

Recall that $[g^*(\gamma_n)]$ is the isomorphism class of the pull-back of the universal bundle $EO(n)$ over $BO(n)$. This idea of representing functors is a central idea of mathematics.

Now, given any map

$$f\colon\ X \to X',$$

we obtain a commutative diagram

$$\begin{array}{ccc} [X', BO(n)] & \xrightarrow{f^* \equiv [f, \mathrm{id}]} & [X, BO(n)] \\ \downarrow{\scriptstyle \Phi_{X'}} & & \downarrow{\scriptstyle \Phi_X} \\ \mathcal{E}_n(X') & \xrightarrow{f^* \equiv \mathcal{E}_n(f)} & \mathcal{E}_n(X). \end{array}$$

The commutativity of this diagram states that pulling back $EO(n)$ to X' using a map g, then pulling that back to X using f, yields a bundle which is isomorphic to the pullback of $EO(n)$ by the composite $f \circ g$. This establishes the fact that Φ is a natural transformation. In fact, we are simply using the general fact that

$$\begin{array}{ccccc} (f^*(g^*E) \cong (g \circ f)^*E & & \longrightarrow & & E \\ \downarrow & & & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z, \end{array}$$

which can be checked from the construction of the two bundles on the top left corner, or by the universal property of pullbacks.

This natural isomorphism of functors gives us two different methods for studying vector bundles. It remains to prove three things.

- First, if $f \simeq g\colon\ X \to Y$ and $p\colon\ E \to Y$ is an n-plane bundle, then $f^*E \cong g^*E$.
- Next, Φ is surjective.
- Lastly, Φ is injective.

This will complete the proof that

$$\Phi\colon\ [X, BO(n)] \to \mathcal{E}_n(X)$$

is a natural isomorphism. We begin by proving surjectivity.

Proof of Surjectivity. The idea is to use a variant of the Whitney embedding theorem. Say M is a manifold embedded in $\mathbb{R}^q$, e.g. the sphere S^{q-1}. We can translate the tangent plane at each point to a plane at the origin, which gives a map

$$\begin{array}{ccc} \tau(M) & \longrightarrow & E(\gamma_n^q) \\ \downarrow & & \downarrow \\ M & \longrightarrow & G_n(\mathbb{R}^q). \end{array}$$

which is called the Gauss map of the tangent bundle. If we take every point to its normal plane, we obtain the Gauss map of the normal bundle. The idea is to generalize this to arbitrary bundles.

For simplicity, let us assume that our space is compact. This allows us to work with finite covers. Suppose we have such a bundle E over X. Let $\mathcal{O} = \{U_1, \ldots, U_m\}$ denote the cover over which we have trivializations

$$\begin{array}{ccc} U_i \times \mathbb{R}^n & \longrightarrow & p^{-1}(U_i) \\ & \searrow \quad \swarrow & \\ & U_i & \end{array} .$$

Now, recall that $E(\gamma_n^q) \subset G_n(\mathbb{R}^q) \times \mathbb{R}^q$. So we would like a continuous map of the total space

$$\hat{g} \colon E \to \mathbb{R}^q$$

such that the restriction of $\hat{g}$ to every fibre is an injective linear map. Using the planes given by the images of the fibres, this would define a map $X \to G_n(\mathbb{R}^q)$, and, by construction, we would have a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & E(\gamma_n^q) \\ \downarrow & & \downarrow \\ X & \longrightarrow & G_n(\mathbb{R}^q), \end{array}$$

having the property that the top map is an isomorphism on fibres, which establishes the fact that E is isomorphic to the pullback of the universal bundle.

We need one fact from point set topology.

Lemma 9.2 (Urysohn's Lemma). *Under appropriate conditions on the topological space B, there exists a map*

$$\lambda_i : B \to I$$

such that $\lambda_i^{-1}((0,1]) = U_i$.

Let $q = m \times n$, i.e: the size of the fibre times the number of open sets in the cover. We think of $\mathbb{R}^q$ as

$$\mathbb{R}^n \oplus \mathbb{R}^n \cdots \oplus \mathbb{R}^n$$

with the ith factor corresponding to U_i. We now define $\hat{g}$ by

$$g = (g_1, \ldots, g_m)\colon\ E \to \mathbb{R}^n \oplus \mathbb{R}^n \cdots \oplus \mathbb{R}^n$$

where

$$g_i(e) = \lambda_i(p(e)) \cdot \pi_2(\phi_{U_i}^{-1}(e))$$

where π_2 is the projection onto the $\mathbb{R}^n$ factor of

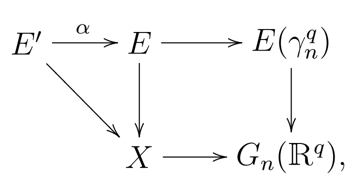
$$U_i \times \mathbb{R}^n.$$

Note that g_i is clearly a monomorphism on each fibre $p^{-1}(U_i)$ since it simply identifies every fibre with the corresponding copy of $\mathbb{R}^n$ coming from the trivialization ϕ_{U_i}. Since every point lies in some U_i, we conclude that g is globally a monomorphism on fibres. This completes the proof of surjactivity of Φ. □

Proof of Injectivity. We must prove that $\Phi[f] = \Phi[f']$ implies that f and f' are homotopic. Let

$$E = f^*(\gamma_n^q) \cong f'^*(\gamma_n^q) = E'.$$

We have an isomorphism in the diagram



and we would like to claim that f and f' are homotopic. But we know that the isomorphism class of the pullback is determined by the Gauss maps. In fact, if the Gauss maps of E and E' were linearly independent at each point, we could simply use the homotopy

$$\hat{h}(e, t) = t\hat{g} + (1 - t)\hat{g}',$$

which would construct the desired homotopy between f and f'.

In order to achieve the hypothesis that the Gauss maps of E and E' have linearly independent images, we use two maps

$$\begin{aligned} \alpha\colon\ \mathbb{R}^\infty &\to \mathbb{R}^\infty \\ e_q &\mapsto e_{2q} \\ \beta\colon\ \mathbb{R}^\infty &\to \mathbb{R}^\infty \\ e_q &\mapsto e_{2q+1}, \end{aligned}$$

which are both clearly homotopic to the identity. Note that regardless of what $\hat{g}$ and $\hat{g}'$ are, the maps $\alpha\hat{g}$ and $\beta\hat{g}'$ are necessarily linearly independent. We can therefore use the above linear homotopy, and conclude that f and f' are homotopic. $\square$

LECTURE 10

Monday, July 24[1]

1. Pullback

Suppose $f : X \to Y$ be a continuous map. This induces a map $f^* : \mathcal{E}_n(X) \to \mathcal{E}_n(Y)$, where $\mathcal{E}_n(X)$ denotes the set of isomorphism classes of n-vector bundles on X. Our goal is to finish the proof of the following theorem.

THEOREM 10.1. *Homotopic maps induce isomorphic pullbacks of vector bundles.*

We need some preliminary lemmas.

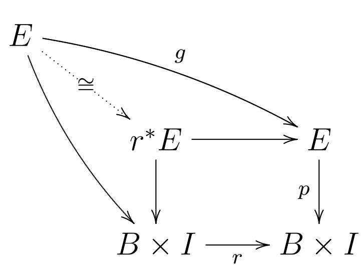
LEMMA 10.2. *Set $p : E \to U \times [a, c]$ with $a < b < c$. If $p|_{U\times[a,b]}$ and $p|_{U\times[b,c]}$ are trivial, then p is trivial.*

PROOF. The idea is to glue the two trivializations for the sub-bundles together to explicitly construct a trivialization for the whole bundle. □

LEMMA 10.3. *There exists a finite open cover $\{U_1, ..., U_n\}$ of B such that $p|_{U_i\times I}$ is trivial for $1 \le i \le n$.*

PROOF. Use the compactness of B and an inductive argument using the previous lemma. □

PROPOSITION 10.4. *Suppose $p : E \to B \times I$ is an n-plane bundle, where B is compact. Let $r : B\times I \to B\times I$ be defined by $r(b, t) = r(b, 1)$. Then there exists a map of bundles $g : E \to E$ such that the following diagram commutes.*



[1]These notes were taken and TEXed by Masoud Kamgarpour.

Thus, $E \cong r^*E$.

PROOF. Observe that there exists $\lambda_i : B \to I$ such that $\lambda_i^{-1}(0,1] = U_i$. Let $\nu_i : B \to I$ be defined by $\nu_i(b) = \lambda_i(b)/max\{\lambda_1(b), ..., \lambda_m(b)\}$. Then we have $max_i\,\{v_i(b)\} = 1$. Let

$$r_i(b,t) := (b,\, max(\nu_i b, t)), \qquad g_i(\phi_i(b,t,v) := \phi_i(b, r_i(b,t), v)$$

Then g_i and r_i fit into the following commutative diagram.

$$\begin{array}{ccc} E & \xrightarrow{g_i} & E \\ \downarrow & & \downarrow p \\ B \times I & \xrightarrow[r_i]{} & B \times I \end{array}$$

Then $r_m \circ ... \circ r_1 = r$ and we can define $g_m \circ ... \circ g_1 = g$. ☐

PROOF. (of Theorem 10.1) Suppose f_0 and f_1 are maps $A \to B$, which are homotopic via the homotopy h. Write $f_0 = h \circ i_0$ and $f_1 = h \circ i_1 : A \to B$. Now by functoriality we have:

$$f_0^*E = (h \circ i_0)^*E = i_0^*h^*E \cong i_0^*r^*h^*E \cong i_1^*h^*E \cong f_1^*E$$

☐

2. Stably Equivalent Bundles

Write $BO(n)$ as the union $\cup_q G_n(\mathbb{R}^n \oplus \mathbb{R}^q)$. Let $i_n : BO(n) \to BO(n+1)$ be the canonical map. (Geometrically this corresponds to adding ϵ, the trivial one dimensional bundle).

DEFINITION 10.5. Two vector bundles E and D on X are said to be **stably equivalent** if there exists an isomorphism $D \oplus \epsilon^m \cong E \oplus \epsilon^n$ for some non-negative integers m and n. Let $\mathcal{E}_{st}(X)$ be the set of stable equivalence classes of vector bundles.

Note that if we want to consider based maps, then we add a disjoint base point to X, to make it a based space X_+. In this case, the homotopies and the maps we are considering will also be based. Next let $BO = \cup_n BO(n)$. We have:

THEOREM 10.6. *If* X *is compact,* $[X_+, BO] \cong \mathcal{E}_{st}(X)$.

PROOF. Because X is compact, the image of any map from X to BO lands in some $BO(n)$. Now use $[X_+, BO(n)] \cong \mathcal{E}_n(X)$. ☐

PROPOSITION 10.7. *Let* X *be a compact space. Then for any bundle* E*, there exists a bundle* D *such that* $E \oplus D$ *is trivial.*

PROOF. The idea is to use a Gauss map to construct an orthogonal complement. (See the section on the sum of tangent bundle and normal bundle) □

COROLLARY 10.8. *Every element $\zeta - \nu$ of $K(X)$ can be written of the form $\alpha - q.\epsilon$ for some integer q and some $\alpha \in \mathcal{E}(X)$.*

COROLLARY 10.9. *Let X be connected and compact. Then, $\mathcal{E}_{st}(X)$ is naturally isomorphic to $\tilde{K}(X)$.*

PROOF. Let $\zeta - q$ be an element of $\tilde{K}(X)$. Then $dim\zeta = q$. Define a map $\tilde{K}(X) \to \mathcal{E}_{st}(X)$ by $\zeta - dim\zeta \mapsto [\zeta]$. It is to see that this is an isomorphism. □

COROLLARY 10.10. *Under the same assumptions on X we have:*
(1) $[X_+, BO \times \mathbb{Z}] \cong KO(X)$.
(2) $[X_+, BU \times \mathbb{Z}] \cong KU(X)$.

It is not hard to show that $\tilde{K}(S^2) \cong \mathbb{Z}$. This boils down to the fact that $\pi_2(BU) = \pi_2(\mathbb{C}P^\infty) \cong \mathbb{Z}$. A surprising and fundamental fact is the following theorem:

THEOREM 10.11. **(Bott Periodicity)** *The canonical map*

$$\otimes : K(X) \otimes K(S^2) \to K(X \times S^2)$$

is an isomorphism.

LECTURE 11

Wednesday, July 26[1]

Recall that last time we defined the important notion of stable vector bundles on a topological space X, and showed that for compact spaces X, we have a canonical isomorphism:

$$\mathcal{E}_{st}(X) = [X_+, BO].$$

Similarly for complex vector bundle on X we have the canonical isomorphism

$$\tilde{K}(X) = [X_+, BU].$$

Next observe that the (exterior) tensor product, gives us a map $K(X) \otimes K(Y) \to K(X \times Y)$. Bott Periodicity for complex vector bundles states that this map is an isomorphism for $Y = S^2$. Thus, we see that it's fundamental to understand the K-theory of sphere. This will lead us to the 'Hopf invariant one' problem.

1. Digression on representable functors and smash products

Let $X \vee Y = X \times \{*\} \cup \{*\} \times Y$. Define $X \wedge Y = X \times Y / X \vee Y$. $X \wedge Y$ is known as the **smash product** of the spaces X and Y. For example, $S^1 \wedge S^1 = S^2$. In fact, it is easy to show that $S^m \wedge S^n = S^{m+n}$. (Do this). This trivial result is at the foundation of the stable homotopy theory of spheres.

1.1. Cones and Suspension. Let $f : X \to Y$ be a continuous map. We define the cofiber of f, denoted by Cf, to be the topological space $Y \cup_f CX$, where CX is the cone of X, defined by

$$CX = X \times I/(X \times \{*\} \cup \{*\} \times I).$$

We thus get a sequence

$$X \to Y \to Cf \to \Sigma X \to \Sigma Y \to ...$$

This is an 'exact sequence' of topological space. Recall that a sequence of abelian groups is exact if the image of each map is equal to the kernel of the next.

[1] These notes were taken and TEXed by Masoud Kamgarpour.

EXAMPLE 11.1. The sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

is an example of a non-split short exact sequence in the category of abelian groups.

Next consider the contravariant functor $Z \to [-, Z]$ applied to the above exact sequence of topological spaces. Then we get an exact sequence of based-sets

$$[X, Z] \leftarrow [Y, Z] \leftarrow [Cf, Z] \leftarrow [\Sigma X, Z] \leftarrow \dots.$$

EXERCISE 11.2. Suppose we have a diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Cf & \longrightarrow & \Sigma X \\
\Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} & & \vdots{\scriptstyle \gamma} & & \Big\downarrow \\
X & \xrightarrow[f']{} & Y' & \longrightarrow & Cf' & \longrightarrow & \Sigma X'
\end{array}$$

Such that the left hand square homotopy commutes. Show that there exists a map $\gamma : Cf \to Cf'$ which makes the two right squares homotopy commutative. (However, γ is NOT unique, why?)

Note that in the above exact sequence we can replace $[\Sigma X, Z]$ by the isomorphic based set $[Ci, Z]$. Furthermore, one can show that $[\Sigma X, Z]$ is a group. Applying this construction twice, one gets that $[\Sigma^2 X, Z]$ is an **abelian** group. (Exercise! The proofs of these facts are parallels of the proof that the homotopy groups are actually groups, and that higher homotopy groups are abelian.)

Next note that the cofibration

$$X \vee Y \to X \times Y \to X \wedge Y$$

gives us an isomorphism:

$$K(X \times Y) \cong K(X \wedge Y) \oplus K(X) \oplus (Y).$$

Using the above isomorphism, it is easy to show that

$$\tilde{K}(S^q) = \begin{cases} \mathbb{Z} & q \ even \\ 0 & q \ odd \end{cases}$$

Our goal is to prove Hopf invariant one, which is one of the fundamental results in mathematics. Suppose n is an even integer and let $f : S^{2n-1} \to S^n$ be a map. Denote by $X = Cf$ the cofiber of f. We attach to this datum a number $h(f)$ known as the Hopf invariant of f. The exact sequence of spaces

$$S^{2n-1} \longrightarrow S^n \longrightarrow X = Cf \longrightarrow S^{2n} \longrightarrow S^{n-1}$$

gives rise to maps

$$\tilde{K}(S^{2n-1}) = 0 \leftarrow \tilde{K}(S^n) = \mathbb{Z}i_n \leftarrow \tilde{K}(X) = \mathbb{Z}a \oplus \mathbb{Z}b \leftarrow \tilde{K}(S^{2n}) = \mathbb{Z}i_{2n} \leftarrow \tilde{K}(S^{n+1}) = 0.$$

Here, $a \mapsto i_n$ and $b \mapsto i_{2n}$. As $i_n^2 = i_{2n}^2 = b^2 = 0$, we see that a^2 has to be an integer multiple of b. This multiple is known as the **Hopf invariant** of f. Our goal is to sketch the proofs of two theorems. The first one is easy. The second one is very deep.

THEOREM 11.3. *Given a map $\phi : S^{n-1} \times S^{n-1} \to S^{n-1}$ of bi-degree (p, q). Then there exists a map $f = H(\phi) : S^{2n-1} \to S^n$ such that $h(f) = \pm pq$.*

THEOREM 11.4. **Hopf Invariant One** *If $f : S^{2n-1} \to S^n$ has $h(f) = \pm 1$; then, $n = 2, 4,$ or 8.*

LECTURE 12

Friday, July 28[1]

Aside: The following theorem is very useful.

Theorem 12.1. *F is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

Note however that this result depends on the Axiom of Choice.

1. Division Algebras over $\mathbb{R}$

Let $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be such that $\phi(x, y) = 0$ implies that $x = 0$ or $y = 0$. Assume further that there exists $e \in \mathbb{R}^n$ such that $\phi(e, y) = y = \phi(y, e)$. We have the following surprising theorem.

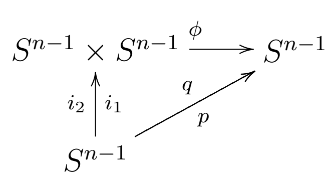
Theorem 12.2. *If ϕ is as above, then $n = 1, 2, 4, or 8$.*

The only known proofs for this purely algebraic result are topological. We now sketch a proof of this theorem based on Hopf Invariant One theorem.

Definition 12.3. Let X be a topological space. X is said to be an H-space, if there is a continuous multiplication map $\mu : X \times X \to X$ and an 'identity' element $e \in X$, such that the two maps $X \to X$ given by $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are homotopic to the identity by homotopies based at e.

Exercise: The multiplication ϕ induces the structure of an H-space on S^{n-1}.

Thus, we are reduced to classify which spheres are H-spaces. Let (p, q) be the bi-degree defined by the following diagram:



[1]These notes were taken and TeXed by Masoud Kamgarpour.

Where i_1 and i_2 are the two inclusions, and p and q are the maps (or more precisely, the degree of the maps) induced. We have the following theorem:

THEOREM 12.4. *If $\phi : S^{n-1} \times S^{n-1} \to S^{n-1}$ is given, and n is odd, then $p = 0$ or $q = 0$.*

PROOF. ϕ induces a map on K-theory,

$$\phi^* : K(S^{n-1}) \to K(S^{n-1} \times S^{n-1}) = K(S^{n-1}) \otimes K(S^{n-1}).$$

One can check that

$$\phi^*(k) = p(i \otimes 1) + q(1 \otimes j) + r(i \otimes j)$$

where i, j, k are generators of the appropriate K-groups. Now last time we saw that $i^2 = j^2 = 0$; This shows at once that $(\phi^*(k))^2 = 2pq(i \otimes j)$. On the other hand, as $k^2 = 0$, we see that $\phi^*(k^2) = (\phi^*)^2(k) = 0$. It follows that $p = 0$ or $q = 0$. □

2. Towards Hopf Invariant One

From now on we assume n is even and $n > 2$.

THEOREM 12.5. *Let $\phi : S^{n-1} \times S^{n-1} \to S^{n-1}$ be given with bidegree (p, q). There exists a map $f = H(\phi) : S^{2n-1} \to S^n$ such that $h(f) = \pm pq$.*

PROOF. To construct f it's enough to construct a map on the homotopic spaces:

$$f : CS^{n-1} \times S^{n-1} \cup S^{n-1} \times CS^{n-1} \longrightarrow \Sigma S^{n-1}$$

For this, we write

$$f([x, t], y) = [\phi(x, y), (1 + t)/2], \qquad f(x, [y, t]) = [\phi(x, y), (1 - t)/2]$$

The essential point is that we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \wedge X \\ & & \uparrow{\scriptstyle \alpha\wedge\alpha} \\ \downarrow{\scriptstyle \beta} & & S^n \wedge S^n \\ & & \uparrow{\scriptstyle p\wedge q} \\ S^{2n} & \xrightarrow{=} & S^{2n} \end{array}$$

It just remains to chase the diagram

$$
\begin{array}{ccc}
a^2 & \longleftrightarrow & a \otimes a \\
\uparrow & & \uparrow \\
 & & i_n \otimes i_n \\
 & & \uparrow \\
pqi_{2n} & \longleftarrow & pqi_n \otimes i_n
\end{array}
$$

But by the definition of the Hopf invariant pqi_{2n} should map to $pqH(f)$ in such a commutative diagram. It follows that $H(f) = pq$. $\square$

2.1. Adams Operations on K-theory. There exists natural ring homomorphisms $\psi^k : K(X) \to K(X)$ for k a positive integer, satisfying:

(1) $\psi^1 = id$.
(2) $\psi^p(x) \equiv x^p \pmod p$.
(3) $\psi^k(x) = n^k x$ if $x \in \tilde{K}(S^{2n})$.
(4) $\psi^k \psi^l = \psi^{kl} = \psi^l \psi^k$.
(5) If ζ is a line bundle, then $\psi^k(\zeta) = \zeta^k$.

Given these operations, let us see how we can prove Hopf invariant one:

Theorem 12.6. *If $h(f) = \pm 1$ then $n = 2, 4, or 8$.*

Proof. Let $n = 2m$. Then $\phi^k(a) = k^m a + \mu_k b$, and $\phi^k(b) = k^{2m} b$. Furthermore, $\psi^2(a) \equiv a^2 \pmod 2$. As $a^2 = h(f)b$, we see that

$$\mu_2 \equiv h(f) \pmod 2.$$

Now

$$\psi^2 \psi^k(a) = \psi^2(k^m a + \mu_k b) = 2^m k^m a + k^m \mu_2 b + 2^{2m} \mu_k b.$$

On the other hand, we have:

$$\psi^2 \psi^k = \psi^k \psi^2(a) = \psi^k(2^m a + \mu_2 b) = 2^m k^m a + 2^m \mu_k b + b^{2m} \mu_2 b.$$

It follows that

$$k^m \mu_2 + 2^{2m} \mu_k = 2\mu_k + k^{2m} \mu_2$$

which in turn implies that

$$k^m (k^m - 1) \mu_2 = 2^m (2^m - 1) \mu_k.$$

But now, μ_2 is odd; thus, $2^m | k^m - 1$ for ALL odd k. The rest is easy number theory. It is immediate that m has to be even. Let $k = 1 + 2^{m/2}$. Then,

$$k^m \equiv 1 + m2^{m/2} \quad (mod\, 2^m).$$

It follows that

$$2^m | m2^{m/2}$$

But this last equation is true if and only if $m = 2, 4$. □

LECTURE 13

Wednesday, August 2[1]

This lecture was mostly review so I shall give outlines and references to the points in the notes where expanded versions can be found.

1. Vector Bundles

This mostly follows the notes from Lecture 7.

Let $p : E \to B$ be a vector bundle. Then we write $p^{-1}(b) = F_b \cong \mathbb{R}^n$ (or $\mathbb{C}^n$) for the fibre over a point $b \in B$. Vector bundles are locally trivial meaning that they have an open cover $\{U\}$ with

$$\begin{array}{ccc} U \times \mathbb{R}^n & \xrightarrow{\phi_U} & p^{-1}(U) \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

and

$$U \bigcap V \xrightarrow{\phi_V^{-1}\phi_U} GL_n(\mathbb{R}) \ .$$

Let M be a manifold. By Whitney's theorem, there is some $q \in \mathbb{N}$ such that we can embed M into $\mathbb{R}^q$. Recall the tangent space $\tau(M)$ and the normal space $\nu(M)$ consisting of, respectively, all the tangent or normal vectors over each point of M. Then

$$\tau(M) \oplus \nu(M) \cong M \times \mathbb{R}^q$$

which is the trivial bundle over M.

We are interested in the collection of all vector bundles over a compact, connected base space B. Define $\mathcal{E}_n(B)$ to be the set of all equivalence classes of n-plane bundles over B. Define $\text{Vect}(B) = \amalg_{n\geq 0}\mathcal{E}_n(B)$. Notice that $\text{Vect}(B)$ is a semiring under $\oplus$, $\otimes$ with $0 = [\epsilon^0]$ and $1 = [\epsilon^1]$. It is worth noting that anything we can do to vector spaces, we can do fibrewise to vector bundles. For example,

$$(D \oplus E)_b = D_b \oplus E_b \quad (D \otimes E)_b = D_b \otimes E_b.$$

[1]TeXed by Emma Smith.

Applying the Grothendieck construction to Vect(B), in the real case we denote the result by $KO(B)$ and in the complex case by $KU(B)$ or simply $K(B)$.

1.1. Lifting vector bundles. Let $p : E \to B$ be a vector bundle and $f : A \to B$ a map of spaces. Then we can define the lift of p to a vector bundle over A as follows.

$$\begin{array}{ccc} \{(a,e) \mid f(a) = p(e)\} = f^{-1}(E) & \xrightarrow{\ g\ } & E \\ \downarrow q & & \downarrow p \\ A & \xrightarrow[\ f\ ]{} & B. \end{array}$$

Note that $q^{-1}(a) \cong p^{-1}(f(a)) \cong \mathbb{R}^n$. Hence, we have

$$\begin{aligned} f^* : \mathcal{E}_n(B) &\to \mathcal{E}_n(A) \\ \mathrm{Vect}(B) &\to \mathrm{Vect}(A) \\ KO(B) &\to KO(A). \end{aligned}$$

Similarly,

$$\begin{array}{ccccc} (ff')^{-1}(E) \cong (f')^{-1}(f^{-1}(E)) & \longrightarrow & f^{-1}(E) & \xrightarrow{\ g\ } & E \\ \downarrow & & \downarrow q & & \downarrow p \\ A' & \xrightarrow{\ f'\ } & A & \xrightarrow{\ f\ } & B \end{array}$$

or in other words, $f'^* f^* = (ff')^*$.

1.2. Universal Bundle. This is almost directly out of *A Concise Course in Algebraic Topology*, Chapter 23, Section 1.

Let $V_n(\mathbb{R}^q)$ be n-tuples of orthonormal vectors in $\mathbb{R}^q$ and $G_n(\mathbb{R}^q)$ be the grassmanian of n-planes in $\mathbb{R}^q$. Then we have

$$\begin{array}{ccc} V_n(\mathbb{R}^q) & \hookrightarrow & \mathbb{R}^N \\ \downarrow \gamma_n^q & & \\ G_n\mathbb{R}^q & & \end{array}$$

where γ_n^q sends the n-tuple of vectors to the plane that they span. There are natural maps increasing each index by adding the canonical basis vector of the new dimension.

$$\begin{array}{ccccc} V_{n+1}(\mathbb{R}^{n+1} \oplus \mathbb{R}^q) & \longleftarrow & V_n(\mathbb{R}^n \oplus \mathbb{R}^q) & \longrightarrow & V_n(\mathbb{R}^n \oplus \mathbb{R}^{q+1}) \\ \downarrow & & \downarrow & & \downarrow \\ G_{n+1}(\mathbb{R}^{n+1} \oplus \mathbb{R}^q) & \longleftarrow & G_n(\mathbb{R}^n \oplus \mathbb{R}^q) & \longrightarrow & G_n(\mathbb{R}^n \oplus \mathbb{R}^{q+1}) \end{array}$$

Define $BO_n = \bigcup_q G_n(\mathbb{R}^n \oplus \mathbb{R}^q)$ and $EO_n = \bigcup_q E_n(\mathbb{R}^n \oplus \mathbb{R}^q)$ with $\gamma_n : EO_n \to BO_n$ and $E_n(\mathbb{R}^n \oplus \mathbb{R}^q) = \{(x, v) \mid x$ is an n-plane and $v \in x$ a vector$\}$.

Technically, it is usual to assume that B is paracompact, but we shall require a numerable open cover. This means that there are continuous maps $\lambda_U : B \to I$ such that $\lambda_U^{-1}(I) = U$ and that the cover is locally finite in the sense that each $b \in B$ is a point of only finitely many open sets in our cover. Any open cover of a paracompact space has a numerable refinement.

Last time we used the notation $[X, Y]$ it meant the space of unbased maps from X to Y. Now we shall change notation. From now on, $[X, Y]$ is the space of based homotopy classes of based maps. We also define $X_+ = X \amalg \{*\}$. Hence, $[X_+, Y]$ is the space of unbased homotopy classes of unbased maps $X \to Y$ since the disjoint point is sent to the basepoint of Y and then X can be mapped anywhere.

THEOREM 13.1. *Define $\Phi : [B_+, BO_n] \to \mathcal{E}_n(B)]$ by taking $f : [B, BO_n] \mapsto [f^*\gamma_n]$. Let $t : A \to B$ be any continous map. Then the following diagram commutes.*

$$\begin{array}{ccccc} \mathcal{E}_n(B) & \xleftarrow{\Phi} & [B_+, BO_n] & \qquad & [f] \\ & & \downarrow t^* & & \downarrow \\ \mathcal{E}_n(A) & \xleftarrow{\Phi} & [A_+, BO_n] & & [f \circ t] \end{array}$$

DEFINITION 13.2. A functor F is called **representable** or **represented by Y** if there exists an object Y such that $F(-) = [-, Y]$.

As an example, the functor $\mathcal{E}_n$ is represented by BO_n.

PROOF. We need to construct a map $(g, f) : E \to E_n$ of vector bundles that is an isomorphism on fibres.

$$\begin{array}{ccc} E & \xrightarrow{g} & E_n(\mathbb{R}^r) \\ \downarrow p & & \downarrow \\ B & \xrightarrow{f} & G_n(\mathbb{R}^r) \end{array}$$

It will follow that E is equivalent to f^*E_n, thus showing that Φ is surjective. It suffices to construct a Gauss map $\hat{g} : E \to \mathbb{R}^\infty$ that is a linear monomorphism on fibres, since we can then define $f(e)$ be the image under $\hat{g}$ of the fibre through e and can define $g(e) = (f(e), \hat{g}(e))$. Going the other way, given a bundle map (g, f) we can define our Gauss map by $\hat{g}(e) = \pi_2(g(e))$ where π_2 is projection onto the second component. Suppose we have coordinate charts

$$\begin{array}{ccc} \phi_i : U_i \times \mathbb{R}^n & \xrightarrow{\cong} & p^{-1}(U_i) \\ & \searrow \quad \swarrow & \\ & U_i \subset B. & \end{array}$$

Since B is compact, choose a finite open cover $\{U_1, \dots, U_m\}$ and $\lambda_i : B \to I$ such that $\lambda_i^{-1}(0,1] = U_i$. Define $\hat{g}_i(e) = \begin{cases} 0 & \text{if } e \notin p^{-1}(U_i) \\ \lambda_i(e)\pi_2\phi^{-1}(e) & \text{if } e \in p^{-1}(U_i) \end{cases}$. Take a direct sum of m copies of $\mathbb{R}^n$. Call this new space $\mathbb{R}^r$ where $r = mn$. Then $\hat{g}(e) = \hat{g_1}(e) \oplus \hat{g_2}(e) \oplus \cdots \oplus \hat{g_m}(e)$.

Given two maps $f, f' : B \to BO_n$ we want $\Phi[f] = \Phi[f'] \Rightarrow f \simeq f'$. This will give us a map of bundles

$$\begin{array}{ccc} E \times I & \longrightarrow & E \\ \downarrow & & \downarrow \\ B \times I & \longrightarrow & B. \end{array}$$

Let $(f')^*EO_n = E'$ and $f^*EO_n = E$. Then look at

$$\begin{array}{ccccc} E' & \xrightarrow[\alpha]{\cong} & E & \longrightarrow & EO_n \\ & \searrow & \downarrow & & \downarrow \gamma_n \\ & & B & \xrightarrow[f']{f} & BO_n \end{array}$$

To get a homotopy between these bundles we need a homotopy between the corresponding Gauss maps. Let $\hat{g} : E \to \mathbb{R}^\infty$ and $\hat{g}' : E' \to \mathbb{R}^\infty$. Notice that we also have $\hat{g}\alpha : E' \to \mathbb{R}^\infty$. Suppose the images of $\hat{g}\alpha$ and $\hat{g}'$ are linearly independent in $\mathbb{R}^\infty$. Then we can write an explicit homotopy as

$$\hat{h}(e,t) = t\hat{g}\alpha(e) + (1-t)\hat{g}'(e), \qquad \hat{h} : E' \times I \to \mathbb{R}^\infty.$$

However, if we define $\chi_1 : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by $e_i \mapsto e_{2i}$ and $\chi_2 : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by $e_i \mapsto e_{2i+1}$ then $\chi_1\hat{g}\alpha$ and $\chi_2\hat{g}'$ will be linearly independent in $\mathbb{R}^\infty$. □

Say we have

$$\begin{array}{ccc} & & E \\ & & \downarrow p \\ A & \xrightarrow{f_0 \cong f_1} & B \end{array}$$

then our functor is homotopy invariant if this implies that $f_0^*E \cong f_1^*E$. Define a map $r : B \times I \to B \times I$ by the rule $r((b,t)) = (b,1)$. Let $h : f_0 \cong f_1$ be a homotopy. Then

$$\begin{array}{ccc} r^*h^*E & \longrightarrow & h^*E \\ \downarrow & & \downarrow \\ B \times I & \longrightarrow & B \times I \end{array}$$

but we claim that for any bundle E over $B \times I$ there exists a g such that

$$\begin{array}{ccc} E & \longrightarrow & E \\ \downarrow & & \downarrow \\ B \times I & \longrightarrow & B \times I \end{array}$$

or in other words, $r^*E \simeq E$. Given this claim we have

$$\begin{aligned} f_0^*E &\simeq (h \circ i_0)^*E \\ &\simeq i_0^*h^*E \\ &\simeq i_0^*r^*h^*E \\ &\simeq i_1^*h^*E \\ &\simeq f_1^*E \end{aligned}$$

as desired.

Proof of claim. Take a bundle $E \to U \times [a,c]$. Then if it is trivial over $U \times [a,b]$ and $U \times [b,c]$ then it is trivial over $U \times [a,c]$. Consider then diagrams

$$\begin{array}{ccc} U \times [a,b] \times \mathbb{R}^n & \longrightarrow & p^{-1}(U \times [a,b]) \\ \downarrow & \swarrow & \\ U \times [a,b] & & \end{array} \qquad \begin{array}{ccc} U \times [b,c] \times \mathbb{R}^n & \longrightarrow & p^{-1}(U \times [b,c]) \\ \downarrow & \swarrow & \\ U \times [b,c] & & \end{array}$$

Now, if we have $\{U_1, \ldots, U_m\}$ and $0 = a_0 < a_1 < \ldots < a_q = 1$ such that $U_i \times [a_{j-1}, a_j]$ is trivial, fix U_i and apply the above q times. Then we know that our bundle is trivial over $U_i \times I$. Let $r_i : I \to [a_i, 1]$ for $1 \leq i \leq m$, and set $r = r_1 \circ \cdots \circ r_m$ and $g = g_1 \circ \cdots \circ g_m$. Remember that we have $\lambda_i : B \to I$ such that $\lambda_i^{-1}(0,1] = U_i$. Define

$$\nu_i(b) = \frac{\lambda_i(b)}{\max \lambda_i(b)} \leq 1 \quad \max_i \nu_i(b) = 1.$$

Then we have $r_i(b,t) = (b, \max(\nu_i(b), t))$ and $g_i = \mathrm{id}$ outside $p^{-1}(U_i \times I)$, $g_i(\phi_i(b,t,v)) = \phi_i(r_i(b,t), v)$ inside $p^{-1}(U_i \times I)$. $\square$