FINITE TOPOLOGICAL SPACES

NOTES FOR REU BY J.P. MAY

Contents

1. Alexandroff spaces and finite spaces

It is a standard saying that one picture is worth a thousand words. Since the author is not good at drawing pictures, there will not be as many as there should be. The reader should draw lots of them!

In mathematics, it is perhaps fair to say that one good definition is worth a thousand calculations. The author likes to make up definitions and to see relations between seemingly unrelaed concepts, so we will do lots of that.

However, to quote a slogan from a T-shirt worn by one of the author's students, "calculation is the way to the truth". There is a need for more calculational understanding of the subject here, and the author, being too old and lazy to compute himself, hopes that readers will be inspired.

The intuitive notion of a set in which there is a prescribed description of nearness of points is obvious. Formulating the "right" general abstract notion of what a "topology" on a set should be is not. Distance functions lead to metric spaces, which is how we usually think of spaces. Hausdorff came up with a much more abstract and general notion that is now universally accepted.

Definition 1.1. A *topology* on a set X consists of a set $\mathscr{U}$ of subsets of X, called the "open sets of X in the topology $\mathscr{U}$", with the following properties.

(i) The empty set $\emptyset$ and the set X are in $\mathscr{U}$.
(ii) A finite intersection of sets in $\mathscr{U}$ is in $\mathscr{U}$.
(iii) An arbitrary union of sets in $\mathscr{U}$ is in $\mathscr{U}$.

A complement of an open set is called a closed set. The closed sets include $\emptyset$ and X and are closed under finite unions and arbitrary intersections.

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There are many standard operations on spaces that we shall have occasion to use. We record three of them now and will come back to others later.

Definition 1.2. The *subspace topology* on $A \subset X$ is the set of all intersections $A \cap U$ for open sets U of X.

Definition 1.3. The *topology of the union* on $X \amalg Y$ has as open sets the unions of an open set of X and an open set of Y.

Definition 1.4. The *product topology* on $X \times Y$ is the topology with basis the products $U \times V$ of an open set U in X and an open set V in Y.

It is very often interesting to see what happens when one takes a standard definition and tweaks it a bit. The following tweaking of the notion of a topology is due to Alexandroff [1], except that he used a different name for the notion.

Definition 1.5. A topological space X is an *A-space* if the set $\mathscr{U}$ is closed under arbitrary intersections.

Remark 1.6. The notion of an A-space has a pleasing complementarity. If X is an A-space, then the closed subsets of X give it a new A-space topology. We write X^{op} for X with this opposite topology. Then $(X^{op})^{op}$ is the space X back again.

A space is *finite* if the set X is finite, and the following observation is clear.

Lemma 1.7. *A finite space is an A-space.*

It turns out that a great deal of what can be proven for finite spaces applies equally well more generally to A-spaces. However, the finite spaces have recently captured people's attention. Since digital processing and image processing start from finite sets of observations and seek to understand pictures that emerge from a notion of nearness of points, finite topological spaces seem a natural tool in many such scientific applications. There are many papers on the subject, but few of any mathematical depth, dating from the 1980's and 1990's.

There was a brief early flurry of beautiful mathematical work on this subject. Two independent papers, by McCord and Stong [11, 15], both published in 1966, are especially interesting. We will work through them. We are especially interested in questions raised by the union of these papers that are answered in neither and were not pursued until quite recently. We are also interested in calculational questions about the enumeration of finite topologies.

There is a hierarchy of "separation properties" on spaces, and intuition about finite spaces is impeded by too much habituation to the stronger of them.

Definition 1.8. Let $(X, \mathscr{U})$ be a topological space.

(i) X is a *T_0-space* if for any two points of X, there is an open neighborhood of one that does not contain the other.
(ii) X is a *T_1-space* if each point of X is a closed subset.
(iii) X is a *T_2-space*, or *Hausdorff space*, if any two points of X have disjoint open neighborhoods.[1]

Lemma 1.9. $T_2 \Longrightarrow T_1 \Longrightarrow T_0$.

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[1]The German word for separation is "Trennung", hence the letter T for the hierarchy of separation properties.

We shall omit proofs of standard and elementary results, such as this, that are part of basic point-set topology. However, for the reader's convenience, we give a summary outline of that subject in Chapter **??**.

In most of topology, the spaces considered are Hausdorff. For example, metric spaces are Hausdorff. Intuition gained from thinking about such spaces is rather misleading when one thinks about finite spaces.

Definition 1.10. The *discrete topology* on X is the topology in which all sets are open. The *trivial* or *coarse* topology on X is the topology on X in which $\emptyset$ and X are the only open sets. We write D_n and C_n for the discrete and coarse topologies on a set with n elements. They are the largest and the smallest possible topologies (in terms of the number of open subsets).

Lemma 1.11. *If a finite space is T_1, then it is discrete.*

Proof. Every subset is a union of finitely many points, hence is closed. Therefore every set is open. □

In contrast, finite T_0-spaces are very interesting. The following problem might be a bit difficult right now, but its solution will shortly become evident.

Exercise 1.12. *Show (by induction) that a finite T_0 space has at least one point which is a closed subset.*

Finite spaces have canonical minimal "bases", which we describe next.

Definition 1.13. A *basis* $\mathscr{B}$ for a topological space X is a set of open sets, called *basic open sets*, with the following properties.

(i) Every point of X is in some basic open set.
(ii) If x is in basic open sets B_1 and B_2, then x is in a basic set B_3 that is contained in $B_1 \cap B_2$.

If $\mathscr{B}$ is a set satisfying these two properties, the *topology generated by* $\mathscr{B}$ is the set $\mathscr{U}$ of subsets U of X such that, for each point $x \in U$, there is a set B in $\mathscr{B}$ such that $x \in B \subset U$.

Example 1.14. The set of singleton sets $\{x\}$ is a basis for the discrete topology on X. The set of disks $D_r(x) = \{y | d(x,y) < r\}$ is a basis for the topology on a metric space X.

Lemma 1.15. *$\mathscr{B}$ is a basis for a topology $\mathscr{U}$ if and only if for each $x \in U \in \mathscr{U}$, there is a $B \in \mathscr{B}$ such that $x \in B \subset U$.*

Definition 1.16. Let X be an Alexandroff space. For $x \in X$, define U_x to be the intersection of the open sets that contain x. Define a relation $\leq$ on the set X by $x \leq y$ if $x \in U_y$ or, equivalently, $U_x \subset U_y$. Write $x < y$ if the inclusion is proper.

Lemma 1.17. *The set of open sets U_x is a basis for X. Indeed, it is the unique minimal basis for X.*

Proof. The first statement is clear. If $\mathscr{C}$ is another basis and $x \in X$, there is a $C \in \mathscr{C}$ such that $x \in C \subset U_x$. This implies $C = U_x$, so that $U_x \in \mathscr{C}$. □

2. Alexandroff spaces, preorders, and partial orders

Here we relate Alexandroff spaces to the combinatorial notions of preorder and partial order.

Definition 2.1. A *preorder* on a set X is a reflexive and transitive relation $\leq$; thus $x \leq x$ and if $x \leq y$ and $y \leq z$, then $x \leq z$. A preorder is a *partial order* if it is antisymmetric, which means that $x \leq y$ and $y \leq x$ implies $x = y$.

Lemma 2.2. *The relation $\leq$ on an A-space X is reflexive and transitive, hence $(X \leq)$ is a preorder; it is a partial order if and only if the space X is T_0.*

Proof. The first statement is clear. For the second, $x \leq y$ and $y \leq x$ means that $U_x = U_y$. This holds if and only if every open set that contains either x or y also contains the other. □

Lemma 2.3. *A preorder $(X, \leq)$ determines a topology $\mathscr{U}$ with basis the set of all sets $U_x = \{y|y \leq x\}$, and $(X, \mathscr{U})$ is an A-space; it is T_0 if and only if $(X, \leq)$ is a partial order.*

Proof. If $x \in U_y$ and $x \in U_z$, then $x \leq y$ and $x \leq z$, hence $x \in U_X \subset U_y \cap U_z$. Therefore $\{U_x\}$ is a basis for a topology. The intersection U of a set $\{U_i\}$ of open subsets is open since if $x \in U$, then $U_x \subset U_i$ for each i and therefore U is the union of these U_x. For the second statement, $x \leq y$ and $y \leq x$ if and only if $U_x = U_y$, and the T_0 property then gives that $x = y$. □

We put things together to obtain the following conclusion.

Proposition 2.4. *For a set X, the A-space topologies on X are in bijective correspondence with the preorders on X. The topology $\mathscr{U}$ corresponding to $\leq$ is T_0 if and only if the relation $\leq$ is a partial order.*

Remark 2.5. If $\leq$ is a preorder on X, the opposite preorder is given by $x \leq^{op} y$ if and only if $y \leq x$. The corresponding A-space is X^{op}.

3. Continuous maps and order-preserving functions

The real force of the comparison between A-spaces and preorders comes from the fact that, with the appropriate definitions, continuous maps correspond precisely to order-preserving functions.

Definition 3.1. Let X and Y be topological spaces. A function $f\colon X \longrightarrow Y$ is *continuous* if $f^{-1}(V)$ is open in X for each open set V in Y. We call continuous functions "maps". A map f is a *homeomorphism* if f is one–to-one and onto and its inverse function is continuous.

Definition 3.2. Let X and Y be preorders. A function $f\colon X \longrightarrow Y$ is order-preserving if $x \leq y$ implies $f(x) \leq f(y)$.

Lemma 3.3. *A function $f\colon X \longrightarrow Y$ between A-spaces is continuous if and only if it is order preserving.*

Proof. Let f be continuous and suppose that $x \leq y$. Then $x \in U_y \subset f^{-1}U_{f(y)}$ and thus $f(x) \in U_{f(y)}$, which means that $f(x) \leq f(y)$. For the converse, let V be open in Y. If $f(y) \in V$, then $U_{f(y)} \subset V$. If $x \in U_y$, then $x \leq y$ and thus $f(x) \leq f(y)$ and $f(x) \in U_{f(y)} \subset V$, so that $x \in f^{-1}(V)$. Thus $f^{-1}(V)$ is the union of these U_y and is therefore open. □

4. Finite spaces and homeomorphisms

From now on, except where stated otherwise, X is a finite space. We write $|X|$ for the number of points in X. We have chosen to work with finite spaces for simplicity and clarity. However, just as in the sections above, most of what we do applies verbatim, or with minor changes, to A-spaces. At first sight, one might think that finite spaces are uninteresting since they are just finite preorders in disguise, but that turns out to be far from the case.

Topologists are only interested in spaces up to homeomorphism, and we proceed to classify finite spaces up to homeomorphism. Let X and Y be finite spaces in what follows.

Lemma 4.1. *A map $f\colon X \longrightarrow X$ is a homeomorphism if and only if f is either one–to–one or onto.*

Proof. By finiteness, one–to–one and onto are equivalent. Assume they hold. Then f induces a bijection 2^f from the set 2^X of subsets of f to itself. Since f is continuous, if $f(U)$ is open, then so is U. Therefore the bijection 2^f must restrict to a bijection from the topology $\mathscr{U}$ to itself. $\square$

The previous lemma fails if we allow different topologies on X: there are continuous bijections between different topologies. We proceed to describe how to enumerate the distinct topologies up to homeomorphism. There are quite a few papers on this enumeration problem in the literature, although some of them focus on enumeration of all topologies, rather than homeomorphism classes of topologies [3, 4, 6, 5, 9, 7, 8, 10, 13, 14]. The difference already appears for two point spaces, where there are four distinct topologies but three inequivalent topologies, that is three non-homeomorphic two point spaces. Here is a table lifted straight from Wikipedia that gives an idea of the enumeration.

n	Distinct topologies	Distinct T_0-topologies	Inequivalent topologies	Inequivalent T_0-topologies
1	1	1	1	1
2	4	3	3	2
3	29	19	9	5
4	355	219	33	16
5	6942	4231	139	63
6	209,527	130,023	718	318
7	9,535,241	6,129,859	4,535	2,045
8	642,779,354	431,723,379	35,979	16,999
9	63,260,289,423	44,511,042,511	363,083	183,231
10	8,977,053,873,043	6,611,065,248,783	4,717,687	2,567,284

Through $n = 9$, a published source for the fourth column is [9]. However, this is not the kind of enumeration problem for which one expects to obtain a precise answer for all n. Rather, one expects bounds and asymptopics. There is a precise formula relating the second column to the first column, but we are really only interested in the last column. In fact, we are far more interested in refinements of the last column that shrink its still inordinately large numbers to smaller numbers of far greater interest to an algebraic topologist.

We shall explain how to reduce the determination of the fourth column to a matrix computation, using minimal bases. For this purpose, it is convenient to describe minimal bases for the topology on X without reference to their enumeration by the elements $x \in X$, since the latter can give redundant information.

Lemma 4.2. *A set $\mathscr{B}$ of nonempty subsets of X is the minimal base for a topology if and only if*

(i) *Every point of X is in some set B in $\mathscr{B}$.*
(ii) *The intersection of two sets in $\mathscr{B}$ is a union of sets in $\mathscr{B}$.*
(iii) *If a union of sets B_i in $\mathscr{B}$ is again in $\mathscr{B}$, then the union is equal to one of the B_i.*

Proof. Conditions (i) and (ii) are equivalent to saying that $\mathscr{B}$ is a basis, and then the minimal basis is contained in $\mathscr{B}$. If (iii) also holds, then each B in $\mathscr{B}$, being a union of sets of the form U_x, must be one of the U_x. Conversely, if $\mathscr{B}$ is the minimal basis and U_x is in $\mathscr{B}$ and is the union of sets U_y, then x is in U_y for some y and thus $U_x = U_y$, so (iii) holds. □

This result implies the following descriptions of the relationship between minimal bases and subspaces, disjoint unions, and products.

Lemma 4.3. *If A is a subspace of X, the minimal basis of A consists of the intersections $A \cap U$, where U is in the minimal basis of X.*

Lemma 4.4. *The minimal basis of $X \amalg Y$ is the union of the minimal basis of X and the minimal basis of Y.*

Lemma 4.5. *The minimal basis of $X \times Y$ is the set of products $U \times V$, where U and V are in the minimal bases of X and Y.*

Definition 4.6. Consider square matrixes $M = (a_{i,j})$ with integer entries that satisfy the following properties.

(i) $a_{i,i} \geq 1$.
(ii) $a_{i,j}$ is -1, 0, or 1 if $i \neq j$.
(iii) $a_{i,j} = -a_{j,i}$ if $i \neq j$.
(iv) $a_{i_1,i_s} = 0$ if there is a sequence of distinct indices $\{i_1, \cdots, i_s\}$ such that $s > 2$ and $a_{i_k,i_{k+1}} = 1$ for $1 \leq k \leq s-1$.

Say that two such matrices M and N are equivalent if there is a permutation matrix T such that $T^{-1}MT = N$ and let $\mathscr{M}$ denote the set of equivalence classes of such matrices.

Theorem 4.7. *The homeomorphism classes of finite spaces are in bijective correspondence with $\mathscr{M}$. The number of sets in a minimal basis for a finite space X determines the size of the corresponding matrix, and the trace of the matrix is the number of elements of X. Moreover, X is a T_0-space if and only if the diagonal entries $a_{i,i}$ are all one.*

Proof. We work with minimal bases for the topologies rather than with elements of the set. For a minimal basis $U_1, \cdots, U_r$ of a topology $\mathscr{U}$ on a finite set X, define an $r \times r$ matrix $M = (a_{i,j})$ as follows. If $i = j$, let $a_{i,i}$ be the number of elements $x \in X$ such that $U_x = U_i$. Define $a_{i,j} = 1$ and $a_{j,i} = -1$ if $U_i \subset U_j$ and there is no k (other than i or j) such that $U_i \subset U_k \subset U_j$. Define $a_{i,j} = 0$ otherwise. Clearly (i)–(iv) hold, and a reordering of the basis results in a permutation matrix that conjugates M into the matrix determined by the reordered basis. Thus X determines an element of $\mathscr{M}$.

If $f\colon X \longrightarrow Y$ is a homeomorphism, then f determines a bijection from the basis for X to the basis for Y that preserves inclusions and the number of elements that determine corresponding basic sets, hence X and Y determine the same element of $\mathscr{M}$. Conversely, suppose that X and Y have minimal bases $\{U_1, \cdots, U_r\}$ and $\{V_1, \cdots, V_r\}$ that give rise to the same element of $\mathscr{M}$. Reordering bases if necessary, we can assume that they give rise to the same matrix. For each i, choose a bijection f_i from the set of elements $x \in X$ such that $U_x = U_i$ and the set of elements $y \in Y$ such that $V_y = V_i$. We read off from the matrix that the f_i together specify a homeomorphism $f\colon X \longrightarrow Y$. Therefore our mapping from homeomorphism classes to $\mathscr{M}$ is one–to–one.

To see that our mapping is onto, consider an $r \times r$-matrix M of the sort under consideration and let X be the set of pairs of integers (u, v) with $1 \leq u \leq r$ and $1 \leq v \leq a_{i,i}$. Define subsets U_i of X by letting U_i have elements those $(u, v) \in X$ such that either $u = i$ or $u \neq i$ but $u = i_1$ for some sequence of distinct indices $\{i_1, \cdots, i_s\}$ such that $s \geq 2$, $a_{i_k, i_{k+1}} = 1$ for $1 \leq k \leq s-1$, and $i_s = i$. We see that the U_i give a minimal basis for a topology on X by verifying the conditions specified in Lemma 4.2. Condition (i) is clear since $(u, v) \in U_u$. To verify (ii) and (iii), we observe that if $(u, v) \in U_i$ and $u \neq i$, then $U_u \subset U_i$. Indeed, we certainly have $(u, v) \in U_i$ for all v, and if $(k, v) \in U_u$ with $k \neq u$, we must have a sequence connecting k to u and a sequence connecting u to i which can be concatenated to give a sequence connecting k to i that shows that (k, v) is in U_i. To see (ii), if $(u, v) \in U_i \cap U_j$, then $U_u \subset U_i \cap U_j$, which implies that $U_i \cap U_j$ is a union of sets U_u. To see (iii), if a union of sets U_i is a set U_j, there is an element of U_j in some U_i and then $U_j \subset U_i$, so that $U_j = U_i$. A counting argument for the diagonal entries and consideration of chains of inclusions show that the matrix associated to the topology whose minimal basis is $\{U_i\}$ is the matrix M that we started with. □

5. Spaces with at most four points

We describe the homeomorphism classes of spaces with at most four points, with just a start on taxonomy.

There is a unique space with one point, namely $C_1 = D_1$.

There are three spaces with two points, namely C_2, $P_2 = \mathbb{C}D_1$, D_2.

Proper subsets of X are those not of the form $\emptyset$ or X. We often restrict to proper subsets when specifying topologies. The following definitions prescribe the two names for the second space in the short list just given.

Definition 5.1. For a set with n elements, let $P_n = P_{1,n}$ be the space (unique up to homeomorphism) which has only one proper open set, containing only one point; for $1 < m < n$, let $P_{m,n}$ be the space whose proper open subsets are the non-empty subsets of a given subset with m elements.

Definition 5.2. For a space X define the *non-Hausdorff cone* $\mathbb{C}X$ by adjoining a new point $*$ and letting the proper open subsets of $\mathbb{C}X$ be the non-empty open subsets of X. Thus, if $|X| = n-1$, then $\mathbb{C}X \cong P_{n-1,n}$.

We shall see that $\mathbb{C}X$ is contractible in Lemma 8.2 below.

Here is a table of the nine homeomorphism classes of topologies on a three point set $X = \{a, b, c\}$.

Proper open sets	Name	T_0?	connected?
all	D_3	yes	no
a, b, (a,b), (b,c)	$D_1 \amalg P_2$	yes	no
a, b, (a,b)	$P(2,3) \cong \mathbb{C}D_2$	yes	yes
a	P_3	no	yes
a, (a,b)	$\mathbb{C}P_2 \cong (\mathbb{C}P_2)^{op}$	yes	yes
a, (b,c)	$D_1 \amalg C_2$	no	no
a, (a,b), (a,c)	$(\mathbb{C}D_2)^{op}$	yes	yes
(a,b)	$\mathbb{C}C_2 \cong P_3^{op}$	no	yes
none	$C_3 = D_3^{op}$	no	yes

Here is a tabulation of *bases* for the proper open subsets of the 33 homeomorphism classes of topologies on a four point space $X = \{a, b, c, d\}$. That is, the topologies are obtained by adding in the empty set, the whole set, and all unions of the listed sets. The list is ordered by decreasing number of singleton sets in the topology, and, when that is fixed, by increasing number of two-point subsets and then by increasing number of three-point subsets.

1	all
2	a, b, c, (a,b), (a,c), (b,c), (a,b,c)
3	a, b, c, (a,b), (a,c), (b,c), (a,b,c), (a,b,d)
4	a, b, c, (a,b), (a,c), (b,c), (a,d), (a,b,c), (a,b,d), (a,c,d)
5	a, b, (a,b)
6	a, b, (a,b), (a,b,c)
7	a, b, (a,b), (a,c,d)
8	a, b, (a,b), (a,b,c), (a,b,d)
9	a, b, (a,b), (a,c), (a,b,c)
10	a, b, (a,b), (a,c), (a,b,c), (a,c,d)
11	a, b, (a,b), (a,c), (a,b,c), (a,b,d)
12	a, b, (a,b), (c,d), (a,c,d), (b,c,d)
13	a, b, (a,b), (a,c), (a,d), (a,b,c), (a,b,d)
14	a, b, (a,b), (a,c), (a,d), (a,b,c), (a,b,d), (a,c,d)
15	a
16	a, (a,b)
17	a, (a,b), (a,b,c)
18	a, (b,c), (a,b,c)
19	a, (a,b), (a,c,d)
20	a, (a,b), (a,b,c), (a,b,d)
21	a, (b,c), (a,b,c), (b,c,d)
22	a, (a,b), (a,c), (a,b,c)
23	a, (a,b), (a,c), (a,b,c), (a,b,d)
24	a, (c,d), (a,b), (a,c,d)
25	a, (a,b), (a,c), (a,d), (a,b,c), (a,b,d), (a,c,d)
26	a, (a,b,c)
27	a, (b,c,d)
28	(a,b)
29	(a,b), (c,d)
30	(a,b), (a,b,c)
31	(a,b), (a,b,c), (a,b,d)
32	(a,b,c)
33	none

Problem 5.3. *Determine which of these spaces are T_0 and which are connected. Give a taxonomy in terms of explicit general constructions that accounts for all of these topologies. That is, determine appropriate "names" for all of these spaces.*

6. Connectivity and path connectivity

We begin the exploration of homotopy properties of finite spaces by discussing connectivity and path connectivity. We recall the general definitions.

Definition 6.1. A space X is *connected* it is not the disjoint union of two non-empty open subsets. Equivalently, X is connected if the only open and closed subsets of X are $\emptyset$ and X. Define an equivalence relation $\sim$ on X by $x \sim y$ if x and y are elements of some connected subspace of X. An equivalence class under $\sim$ is called a *component* of X.

Lemma 6.2. *The components of X are connected, X is the disjoint union of its components, and any connected subspace of X is contained in a component.*

Proof. Left as an exercise. (Or see Munkres [12, 3.3.1].) $\square$

Lemma 6.3. *If $f\colon X \longrightarrow Y$ is a map and X is connected, then $f(X)$ is a connected subspace of Y.*

Proof. Left as an exercise. (Or see Munkres [12, 3.1.5].) $\square$

Let $I = [0,1]$ with its usual metric topology as a subspace of $\mathbb{R}^n$. It is a connected space, hence so is its image under any map. A map $p\colon I \longrightarrow X$ is called a *path* from $p(0)$ to $p(1)$ in X.

Definition 6.4. A space X is *path connected* if any two points can be connected by a path. Define a second equivalence relation $\simeq$ on X by $x \simeq y$ if there is a path connecting x to y. An equivalence class under $\simeq$ is called a *path component* of X. Note that $x \simeq y$ implies $x \sim y$, but not conversely in general.

Lemma 6.5. *The path components of X are path connected, X is the disjoint union of its path components, and any path connected subspace of X is contained in a path omponent. Each path component is contained in a component.*

Proof. Left as an exercise. (Or see Munkres [12, 3.3.2].) $\square$

Now return to finite spaces X. At first sight, one might imagine that there are no continuous maps from I to a finite space, but that is far from the case. The most important feature of finite spaces is that they are surprisingly richly related to the "real" spaces that algebraic topologists care about.

Lemma 6.6. *Each U_x is connected. If X is connected and $x, y \in X$, there is a sequence of points z_i, $1 \leq i \leq s$, such that $z_1 = x$, $z_s = y$ and either $z_i \leq z_{i+1}$ or $z_{i+1} \leq z_i$ for $i < s$.*

Proof. If $U_x = A \amalg B$, A and B open, say $x \in A$, then $U_x \subset A$ and therefore $B = \emptyset$. Fix x and consider the set A of points y that are connected to x by some sequence z_i. We see that A is open since $z \leq z'$ implies $U_z \subset U_{z'}$. We see that A is closed since if y is not so connected to x, then neither is any point of U_y, so that the complement of A is open. Since X is connected, it follows that $A = X$. $\square$

Lemma 6.7. *If $x \leq y$, then there is a path p connecting x and y.*

Proof. Define $p(t) = x$ if $t < 1$ and $p(1) = y$. We claim that p is continuous. Let V be an open set of X. If neither x nor y is in V, then $p^{-1}(V) = \emptyset$. If x is in V and y is not in V, then $p^{-1}(V) = [0,1)$. If y is in V, then x is in $V_y \subset V$ since $x \leq y$, hence $p^{-1}(V) = I$. $\square$

Proposition 6.8. *A finite space is connected if and only if it is path connected.*

Proof. The previous two lemmas imply that $x \sim y$ if and only if $x \simeq y$. $\square$

7. Function spaces and homotopies

Definition 7.1. A space is *compact* if every open cover has a finite subcover.

Definition 7.2. Let X and Y be spaces and consider the set Y^X of maps $X \longrightarrow Y$. The *compact–open topology* on Y^X is the topology in which a subset is open if and only if it is a union of finite intersections of sets $W(C,U) = \{f|f(C) \subset U\}$, where C is compact in X and U is open in Y. This means that the set of all $W(C,U)$ is a *subbasis* for the topology.

We insert a small but non-standard technical condition. Experts will recognize that it codifies a standard property of locally compact Hausdorff spaces, but it is also true trivially for all finite spaces.

Definition 7.3. A space X is locally compact if for each $x \in X$, there is a compact subspace C of X and an open subspace U such that $x \in U \subset C$; X is *smally compact* if every open subset V is locally compact. When X is finite, X is smally compact since every subset is compact and we can take $U = C = V$.

Ignoring topology, for sets X, Y, and Z, functions $f\colon X\times Y \longrightarrow Z$ are in bijective correspondence with functions $\hat{f}\colon X \longrightarrow Z^Y$ via the relation $f(x,y) = f(x)(y)$. Returning to topology, and so restricting Z^Y to consist of the continuous maps $Y \longrightarrow Z$, inspection of the proof of similar statements in any standard text, for example [12, 7.5.3, 7.5.4], shows that the following result holds.

Proposition 7.4. *For spaces X, Y, and Z such that X is smally compact, a function $f\colon X \times Y \longrightarrow Z$ is continuous if and only if $\hat{f}\colon X \longrightarrow Z^Y$ is continuous.*

Definition 7.5. A *homotopy* $h\colon f \simeq g$ is a map $h\colon X \times I \longrightarrow Y$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$. Two maps are homotopic, written $f \simeq g$ if there is a homotopy between them.

Lemma 7.6. *If X is smally compact, then homotopies $h\colon X \times I \longrightarrow Y$ correspond bijectively to continuous maps $j\colon X \longrightarrow Y^I$ via $h \leftrightarrow j$ if $h(x,t) = j(x)(t)$. The homotopy classes of maps $X \longrightarrow Y$ are in canonical bijective correspondence with the path components of Y^X.*

Definition 7.7. If Y is finite, define the pointwise ordering of maps $X \longrightarrow Y$ by $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$.

Proposition 7.8. *If Y is finite, then the intersection of the open sets in Y^X that contain a map g is $\{f|f \leq g\}$.*

Proof. Let V_g be the cited intersection and let $Z_g = \{f|f \leq g\}$. Let $f \in V_g$ and $x \in X$. Since $g \in W(\{x\}, U_{g(x)})$, $f \in W(\{x\}, U_{g(x)})$, so $f(x) \in U_{g(x)}$ and $f(x) \leq g(x)$. Since x was arbitrary, f is in Z_g. Conversely, let $f \leq g$. Consider any $W(C,U)$ which contains g and let $x \in C$. Then $g(x) \in U$ and, since $f(x) \leq g(x)$, $f(x) \in U_{g(x)} \subset U$. Therefore $f \in W(C,U)$ and f is in all open subsets of Y^X that contain g. □

Corollary 7.9. *If X and Y are finite, then the pointwise ordering on Y^X coincides with the ordering given by its compact open topology.*

Proposition 7.10. *If Y is finite and $f \leq g$, then $f \simeq g$ by a homotopy h such that $h(x,t) = f(x)$ for all t and all points $x \in X$ such that $f(x) = g(x)$.*

Proof. We have the path p connecting f to g in Y^X specified by $p(t) = f$ if $t < 1$ and $p(1) = g$. Indeed, with $V = W(C, U)$, the proof that p is continuous is a direct adaptation of the proof of Lemma 6.7, the key point being that if $g \in V$, then $f \in V$ by Proposition 7.8. $\square$

8. Homotopy equivalences

We have seen that enumeration of finite sets with reflexive and transitive relations $\leq$ amounts to enumeration of the topologies on finite sets. We have refined this to consideration of homeomorphism classes of finite spaces. We are much more interested in the enumeration of the homotopy types of finite spaces. We will come to a still weaker and even more interesting enumeration problem later.

Definition 8.1. Two spaces X and Y are *homotopy equivalent* if there are maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. A space is *contractible* if it is homotopy equivalent to a point.

This relationship can change the number of points. We have a first example.

Lemma 8.2. *If X is a space containing a point y such that the only open (or only closed) subset of X containing y is X itself, then X is contractible. In particular, the non-Hausdorff cone $\mathbb{C}X$ is contractible for any X.*

Proof. This is a variation on a theme we have already seen twice. Let $*$ denote a space with a single point, also denoted $*$. Define $r\colon X \longrightarrow *$ by $r(x) = *$ for all x and define $i\colon * \longrightarrow X$ by $i(*) = y$. Clearly $r \circ i = \mathrm{id}$. Define $h\colon X \times I \longrightarrow X$ by $h(x,t) = x$ if $t < 1$ and $h(x,1) = y$. Then h is continuous. Indeed, let U be open in X. If $y \in U$, then $U = X$ and $h^{-1}(U) = X \times I$, while if $y \notin U$, then $h^{-1}(U) = U \times [0,1)$. The argument when X is the only closed subset containing y is the same. Clearly h is a homotopy $\mathrm{id} \simeq i \circ r$. $\square$

Corollary 8.3. *If X is finite, then U_x is contractible.*

Proof. The only open subset of U_x that contains x is U_x itself. $\square$

The following result of McCord [11, Thm. 4] says that, when studying finite spaces up to homotopy type, there is no loss of generality if we restrict attention to T_0-spaces, that is, to finite posets (poset = partially ordered set).

Theorem 8.4. *Let X be a finite space. There is a quotient T_0-space X_0 such that the quotient map $q_X\colon X \longrightarrow X_0$ is a homotopy equivalence. For a map $f\colon X \longrightarrow Y$ of finite spaces, there is a unique map $f_0\colon X_0 \longrightarrow Y_0$ such that $q_Y \circ f = f_0 \circ q_X$.*

Proof. Define $x \sim y$ if $U_x = U_y$, or, equivalently, if $x \leq y$ and $y \leq x$. Let X_0 be the set of equivalence classes and let $q = q_X$ send x to its equivalence class $[x]$. Give X_0 the quotient topology. This means that a subset V of X_0 is open if and only if $q^{-1}(V)$ is open in X. Clearly q is continuous. The relation $\leq$ on X induces a relation $\leq$ on X_0. Since X_0 is finite, we have the open set $U_{q(x)}$ for $x \in X$. Observe that $q^{-1}q(U_x) = U_x$ since if $q(y) = q(z)$ where $z \in U_x$, then $y \in U_y = U_z \subset U_x$. Therefore $q(U_x)$ is open, hence contains $U_{q(x)}$. Conversely, $U_x \subset q^{-1}(U_{q(x)})$ by continuity and thus $q(U_x) \subset U_{q(x)}$. This proves that $q(U_x) = U_{q(x)}$. It follows that $[x] \leq [y]$ if and only if $x \leq y$. Indeed, $q(x) \leq q(y)$ implies $q(x) \in U_{q(y)} = q(U_y)$. Thus $q(x) = q(z)$ for some $z \in U_y$ and $U_x = U_z \subset U_y$, so that $x \leq y$. Conversely, if $x \leq y$, then $U_x \subset U_y$ and therefore $U_{q(x)} \subset U_{q(y)}$, so that $q(x) \leq q(y)$. It follows that $\leq$ is antisymmetric on X_0, so that X_0 is a T_0-space.

We must prove that q is a homotopy equivalence. Let $f\colon X_0 \longrightarrow X$ be any function such that $q \circ f = \mathrm{id}$. That is, we choose a point from each equivalence class. By what we have just proven, f preserves $\leq$ and is therefore continuous. Let $g = f \circ q$. We must show that g is homotopic to the identity. We see that g is obtained by first choosing one x_u with $U_{x_u} = U$ for each U in the minimal basis for X and then letting $g(x) = x_u$ if $U_x = U$. Thus $U_{g(x)} = U_x$ and $g(x) \in U_x$, which means that $g \leq \mathrm{id}$. Now Proposition 7.10 gives the required homotopy $h\colon \mathrm{id} \simeq g$. Note that $h(g(x), t) = g(x)$ for all t.

For the last statement, a map $f\colon X \longrightarrow Y$ is a function that preserves $\leq$, and it follows that it induces a unique function $f_0\colon X_0 \longrightarrow Y_0$ such that $q_Y \circ f = f_0 \circ q_X$. Clearly f_0 preserves $\leq$ and is thus continuous. □

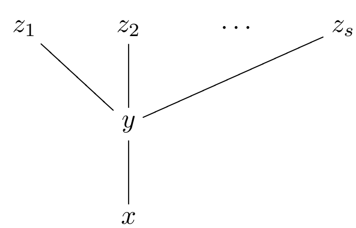
The space T_0 is called the Kolmogorov quotient of T. The construction is classical and has many other applications. We conclude that to classify finite spaces up to homotopy equivalence, it suffices to classify T_0-spaces up to homotopy equivalence. Stong [15, §4] has given an interesting way of studying this. We change his language a bit in the following exposition.

Definition 8.5. Let X be a finite space.

(a) A point $x \in X$ is *upbeat* if there is a $y > x$ such that $z > x$ implies $z \geq y$.
(b) A point $x \in X$ is *downbeat* if there is a $y < x$ such that $z < x$ implies $z \leq y$.

X is a minimal finite space if it is a T_0-space and has no upbeat or downbeat points. A *core* of a finite space X is a subspace Y that is a minimal finite space and a deformation retract of X. That is, if $i\colon Y \longrightarrow X$ is the inclusion, there is a map $r\colon X \longrightarrow Y$ such that $r \circ i = \mathrm{id}$ together with a homotopy $h\colon X \times I \longrightarrow X$ from id to $i \circ r$ such that $h(y, t) = y$ if $y \in Y$.

Remark 8.6. If we draw a graph of a poset by drawing a line upwards from x to y if $x < y$, we see that, above an upbeat point x, the graph looks like



Turning the picture upside down, we see what the graph below a downbeat point looks like.

Intuitively, identifying x and y and erasing the line between them should not change the homotopy type. We say this another way in the proof of the following result, looking at inclusions rather than quotients in accordance with our definition of a core.

Theorem 8.7. *Any finite (or finite based) space X has a core.*

Proof. With the notations of the proof of Theorem 8.4, identify X_0 with its image $g(X_0) \subset X$. The proof of Theorem 8.4 shows that X_0, so interpreted, is a deformation retract of X. Thus we may as well assume that X is T_0. Suppose that

X has an upbeat point x. We claim that the subspace $X - \{x\}$ is a deformation retract of X. To see this define $f\colon X \longrightarrow X - \{x\} \subset X$ by $f(z) = z$ if $z \neq x$ and $f(x) = y$, where $y > x$ is such that $z > x$ implies $z \geq y$. Clearly $f \geq \mathrm{id}$. We claim that f preserves order and is therefore continuous. Thus suppose that $u \leq v$. We must show that $f(u) \leq f(v)$. If $u = v = x$ or if neither u nor v is x, there is nothing to prove. When $u = x < v$, $f(u) = y$ and $f(v) = v \geq y$. When $u < x = v$, $f(u) = u < x < y = f(v)$. Now Proposition 7.10 gives the required deformation. A similar argument applies to show that $X - \{x\}$ is a deformation retract of X if x is a downbeat point. Starting with X_0, define X_i from X_{i-1} by deleting one upbeat or downbeat point. After finitely many stages, there are no more upbeat or downbeat points left, and we arrive at the required core. □

Theorem 8.8. *If X is a minimal finite space and $f\colon X \longrightarrow X$ is homotopic to the identity, then f is the identity.*

Proof. First suppose that $f \geq \mathrm{id}$. For all x, $f(x) \geq x$. If x is a maximal point, then $f(x) = x$. Let x be any point of X and suppose inductively that $f(z) = z$ for all $z > x$. Then, by continuity, $z > x$ implies $z = f(z) \geq f(x) \geq x$. If $f(x) \neq x$, this implies that x is an upbeat point, contradicting the minimality of X. Therefore $f(x) = x$. By induction, $f(x) = x$ for all x. A similar argument shows that $f \leq \mathrm{id}$ implies $f = \mathrm{id}$. By Lemma 6.6, it now follows that the component of the identity map in the finite space X^X consists only of the identity map. That is, any map homotopic to the identity is the identity. □

Corollary 8.9. *If $f\colon X \longrightarrow Y$ is a homotopy equivalence of minimal finite spaces, then f is a homeomorphism.*

Proof. If $g\colon Y \longrightarrow X$ is a homotopy inverse, then $g \circ f \simeq \mathrm{id}$ and $f \circ g \simeq \mathrm{id}$. By the theorem, $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. □

Corollary 8.10. *Finite spaces X and Y are homotopy equivalent if and only if they have homeomorphic cores. In particular, the core of X is unique up to homeomorphism.*

Proof. This is immediate since the cores of X and Y are minimal finite spaces that are homotopy equivalent to X and Y. □

Remark 8.11. In any homotopy class of finite spaces, there is a representative with the least possible number of points. This representative must be a minimal finite space, since its core is a homotopy equivalent subspace. The minimal representative is homeomorphic to a core of any finite space in the given homotopy class.

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