Representations of Algebras and Finite Groups: An Introduction

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Preface

These notes describe the basic ideas of the theory of representations of finite groups. Most of the essential structural results of the theory follow immediately from the structure theory of semisimple algebras, and so this topic occupies a long chapter.

Material not covered here include the theory of induced representations. The arithmetic properties of group characters are also not dealt with in detail.

It is not the purpose of these notes to give comprehensive accounts of all aspects of the topics covered. The objective is to see the theory of representations of finite groups as a coherent narrative, building some general structural theory and applying the ideas thus developed to the case of the symmetric group S_n. It is also not the objective to present a very efficient and fast route through the theory. For many of the ideas we pause the examine the same set of results from several different points of view. For example, in Chapter 10, we develop the the theory of decomposition of a module with respect to the commutant ring in three distinct ways.

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Chapter 1

Basic Concepts and Constructions

A group is an abstract mathematical object, a set with elements and an operation satisfying certain axioms. A representation of a group realizes the elements of the group concretely as geometric symmetries. The same group will generally have many different such representations. Thus, even a group which arises naturally and is defined as a set of symmetries may have representations as geometric symmetries at different levels.

In quantum physics the group of rotations in three-dimensional space gives rise to symmetries of a complex Hilbert space whose rays represent states of a physical system; the same abstract group appears once, classically, in the avatar of rotations in space and then expresses itself at the level of a more 'implicate order' in the quantum theory as unitary transformations on Hilbert spaces.

In this chapter we (i) introduce the basic concepts, defining group representations, irreducibility and characters, (ii) carry out certain useful standard constructions with representations, and (iii) present a result or two of interest which follow very quickly from the basic notions.

All through this chapter G denotes a group, and $\mathbb{F}$ a field. We will work with vector spaces, denoted V, W, E, F, over the field $\mathbb{F}$.

1.1 Representations of Groups

A representation ρ of a group G on a vector space V associates to each element $x \in G$ an invertible linear map

$$\rho(x) : V \to V : v \mapsto \rho(x)v$$

such that

$$\begin{aligned} \rho(xy) &= \rho(x)\rho(y) \qquad \text{for all } x, y \in G. \\ \rho(e) &= I, \end{aligned} \tag{1.1}$$

where $I : V \to V$ is the identity map. Here, our vector space V is over a field $\mathbb{F}$. We denote by

$$\mathrm{End}_{\mathbb{F}}(V)$$

the ring of endomorphisms of a vector space V. A representation ρ of G on V is thus a map

$$\rho : G \to \mathrm{End}_{\mathbb{F}}(V)$$

satisfying (1.1) and such that $\rho(x)$ is invertible for every $x \in G$.

A *complex representation* is a representation on a vector space over the field $\mathbb{C}$ of complex numbers.

The homomorphism condition (1.1) implies

$$\rho(e) = I, \quad \rho(x^{-1}) = \rho(x)^{-1} \quad \text{for all } x \in G.$$

We will often say 'the representation E' instead of 'the representation ρ on the vector space E'.

If V is finite-dimensional with basis $b_1, ..., b_n$, then the matrix

$$\begin{bmatrix} \rho(g)_{11} & \rho(g)_{12} & \dots & \rho(g)_{1n} \\ \rho(g)_{21} & \rho(g)_{22} & \dots & \rho(g)_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \rho(g)_{n1} & \rho(g)_{n2} & \dots & \rho(g)_{nn} \end{bmatrix} \tag{1.2}$$

of $\rho(g) \in \mathrm{End}_{\mathbb{F}}(V)$ is often useful to explicitly express the representation or to work with it. Indeed, with some basis in mind, we will often not make a distinction between $\rho(x)$ and its matrix form.

Sometimes the term 'matrix element' is used to mean a function on G which arises from a reprsentation ρ as

$$G \to \mathbb{F} : x \mapsto \langle f|\rho(g)|v\rangle,$$

where $|v\rangle$ is a vector in the representation space of ρ, and $\langle f|$ is in the dual space.

Consider the group S_n of permutations of $[n] = \{1, ..., n\}$, acting on the vector space $\mathbb{F}^n$ by permutation of coordinates:

$$S_n \times \mathbb{F}^n \to \mathbb{F}^n : \big(\sigma, (v_1, ..., v_n)\big) \mapsto R(\sigma)(v_1, ..., v_n) \stackrel{\text{def}}{=} (v_{\sigma^{-1}(1)}, ..., v_{\sigma^{-1}(n)}).$$

Another way to understand this is by specifying

$$R(\sigma)e_j = e_{\sigma(j)} \qquad \text{for all } j \in [n].$$

Here e_j is the j-th vector in the standard basis of $\mathbb{F}^n$; it has 1 in the j-th entry and 0 in all other entries. Thus, for example, for S_4 acting on $\mathbb{F}^4$, the matrix for $R\big((134)\big)$ relative to the standard basis of $\mathbb{F}^4$, is

$$R\big((134)\big) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

1.2 Representations and their Morphisms

If ρ and ρ' are representations of G on vector spaces E and E' over $\mathbb{F}$, and

$$A : E \to E'$$

is a linear map such that

$$\rho'(x) \circ A = A \circ \rho(x) \qquad \text{for all } x \in G \tag{1.3}$$

then we can consider A to be a *morphism* from the representation ρ to the representation ρ'. For instance, the identity map $I : E \to E$ is a morphism from ρ to itself.

The composition of two morphisms is clearly also a morphism, and the inverse of an invertible morphism is again a morphism.

Two representations ρ and ρ' of G are *isomorphic* or *equivalent* if there is an invertible *intertwining operator* between them, i.e. a linear isomorphism

$$A : E \to E'$$

for which

$$A\rho(x) = \rho'(x)A \text{ for all } x \in G. \tag{1.4}$$

1.3 Sums, Products, and Tensor Products

If ρ_1 and ρ_2 are representations of G on E_1 and E_2, respectively, then we have the direct sum

$$\rho_1 \oplus \rho_2$$

representation on $E \oplus E'$:

$$(\rho_1 \oplus \rho_2)(x) = (\rho_1(x), \rho_2(x)) \in \mathrm{End}_{\mathbb{F}}(E_1 \oplus E_2)$$

If bases are chosen in E_1 and E_2 then the matrix for $(\rho \oplus \rho')(x)$ is block diagonal, with the blocks $\rho_1(x)$ and $\rho_2(x)$ on the diagonal:

$$x \mapsto \begin{bmatrix} \rho_1(x) & 0 \\ 0 & \rho_2(x) \end{bmatrix}$$

This notion clearly generalizes to a direct sum (or product) of any family of representations.

We also have the *tensor product* $\rho_1 \otimes \rho_2$ of the representations, acting on $E_1 \otimes E_2$, specified through

$$(\rho_1 \otimes \rho_2)(x) = \rho_1(x) \otimes \rho_2(x) \tag{1.5}$$

1.4 Change of Field

There is a more subtle operation on vector spaces, which involves changing the ground field over which the vector space is defined. Suppose then that V is a vector space over a field $\mathbb{F}$, and let $\mathbb{F}' \supset \mathbb{F}$ be a field which contains $\mathbb{F}$ as a sub-field. Then V specifies an $\mathbb{F}'$-vector-space

$$V_{\mathbb{F}'} = \mathbb{F}' \otimes_{\mathbb{F}} V \tag{1.6}$$

Here we have, on the surface, a tensor product of two $\mathbb{F}$-vector-spaces: $\mathbb{F}'$, treated as a vector space over the subfield $\mathbb{F}$, and V itself. But $V_{\mathbb{F}'}$ acquires the structure of a vector space over $\mathbb{F}'$ by the multiplication rule

$$c(a \otimes v) = (ca) \otimes v,$$

for all $c, a \in \mathbb{F}'$ and $v \in V$. More concretely, if $V \neq 0$ has a basis B then the same set B is a basis for the $\mathbb{F}'$-vector-space $V_{\mathbb{F}'}$, simply by using coefficients from the field $\mathbb{F}'$.

Now suppose ρ is a representation of a group G on a vector space over V. Then there is, naturally induced, the reprsentation $\rho_{\mathbb{F}'}$ on $V_{\mathbb{F}'}$ as follows:

$$\rho_{\mathbb{F}'}(x)(a \otimes v) = a \otimes \rho(x)v \tag{1.7}$$

for all $a \in \mathbb{F}'$ and $v \in V$.

To get a concrete feel for $\rho_{\mathbb{F}'}$ let us look at the matrix form. Choose a basis $b_1, ..., b_n$ for V, assumed finite-dimensional and non-zero. Then, almost by definition, this is also a basis for $V_{\mathbb{F}'}$, only with scalars to be drawn from $\mathbb{F}'$. Thus, *the matrix for $\rho_{\mathbb{F}'}(x)$ is exactly the same as the matrix for $\rho(x)$*, for every $x \in G$. The difference is only that we should think of this matrix now as a matrix over $\mathbb{F}'$ whose entries happen to lie in the subfield $\mathbb{F}$.

This raises a fundamental and deep question. Given a representation ρ, is it possible to find a basis of the vector space such that all entries of all the matrices $\rho(x)$ lie in some proper subfield of the field we started with? A remarkable result of Brauer shows that all irreducible complex representations of a finite group can be realized over a field obtained by adjoining suitable roots of unity to the field $\mathbb{Q}$ of rationals. Thus, in effect, under very simple requirements, the abstract group essentially specifies a certain number field and geometries over this field in which it is represented as symmetries.

1.5 Invariant Subspaces and Quotients

A subspace $W \subset V$ is said to be *invariant* under ρ if

$$\rho(x)W \subset W \text{ for all } x \in G.$$

In this case,

$$x \mapsto \rho(x)|W \in \mathrm{End}_{\mathbb{F}}(W)$$

is a representation of G on W. It is a *subrepresentation* of ρ. Put another way, the inclusion map

$$W \to V : w \mapsto w$$

is a morphism from $\rho|W$ to ρ.

If W is invariant, then there is induced, in the natural way, a representation on the quotient space

$$V/W$$

given by

$$\rho_{V/W}(x) : a + W \mapsto \rho(x)a + W, \qquad \text{for all } a \in V \tag{1.8}$$

The following result is readily checked:

Proposition 1.5.1 *Suppose V is a representation of a group G, and $W \subset V$ a subrepresentation. Then the map*

$$W \oplus (V/W) \to V : (w, v + W) \mapsto w + v$$

gives an isomorphism of representations.

1.6 Dual Representations

For a vector space V over a field $\mathbb{F}$, let V' be the dual space of all linear mappings of V into $\mathbb{F}$:

$$V' = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F}). \tag{1.9}$$

If ρ is a representation of a group G on V there is induced a representation ρ' on V' specified as follows:

$$\rho'(x)f = f \circ \rho(x)^{-1} \qquad \text{for all } x \in G. \tag{1.10}$$

When working with a vector space and its dual, there is a visually appealing notation due to Dirac. A vector in V is denoted

$$|v\rangle$$

and is called a 'ket', while an element of the dual V' is denoted

$$\langle f|$$

and called a 'bra.' The evaluation of the bra on the ket is then, conveniently, the 'bra-ket'

$$\langle f|v\rangle \in \mathbb{F}.$$

Suppose now that V is finite-dimensional, with a basis $|b_1\rangle, ..., |b_n\rangle$. Then there is a corresponding *dual basis* of V' made up of the elements $\langle b_1|, ..., \langle b_n| \in V'$ which are specified by

$$\langle b_j|b_k\rangle = \delta_{jk} = \begin{cases} 1 & \text{if } j = k; \\ 0 & \text{if } j \neq k. \end{cases} \tag{1.11}$$

If $T : V \to V$ is a linear map its matrix relative to the basis $|b_1\rangle, ..., |b_n\rangle$ has entries

$$T_{jk} = \langle b_j|T|b_k\rangle$$

Note that, by convention and definition, T_{jk} is the j-th component of the vector obtained by applying T to the k-th basis vector.

There is one small spoiler: the notation $\langle b_j|$ wrongly suggests that it is determined solely by the vector $|b_j\rangle$, when in fact one needs the full basis $|b_1\rangle, ..., |b_n\rangle$ to give meaning to it.

Let us work out the matrix form of the dual representation ρ' relative to bases. If $|b_1\rangle, ..., |b_n\rangle$ is a basis of the representation space of ρ then

$$\begin{aligned} \rho'(x)_{jk} &= \langle \rho'(x)b_k|b_j\rangle \\ &= \langle b_k|\rho(x)^{-1}|b_j\rangle \\ &= \rho(x^{-1})_{kj} \end{aligned}$$

Thus, the matrix for $\rho'(x)$ is the *transpose* of the matrix for $\rho(x^{-1})$:

$$\rho'(x) = \rho(x^{-1})^{\mathrm{tr}}, \tag{1.12}$$

as matrices.

1.7 Irreducible Representations

A representation ρ on V is *irreducible* if $V \neq 0$ and the only invariant subspaces of V are 0 and V.

Thus, an irreducible representation is a kind of 'atom' (or, even better, 'elementary particle') among representations; there is no smaller representation than an irreducible one, other than the zero representation.

A nice, non-trivial, example of an irreducible representation of the symmetric group S_n can be extracted by looking first at the 'natural' action of S_n on the basis vectors of an n-dimensional space. We have the representation of S_n on the n-dimensional vector space $\mathbb{F}^n$ given by

$$S_n \times \mathbb{F}^n \to \mathbb{F}^n : (\sigma, vx) \mapsto R(\sigma)v = v \circ \sigma^{-1}, \tag{1.13}$$

where $v \in \mathbb{F}^n$ is to be thought of as a map $v : \{1, ..., n\} \to \mathbb{F} : j \mapsto v_j$. Thus, if $e_1, ..., e_n$ is the standard basis of $\mathbb{F}^n$, we have

$$R(\sigma)e_j = e_{\sigma(j)}$$

The subspaces

$$E_0 = \{(v_1, ..., v_n) \in \mathbb{F}^n \,:\, v_1 + \cdots + v_n = 0\} \tag{1.14}$$

and

$$D = \{(v, v, ..., v) \,:\, v \in \mathbb{F}\} \tag{1.15}$$

are clearly invariant subspaces. If $n1_{\mathbb{F}} \neq 0$ in $\mathbb{F}$ then the subspaces D and E_0 have in common only the zero vector, and provide a decomposition of $\mathbb{F}^n$ into a direct sum of proper, invariant subspaces. Moreover, the representations obtained by restricting R to the subspaces D and E_0 are irreducible. (See Exercise 3.)

As we will see later, for a finite group G, for which $|G| \neq 0$ in the field $\mathbb{F}$, every representation is a direct sum of irreducible representations. One of the major tasks of representation theory is to determine the irreducible representations of a group.

A one-dimensional representation is automatically irreducible. Our definitions all the trivial representation on the trivial space $V = \{0\}$ as a representation as well, and so we have to try to be careful everywhere to exclude this silly case as necessary.

1.8 Character of a Representation

The *character* χ_ρ of a representation of a group G on a finite-dimensional vector space E is the function on G given by

$$\chi_\rho(x) \stackrel{\text{def}}{=} \operatorname{tr} \rho(x) \qquad \text{for all } x \in G. \tag{1.16}$$

For instance, for the simplest representation, where $\rho(x)$ is the identity I on E for all $x \in G$, the character is the constant function with value $\dim_{\mathbb{F}} E$.

It may seem odd to single out the trace, and not, say, the determinant or some other such natural function of $\rho(x)$. But observe that if we know the trace of $\rho(x)$, with x running over *all* the elements of G, then we know the traces of $\rho(x^2)$, $\rho(x^3)$, etc., which means that we know the traces of all powers of $\rho(x)$, for every $x \in G$. This is clearly a lot of information about a matrix. Indeed, as we shall see later, $\rho(x)$ can, under some mild conditions, be written as a diagonal matrix with respect to some basis (generally dependent on x), and then knowing traces of all powers of $\rho(x)$ would mean that we would know this diagonal matrix completely, up to permutation of the basis vectors. Thus, knowledge of the character of ρ pretty much specifies each $\rho(x)$ up to basis change. In other words, under some simple assumptions, if ρ_1 and ρ_2 are finite-dimensional non-zero representations with the same character then for each x, there are bases in which the matrix of $\rho_1(x)$ is the same as the matrix of $\rho_2(x)$. This leaves open the possibility, however, that the special choice of bases might depend on x. Remarkably, this is not so! As we shall see much later, in Theorem 6.2.1, the character determines the representation up to equivalence. For now we will be satisfied with a simple observation:

Proposition 1.8.1 *If ρ_1 and ρ_2 are equivalent representations on finite-dimensional vector spaces then*

$$\operatorname{tr} \rho_1(x) = \operatorname{tr} \rho_2(x) \qquad \textit{for all } x \in G.$$

Proof Let $e_1, ..., e_d$ be a basis for the representation space E for ρ_1 (if this space is $\{0\}$ then the result is obviously and trivially true, and so we discard this case). Then in the representation space F for ρ_2, the vectors $f_i = Ae_i$ form a basis, where A is any isomorphism $E \to F$. We take in the present case, the isomorphism A which intertwines ρ_1 and ρ_2:

$$\rho_2(x) = A\rho_1(x)A^{-1} \qquad \text{for all } x \in G.$$

Then, for any $x \in G$, the matrix for $\rho_2(x)$ relative to the basis $Ae_1, ..., Ae_d$ is the same as the matrix of $\rho_1(x)$ relative to the basis $e_1, ..., e_d$. Hence, the trace of $\rho_2(x)$ equals the trace of $\rho_1(x)$. $\boxed{\text{QED}}$

The following observations are readily checked by using bases:

Proposition 1.8.2 *If ρ_1 and ρ_2 are equivalent representations on finite-dimensional vector spaces then*

$$\begin{aligned} \chi_{\rho_1 \oplus \rho_2} &= \chi_{\rho_1} + \chi_{\rho_2} \\ \chi_{\rho_1 \otimes \rho_2} &= \chi_{\rho_1} \chi_{\rho_2} \end{aligned} \tag{1.17}$$

Characters provide an extremely useful tool to understand representations. As we shall see in examples, it is often possible to work out the character of a representation without first having to work out the representation itself explictly.

Let us work out the character of the representation of the permutation group S_n on $\mathbb{F}^n$, and on the subspaces D and E_0 given in (1.14) and (1.15), discussed earlier in section 1.7. Recall that for $\sigma \in S_n$, and any standard-basis vector e_j of $\mathbb{F}^n$,

$$\rho_n(\sigma)e_j \stackrel{\text{def}}{=} e_{\sigma(j)}$$

Hence,

$$\chi_{\rho_n}(\sigma) = \text{number of fixed points of } \sigma. \tag{1.18}$$

Now consider the restriction ρ_D of this action to the 'diagonal' subspace $D = \mathbb{F}(e_1 + \cdots + e_n)$. Clearly, $\rho_D(\sigma)$ is the identity map for every $\sigma \in S_n$, and so

$$\chi_{\rho_D}(\sigma) = 1 \qquad \text{for all } \sigma \in S_n$$

Then we can readily deduce the character of the representation $\rho_{E_0} = \rho(\cdot)|E_0$:

$$\chi_{\rho_0}(\sigma) = \chi_\rho(\sigma) - \chi_D(\sigma) = |\{j : \sigma(j) = j\}| - 1 \tag{1.19}$$

Next, for $n \in \{2, 3, ...\}$, consider the 1-dimensional representation ϵ of S_n specified by requiring that a permutation σ act through multiplication by the signature $\epsilon(\sigma)$ of the permutation σ; recall that $\epsilon(\sigma)$ is specified by requiring that

$$\prod_{1 \le j < k \le n}^{n} (X_{\sigma(j)} - X_{\sigma(k)}) = \epsilon(\sigma) \prod_{1 \le j < k \le n}^{n} (X_j - X_k) \tag{1.20}$$

for any formal variables $X_1, ..., X_n$. Then we have the tensor product representation $\epsilon \otimes \rho_{E_0}$ on the $(n-1)$-dimensional space E_0.

Characters could get confusing when working with representations over different fields at the same time. Fortunately, at least in the simplest natural situation there is no confusion:

Proposition 1.8.3 *If ρ is a representation of a finite group G on a finite-dimensional vector space V over a field $\mathbb{F}$, and $\rho_{\mathbb{F}'}$ is the corresponding representation on $V_{\mathbb{F}'}$, where $\mathbb{F}'$ is a field containing $\mathbb{F}$ as a subfield, then*

$$\chi_{\rho_{\mathbb{F}'}} = \chi_\rho.$$

Proof As seen in section 1.4, $\rho_{\mathbb{F}'}$ has exactly the same matrix as ρ, relative to suitable bases. Hence the characters are the same. $\boxed{\text{QED}}$

One last remark about characters. If ρ_1 is a one-dimensional representation of a group G then, for each $x \in G$, the operator $\rho_1(x)$ is simply multiplication by a scalar, which we will always denote again by $\rho_1(x)$. Then the character of ρ_1 is ρ_1 itself! In the converse direction, if a character χ of G is a homomorphism of G into the multiplicative group of invertible elements in the field then χ provides a one-dimensional representation.

1.9 Unitarity

Let G be a finite group and ρ a representation of G on a finite-dimensional vector space V over a field $\mathbb{F}$. Remarkably, under some mild conditions on the field $\mathbb{F}$, every element $\rho(x)$ can be expressed as a diagonal matrix relative to some basis (depending on x) in V, with the diagonal entries being roots of unity in $\mathbb{F}$:

$$\rho(x) = \begin{bmatrix} \zeta_1(x) & 0 & 0 & \dots & 0 & \\ 0 & \zeta_2(x) & 0 & \dots & 0 & \\ \vdots & \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & 0 & & \cdots & \zeta_d(x) \end{bmatrix}$$

where each $\zeta_j(x)$, when raised to the $|G|$-th power, gives 1.

An endomorphism in a vector space which has such a matrix relative to some basis is said to be *unitary*. (This terminology is generally used when the field is $\mathbb{C}$.) A representation ρ is said to be unitary if $\rho(x)$ is unitary for all x in the group. Thus, what we shall show is that, under some minimal conditions on the field, all representations of finite groups are unitary.

An m-th root of unity in a field $\mathbb{F}$ is an element $\zeta \in \mathbb{F}$ for which $\zeta^m = 1$. We will often work with fields which contain all the m-th roots of unity, where m is some fixed positive integer. This means that there exist $\eta_1, ..., \eta_m \in \mathbb{F}$

such that

$$X^m - 1 = \prod_{j=1}^{m} (X - \eta_m).$$

Proposition 1.9.1 *Suppose $\mathbb{F}$ is a field which contains m distinct m-th roots of unity, for some $m \in \{1, 2, 3, ..\}$. If $V \neq 0$ is a vector space over $\mathbb{F}$ and $T : V \to V$ is a linear map for which $T^m = I$, then there is a basis of V relative to which the matrix for T is diagonal and each diagonal entry is an m-th root of unity.*

Proof. Let $\eta_1, ..., \eta_m$ be the distinct elements of $\mathbb{F}$ for which the polynomial $X^m - 1$ factors as

$$X^m - 1 = (X - \eta_1)...(X - \eta_m)$$

Then

$$(T - \eta_1 I)...(T - \eta_m I) = T^m - I = 0$$

and so there is a smallest $d \in \{1, ..., m\}$ such that

$$(T - \eta_{j_1} I)...(T - \eta_{j_d} I) = 0 \tag{1.21}$$

for distinct $j_1, ..., j_d \in \{1, ..., m\}$. Let

$$V_a = \ker(T - \eta_{j_a})$$

Clearly, each V_a is mapped into itself by T. In fact, linear algebra tells us that V is the direct sum of the subspaces V_a, each of which is non-zero:

$$V = \bigoplus_{a=1}^{d} V_a.$$

But, on V_a, the mapping T is simply multiplication by the scalar η_{j_a}. Thus, choosing a basis in each V_a, and putting them together into a basis for V, we see that T has the desired property with respect to this basis. $\boxed{\text{QED}}$

As consequence we have:

Proposition 1.9.2 *Suppose G is a group in which $x^m = e$ for all $x \in G$, for some positive integer m. Let $\mathbb{F}$ be a field which contains m distinct m-th roots of unity. Then, for any representation ρ of G on a vector space $V_\rho \neq 0$ over $\mathbb{F}$, for each $x \in G$ there is a basis of V_ρ with respect to which the matrix of $\rho(x)$ is diagonal and the diagonal entries are each m-th roots of unity in $\mathbb{F}$. If V_ρ is finite-dimensional then $\chi_\rho(x)$ is a sum of m-th roots of unity, for every $x \in G$.*

There is a way to bootstrap our way up to a stronger form of the preceding result. Suppose that it is not the field $\mathbb{F}$, but rather an extension, a larger field $\mathbb{F}' \supset \mathbb{F}$ which contains m distinct m-th roots of unity; for instance, $\mathbb{F}$ might be the reals $\mathbb{R}$ and $\mathbb{F}'$ is the field $\mathbb{C}$. The representation space V can be dressed up to $V_{\mathbb{F}'} = \mathbb{F}' \otimes_{\mathbb{F}} V$, which is a vector space over $\mathbb{F}'$, and then a linear map $T : V \to V$ yields the linear map

$$T_{\mathbb{F}'} : V_{\mathbb{F}'} \to V_{\mathbb{F}'} : 1 \otimes v \mapsto 1 \otimes Tv.$$

If B is a basis of V then $\{1 \otimes w : w \in B\}$ is a basis of $V_{\mathbb{F}'}$, and the matrix of $T_{\mathbb{F}'}$ relative to this basis is the same as the matrix of T relative to B, and so

$$\operatorname{tr} T_{\mathbb{F}'} = \operatorname{tr} T.$$

Consequently, if in Proposition 1.9.2 we require simply that there be an extension field of $\mathbb{F}$ in which there are m distinct m-th roots of unity and ρ is a finite-dimensional representation over $\mathbb{F}$ then the values of the character χ_ρ are again sums of m-th roots of unity (which, themselves, need not lie in $\mathbb{F}$).

There is another aspect of unitarity which is very useful. Suppose the field $\mathbb{F}$ has an automorphism, call it *conjugation*,

$$\mathbb{F} \to \mathbb{F} : z \mapsto \overline{z}$$

which takes each root of unity to its inverse; let us call self-conjugate elements *real*. For instance, if $\mathbb{F}$ is a subfield of $\mathbb{C}$ then the usual complex conjugation provides such an automorphism. Then, under the hypotheses of Proposition 1.9.2, for each $x \in G$ and representation ρ of G on a finite-dimensional vector space $V_\rho \neq 0$, there is a basis of V_ρ relative to which the matrix of $\rho(x)$ is diagonal with entries along the diagonal being roots of unity; hence, $\rho(x^{-1})$, relative to the same basis, has diagonal matrix, with the diagonal entries being the conjugates of those for $\rho(x)$. Hence

$$\chi_\rho(x^{-1}) = \overline{\chi_\rho(x)}. \tag{1.22}$$

In particular, if an element of G is conjugate to its inverse, then the value of any character on such an element is real. In the symmetric group S_n, every element is conjugate to its own inverse, and so:

the characters of all complex representations of S_n are *real-valued*.

This is an amazing, specific result about a familiar concrete group which falls out immediately from some of the simplest general observations. Later, with greater effort, it will become clear that the characters of S_n in fact have integer values!

Exercises

1. Give an example of a representation ρ of a finite group G on a finite-dimensional vector space V over a field of characteristic 0, such that there is an element $g \in G$ for which $\rho(g)$ is not diagonal in any basis of V.

2. Prove Proposition 1.8.2.

3. Let $n \geq 2$ be a positive integer, $\mathbb{F}$ a field in which $n1_{\mathbb{F}} \neq 0$, and consider the representation R of S_n on $\mathbb{F}^n$ given by

   $$R(\sigma)(v_1, ..., v_n) = (v_{\sigma^{-1}(1)}, ..., v_{\sigma^{-1}(n)}) \qquad \text{for all } (v_1, ..., v_n) \in \mathbb{F}^n \text{ and } \sigma \in S_n.$$

   Let

   $$D = \{(v, ..., v) : v \in \mathbb{F}\} \subset \mathbb{F}^n, \text{ and } E_0 = \{(v_1, ..., v_n) : v_1 + \cdots + v_n = 0\}.$$

   Consider a non-zero vector $w = (w_1, ..., w_n) \in E_0$.

   (i) Show that $w \notin D$.

   (ii) Show that each vector $e_1 - e_j$ lies in the span of $\{R(\sigma)w : \sigma \in S_n\}$, where $e_1, ..., e_n$ is the standard basis of $\mathbb{F}^n$, with e_k having 1 in the k-th entry and 0 elsewhere.

   (iii) Show that the restriction R_0 of R to the subspace E_0 is an irreducible representation of S_n.

   Now examine what happens if $n1_{\mathbb{F}} = 0$.

4. Determine all one-dimensional representations of S_n over any field.

5. Let $n \in \{3, 4, ...\}$, and $n1_{\mathbb{F}} \neq 0$ in a field $\mathbb{F}$. Denote by R_0 be the restriction of the representation of S_n on $\mathbb{F}^n$ to the subspace $E_0 = \{x \in \mathbb{F}^n : x_1 + \cdots + x_n = 0\}$. Let ϵ be the one-dimensional representation of S_n on $\mathbb{F}$ given by the signature, i.e., $\sigma \in S_n$ acts by multiplication by

the signature $\epsilon(\sigma) \in \{+1, -1\}$. Show that $R_1 = R_0 \otimes \epsilon$ is an irreducible representation of S_n. Then work out the sum

$$\sum_{\sigma \in S_n} \chi_{R_0}(\sigma)\chi_{R_1}(\sigma^{-1}).$$

6. Consider S_3, which is generated by the cyclic permutation $c = (123)$ and the transposition $\tau = (12)$, which satisfy the relations

$$c^3 = \iota, \qquad \tau^2 = \iota, \qquad \tau c \tau^{-1} = c^2.$$

   Let $\mathbb{F}$ be a field. The group S_3 acts on $\mathbb{F}^3$ by permutation of coordinates, and preserves the subspace $E_0 = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$; the restriction of the action to E_0 is a 2-dimensional representation R_0 of S_3. Work out matrices for $R_0(\tau)$ and $\rho(c)$ relative to the basis $u_1 = (1, 0, -1)$ and $u_2 = (0, 1, -1)$ of E_0. Work out the values of the character χ_{R_0} on all the six elements of S_3 and then work out

$$\sum_{\sigma \in S_3} \chi_0(\sigma)\chi_0(\sigma^{-1}).$$

7. Consider A_4, the group of even permutations on $\{1, 2, 3, 4\}$, acting through permutation of coordinates of $\mathbb{F}^4$, where $\mathbb{F}$ is a field. This action preserves the subspace $E_0 = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$; Work out the values of the character χ of R_0 on all elements of A_4.

8. Let $V \neq 0$ be a vector space over $\mathbb{Z}_2$, and $T : V \to V$ a linear map for which $T^2 = I$. Show that T has an eigenvector, i.e. there is nonzero $v \in V$ for which $Tv = v$. Produce an example of such V and T for which T is *not* diagonal relative to any basis of V.

9. Suppose ρ is an irreducible representation of a finite group G on a vector space V over a field $\mathbb{F}$. If $\mathbb{F}' \supset \mathbb{F}$ is an extension field of $\mathbb{F}$, is the representation $\rho_{\mathbb{F}'}$ on $V_{\mathbb{F}'}$ irreducible?

10. Let ρ_1 and ρ_2 be representations of a group G on vector spaces V_1 and V_2, respectively, over a common field. For $x \in G$, let $\rho_{12}(x) : \mathrm{Hom}(V_1, V_2) \to \mathrm{Hom}(V_1, V_2)$ be given by

$$\rho_{12}(x)T = \rho_2(x)T\rho_1(x)^{-1}.$$

Show that ρ_{12} is a representation of G. Taking V_1 and V_2 to be finite-dimensional, show that this representation is equivalent to the tensor product representation $\rho_1' \otimes \rho_2$ on $V_1' \otimes V_2$.

11. Let ρ be a representation of a group G on a vector space V. Show that the subspace $V^{\hat{\otimes}2}$ consisting of symmetric tensors in $V \otimes V$ is invariant under the tensor product representation $\rho \otimes \rho$. Assuming that the ground field is $\mathbb{C}$, work out the character of the representation ρ_s which is given by the restriction of $\rho \otimes \rho$ to $V^{\hat{\otimes}2}$. (Hint: Use unitarity.)

Chapter 2

Basic Examples

We will work through some examples in this chapter, looking at representations, and their characters, of some familiar finite groups. For ease of reading, and to maintain sanity, we will work with the field $\mathbb{C}$ of complex numbers. Of course, any algebraically closed field of characteristic zero could be substituted for $\mathbb{C}$.

Recall that the character χ_ρ of a finite-dimensional representation ρ of a group G is the function on the group specified by

$$\chi_\rho(x) = \mathrm{Tr}\,\rho(x). \tag{2.1}$$

Clearly, $\chi(x)$ remains unchanged if x is replaced by a conjugate yxy^{-1}. Thus, characters are constant on conjugacy classes.

Let $\mathcal{C}_G$ be the set of all conjugacy classes in G. If C is a conjugacy class then we denote by C^{-1} the conjugacy class consisting of the inverses of the elements in C. We have seen before (1.22) that

$$\chi_\rho(x^{-1}) = \overline{\chi_\rho(x)} \qquad \text{for all } x \in G. \tag{2.2}$$

It will be useful, while going through examples, to keep at hand some facts about characters that we will prove later in Chapter 6. The most fundamental facts are: (i) a finite group G has only finitely many inequivalent irreducible representations and these are all finite-dimensional; and (ii) two finite-dimensional representations are equivalent if and only if they have the same character. Moreover, a representation is irreducible if and only if its character χ_ρ satisfies

$$\sum_{C \in \mathcal{C}_G} |C||\chi_\rho(C)|^2 = 1 \tag{2.3}$$

The number of conjugacy classes in G exactly matches the number of inequivalent irreducible complex representations. Let $\mathcal{R}_G$ be a maximal set of inequivalent irreducible complex representations of G.

In going through the examples in this chapter we will sometimes pause to use or verify some standard properties of characters, which we prove in generality later. The properties are summarized in the Schur orthogonality relations:

$$\begin{aligned} \sum_{y\in G}\chi_\rho(xy)\chi_{\rho'}(y^{-1}) &= |G|\chi_\rho(x)\delta_{\rho\rho'} \\ \sum_{\rho\in\mathcal{R}_G}\chi_\rho(C')\chi_\rho(C^{-1}) &= \frac{|G|}{|C|}\delta_{C'C} \end{aligned} \tag{2.4}$$

where δ_{ab} is 1 if $a=b$ and is 0 otherwise, the relations above being valid for all $\rho,\rho'\in\mathcal{R}_G$, all conjugacy classes $C,C'\in\mathcal{C}$, and all elements $x\in G$. Specializing this to specific assumptions (such as $\rho=\rho'$, or $x=e$), we have:

$$\begin{aligned} \sum_{\rho\in\mathcal{R}_G}(\dim\rho)^2 &= |G| \\ \sum_{\rho\in\mathcal{R}_G}\dim\rho\,\chi_\rho(y) &= 0 \qquad \text{if } y\neq e \\ \sum_{y\in G}\chi_\rho(y)\chi_{\rho'}(y^{-1}) &= |G|\delta_{\rho,\rho'}\dim\rho \qquad \text{for } \rho,\rho'\in\mathcal{R}_G \end{aligned} \tag{2.5}$$

2.1 Cyclic Groups

Let us work out all irreducible representations of a cyclic group C_n containing n elements. Being cyclic, C_n contains a *generator* c, which is an element such that C_n consists exactly of the power $c,c^2,...,c^n$, where c^n is the identity e in the group.

Let ρ be a representation of C_n on a complex vector space $V\neq 0$. By Proposition 1.9.2, there is a basis of V relative to which the matrix of $\rho(c)$ is diagonal, with each entry being an n-th root of unity:

$$\text{matrix of } \rho(c) = \begin{bmatrix} \eta_1 & 0 & 0 & \dots & 0 \\ 0 & \eta_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \eta_d \end{bmatrix}$$

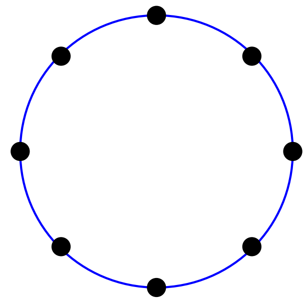

Figure 2.1: A picture for the cyclic group C_8

Since c generates the full group C_n, the matrix for ρ is diagonal on all the elements c^j in C_n. Thus, V is a direct sum of one-dimensional subspaces, each of which provides a representation of C_n. Of course, any one-dimensional representation is automatically irreducible.

Let us summarize our observations:

Theorem 2.1.1 *Let C_n be a cyclic group of order $n \in \{1, 2, ...\}$. Every complex representation of C_n is a direct sum of irreducible representations. Each irreducible representation of C_n is one-dimensional, specified by the requirement that a generator element $c \in G$ act through multiplication by an n-th root of unity. Each n-th root of unity provides an irreducible representation of C_n, and these representations are mutually inequivalent.*

Thus, there are exactly n inequivalent irreducible representations of C_n.

Everything we have done here goes through for representations of C_n over a field which contains n distinct roots of unity.

Let us now take a look at what happens when the field does not contain the requisite roots of unity. Consider, for instance, the representations of C_3 over the field $\mathbb{R}$ of real numbers. There are three geometrically apparent representations:

(i) the one-dimensional ρ_1 representation associating the identity operator (multiplication by 1) to every element of C_3;

(ii) the two-dimensional representation ρ_2^+ on $\mathbb{R}^2$ in which c is associated with rotation by 120^0;

(iii) the two-dimensional representation ρ_2^- on $\mathbb{R}^2$ in which c is associated with rotation by -120^0.

These are clearly all irreducible. Moreover, any irreducible representation of C_3 on $\mathbb{R}^2$ is clearly either (ii) or (iii).

Now consider a general real vector space V on which C_3 has a representation ρ. Choose a basis B in V, and let $V_{\mathbb{C}}$ be the complex vector space with B as basis (put another way, $V_{\mathbb{C}}$ is $\mathbb{C} \otimes_{\mathbb{R}} V$ viewed as a complex vector space). Then ρ gives, naturally, a representation of C_3 on $V_{\mathbb{C}}$. Then $V_{\mathbb{C}}$ is a direct sum of complex one-dimensional subspaces, each invariant under the action of C_3. Since a complex one-dimensional vector space is a real two-dimensional space, and we have already determined all two-dimensional real representations of C_3, we are done with classifying all real representations of C_3. Too fast, you say? Proceed to Exercise 7!

2.2 Dihedral Groups

The dihedral group D_n, for n any positive integer, is a group of $2n$ elements generated by two elements c and r, where c has order n, r has order 2, and conjugation by r turns c into c^{-1}:

$$c^n = e, \qquad r^2 = e, \qquad rcr^{-1} = c^{-1} \tag{2.6}$$

Geometrically, think of c as rotation in the plane by the angle $2\pi/n$ and r as reflection across a fixed line through the origin. The elements of D_n are

$$c, cr, c^2, c^2r, ..., c^n, c^nr,$$

where, of course, c^n is the identity element e.

The geometric view of D_n immediately yields a real two-dimensional representation: let c act on $\mathbb{R}^2$ through rotation by angle $2\pi/n$ and r through reflection across the x-axis. Complexifying this gives a two-dimensional complex representation ρ_1 on $\mathbb{C}^2$:

$$\rho_1(c) = \begin{bmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{bmatrix}, \qquad \rho_1(r) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where η is a primitive n-th root of unity, say

$$\eta = e^{2\pi i/n}.$$

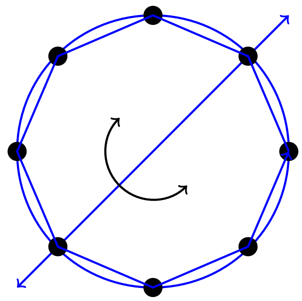

Figure 2.2: A picture for the dihedral group D_4

More generally, we have the representation ρ_m specified by requiring

$$\rho_m(c) = \begin{bmatrix} \eta^m & 0 \\ 0 & \eta^{-m} \end{bmatrix}, \qquad \rho_m(r) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

for $m \in \mathbb{Z}$; of course, to avoid repetition, we may focus on $m \in \{1, 2, ..., n-1\}$. The values of ρ_m on all elements of D_n are given by:

$$\rho_m(c^j) = \begin{bmatrix} \eta^{mj} & 0 \\ 0 & \eta^{-mj} \end{bmatrix}, \qquad \rho_m(c^j r) = \begin{bmatrix} 0 & \eta^{mj} \\ \eta^{-mj} & 0 \end{bmatrix}$$

Having written this, we note that this representation makes sense over any field $\mathbb{F}$ containing n-th roots of unity. However, we stick to the ground field $\mathbb{C}$, or at least $\mathbb{Q}$ with any primitive n-th root of unity adjoined.

Clearly, ρ_m repeats itself whem m changes by multiples of n. Thus we need only focus on $\rho_1, ..., \rho_{n-1}$.

Is ρ_m irreducible? Yes if and only if there is a non-zero vector $v \in \mathbb{F}^2$ fixed by $\rho_m(r)$ and $\rho_m(c)$. Being fixed by $\rho_m(r)$ means that such a vector must be a multiple of $(1, 1)$ in $\mathbb{C}^2$. But $\mathbb{C}(1, 1)$ is also invariant under $\rho_m(c)$ if and only if η^m is equal to η^{-m}, i.e., if and only if $n = 2m$.

Thus, ρ_m, for $m \in \{1, ..., n-1\}$, is irreducible if $n \neq 2m$, and is reducible if $n = 2m$.

Are we counting things too many times? Indeed, the representations ρ_m are not all inequivalent. Interchanging the two axes, converts ρ_m into $\rho_{-m} = \rho_{n-m}$. Thus, we can narrow our focus onto ρ_m for $1 \leq m < n/2$.

We have now identified $n/2 - 1$ irreducible two-dimensional representations if n is even, and $(n-1)/2$ irreducible two-dimensional representations if n is odd.

The character χ_m of ρ_m is obtained by taking the trace of ρ_m on the elements of the group D_n:

$$\chi_m(c^j) = \eta^{mj} + \eta^{-mj}, \qquad \chi_m(c^j r) = 0.$$

Now consider a one-dimensional representation θ of D_n (over any field). First, from $\theta(r)^2 = 1$, we see that $\theta(r) = \pm 1$. Applying θ to the relation that rcr^{-1} equals c^{-1} it follows that $\theta(c)$ must also be ± 1. But then, from $c^n = e$, it follows that $\theta(c)$ can be -1 only if n is even. Thus, we have the one-dimensional representations specified by:

$$\begin{aligned} \theta_{+,\pm}(c) = 1, \quad \theta_{+,\pm}(r) = \pm 1 \qquad &\text{if } n \text{ is even or odd} \\ \theta_{-,\pm}(c) = -1, \quad \theta_{-,\pm}(r) = \pm 1 \qquad &\text{if } n \text{ is even.} \end{aligned} \tag{2.7}$$

This gives us 4 one-dimensional representations if n is even, and 2 if n is odd.

Thus, for n even we have identified a total of $3+n/2$ irreducible representations, and for n odd we have identified $(n+3)/2$ irreducible representations.

According to results we will prove later, the sum

$$\sum_{\chi} d_\chi^2$$

over all distinct complex irreducible characters is the total number of elements in the group, i.e., in this case the sum should be $2n$. Working out the sum over all the irreducible characters χ we have determined, we obtain:

$$\begin{aligned} \left(\frac{n}{2} - 1\right) 2^2 + 4 = 2n \qquad &\text{for even } n; \\ \left(\frac{n-1}{2}\right) 2^2 + 2 = 2n \qquad &\text{for odd } n. \end{aligned} \tag{2.8}$$

Thus, our list of irreducible complex representations contains all irreducible representations, up to equivalence.

Our objective is to work out all characters of D_n. Characters being constant on conjugacy classes, let us first determine the conjugacy classes in D_n.

Since rcr^{-1} is c^{-1}, it follows that

$$r(c^j r)r^{-1} = c^{-j} r = c^{n-j} r.$$

This already indicates that the conjugacy class structure is different for n even and n odd. In fact notice that conjugating $c^j r$ by c results in increasing j by 2:

$$c(c^j r)c^{-1} = c^{j+1}\, rc^{-1}r^{-1}\, r = c^{j+1}cr = c^{j+2}r.$$

If n is even, the conjugacy classes are:

$$\begin{aligned} &\{e\}, \{c, c^{n-1}\}, \{c^2, c^{n-2}\}, ..., \{c^{n/2-1}, c^{n/2+1}\}, \{c^{n/2}\}, \\ &\qquad\qquad \{r, c^2r, ..., c^{n-2}r\}, \{cr, c^3r, ..., c^{n-1}r\} \end{aligned} \tag{2.9}$$

Note that there are $3 + n/2$ conjugacy classes, and this exactly matches the number of inequivalent irreducible representations obtained earlier.

To see how this plays out in practice let us look at D_4. Our analysis shows that there are five conjugacy classes:

$$\{e\}, \{c, c^3\}, \{c^2\}, \{r, c^2r\}, \{cr, c^3r\}.$$

There are four one-dimensional representations $\theta_{\pm,\pm}$, and one irreducible two-dimensional representation ρ_1 specified through

$$\rho_1(c) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \qquad \rho_1(r) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

In Table 2.1 we list the values of the characters of D_4 on the various conjugacy classes. The latter are displayed in a row (second from top), each conjugacy class identified by an element which it contains; above each conjugacy class we have listed the number of elements it contains. Each row in the main body of the table displays the values of a character on the conjugacy classes.

The case for odd n proceeds similarly. Take, for instance, $n = 3$. The group D_3 is generated by elements c and r subject to the relations

$$c^3 = e, \quad r^2 = e, \qquad rcr^{-1} = c^{-1}.$$

The conjugacy classes are:

$$\{e\}, \{c, c^2\}, \{r, cr, c^2r\}$$

The irreducible representations are: $\theta_{+,+}$, $\theta_{+,-}$, ρ_1. The character table is produced in Table 2.2, where the first row displays the number of elements in the conjugacy classes listed (by choice of an element) in the second row.

	1	2	1	2	2
	e	c	c^2	r	cr
$\theta_{+,+}$	1	1	1	1	1
$\theta_{+,-}$	1	1	1	-1	-1
$\theta_{-,+}$	1	-1	1	1	-1
$\theta_{-,-}$	1	-1	1	-1	1
χ_1	2	0	-2	0	0

Table 2.1: Character Table for D_4

	1	2	3
	e	c	r
$\theta_{+,+}$	1	1	1
$\theta_{+,-}$	1	1	-1
χ_1	2	-1	0

Table 2.2: Character Table for D_3

The dimensions of the representations can be read off from the first column in the main body of the table. Observe that the sum of the squares of the dimensions of the representations of S_3 listed in the table is

$$1^1 + 1^2 + 2^2 = 6,$$

which is exactly the number of elements in D_3. This verifies the first property listed earlier in (2.5).

2.3 The symmetric group S_4

The symmetric group S_3 is isomorphic to the dihedral group D_3, and we have already determined the irreducible representations of D_3 over the complex numbers.

Let us turn now to the symmertic group S_4, which is the group of permutations of $\{1, 2, 3, 4\}$. Geometrically, this is the group of rotational symmetries of a cube.

Two elements of S_4 are conjugate if and only if they have the same cycle structure; thus, for instance, (134) and (213) are conjugate, and these are not conjugate to (12)(34). The following elements then belong to all the distinct conjugacy classes:

$$\iota, \qquad (12)(34), \qquad (123), \qquad (1234), \qquad (12)(34)$$

Number of elements	1	6	8	6	3
Conjugacy class of	ι	(12)	(123)	(1234)	(12)(34)

Table 2.3: Conjugacy classes in S_4

where ι is the identity permutation. The conjugacy classes, each identified by one element it contains, are listed with the number of elements in each conjugacy class, in Table 2.3.

There are two one-dimensional representations of S_4 we are familiar with: the trivial one, associating 1 to every element of S_4, and the signature representation ϵ whose value is $+1$ on even permutations and -1 on odd ones.

We also have seen a three-dimensional irreducible representation of S_4; recall the representation R of S_4 on $\mathbb{C}^4$ given by permutation of coordinates:

$$(x_1, x_2, x_3, x_4) \mapsto (x_{\sigma^{-1}(1)}, ..., x_{\sigma^{-1}(4)})$$

Equivalently,

$$R(\sigma)e_j = e_{\sigma(j)} \qquad j \in \{1, 2, 3, 4\}$$

where $e_1, ..., e_4$ are the standard basis vectors of $\mathbb{C}^4$. The three-dimensional subspace

$$E_0 = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 : \, x_1 + x_2 + x_3 + x_4 = 0\}$$

is mapped into itself by the action of R, and the restriction to E_0 gives an irreducible representation R_0 of S_4. In fact,

$$\mathbb{C}^4 = E_0 \oplus \mathbb{C}(1, 1, 1, 1)$$

decomposes $\mathbb{C}^4$ into complementary irreducible representation subspaces, where the subspace $\mathbb{C}(1, 1, 1, 1)$ carries the trivial representation. Examining the effect of the group elements on the standard basis vectors, we can work out the character of R. For instance, $R((12))$ interchanges e_1 and e_2, and leaves e_3 and e_4 fixed, and so its matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Conjugacy class of	ι	(12)	(123)	(1234)	(12)(34)
χ_R	4	2	1	0	0
χ_0	3	1	0	-1	-1
χ_1	3	-1	0	1	-1

Table 2.4: The characters χ_R and χ_0 on conjugacy classes

and the trace is

$$\chi_R((12)) = 2.$$

Subtracting off the trivial character, which is 1 on all elements of S_4, we obtain the character χ_0 of the representation R_0. All this is displayed in the first three rows of Table 2.4.

We can manufacture another three-dimensional representation R_1 by tensoring R_0 with the signature ϵ:

$$R_1 = R_0 \otimes \epsilon.$$

The character χ_1 of R_1 is then written down by taking products, and is displayed in the fourth row in Table 2.4.

Since R_0 is irreducible and R_1 acts by a simple ± 1 scaling of R_0, it is clear that R_1 is also irreducible. Thus, we now have two one-dimensional representations and two three-dimensional irreducible representations. The sum of the squares of the dimensions is

$$1^2 + 1^2 + 3^2 + 3^2 = 20.$$

From the first relation in (2.5) we know that the sum of the squares of the dimensions of all the inequivalent irreducible representations is $|S_4| = 24$. Thus, looking at the equation

$$24 = 1^2 + 1^2 + 3^2 + 3^2 + ?^2$$

we see that we are missing a two-dimensional irreducible representation R_2. Leaving the entries for this blank, we have the following character table:

	1	6	8	6	3
	ι	(12)	(123)	(1234)	(12)(34)
trivial	1	1	1	1	1
ϵ	1	-1	1	-1	1
χ_0	3	1	0	-1	-1
χ_1	3	-1	0	1	-1
χ_2	2	?	?	?	?

Table 2.5: Character Table for S_4 with missing row

As an illustration of the power of the character method, let us work out the character χ_2 of this 'missing' representation R_2, without even bothering to search for the representation itself. Recall from (2.5) the relation

$$\sum_\rho \dim \rho \, \chi_\rho(\sigma) = 0, \qquad \text{if } \sigma \neq \iota,$$

where the sum runs over a maximal set of inequivalent irreducible complex representations of S_4 and σ is any element of S_4. More geometrically, this means that *the vector formed by the first column* in the main body of the table (i.e., the column for the trivial conjugacy class) *is orthogonal to the vectors* formed by the columns *for the other conjugacy classes*. Using this we can work out the missing entries of the character table. For instance, taking $\sigma = (12)$, we have

$$2\chi_2((12)) + 3 * \underbrace{(-1)}_{\chi_1((12))} + 3 * 1 + 1 * (-1) + 1 * 1 = 0$$

which yields

$$\chi_2((12)) = 0.$$

For $\sigma = (123)$, we have

$$2\chi_2((123)) + 3 * \underbrace{0}_{\chi_1((123))} + 3 * 0 + 1 * 1 + 1 * 1 = 0$$

	1	6	8	6	3
	ι	(12)	(123)	(1234)	(12)(34)
trivial	1	1	1	1	1
ϵ	1	-1	1	-1	1
χ_0	3	1	0	-1	-1
χ_1	3	-1	0	1	-1
χ_2	2	0	-1	0	2

Table 2.6: Character Table for S_4

which produces

$$\chi_2((123)) = -1.$$

Filling in the entire last row of the character table in this way produces Table 2.6.

Just to be sure that the indirectly detected character χ_2 is irreducible let us run the check given in (2.3) for irreducible characters: the sum of the quantities $|C||\chi_2(C)|^2$ over all the conjugacy classes C should work out to 1. Indeed, we have

$$\sum_C |C||\chi_2(C)|^2 = 1 * 2^2 + 6 * 0^2 + 8 * (-1)^2 + 6 * 0^2 + 3 * 2^2 = 24 = |S_4|,$$

a pleasant affirmation of the power of the theory and tools promised to be developed in the chapters ahead.

Exercises

1. Work out the character table of D_5.

2. If H is a normal subgroup of a finite group G, and ρ a representation of the group G/H then let ρ_G be the representation of G specified by

$$\rho_G(x) = \rho(xH) \qquad \text{for all } x \in G.$$

Show that ρ_G is irreducible if and only if ρ is irreducible. Work out the character of ρ_G in terms of the character of ρ.

3. Let V_4 be the subgroup of S_4 consisting of the identity ι along with the order-2 permutations in the conjugacy class containing $(12)(34)$. Explicitly, $V_4 = \{\iota, (12)(34), (13)(24), (14)(23)\}$. Being a union of conjugacy classes, V_4 is invariant under conjugations, i.e., is a normal subgroup of S_4. Now view S_3 as a subgroup of S_4, consisting of the permutations fixing 4. Thus, $V_4 \cap S_3 = \{\iota\}$. Show that the mapping

$$S_3 \to S_4/V_4 : \sigma \mapsto \sigma V_4$$

is an isomorphism.

4. Obtain an explicit form of a two-dimensional irreducible complex representation of S_4 for which the character is χ_2 as given in Table 2.6.

5. In S_3 there is the cyclic group C_3 generated by (123), which is a normal subgroup. The quotient $S_3/C_3 \simeq S_2$ is a two-element group. Work out the one-dimensional representation of S_3 which arises from this by the method of Problem 2 above.

6. The alternating group A_4 consists of all even permutations inside S_4. It is generated by the elements

$$c = (123), \quad x = (12)(34), \quad y = (13)(24), \quad z = (14)(23)$$

satisfying the relations

$$cxc^{-1} = z, \quad cyc^{-1} = x, \quad cxc^{-1} = y, \quad c^3 = \iota, \quad xy = yx = z.$$

   (i) Show that the conjugacy classes are

   $$\{\iota\},\ \{x, y, z\},\ \{c, cx, cy, cz\},\ \{c^2, c^2x, c^y, c^z\}.$$

   Note that c and c^2 are in different conjugacy classes in A_4, even though in S_4 they are conjugate.

   (ii) Show that the group A_4 generated by all commutators $aba^{-1}b^{-1}$ is $V_4 = \{\iota, x, y, z\}$.

	1	3	4	4
	ι	(12)(34)	(123)	(132)
ψ_0	1	1	1	1
ψ_1	1	1	ω	ω^2
ψ_2	1	1	ω^2	ω
χ_1	?	?	?	?

Table 2.7: Character Table for A_4

(iii) Check that there is an isomorphism given by

$$C_3 \mapsto A_4/V_4 : c \mapsto cV_4.$$

(iv) Obtain 3 one-dimensional representations of A_4.

(v) The group $A_4 \subset S_4$ acts by permutation of coordinates on $\mathbb{C}^4$ and preserves the three-dimensional subspace $E_0 = \{(x_1, ..., x_4) : x_1 + \cdots + x_4 = 0\}$. Work out the character χ_3 of this representation of A_4.

(vi) Work out the full character table for A_4, by filling in the last row of Table 2.7.

7. Let V be a real vector space and $T : V \to V$ a linear mapping with $T^m = I$, for some positive integer m. Choose a basis B of V, and let $V_{\mathbb{C}}$ be the complex vector space with basis B. Define the *conjugation* map $C : V_{\mathbb{C}} \to V_{\mathbb{C}} : v \mapsto \overline{v}$ to be given by

$$C(\sum_{b\in B} v_b b) = \sum_{b\in B} \overline{v_b} b$$

where each $v_b \in \mathbb{C}$, and on the right we just have the ordinary complex conjugates $\overline{v}_b$. Show that $x = v + Cv$ and $y = i(v - Cv)$ are in V for every $v \in V_{\mathbb{C}}$. If v is an eigenvector of T, i.e., $Tv = \alpha v$ for some $\alpha \in \mathbb{C}$, show that T maps the subspace of V spanned by x and y into itself.

Chapter 3

The Group Algebra

All through this chapter G is a finite group, and $\mathbb{F}$ a field.

As we will see, representations of G correspond to representations of an algebra, called the *group algebra*, formed from $\mathbb{F}$ and G. A vast amount of information about the representations of G will fall out of studying representations of such algebras.

For us, a *ring* is always a ring with a unit element $1 \neq 0$.

3.1 Definition of $\mathbb{F}[G]$

It is extremely useful to introduce the *group algebra*

$$\mathbb{F}[G].$$

As a set, this consists of all formal linear combinations

$$a_1x_1 + \cdots + a_mx_m$$

with any $m \in \{1, 2, ...\}$, $a_1, ..., a_m \in \mathbb{F}$, and $x_1, ..., x_m \in G$. We add and multiply these new objects in the only natural way that is sensible. For example,

$$(2x_1 + 3x_2) + (-4x_1 + 5x_3) = (-2)x_1 + 3x_2 + 5x_3$$

and

$$(2x_1 - 4x_2)(x_4 + x_3) = 2x_1x_4 + 2x_1x_3 - 4x_2x_4 - 4x_2x_3.$$

Officially, $\mathbb{F}[G]$ consists of all maps

$$a : G \mapsto \mathbb{F} : x \mapsto a_x$$

If G is allowed to be infinite, then a_x is required to be 0 for all except finitely many $x \in G$; thus, $\mathbb{F}[G]$ is the direct sum copies of the field $\mathbb{F}$, one copy for each element of G. The function $a \in \mathbb{F}[G]$ is more conveniently written as

$$a = \sum_{x \in G} a_x x.$$

Addition and multiplication, as well as multiplication by elements $t \in \mathbb{F}$, are defined in the obvious way:

$$\sum_{x\in G} a_x x + \sum_{x\in G} b_x x = \sum_{x\in G} (a_x + b_x) x \tag{3.1}$$

$$\sum_{x\in G} a_x x \sum_{x\in G} b_x x = \sum_{x\in G} \left(\sum_{y\in G} a_y b_{y^{-1}x} \right) x \tag{3.2}$$

$$t \sum_{x\in G} a_x x = \sum_{x\in G} t a_x x \tag{3.3}$$

It is readily checked that $\mathbb{F}[G]$ is an algebra over $\mathbb{F}$, i.e. it is a ring and a $\mathbb{F}$-module, and the multiplication

$$\mathbb{F}[G] \times \mathbb{F}[G] \to \mathbb{F}[G] : (a, b) \mapsto ab$$

is $\mathbb{F}$-bilinear.

Sometimes it is useful to think of G as a subset of $\mathbb{F}[G]$, by identifying $x \in G$ with the element $1x \in \mathbb{F}[G]$. The multiplicative unit $1e_G$ in $\mathbb{F}[G]$ will be denoted 1, and in this way $\mathbb{F}$ can be viewed as a subset of $\mathbb{F}[G]$:

$$\mathbb{F} \to \mathbb{F}[G] : t \mapsto te_G.$$

3.2 Representations of G and $\mathbb{F}[G]$

The utility of the algebra $\mathbb{F}[G]$ stems from the observation that any representation

$$\rho : G \to \mathrm{End}_{\mathbb{F}}(E)$$

defines, in a unique way, a representation of $\mathbb{F}[G]$ in terms of operators on E. More specifically, we have, for each element

$$a = \sum_x a_x x \in \mathbb{F}[G]$$

an element

$$\rho(a) \stackrel{\text{def}}{=} \sum_x a_x \rho(x) \in \mathrm{End}_{\mathbb{F}}(E) \tag{3.4}$$

This induces a left $\mathbb{F}[G]$-module structure on E:

$$\left(\sum_{x\in G} a_x x\right) v = \sum_{x\in G} a_x \rho(x) v \tag{3.5}$$

It is very useful to look at representations in this way.

Put another way, we have an extension of ρ to an algebra-homomorphism

$$\rho : \mathbb{F}[G] \to \mathrm{End}_{\mathbb{F}}(E) : \sum_{x\in G} a_x x \mapsto \sum_{x\in G} a_x \rho(x) \tag{3.6}$$

Thus, *a representation of G corresponds to a module over the ring* $\mathbb{F}[G]$.

A subrepresentation or invariant subspace corresponds to a submodule, and the notion of direct sum of representations corresponds to direct sums of modules. A morphism of representations corresponds to a $\mathbb{F}[G]$-linear map, and an isomorphism of representations is simply an isomorphism of $\mathbb{F}[G]$-modules.

Conversely, if E is a $\mathbb{F}[G]$-module, then we have a representation of G on E, by simply restricting multiplication to the elements in $\mathbb{F}[G]$ which are in G.

3.3 The center of $\mathbb{F}[G]$

It is easy to determine the center

$$Z(\mathbb{F}[G])$$

of the algebra $\mathbb{F}[G]$. An element

$$a = \sum_{x\in G} a_x x$$

belongs to the center if and only if it commutes with every $g \in G$, i.e. if and only if

$$gag^{-1} = a,$$

i.e. if and only if

$$\sum_{x \in G} a_x gxg^{-1} = \sum_{x \in G} a_x x.$$

This holds if and only if

$$a_{g^{-1}xg} = a_x \qquad \text{for every } x \in G \tag{3.7}$$

This means that the function a is constant on conjugacy classes in G. Thus, a is in the center if and only if it can be expressed as a linear combination of the elements

$$b_C = \sum_{x \in C} x, \qquad C \text{ a (finite) conjugacy class in } G. \tag{3.8}$$

We are primarily interested in finite groups, and then the added qualifier of finiteness of the conjugacy classes is is not needed.

If C and C' are distinct conjugacy classes then b_C and $b_{C'}$ are sums over disjoint sets of elements of G, and so the collection of all such b_C is linearly independent. Thus, we have a simple but important observation:

Proposition 3.3.1 *Suppose G is a finite group. Then the center of $\mathbb{F}[G]$ is a vector space with basis given by the elements b_C, with C running over all conjugacy classes of G. In particular, the dimension of the center of $\mathbb{F}[G]$ is equal to the number of conjugacy classes in G.*

3.4 A glimpse of some results

We have seen that representations of G correspond to representations of the algebra $\mathbb{F}[G]$. A substantial amount of information about representations of G can be gleaned just by studying the structure of modules over rings, the ring of interest here being $\mathbb{F}[G]$. Of course, $\mathbb{F}[G]$ is not just any ring; in the cases of interest for us, it has a *very special structural property called semisimplicity*, which leads to a great deal of information about modules over it.

Decomposing a representation into irreducible components corresponds to decomposing a $\mathbb{F}[G]$-module into *simple* submodules (i.e. non-zero submodules which have no proper non-zero submodules). A module which is a direct sum of simple submodules is a *semisimple* module.

We will work with a finite group G and a field $\mathbb{F}$ in which $|G| \neq 0$, i.e. the characteristic of $\mathbb{F}$ is not a divisor of the number of elements in G. As we will see, the key fact is that the ring $\mathbb{F}[G]$ itself, viewed as a left module over itself, is semisimple. Such a ring is called a semisimple ring. For a semisimple ring A, we will prove the following wonderful facts:

(i) Every module over A is semisimple; applied to $\mathbb{F}[G]$ this means that every representation of G is a direct sum of irreducible representations.

(ii) There are a finite number of simple left ideals $L_1, ..., L_s$ in A such that every simple A-module is isomorphic to exactly one of the L_j; thus, *there are only finitely many isomorphism classes of irreducible representations* of G, and *each irreducible representation is isomorphic to a copy inside* $\mathbb{F}[G]$. In particular, *every irreducible representation of G is finite dimensional.*

(iii) If A_j is the sum of all left ideals in A isomorphic to L_j (as in (ii)) then A_j is a two-sided ideal. As a ring, A is isomorphic to the product $\prod_{j=1}^{s} A_j$. Each A_j can be expressed also as a *direct* sum of left ideals isomorphic to L_j. Applied to the algebra $\mathbb{F}[G]$, where $\mathbb{F}$ is algebraically closed, the ring A_j is isomorphic, as a $\mathbb{F}$-algebra, to $\mathrm{End}_{\mathbb{F}}(L_j)$, and so has dimension $(\dim_{\mathbb{F}} L_j)^2$. Thus, in this case, the dimension of the $\mathbb{F}$-vector space $\mathbb{F}[G]$ is

$$|G| = \sum_{L_j} \left(\dim_{\mathbb{F}} L_j\right)^2, \tag{3.9}$$

where the sum is over all the non-isomorphic irreducible representations L_j of G.

(iv) The two-sided ideals A_j are of the form

$$A_j = u_j A$$

where $u_1, ..., u_s$ are *idempotents*:

$$u_j^2 = u_j, \quad u_j u_k = 0 \quad \text{for every } j \neq k, \tag{3.10}$$

$$1 = u_1 + \cdots + u_s, \tag{3.11}$$

and $u_1, ..., u_s$ form a basis of the vector space $Z(A)$, the center of A. In applying to the algebra $\mathbb{F}[G]$, we note that Proposition 3.3.1 says that a basis of the center of $\mathbb{F}[G]$ is given by all elements of the form

$$b_C = \sum_{x \in C} x,$$

with C running over all conjugacy classes in G. Consequently, *the number of distinct irreducible representations of G is equal to the number of conjugacy classes in G.*

ADD DIVISOR PROPERTY FOR $\dim(L_j)$.

3.5 $\mathbb{F}[G]$ is semisimple

In this section we will prove a fundamental structural property of the algebra $\mathbb{F}[G]$ which will yield a large trove of results about representations of G. This property is semisimplicity:

Definition 3.5.1 *A module E over a ring is semisimple if for any submodule F in E there is a complementary submodule F', i.e. a submodule F' for which E is the direct sum of F and F'. A ring is semisimple if it is semisimple as a left module over itself.*

The definition of semisimplicity of a ring here is in terms of viewing itself as a *left* module over itself. It will turn out, eventually, that a ring is 'left semisimple' if and only if it is 'right semisimple'.

Our immediate objective is to prove Maschke's theorem:

Theorem 3.5.1 *Suppose G is a finite group, and $\mathbb{F}$ a field of characteristic not a divisor of $|G|$, i.e. $|G|1_{\mathbb{F}} \neq 0$. Then every module over the ring $\mathbb{F}[G]$ is semisimple. In particular, $\mathbb{F}[G]$ is semisimple.*

Proof. Let E be a $\mathbb{F}[G]$-module, and F a $\mathbb{F}[G]$-submodule. We have then the $\mathbb{F}$-linear inclusion

$$j : F \to E$$

and so, since E and F are vector spaces over $\mathbb{F}$, there is a $\mathbb{F}$-linear map

$$P : E \to F$$

satisfying

$$Pj = \mathrm{id}_F. \tag{3.12}$$

(Choose a basis of F and extend to a basis of E. Then let P be the map which keeps each of the basis elements of F fixed, but maps all the other basis elements to zero.)

All we have to do is modify P to make it $\mathbb{F}[G]$-linear. The action of G on $\mathrm{Hom}_{\mathbb{F}}(F, E)$ given by

$$(x, A) \mapsto xAx^{-1}. \tag{3.13}$$

keeps the inclusion map j invariant. Consequently,

$$xPx^{-1}j = \mathrm{id}_F \qquad \text{for all } x \in G. \tag{3.14}$$

So we have

$$P'j = \mathrm{id}_F,$$

where

$$P' = \frac{1}{|G|}\sum_{x\in G} xPx^{-1};$$

here the division makes sense because $|G|1_{\mathbb{F}} \neq 0$ in $\mathbb{F}$. Clearly, P' is G-invariant and hence $\mathbb{F}[G]$-linear. Therefore, E splits as a direct sum of $\mathbb{F}[G]$-submodules:

$$E = F \oplus F',$$

where

$$F' = \ker P'$$

is also a $\mathbb{F}[G]$-submodule of E.

Thus, every submodule of a $\mathbb{F}[G]$-module has a complementary submodule. In particular, this applies to $\mathbb{F}[G]$ itself, and so $\mathbb{F}[G]$ is semisimple. $\boxed{\text{QED}}$

The map

$$\mathbb{F}[G] \to \mathbb{F}[G] : x \mapsto \hat{x} = \sum_{g\in G} x(g)g^{-1} \tag{3.15}$$

reverses left and right in the sense that

$$\widehat{(xy)} = \hat{y}\hat{x}$$

This makes every right $\mathbb{F}[G]$-module a left $\mathbb{F}[G]$-module by defining the left module structure through

$$g \cdot v = vg^{-1},$$

and then every sub-right-module is a sub-left-module. Thus, $\mathbb{F}[G]$, *viewed as a right module over itself*, is also semisimple.

Exercises

1. Let G be a finite group, $\mathbb{F}$ a field, and G^* the set of all non-zero multiplicative homomorphisms $G \to \mathbb{F}$. For $f \in G^*$, let

$$s_f = \sum_{x \in G} f(x^{-1})x.$$

Show that $\{cs_f \,:\, c \in \mathbb{F}\}$ is an invariant subspace of $\mathbb{F}[G]$.

2. If $\mathbb{F}$ is a field of characteristic $p > 0$, and G a finite group with $|G|$ a multiple of p, show that $\mathbb{F}[G]$ is not semisimple.

3. For $g \in G$, let $T_g : \mathbb{F}[G] \to \mathbb{F}[G] : a \mapsto ga$. Show that

$$\mathrm{Tr}(T_g) = \begin{cases} |G| & \text{if } g = e; \\ 0 & \text{if } g \neq e \end{cases} \tag{3.16}$$

4. For $g, h \in G$, let $T_{(g,h)} : \mathbb{F}[G] \to \mathbb{F}[G] : a \mapsto gah^{-1}$. Show that

$$\mathrm{Tr}(T_{(g,h)}) = \begin{cases} 0 & \text{if } g \text{ and } h \text{ are not conjugate;} \\ \frac{|G|}{|C|} & \text{if } g \text{ and } h \text{ belong to the same conjugacy class } C. \end{cases} \tag{3.17}$$

Chapter 4

Semisimple Modules and Rings: Structure and Representations

In this chapter we will determine the structure of semisimple modules and rings. A large number of results on representations of such algebras will follow easily once the structure theorems have been obtained.

We will be working with modules over a ring A with unit $1 \neq 0$.

4.1 Schur's Lemma

Let A be a ring with unit $1 \neq 0$. Note that A need not be commutative (indeed, for the purposes of this section and the next, A need not even be associative).

Definition 4.1.1 *A module E over a ring is* simple *if it is $\neq 0$ and if its only submodules are 0 and E.*

Suppose

$$f : E \to F$$

is linear, where E is a simple A-module and F an A-module. The kernel

$$\ker f = f^{-1}(0)$$

is a submodule of E and hence is either $\{0\}$ or E itself. If, moreover, F is also simple then $f(E)$, being a submodule of F, is either $\{0\}$ or F.

Thus we have the simple, but powerful, **Schur's Lemma**:

Proposition 4.1.1 *If E and F are simple A-modules then in*

$$\mathrm{Hom}_A(E, F)$$

every non-zero element is a bijection, i.e. an isomorphism of E onto F.

For a simple A-module E, this implies that

$$\mathrm{End}_A(E)$$

is a division ring.

We can now specialize to a case of interest, where A is a finite-dimensional algebra over an algebraically closed field $\mathbb{F}$. We can view $\mathbb{F}$ as a subring of $\mathrm{End}_A(E)$:

$$k \simeq k1 \subset \mathrm{End}_A(E),$$

where 1 is the identity element in $\mathrm{End}_A(E)$. The assumption that $\mathbb{F}$ is algebraically closed implies that $\mathbb{F}$ has no proper finite extension, and this leads to the following consequence:

Proposition 4.1.2 *Suppose A is a finite-dimensional algebra over an algebraically closed field $\mathbb{F}$. Then for any simple A-module E, which is a finite dimensional vector space over $\mathbb{F}$,*

$$\mathrm{End}_A(E) = \mathbb{F},$$

upon identifying $\mathbb{F}$ with $\mathbb{F}1 \subset \mathrm{End}_A(E)$. Moreover, if E and F are simple A-modules, then $\mathrm{Hom}_A(E, F)$ is either $\{0\}$ or a one-dimensional vector space over $\mathbb{F}$.

Proof. Let $x \in \mathrm{End}_A(E)$. Suppose $x \notin \mathbb{F}1$. Note that x commutes with all elements of $\mathbb{F}1$. Since $\mathrm{End}_A(E) \subset \mathrm{End}_{\mathbb{F}}(E)$ is a finite-dimensional vector space over $\mathbb{F}$, there is a smallest natural number $n \in \{1, 2, ...\}$ such that $1, x, ..., x^n$ are linearly dependent over $\mathbb{F}$, i.e. there is a polynomial $p(X) \in \mathbb{F}[X]$, of lowest degree, with $\deg p(X) = n \geq 1$, such that

$$p(x) = 0.$$

Since $\mathbb{F}$ is algebraically closed, $p(X)$ factorizes over $\mathbb{F}$ as

$$p(X) = (X - \lambda)q(X)$$

for some $\lambda \in \mathbb{F}$. Consequently, $x - \lambda 1$ is not invertible, for otherwise $q(x)$, of lower degree, would be 0. Thus, by Schur's lemma, $x = \lambda 1 \in \mathbb{F}1$.

Now suppose E and F are simple A-modules, and suppose there is a non-zero element $f \in \mathrm{Hom}_A(E, F)$. By Proposition 4.1.1, f is an isomorphism. If g is also an element of $\mathrm{Hom}_A(E, F)$, then $f^{-1}g$ is in $\mathrm{End}_A(E, E)$, and so, by the first part of this result, is a multiple of the identity element in $\mathrm{End}_A(E)$. Consequently, g is a multiple of f. $\boxed{\text{QED}}$

4.2 Semisimple Modules

Let A be a ring with unit element $1 \neq 0$, possibly non-commutative. We have in mind, as always, the example of $\mathbb{F}[G]$. Indeed, we will not need A to be associative either; A could, for example, be a Lie algebra. One fact we will, however, need is that for any element x in an A-module M, the subset Ax is also an A-module.

Recall that a module E is *semisimple* if every submodule has a complement, i.e. if F is a submodule of E then there is a submodule F' such that E is the direct sum of F and F'. Below we shall prove that this is equivalent to E being *a direct sum of simple submodules*, but first let us observe:

Proposition 4.2.1 *Submodules and quotient modules of semisimple modules are semisimple.*

Proof. Let E be a semisimple module and F a submodule. Let G be a submodule of F. Then G has a complement G' in E:

$$E = G \oplus G'.$$

If $f \in F$ then we can write this uniquely as

$$f = g + g'$$

with $g \in G$ and $g' \in G'$. Then

$$g' = f - g \in F$$

and so, in the decomposition of $f \in F$ as $g + g'$, both g and g' are in f. We conclude that

$$F = G \oplus (G' \cap F)$$

Thus every submodule of F has a complement inside F. Thus, F is semisimple.

If F' is the complementary submodule to F in E, then

$$E/F \simeq F',$$

and so E/F, being isomorphic to the submodule F', is semisimple. $\boxed{\text{QED}}$

Before turning to the fundamental facts about semisimplicity of modules, let us recall a simple fact from vector spaces: if T is a linearly independent subset of a vector space, and S a subset which spans the whole space, then a basis of the vector space is formed by adjoining to T a maximal subset of S which respects linear independence. A similar idea will be used in the proof below for simple modules.

Theorem 4.2.1 *The following conditions are equivalent for an A-module E:*

(i) *E is a sum of simple submodules*

(ii) *E is a direct sum of simple submodules*

(iii) *Every submodule F of E has a complement, i.e. there is a submodule F' such that $E = F \oplus F'$.*

If $E = \{0\}$ then the sum is the empty sum.

Proof. Suppose $\{E_j\}_{j\in J}$ is a family of simple submodules of E, and F a submodule of E with

$$F \subset \sum_{j\in J} E_j.$$

By Zorn's lemma, there is a maximal subset K of J such that the sum

$$H = F + \sum_{k\in K} E_k$$

is a direct sum. For any $j \in J$, the intersection $E_j \cap H$ is either 0 or E_j. It cannot be 0 by maximality of K. Thus, $E_j \subset H$ for all $j \in J$, and so $\sum_{j\in J} E_j \subset H$. Thus,

$$\sum_{j\in J} E_j = F + \sum_{k\in K} E_k, \quad \text{the latter being a direct sum.}$$

Applying this observation to the case where $\{E_j\}_{j\in J}$ span all of E, and taking $F = 0$, we see that E is a direct sum of some of the simple submodules E_k. This proves that (i) implies (ii).

Applying the result to a family $\{E_j\}_{j\in J}$ which gives a direct sum decomposition of E, and taking F to be any submodule of E, it follows that

$$E = F \oplus F',$$

where F' is a direct sum of some of the simple submodules E_k. Thus, (ii) implies (iii).

Now assume (iii). Let F be the sum of a maximal collection of simple submodules of E. Then $E = F \oplus F'$, by (iii), for a submodule F' of E. We will show that $F' = 0$. Suppose $F' \neq 0$. Then, as we prove below, F' has a simple submodule, and this contradicts the maximality of F. Thus, E is a sum of simple submodules.

It remains to show that if (iii) holds then every non-zero submodule F contains a simple submodule. Since $F \neq 0$, it contains a non-zero element x which generates a submodule Ax. If Ax is simple then we are done. Suppose then that Ax is not simple. We will produce a maximal proper submodule of Ax; its complement inside Ax will then have to be simple. Any increasing chain $\{F_\alpha\}$ of proper submodules of Ax has union $\cup_\alpha F_\alpha$ also a proper submodule of Ax, because x is outside each F_α. Then, by Zorn's lemma, Ax has a maximal submodule M. By assumption (iii) and Proposition 4.2.1, the submodule Ax will also have the property that every submodule has a complement; in particular, M has a complement F' in Ax, which must be non-zero since $M \neq Ax$. The submodule F' cannot have a proper non-zero submodule, because that would contradict the maximality of M. Thus, we have produced a simple submodule inside any given non-zero submodule F in E. $\boxed{\text{QED}}$

Theorem 4.2.1 leads to a full structure theorem for semisimple modules. But first let us oberve something about simple modules, which again is analogous to the situation for vector spaces. Indeed, the proof below is by means of viewing a module as a vector space.

Proposition 4.2.2 *If E is a simple A-module, then E is a vector space over the division ring* $\mathrm{End}_A(E)$. *If $E^n \simeq E^m$ as A-modules, then $n = m$.*

<u>Proof</u>. If E is a simple A-module then, by Schur's lemma,

$$D \stackrel{\text{def}}{=} \mathrm{End}_A(E)$$

is a division ring. Thus, E is a vector space over D. Then E^n is the product vector space over D. If $\dim_D E$ were finite, then we would be done. In the absence of this, the procedure (which seems like a clever trick) is to look at $\mathrm{End}_A(E^n)$. This is a vector space over D, because for any $\lambda \in D$ and A-linear $f : E^n \to E^n$, the map λf is also A-linear. In fact, each element of $\mathrm{End}_A(E^n)$ can be displayed, as usual, as an $n \times n$ matrix with entries in D. Moreover, this effectively provides a basis of the D-vector space $\mathrm{End}_A(E^n)$ consisting of n^2 elements. Thus, $E^n \simeq E^m$ implies $n = m$. $\boxed{\text{QED}}$

Now we can turn to the uniqueness of the structure of semisimple modules of finite type:

Theorem 4.2.2 *Suppose a module E over a ring A can be expressed as*

$$E \simeq E_1^{m_1} \oplus \ldots \oplus E_n^{m_n} \tag{4.1}$$

where $E_1, ..., E_n$, are non-isomorphic simple modules, and each m_i a positive integer. Suppose also that E can be expressed also as

$$E \simeq F_1^{j_1} \oplus \ldots \oplus F_m^{j_m}$$

where $F_1, ..., F_m$, are non-isomorphic simple modules, and each j_i a positive integer. Then $m = n$, and each E_a is isomorphic to one and only one F_b, and then $m_a = j_b$.

Proof. Let G be any simple module isomorphic to a submodule of E. Then composing this map $G \to E$ with the projection $E \to E_r$, we see that there exists an a for which the composite $G \to E_a$ is not zero and hence $G \simeq E_a$. Similarly, there is a b such that $G \simeq F_b$. Thus each E_a is isomorphic to some F_b. The rest follows by Proposition 4.2.2. $\boxed{\text{QED}}$

It is now clear that the ring $\mathrm{End}_A(E)$ can be identified as a ring of matrices:

Theorem 4.2.3 *If E is a semisimple module over a ring A, and E is the direct sum of finitely many simple modules:*

$$E \simeq E_1^{m_1} \oplus \ldots \oplus E_n^{m_n}$$

then the ring $\mathrm{End}_A(E)$ is isomorphic to a product of matrix rings:

$$\mathrm{End}_A(E) \simeq \prod_{i=1}^{n} \mathrm{Matr}_{m_i}(D_i) \tag{4.2}$$

where $\mathrm{Matr}_{m_i}(D_i)$ *is the ring of* $m_i \times m_i$ *matrices over the division ring* $D_i = \mathrm{End}_A(E_i)$.

The endomorphisms of the A-module E are those additive mappings $E \to E$ which commute with the action of all the elements of A. Thus, $\mathrm{End}_A(E)$ is the *commutant* of the ring A acting on E. The preceding result shows that if E is semisimple as an A-module, and is a sum of finitely many simple modules, then the commutant is a direct product of matrix rings. We shall see later that every such ring is semisimple (as a module over itself).

Let us now examine simple modules over semisimple rings.

First consider a left ideal L in A. Then

$$A = L \oplus L',$$

where L' is also a left ideal. Then we can express the multiplicative unit 1 as

$$1 = 1_L + 1_{L'},$$

where $1_L \in L$ and $1_{L'} \in L'$. For any $l \in L$ we then have

$$l = l1 = l1_L + l1_{L'}$$

and $l1_{l'}$ being then in both L' and L must be 0. Consequently,

$$L \subset LL.$$

Of course, L being a left ideal, we also have $LL \subset L$. Thus,

$$LL = L \tag{4.3}$$

Using this we will prove the following convenient characterization of modules isomorphic to a given left ideal.

Lemma 4.2.1 *Let* A *be a semisimple ring,* L *a simple left ideal in* A*, and* E *a simple* A*-module. Then exactly one of the following holds:*

(i) $LE = 0$ *and* L *is not isomorphic to* E*;*

(ii) $LE = E$ *and* L *is isomorphic to* E*.*

<u>Proof</u>. Since LE is a submodule of E, it is either $\{0\}$ or E. We will show that LE equals E if and only if L is isomorphic to E.

Assuming $LE = E$, take a $y \in E$ with $Ly \neq 0$. By simplicity of E, then $Ly = E$. The map

$$L \mapsto E = Ly : a \mapsto ay$$

is an A-linear surjection, and it is injective because its kernel, being a submodule of the simple module L, is $\{0\}$. Thus, if $LE = E$ then L is isomorphic to E.

Now we will show that, conversely, if L is isomorphic to E then $LE = E$.

If $f : L \to E$ is A-linear we have then

$$E = f(L) = f(LL) = Lf(L) = LE$$

Thus, if f is an isomorphism then $E = LE$. $\boxed{\text{QED}}$

Let us note that any two isomorphic left ideals are right translates of each another:

Proposition 4.2.3 *If L and M are isomorphic left ideals in a semisimple ring A then*

$$L = Mx,$$

for some $x \in A$.

<u>Proof</u>. Suppose $F : M \to L$ is an isomorphism. Composing with a projection $p_M : A \to M$, we obtain a map

$$G = F \circ p_M : A \to L$$

which is A-linear. Hence,

$$G(a) = G(a1) = aG(1) = ax,$$

where

$$x = G(1) = F(p_M(1)).$$

Restricting the map G to M we see that

$$G(M) = Mx.$$

But p_M and F are both surjective, and so $Mx = L$. $\boxed{\text{QED}}$

4.3 Structure of Semisimple Rings

In this section we will work with a semisimple ring A. Recall that this means that A is semisimple as a left module over itself.

Recall that, by semisimplicity, A decomposes as a direct sum of simple submodules. A submodule in A is just a left ideal. Thus, we have a decomposition

$$A = \sum \{\text{all simple left ideals of } A\}$$

In this section we will see that if we sum up all those ideals which are isomorphic to each other and call this submodule A_i, then A_i is a two-sided ideal and a subring in A, and A is the direct product of these rings.

Let

$$\{L_i\}_{i \in \mathcal{R}}$$

be a maximal family of non-isomorphic simple left ideals in A. Let

$$A_i = \sum \{L : L \text{ is a left ideal isomorphic to } L_i\}$$

By (4.2.1), we have

$$LL' = 0 \qquad \text{if } L \text{ is not isomorphic to } L'.$$

So

$$A_i A_j = 0 \qquad \text{if } i \neq j \tag{4.4}$$

Since A is semisimple, it is the sum of all its simple submodules, and so

$$A = \sum_{i \in I} A_i.$$

Now each A_i is clearly a left ideal. It is also right ideal because

$$A_i A = A_i \sum_j A_j = A_i A_i \subset A_i.$$

Thus, A is a sum of two-sided ideals A_i. We will soon see that the index set $\mathcal{R}$ is finite.

The unit element $1 \in A$ decomposes as

$$1 = \sum_{i \in \mathcal{R}} u_i \tag{4.5}$$

where $u_i \in A_i$, and the sum is finite, i.e. all but finitely many u_i are 0. For any $a \in A$ we can write

$$a = \sum_{i \in \mathcal{R}} a_i \qquad \text{with each } a_i \text{ in } A_i.$$

Then, on using (4.4),

$$a_j = a_j 1 = a_j u_j = a u_j$$

Thus a determines the 'components' a_j uniquely, and so

the sum $A = \sum_{i \in \mathcal{R}} A_i$ *is a direct sum.*

If some u_j were 0 then all the corresponding a_j would be 0, which cannot be since each A_j is non-zero. Consequently,

the index set $\mathcal{R}$ *is finite.*

Since we also have, for any $a \in A$,

$$a = 1a = \sum_i u_i a,$$

we have from the fact that the sum $A = \sum_i A_i$ is direct,

$$u_i a = a_i = a u_i.$$

Thus A_i is a two-sided ideal. Clearly, u_i is the identity in A_i.

We have arrived at the wonderful structure theorem for semisimple rings:

Theorem 4.3.1 *Suppose A is a semisimple ring. Then there are finitely many left ideals $L_1, ..., L_r$ in A such that every left ideal of A is isomorphic, as a left A-module, to exactly one of the L_j. Furthermore,*

$$A_i = \textit{sum of all left ideals isomorphic to } L_i$$

is a two-sided ideal, with a non-zero unit element u_i, and A is the product of the rings A_i:

$$A \simeq \prod_{i=1}^{r} A_i \tag{4.6}$$

Any simple left ideal in A_i is isomorphic to L_i. Moreover,

$$1 = u_1 + \cdots + u_r \tag{4.7}$$

$$A_i = A u_i \tag{4.8}$$

$$A_i A_j = 0 \qquad \textit{for } i \neq j \tag{4.9}$$

In Theorem 4.3.3 below, we will see that there is a ring isomomorphism

$$A_i \simeq \mathrm{End}_{C_i}(L_i), \quad \text{where } C_i = \mathrm{End}_A(L_i).$$

Thus, any semisimple ring A can be decomposed as a product of endomorphism rings:

$$A \simeq \prod_{i=1}^{r} \mathrm{End}_{C_i}(L_i) \tag{4.10}$$

where $L_1, ..., L_r$ is a maximal collection of non-isomorphic simple left ideals in A, and $C_i = \mathrm{End}_A(L_i)$. An element $a \in A$ is mapped, by this isomorphism, to $(\tilde{a}_i)_{1\leq i\leq r}$, where

$$\tilde{a}_i : L_i :\to L_i : x \mapsto ax. \tag{4.11}$$

The two-sided ideals A_j are, it turns out, minimal two-sided ideals, and every two-sided ideal in A is a sum of certain A_j. We will prove this using some results which we prove later in subsection 4.3.1 below.

Proposition 4.3.1 *Each A_j is a minimal two-sided ideal in A, and every two-sided ideal in A is a sum of some of the A_j.*

Proof. Let I be a two-sided ideal in A. Then $AI \subset I$, but also $I \subset AI$ since $1 \in A$. Hence

$$I = AI = A_1I + \cdots + A_rI$$

Note that A_jI is a two-sided ideal, and $A_jI \subset A_j$. The ring A_j has the special property that every simple left ideal in A_j is isomorphic to the same simple left ideal, L_j. As we prove in Proposition 4.3.3 below, this implies that the only two-sided ideals in A_j are 0 and A_j. Thus, A_jI is either 0 or A_j. Consequently,

$$I = \sum_{j:\, A_jI\neq 0} A_j. \qquad \boxed{\text{QED}}$$

It is useful to summarize the properties of the elements u_i:

Proposition 4.3.2 *The elements $u_1, ..., u_r$ are non-zero, and satisfy*

$$\begin{aligned} u_i^2 &= u_i, \quad u_iu_j = 0 \quad \text{if } i \neq j && (4.12)\\ u_1 + \cdots + u_r &= 1 && (4.13)\end{aligned}$$

Multiplication by u_i in A is the identity on A_i and is 0 on all A_j for $j \neq 1$. If A is a finite dimensional algebra over an algebraically closed field $\mathbb{F}$, then $u_1, ..., u_r$ form a vector space basis of $Z(A)$.

For the last claim above, we use Proposition 4.3.4, which implies that each $Z(A_i)$ is the field $\mathbb{F}$ imbedded in A_i, and so every element in $Z(A_i)$ is a multiple of the unit element u_i. The decomposition $A \simeq \prod_{i=1}^{r} A_i$ then implies that the center of A is the linear span of the u_i.

We will return to a more detailed examination of idempotents later.

4.3.1 Simple Rings

The subrings A_j are *isotypical* or *simple rings*, in that they are sums of simple left ideals which are all isomorphic to the same left ideal L_j.

Definition 4.3.1 *A ring B is simple if it is a sum of simple left ideals which are all isomorphic to each other as left B-modules.*

Since, by Proposition 4.2.3, all isomorphic left ideals are right translates of one another, a simple ring B is a sum of right translates of any given simple left ideal L. Consequently,

$$B = LB \qquad \text{if } B \text{ is a simple ring, and } L \text{ any simple left ideal.} \tag{4.14}$$

As consequence we have:

Proposition 4.3.3 *The only two-sided ideals in a simple ring are* 0 *and the whole ring itself.*

Proof. Let I be a two-sided ideal in a simple ring B, and suppose $I \neq 0$. By simplicity, I is a sum of simple left ideals, and so, in particular, contains a simple left ideal L. Then by (4.14) we see that $LB = B$. But $LB \subset I$, because I is also a right ideal. Thus, $I = B$. $\boxed{\text{QED}}$

For a ring B, any B-linear map $f : B \to B$ is completely specified by the value $f(1)$, because

$$f(b) = f(b1) = bf(1)$$

Moreover, if $f, g \in \mathrm{End}_B(B)$ then

$$(fg)(1) = f(g(1)) = g(1)f(1)$$

and so we have a ring isomorphism

$$\mathrm{End}_B(B) \to B^{\mathrm{opp}} : f \mapsto f(1) \tag{4.15}$$

where B^{opp} is the ring B with multiplication in 'opposite' order:

$$(a, b) \mapsto ba$$

We then have

Theorem 4.3.2 *If B is a simple ring, isomorphic as a module to M^n, for some simple left ideal M and positive integer n, then B is isomorphic to the ring of matrices*

$$B \simeq \mathrm{Matr}_n(D^{\mathrm{opp}}), \tag{4.16}$$

where D is the division ring $\mathrm{End}_B(M)$.

Proof. We know that there are ring isomorphisms

$$B^{\mathrm{opp}} \simeq \mathrm{End}_B(B) = \mathrm{End}_B(M^n) \simeq \mathrm{Matr}_n(D)$$

Taking the opposite ring, we obtain an isomorphism of B with $\mathrm{Matr}_n(D)^{\mathrm{opp}}$. But now consider the transpose of $n \times n$ matrices:

$$\mathrm{Matr}_n(D)^{\mathrm{opp}} \to \mathrm{Matr}_n(D^{\mathrm{opp}}) : A \mapsto A^t.$$

Then, working in components of the matrices, and denoting multiplication in D^{opp} by $*$:

$$(A * B)^t_{ik} = (BA)_{ki} = \sum_{j=1}^{n} B_{kj} A_{ji} = \sum_{j=1}^{n} A_{ji} * B_{kj},$$

which is the ordinary matrix product $A^t B^t$ in $\mathrm{Matr}_n(D^{\mathrm{opp}})$. Thus, the transpose gives an isomorphism $\mathrm{Matr}_n(D)^{\mathrm{opp}} \simeq \mathrm{Matr}_n(D^{\mathrm{opp}})$. $\boxed{\text{QED}}$

The opposite ring often arises in matrix representations of endomorphisms. If M is a 1-dimensional vector space over a division ring D, with a basis element v, then to each $T \in \mathrm{End}_D(M)$ we can associate the 'matrix' element $\hat{T} \in D$ specified through $T(v) = \hat{T}v$. But then, for any $S, T \in \mathrm{End}_D(M)$ we have

$$\widehat{ST} = \hat{T}\hat{S}$$

Thus, $\mathrm{End}_D(M)$ is isomorphic to D^{opp}, via its matrix representation.

There is a more abstract, 'coordinate free' version of Theorem 4.3.2. First let us observe that for a module M over a ring A, the endomorphism ring

$$A' = \mathrm{End}_A(M)$$

is the commutant for A, i.e. all additive maps $M \to M$ which commute with the action of A. Next,

$$A'' = \mathrm{End}_{A'}(M)$$

is the commutant of A'. Since, for any $a \in A$, the multiplication

$$l(a) : M \to M : x \mapsto ax \tag{4.17}$$

commutes with every element of A', it follows that

$$l(a) \in A''$$

Note that

$$l(ab) = l(a)l(b)$$

and l maps the identity element in A to that in A'', and so l is a ring homomorphism. The following result is due to Rieffel:

Theorem 4.3.3 *Let B be a simple ring, L a non-zero left ideal in B,*

$$B' = \mathrm{End}_B(L), \qquad B'' = \mathrm{End}_{B'}(L)$$

and

$$l : B \to B''$$

the natural ring homomorhism given by (4.17). Then l is an isomorphism. In particular, every simple ring is isomorphic to the ring of endomorphisms on a module.

Proof. To avoid confusion, it is useful to keep in mind that elements of B' and B'' are all maps $\mathbb{Z}$-linear maps $L \to L$.

The ring morphism $l : B \to B''$ is given explicitly by

$$l(b)x = bx, \qquad \text{for all } b \in B, \text{ and } x \in L.$$

It maps the unit element in B to the unit element in B'', and so is not 0. The kernel of $l \neq 0$ is a two-sided ideal in a simple ring, and hence is 0. Thus, l is injective.

We will show that $l(B)$ is B''. Since $1 \in l(B)$, it will be sufficient to prove that $l(B)$ is a left ideal in B''.

Since LB contains L as a subset, and is thus not $\{0\}$, and is clearly a two-sided ideal in B, it is equal to B:

$$LB = B.$$

This key fact implies

$$l(L)l(B) = l(B)$$

Thus, it will suffice to prove that $l(L)$ is a left ideal in B''. We can check this as follows: if $f \in B''$ and $x, y \in L$ then

$$\begin{aligned}
\big(fl(x)\big)(y) &= f(xy) \\
&= f(x)y \quad \text{because } L \to L : x \mapsto xy \text{ is in } B' = \mathrm{End}_B(L) \\
&= l\big(f(x)\big)(y),
\end{aligned}$$

thus showing that

$$f \cdot l(x) = l\big(f(x)\big),$$

and hence $l(L)$ is a left ideal in B''. $\boxed{\text{QED}}$

Lastly, let us make an observation about the center of a simple ring:

Proposition 4.3.4 *If B is a simple ring then its center $Z(B)$ is a field. If B is a finite-dimensional simple algebra over an algebraically closed field $\mathbb{F}$, then $Z(B) = \mathbb{F}1$.*

Proof. Let $z \in Z(B)$, and consider the map

$$l_z : B \to B : b \mapsto zb.$$

Because z commutes with all elements of B, this map is B-linear. The kernel $\ker l_z$ is a two-sided ideal; it would therefore be $\{0\}$ if $l_z \neq 0$. Now $l_z(1) = z$, and so $\ker l_z$ must be $\{0\}$ if z is not 0:

$$\ker l_z = \{0\} \quad \text{if } z \neq 0.$$

The image $l_z(B)$ is also a two-sided ideal and so:

$$l_z(B) = B \qquad \text{if } z \neq 0.$$

Thus, l_z is a linear isomorphism if $z \neq 0$, and

$$l : Z(B) \to \mathrm{End}_B(B) : z \mapsto l_z$$

is a $\mathbb{Z}$-linear injection. Moreover,

$$l_z l_w = l_{zw}.$$

For $z \neq 0$ in $Z(B)$, writing $y = l_z^{-1}(1)$, we have

$$yz = zy = l_z(y) = 1.$$

Thus, every non-zero element in $Z(B)$ is invertible. Since $Z(B)$ is commutative and contains $1 \neq 0$, we conclude that it is a field.

Suppose now that B is a finite dimensional $\mathbb{F}$-algebra, and $\mathbb{F}$ is algebraically closed. Then any $z \in Z(B)$ not in $\mathbb{F}$ would give rise to a proper finite extension of $\mathbb{F}$ and this is impossible. In more detail, since $Z(B)$ is a finite-dimensional vector space over $\mathbb{F}$, there is a smallest integer $n \geq 1$ such that 1, z,...,z^n are linearly dependent, and so there is a polynomial $p(X)$ of degree n, with coefficients in $\mathbb{F}$, such that $p(z) = 0$. Since $\mathbb{F}$ is algebraically closed, there is a $\lambda \in k$, and a polynomial $q(X)$ of degree $n-1$ such that $p(X) = (X - \lambda)q(X)$, and so $z - \lambda 1$ is not invertible and therefore $z = \lambda 1$. Thus, $Z(B) = k1$. $\boxed{\text{QED}}$

4.4 Semisimple Algebras as Matrix Algebras

Let us pause to put together some results we have already proved to see that:

(i) *every finite dimensional simple algebra over an algebraically closed field $\mathbb{F}$ is isomorphic to the algebra of all $d \times d$ matrices over $\mathbb{F}$, for some d*;

(ii) *every finite dimensional semisimple algebra over an algebraically closed field $\mathbb{F}$ is isomorphic to an algebra of all matrices of block-diagonal form*, the i-th block running over $d_i \times d_i$ matrices over $\mathbb{F}$;

(iii) *every finite dimensional semisimple algebra A over an algebraically closed field $\mathbb{F}$ is isomorphic to its opposite algebra A^{opp}.*

Observation (i) may be stated more completely, as the following consequence of Theorem 4.3.3:

Proposition 4.4.1 *If an algebra B over an algebraically closed field $\mathbb{F}$ is simple, and L is any simple left ideal in B, then B is isomorphic as a $\mathbb{F}$-algebra to* $\text{End}_{\mathbb{F}}(L)$. *In particular, if B is finite dimensional over $\mathbb{F}$ then*

$$\dim_{\mathbb{F}} B = [\dim_{\mathbb{F}}(L)]^2 \tag{4.18}$$

Observation (ii) then follows from the fact that every semisimple algebra is a product of simple algebras.

Observation (iii) follows from (ii), upon noting that the matrix transpose operation produces an isomorphism between the algebra of all square matrices of a certain degree with its opposite algebra.

4.5 Idempotents

Idempotents play an important role in the structure of semisimple algebras. When represented on a module, an idempotent is a projection map. The decomposition of 1 as a sum of idempotents corresponds to a decomposition of a module into a direct sum of submodules.

Idempotents will be a key tool in constructing representations of S_n in Chapter 8.

Before proceeding to the results, let us note an example. Consider a finite group G and let

$$\tau : G \to \mathbb{F}$$

be a one-dimensional representation of G (for example $\tau(x) = 1$ for all $x \in G$). Then consider

$$u_\tau = \frac{1}{|G|} \sum_{x \in G} \tau(x^{-1})x \in \mathbb{F}[G],$$

assuming that the character of $\mathbb{F}$ does not divide $|G|$. Then it is readily checked that u_τ is an idempotent:

$$u_\tau{}^2 = u_\tau.$$

We have used this idempotent already (in the case $\tau = 1$), in proving semisimplicity of $\mathbb{F}[G]$.

Idempotents generate left ideals, and, conversely, as we see shortly, every left ideal in a semisimple ring is generated by an idempotent. Consider a left ideal L in a semisimple ring A. We have then a complementary ideal L' with

$$A = L \oplus L'$$

and so there is an A-linear projection map

$$p_L : A \to L.$$

But A-linearity puts a serious restriction on this map. Indeed, we have

$$p_L(a) = p(a \cdot 1) = ap_L(1) \tag{4.19}$$

and so p_L is simply multiplication on the right by the 'constant'

$$u_L = p_L(1).$$

The image of p is then

$$p_L(A) = Au_L$$

But p_L, being the projection onto L, is surjective! Thus,

$$L = Au_L. \tag{4.20}$$

Thus, every left ideal is of the form

$$Au_L.$$

Note that

$$u_L = p_L(1) \in A$$

Moreover, since p_L is the identity when restricted to L, we have

$$l = p_L(l) = lu_L, \qquad \text{for all } l \in L \tag{4.21}$$

In particular, applying this to $l = u_L$, we see that u_L is an *idempotent*:

$$u_L^2 = u_L. \tag{4.22}$$

Indeed, this is a reflection of the fundamental property of a projection map:

$$p_L(p_L(a)) = p_L(a) \qquad \text{for all } a \in A.$$

Let us summarize our observations:

Proposition 4.5.1 *Every left ideal L in a semisimple ring A is of the form Au_L for some $u_L \in L$:*

$$L = Au_L \tag{4.23}$$

The element u_L is an idempotent, i.e.

$$u_L^2 = u_L. \tag{4.24}$$

Moreover,

$$yu_L = y \qquad \textit{if and only if } y \in L. \tag{4.25}$$

Conversely, if u is an idempotent then Au is a left ideal and the map

$$A \to Au : x \mapsto xu$$

is an A-linear projection map onto the submodule Au, carrying 1 to the generating element u.

Now suppose

$$M \subset L$$

is a left ideal contained in L. We have then a decomposition

$$L = M \oplus \tilde{M}$$

which yields a decomposition

$$A = L \oplus L' = M \oplus \tilde{M} \oplus L'.$$

Thus, the projection p_{ML} of L onto M composes with p_L to give p_M:

$$p_M = p_{ML} \circ p_L. \tag{4.26}$$

Applying this to the unit element 1, we have:

$$u_M = p_{ML}(u_L) \tag{4.27}$$

On the other hand, applying (4.26) to u_L gives:

$$u_L u_M = p_{ML}(u_L) \tag{4.28}$$

Combining these observations, we have

$$u_L u_M = u_M. \tag{4.29}$$

Similarly,

$$u_L u_{\tilde{M}} = u_{\tilde{M}}.$$

Viewing these idempotents all as projections of the unit element 1 onto the various ideals, we see also that

$$u_L = u_M + u_{\tilde{M}}. \tag{4.30}$$

Consequently,

$$u_{\tilde{M}} u_M = 0 = u_M u_{\tilde{M}}. \tag{4.31}$$

We say that u_M and $u_{\tilde{M}}$ are *orthogonal*.

Thus,

Proposition 4.5.2 *If $M \subset L$ are left ideals in a semisimple ring A, then there are idempotents u_L, u_M and u' in A, such that*

(i) $L = Au_L$,

(ii) $M = Au_M$,

(iii) $u_L = u_M + u'$, *and*

(iv) $u_M u' = 0 = u' u_M$.

Thus, L is the direct sum of the ideals M and Au', which have orthogonal idempotent generators u_M and u'.

Note that it may well be that M and a complementary module M' (with $L = M + M'$ as a direct sum) have other non-orthogonal idempotent generators. (See Exercise 3.3.)

In the converse direction we have:

Proposition 4.5.3 *Suppose u decomposes into a finite sum of orthogonal idempotents v_i:*

$$u = v_1 + \cdots + v_m, \quad v_j^2 = v_j, \quad v_j v_k = 0 \quad \text{when } j \neq k.$$

Then u is an idempotent:

$$u^2 = u,$$

and Au is the internal direct sum of the submodules Av_j:

$$Au = \sum_{j=1}^{m} Av_j.$$

<u>Proof</u>. Squaring u gives

$$u^2 = \sum_j v_j^2 + \sum_{j \neq k} v_j v_k = \sum_j v_j = u.$$

Thus, u is an idempotent.

It is clear that

$$Au \subset \sum_j Av_j$$

For the converse direction, we have

$$v_j u = v_j^2 + \sum_{k \neq j} v_j v_k = v_j^2 = v_j, \tag{4.32}$$

and so

$$\begin{aligned} \sum_j A v_j &= \sum_j A v_j u \\ &\subset Au. \end{aligned}$$

Next, suppose

$$\sum_j a_j v_j = 0,$$

for some $a_j \in A$. Multiplying on the right by v_i gives:

$$0 = \sum_j a_j v_j v_i = a_i v_i^2 + 0 = a_i v_i,$$

and so each $a_i v_i$ is 0. $\boxed{\text{QED}}$

This result suggests that we could start with the unit element $1 \in A$ and keep splitting it into orthogonal idempotents, as long as possible. Thus we would aim to write

$$1 = u_1 + \cdots + u_r,$$

where $u_1, \ldots, u_r$ are idempotents, and

$$u_i u_j = 0 \quad \text{when } i \neq j,$$

in such a way that this process cannot be continued further. This leads us to the following natural concept:

Definition 4.5.1 *A* <u>*primitive idempotent*</u> *in a ring A is an element $u \in A$ which is an idempotent, i.e. satisfies*

$$u^2 = u,$$

and is primitive in the sense that $u \neq 0$ and if

$$u = v + w$$

with v and w being also idempotents, such that

$$vw = 0 = wv,$$

then v or w is 0.

Note that we require that a primitive idempotent be *non-zero*.

The following result is clear from Propositions 4.5.3 and 4.5.2.

Proposition 4.5.4 *A left ideal $L = Au$, where u is an idempotent, is simple if and only if u is primitive.*

(Note that the definition of primitive idempotent does not involve the qualifier 'left', and so the conclusion holds for the right ideal uA as well.)

The decomposition of a semisimple ring A as a direct sum of simple left ideals Ae_j corresponds then to a decomposition of the unit element 1 into a sum of primitive idempotents:

$$1 = e_1 + \cdots + e_N.$$

Decomposing each Ae_k further, we have

$$A = \sum_{j,k} e_j A e_k.$$

Lemma 4.5.1 *Suppose u and u' are idempotents in a semisimple ring A. Then every A-linear map $Au' \to Au$ is of the form*

$$f_x : Au' \to Au : y \mapsto yu'xu = yxu,$$

for some $x \in A$. The element $u'xu$, being the image of u' in Au, depends on x only through the map f_x.

Proof. Let

$$F : Au' \to Au$$

be A-linear. Then:

$$F(au') = aF(u') = axu, \quad \text{where } x \in A \text{ is such that } F(u') = xu.$$

It is convenient to observe that

$$F(au') = F(au'u') = au'F(u') = au'xu = au'u'xu,$$

which allows us to write F cleanly as

$$F(y) = yu'xu \qquad \text{for all } y \in Au'.$$

Thus, $F = f_x$. $\boxed{\text{QED}}$

The following result summarizes many of the key features of idempotents which will be useful in constructing the irreducible representations of S_n in Chapter 8. In particular, (4.33) gives a condition for deciding when an idempotent is primitive, and (4.34) decides when two primitive idempotents generate non-isomorphic left ideals. Some of the results here are reformulations of results we have already proved for simple modules.

Theorem 4.5.1 *Suppose u and u' are non-zero idempotents in a semisimple algebra A over a field $\mathbb{F}$. Then:*

(i) *If*

$$uxu \text{ is a } \mathbb{F}\text{-multiple of } u \text{ for every } x \in A \tag{4.33}$$

then the idempotent u is primitive.

(ii) *If u is a primitive idempotent, and $\mathbb{F}$ is algebraically closed, then (4.33) holds.*

(iii) *If u and u' are primitive idempotents then:*

$$Au \text{ is not isomorphic with } Au' \text{ if and only if } u'xu = 0 \text{ for all } x \in A. \tag{4.34}$$

(iv) *If u and u' are primitive idempotents, and $\mathbb{F}$ is algebraically closed, and if Au is isomorphic to Au' then $\{u'xu : x \in A\}$ is a one dimensional vector space over $\mathbb{F}$, i.e. $u'xu$ is of the form $\lambda_x u'u$, for some $\lambda_x \in k$, and $u'u \neq 0$.*

The peculiar condition in (i) for u being primitive doesn't seem natural, but it is easy to prove and very powerful. An added bonus is that it doesn't require any special conditions at all on the field $\mathbb{F}$.

Proof. (i) Assume that the idempotent u satisfies (4.33). Suppose we have a decomposition of u into idempotents:

$$u = v + w,$$

where v and w are orthogonal idempotents, i.e.

$$v^2 = v, \quad w^2 = w, \quad vw = wv = 0.$$

Then, taking $x = v$ in (4.33), we see that

$$uvu = (v + w)v(v + w) = v + 0 = v$$

and so, by (4.33), it follows that v is a multiple of u:

$$v = \lambda u \quad \text{for some } \lambda \in k.$$

Since both u and v are idempotents, it follows that

$$\lambda^2 = \lambda$$

and so λ is 0 or 1. Hence, u is primitive. The idea in the argument here is that one can recover any idempotent 'subcomponent' v of an idempotent u as uvu, i.e. $\{uxu : x \in A\}$ contains all $\mathbb{F}$-multiples of all 'subcomponents' of u, and so if $\{uxu : x \in A\}$ is ku then any 'subcomponent' of u must be a multiple of u itself, and hence is either 0 or u.

(ii) For any $x \in A$, we have the A-linear map

$$f_x : Au \to Au : au \mapsto auxu$$

If u is primitive then Au is simple and so, by Schur's lemma and the assumption that $\mathbb{F}$ is algebraically closed, this mapping is a multiple of the identity mapping. In particular, there is a $\lambda \in k$, for which $f_x(u) = \lambda u$, i.e. $uxu = \lambda u$.

(iii) Assume that u and u' are primitive idempotents. Consider, for any $x \in A$, the map

$$f_x : Au' \to Au : y \mapsto yxu \tag{4.35}$$

Since both Au and Au' are simple left A-modules, this map is either 0 or an isomorphism. So if Au is not isomorphic to Au', then $f_x = 0$; applying f_x to the element u' we see that

$$u'xu = 0.$$

Conversely, suppose $F : Au' \to Au$ is an isomorphism. Then we know that there is an $x \in A$ such that

$$F(u') = xu$$

and so

$$F(u') = F(u'u') = u'xu,$$

and this is not 0, because that would imply that F is 0.

(iv) Now suppose $Au' \simeq Au$, and assume that A is a finite dimensional algebra over an algebraically closed field $\mathbb{F}$. Then by Schur's lemma, $\mathrm{Hom}_A(Au', Au)$ is a one-dimensional vector space over $\mathbb{F}$. Since $u'xu$ is uniquely determined by f_x, it follows that $\{u'xu : x \in A\}$ is a one-dimensional vector space over $\mathbb{F}$. $\boxed{\text{QED}}$

Here is yet another point of view on idempotents and primitive idempotents:

Proposition 4.5.5 *Let A be a semisimple algebra, $\{L_i\}_{i\in\mathcal{R}}$ a maximal collection of non-isomorphic simple left ideals in A, and A_i the sum of all left ideals isomorphic to L_i. Let $C_i = \mathrm{End}_{A_i}(L_i)$. An element $a \in A$ is an idempotent if and only if its representative block diagonal matrix in $\prod_{i\in\mathcal{R}} \mathrm{End}_{C_i}(L_i)$ is a projection matrix (i.e. an idempotent). It is a primitive idempotent if and only if the matrix is a projection matrix of rank* 1.

<u>Proof</u>. Recall that

$$A \simeq \prod_{i\in\mathcal{R}} \mathrm{End}_{C_i}(L_i) : a \mapsto [a_i]_{i\in\mathcal{R}}$$

an isomorphism of rings. Thus $a \in A$ is an idempotent if and only if each of its components $a_i \in \mathrm{End}_{C_i}(L_i)$ is an idempotent, i.e. a projection map. If the rank of the block matrix $[a_i]$ were not 1, then we could write a_i as a sum of two distinct non-zero projections, and so a would not be primitive. Conversely, if the rank of $[a_i]_{i\in\mathcal{R}}$ is 1 then a is clearly primitive. $\boxed{\text{QED}}$

4.6 Modules over Semisimple Rings

We will now see how the decomposition of a semisimple ring A yields a decomposition of any A-module E.

Let A be a semisimple ring. Recall that there is a finite number of non-isomorphic simple left ideals

$$L_1, ..., L_r \subset A$$

such that every simple left ideal is isomorphic to one of these. Moreover,

$$A_i \stackrel{\text{def}}{=} \text{sum of all left ideals isomorphic to } L_i$$

is a two-sided ideal in A, and A is the direct sum of these ideals as well as being isomorphic to their product:

$$A \simeq \prod_{i=1}^{r} A_i$$

Recall that each A_i has a unit element u_i, and

$$u_1 + \cdots + u_r = 1.$$

Every $a \in A$ decomposes uniquely as

$$a = \sum_{i=1}^{r} a_i,$$

where

$$au_i = a_i = u_i a \in A_i.$$

Consider now any left A-module E. Any element $x \in E$ can then be decomposed as

$$x = 1x = \sum_{j=1}^{r} u_j x$$

Note that

$$u_j x \in E_j \stackrel{\text{def}}{=} A_j E, \tag{4.36}$$

and E_j is a submodule of E. Observe also that since

$$A_j = u_j A,$$

we have

$$E_j = u_j E.$$

Moreover,

$$E_j = A_j E = \sum_{\text{left ideal } L \simeq L_j} LE$$

Before stating our observations formally, note that we have

Lemma 4.6.1 *If A is a semisimple ring and $E \neq \{0\}$ an A-module then E has a submodule isomorphic to some simple left ideal in A.*

Proof. Observe that $E = AE \neq \{0\}$. Now A is the sum of its simple left ideals. Thus, there is a simple left ideal L in A, and an element $v \in E$, such that $Lv \neq \{0\}$. The map

$$L \to Lv : x \mapsto xv$$

is surjective and, by simplicity of L, is also injective. Thus, $L \simeq Lv$, and Lv therefore a simple submodule of E. $\boxed{\text{QED}}$

Now we can state the decomposition result for modules over semisimple rings. To recall the notation briefly: the finite set $\mathcal{R}$ labels a maximal set $\{L_i\}_{i\in\mathcal{R}}$ of non-isomorphic simple left ideals in A, and, for each $i \in \mathcal{R}$, we have the two-sided ideal

$$A_i = \sum_{L \simeq L_i} L,$$

the sum running over all left ideals L isomorphic to L_i. Recall also that any simple left A-module is isomorphic to L_i for exactly one $i \in \mathcal{R}$.

Theorem 4.6.1 *Suppose A is a semisimple ring, and E a left A-module. Then, with notation as above,*

$$E = \bigoplus_{i\in\mathcal{R}} E_i,$$

where

$$E_i = A_iE = u_iE$$

is the sum of all simple submodules of E isomorphic to L_j, this sum being taken to be $\{0\}$ when there is no such submodule.

Proof. Let F be a simple submodule of E. We know that it must be isomorphic to one of the simple ideals L_j in A. Then, since $L'F = 0$ whenever L' is a simple ideal not isomorphic to L_j, we have

$$F = AF = A_jF \subset E_j.$$

Thus, every every submodule isomorphic to L_j is contained in E_j. On the other hand, A_j is the sum of simple left ideals isomorphic to L_j, and so $E_j = A_jE$ is a sum of simple submodules isomorphic to L_j. $\boxed{\text{QED}}$

Let us look at another perspective on the structure of a module over a semisimple ring. Suppose E is a left module over a semisimple ring A,

L_i is a simple left ideal in A, and D_i is the division ring $\mathrm{End}_A(L_i)$. The elements of D_i are A-linear maps $L_i \to L_i$ and so L_i is, naturally, a left D_i-module. On the other hand, D_i acts naturally on the right on $\mathrm{Hom}_A(L_i, E)$ by taking $(f, d) \in \mathrm{Hom}_A(L_i, E) \times D_i$ to the element $fd = f \circ d \in \mathrm{Hom}_A(L_i, A)$. Thus, $\mathrm{Hom}_A(L_i, E)$ is a right D_i-module. Hence there is a well-defined tensor product

$$\mathrm{Hom}_A(L_i, E) \otimes_{D_i} L_i$$

which, for starters, is just a $\mathbb{Z}$-module. However, the left A-module structure on L_i, which commutes with the D_i-module structure, induces naturally a left A-module structure on $\mathrm{Hom}_A(L_i, E) \otimes_{D_i} L_i$ with multiplications on the second factor. We use this in the following result.

Theorem 4.6.2 *If E is a left module over a semisimple ring A, and $L_1, ..., L_r$ a maximal set of non-isomorphic simple left-ideals in A, then there is an isomorphism of A-modules:*

$$E \simeq \bigoplus_{i=1}^{r} \mathrm{Hom}_A(L_i, E) \otimes_{D_i} L_i \tag{4.37}$$

Here the tensor product is taken in the sense of left and right D_i-modules, where D_i is the division ring $\mathrm{Hom}_A(L_i, L_i)$; it has an A-module structure from that on the second factors L_i.

Proof. The module E is a direct sum of simple submodules, each isomorphic to some L_i:

$$E = \bigoplus_{i=1}^{r} \bigoplus_{j \in R_i} E_{ij}$$

where $E_{ij} \simeq L_i$, as A-modules, for each i and $j \in R_i$ (which might be $\emptyset$). In the following we will, as we may, simply assume that $R_i \neq \emptyset$, since $\mathrm{Hom}_A(L_i, E)$ is 0 for all other i. Because L_i is simple, Schur's Lemma implies that $\mathrm{Hom}_A(L_i, E_{ij})$ is a one-dimensional vector space over the division ring D_i, and a basis is given by any fixed non-zero element ϕ_{ij}. For any $f_i \in \mathrm{Hom}_A(L_i, E)$ let

$$f_{ij} : L_i \to E_{ij}$$

be the composition of f_i with the projection onto E_{ij}. Then

$$f_{ij} = \phi_{ij} d_{ij},$$

for some $d_{ij} \in D_i$. Any element of $\mathrm{Hom}_A(L_i, E) \otimes_{D_i} L_i$ is of the form

$$\sum_{j \in R_i} \phi_{ij} \otimes x_{ij}$$

for some $x_{ij} \in L_i$. Consider now the A-linear map

$$J : \bigoplus_{i=1}^{r} \mathrm{Hom}_A(L_i, E) \otimes_{D_i} L_i \to E$$

specified by requiring that for each $i \in \{1, ..., r\}$,

$$J\left(\sum_{i=1}^{r} \sum_{j \in R_i} \phi_{ij} \otimes x_{ij}\right) = \sum_{i=1}^{r} \sum_{j \in R_i} \phi_{ij}(x_{ij}).$$

If this value is 0 then each $\phi_{ij}(x_{ij}) \in E_{ij}$ is 0 and then, since ϕ_{ij} is an isomorphism, x_{ij} is 0. Thus, J is injective. The decomposition of E into the simple submodules E_{ij} shows that J is also surjective. $\boxed{\text{QED}}$

Now consider the case where A is a finite dimensional semisimple algebra over an algebraically closed field $\mathbb{F}$. If

$$1 = e_1 + \cdots + e_n$$

is a decomposition of 1 into non-zero orthogonal idempotents e_j, then $e_1, ..., e_n$ are linearly independent over $\mathbb{F}$ and so $n \leq \dim_{\mathbb{F}} A$. Taking such a decomposition for which n is the largest possible, it follows that each e_j is a primitive idempotent. This shows again that such A can be decomposed as a direct sum of simple left-ideals. Grouping together those e_j with isomorphic Ae_j, produces the decomposition

$$1 = u_1 + \cdots + u_r, \tag{4.38}$$

where $u_1, ..., u_r$ are idempotents forming a basis of $Z(A)$ over $\mathbb{F}$.

Let us decompose each u_i into a sum of primitive idempotents as:

$$u_i = u_{i1} + \cdots + u_{in_i} \tag{4.39}$$

Any element $x \in A$ decomposes as

$$x = \sum_{i=1}^{r} xu_i = \sum_{i=1}^{r} u_i x u_i = \sum_{i=1}^{r} \sum_{1 \leq \alpha, \beta \leq n_i} u_{i\alpha} x u_{i\beta} \tag{4.40}$$

and so x corresponds to a block diagonal matrix, with the i-th block being the matrix

$$[u_{i\alpha}xu_{i\beta}]_{1\leq\alpha,\beta\leq n_i}$$

which may be viewed as an $n_i \times n_i$ matrix with the $\alpha\beta$ entry in

$$\mathrm{Hom}_A(Au_{i\alpha}, Au_{i\beta}) \simeq k.$$

Exercises

1. Let $\tau : G \to \mathbb{F}$ be a homomorphism of the finite group G into the group of invertible elements of the field $\mathbb{F}$, and assume that the characteristic of $\mathbb{F}$ is not a divisor of $|G|$. Let

$$u_\tau = \frac{1}{|G|}\sum_{g\in G}\tau(g^{-1})g$$

   Show that u_τ is a primitive idempotent.

2. Let A be a finite-dimensional semisimple algebra over a field $\mathbb{F}$, and define $\chi^{\mathrm{reg}} : A \to \mathbb{F}$ by

$$\chi^{\mathrm{reg}}(a) = \mathrm{Tr}\big(\rho^{\mathrm{reg}}(a)\big), \quad \text{where } \rho^{\mathrm{reg}}(a) : A \to A : x \mapsto ax. \tag{4.41}$$

   Let $L_1, ..., L_s$ be a maximal collection of non-isomorphic simple left ideals in A, so that $A \simeq \prod_{i=1}^{s} A_i$, where A_i is the two-sided ideal formed by the sum of all left ideals isomorphic to L_i. As usual, let $1 = u_1 + \cdots + u_s$ be the decomposition of 1 into idempotents $u_i \in A_i = Au_i$. Viewing L_i as a vector space over $\mathbb{F}$, define

$${\chi^{\mathrm{reg}}}_i(a) = \mathrm{Tr}(\rho^{\mathrm{reg}}(a)|L_i) \tag{4.42}$$

   Note that since L_i is a left ideal, $\rho^{\mathrm{reg}}(a)(L_i) \subset L_i$. Show that:

   (i) $\chi^{\mathrm{reg}} = \sum_{i=1}^{s} d_i {\chi^{\mathrm{reg}}}_i$, where d_i is the integer for which $A_i \simeq d_i L_i$.

   (ii) ${\chi^{\mathrm{reg}}}_i(u_j) = \delta_{ij}\dim_{\mathbb{F}} L_i$

   (iii) Assume that the character of $\mathbb{F}$ does not divide any of the numbers $\dim_{\mathbb{F}} L_i$. Use (ii) to show that the functions ${\chi^{\mathrm{reg}}}_1, ..., {\chi^{\mathrm{reg}}}_s$ are linearly independent over $\mathbb{F}$.

(iv) Let E be a left A-module, and define $\chi^E : A \to k$ by

$$\chi^E(a) = \mathrm{Tr}\big(\rho^E(a)\big), \quad \text{where } \rho^E(a) : E \to E : x \mapsto ax. \tag{4.43}$$

Show that χ^E is a linear combination of the functions $\chi^{\mathrm{reg}}{}_i$ with non-negative integer coefficients:

$$\chi^E = \sum_{i=1}^{s} n_i \chi^{\mathrm{reg}}{}_i$$

where n_i is the number of copies of L_i in a decomposition of E into simple A-modules.

(v) Under the assumption made in (iii), show that if E and F are left A-modules with $\chi^E = \chi^F$ then $E \simeq F$.

3. Let $B = \mathrm{Matr}_n(\mathbb{F})$ be the algebra of $n \times n$ matrices over the field $\mathbb{F}$.

   (a) Show that for each $j \in \{1, ..., n\}$, the set L_j of all matrices in B which have all entries 0 except possibly those in column j is a simple left ideal.

   (b) Show that if L is a simple left ideal in B then there is a basis $b_1, ..., b_n$ of $\mathbb{F}^n$ such that L consists exactly of those matrices M for which $Mb_i = 0$ whenever $i \neq 1$.

   (c) With notation as in (a), produce orthogonal idempotent generators in $L_1, ..., L_n$.

   (d) Show that L_1 and L_2 also have idempotent generators which are not orthogonal to each other.

4. Show that if u and v are primitive idempotents in a $\mathbb{F}$-algebra A, where $\mathbb{F}$ is algebraically closed, then uv is either 0, or has square equal to 0, or is a $\mathbb{F}$-multiple of a primitive idempotent. What can be said if u and v are commuting primitive idempotents? [Solution: If u and v belong to different A_i then $uv = 0$. Suppose then that u and v both belong to the same A_i. Then we may as well assume that they are $d_i \times d_i$ matrices over $\mathbb{F}$, where $d_i = \dim_{\mathbb{F}}(L_i)$. Since $u^2 = u$, and u has rank 1, we can choose a basis in which u has entry 1 at the top left corner and has all other entries equal to 0. Then, for any matrix v, the product

uv has all entries 0 except those in the top row. Let λ be the top left entry of the matrix uv. Then

$$(uv)^n = \lambda^{n-1}uv$$

If $\lambda = 0$ then $(uv)^2 = 0$. If $\lambda \neq 0$ then $\lambda^{-1}uv$ has 1 as top left entry and all rows below the top one are 0; hence, $\lambda^{-1}uv$ is a rank 1 projection, i.e. a primitive idempotent. Thus, uv is a multiple of a primitive idempotent. If u and v commute and $uv \neq 0$ then $(uv)^2 = u^2v^2 = uv \neq 0$, and so $\lambda^{-1}uv$ is a primitive idempotent for some $\lambda \in k$, and then $\lambda^{-2} = \lambda^{-1}$ and so $\lambda = 1$, i.e. uv is a primitive idempotent.]

5. Prove that if M is a semisimple module over a ring A, and $\mathrm{End}_A(M)$ is abelian, then M is the direct sum of simple submodules, no two of which are isomorphic to each other.

6. Prove that if a module N over a ring is the direct sum of simple submodules, no two of which are isomorphic to each other then every simple submodule of N *is* one of these submodules.

7. Sanity check exercises:

   (a) Is $\mathbb{Z}$ a semisimple ring?

   (b) Is $\mathbb{Q}$ a semisimple ring?

   (c) Is a subring of a semisimple ring also semisimple?

   (d) Show that an abelian simple ring is a field.

Chapter 5

The Regular Representation

We return to the study of representations of a finite group G. In this chapter we essentially restate the conclusions about $\mathbb{F}[G]$ that follow from the results of the previous chapter.

We will work with a field $\mathbb{F}$, and a finite group G. As we have seen, a key role is played by the group algebra

$$\mathbb{F}[G]$$

This is a finite dimensional vector space over $\mathbb{F}$, with

$$\dim_{\mathbb{F}} \mathbb{F}[G] = |G|,$$

with the elements of G giving a basis of $\mathbb{F}[G]$.

The representation ρ^{reg} of G on $\mathbb{F}[G]$ given by left multiplication is the *regular representation.*

We will make the standing assumption that the character of $\mathbb{F}$ is not a divisor of $|G|$. This is needed for semisimplicity of $\mathbb{F}[G]$.

5.1 Structure of the Regular Representation

The algebra $\mathbb{F}[G]$ is semisimple, and contains non-isomorphic simple left modules

$$L_1, ..., L_s$$

and $\mathbb{F}[G]$ splits as a product of two-sided ideals $\mathbb{F}[G]_j$:

$$\mathbb{F}[G] \simeq \prod_{j=1}^{s} \mathbb{F}[G]_j, \tag{5.1}$$

where $\mathbb{F}[G]_j$ is the sum of all simple left ideals isomorphic to L_j; it can also be expressed as a *direct sum* of some of these simple left ideals.

The regular representation ρ^{reg} decomposes into the irreducible subrepresentations ρ_r^{reg} on L_r, for $r \in \{1, ..., s\}$. Viewing L_r as a left $\mathbb{F}[G]$-module, we have, for each $a \in \mathbb{F}[G]$, a $\mathbb{F}$-linear map

$$\rho_r^{\text{reg}}(a) : L_r \to L_r : w \mapsto aw \tag{5.2}$$

The unit element

$$1 \stackrel{\text{def}}{=} 1e$$

of $\mathbb{F}[G]$ can be expressed as a sum

$$1 = u_1 + \cdots + u_s \tag{5.3}$$

and then

$$\mathbb{F}[G] = \sum_{r=1}^{s} u_r \mathbb{F}[G] \tag{5.4}$$

The elements $u_1, ..., u_s$ form a *basis of the center* $Z(\mathbb{F}[G])$, and satisfy the relations

$$u_i^2 = u_i, \quad u_i u_j = 0 \quad \text{if } i \neq j \tag{5.5}$$

Let

$$D_r = \text{End}_{\mathbb{F}[G]}(L_r) \tag{5.6}$$

By Schur's lemma, this is a division ring. Note that $\mathbb{F} \simeq \mathbb{F}1$ is contained in D_r. The important case is

$$D_r = \mathbb{F} \qquad \text{if } \mathbb{F} \text{ is algebraically closed.} \tag{5.7}$$

The representation ρ_r^{reg} produces an *isomorphism* of algebras when restricted to $\mathbb{F}[G]_r$:

$$\rho_r^{\text{reg}} | \mathbb{F}[G]_r \to \text{End}_{D_r}(L_r) \tag{5.8}$$

Thus, in terms of ρ^{reg}, each element of $\mathbb{F}[G]$ may be viewed as a block diagonal matrix, with the r-th diagonal block being an element of $\text{End}_{D_r}(L_r)$.

Note that the two-sided ideal $\mathbb{F}[G]_r$ and $\text{End}_{D_r}(L_r)$ carry both left and right representations of G. The fact that

$$\rho_r^{\text{reg}}(xay) = \rho_r^{\text{reg}}(x)\rho_r^{\text{reg}}(a)\rho_r^{\text{reg}}(y) \quad \text{for all } a \in \mathbb{F}[G]_r, \text{ and } x, y \in G \tag{5.9}$$

shows that ρ_r^{reg} intertwines with both of these representations.

Comparing dimensions of $\mathbb{F}[G]_i$ and $\mathrm{End}_{D_r}(L_r)$, we have

$$\dim_{D_r} \mathbb{F}[G]_i = (\dim_{D_r} L_r)^2 \tag{5.10}$$

Thus, writing

$$d_i = \dim_{D_i} L_i \tag{5.11}$$

we see that $\mathbb{F}[G]_i$ is the direct sum of d_i copies of L_i. Thus, the regular representation splits up as a direct sum

$$\mathbb{F}[G] \simeq \sum_{i=1}^{s} d_i L_i \tag{5.12}$$

Note again that

$$d_i = \dim_{\mathbb{F}}(L_i) \qquad \text{if } \mathbb{F} \text{ is algebraically closed.}$$

If E is a finite dimensional representation of G then E decomposes into a direct sum

$$E = \bigoplus_{i=1}^{s} u_i E, \tag{5.13}$$

and $u_i E$ is the direct sum of all subspaces of E isomorphic to the representation on L_i.

Note that s is the number of distinct isomorphy classes of $\mathbb{F}[G]$ modules.

The center of $\mathbb{F}[G]$ has $u_1, ..., u_s$ as a vector space basis. On the other hand, the center of $\mathbb{F}[G]$ clearly has the following basis: for each conjugacy class C in G take the element

$$b_C = \sum_{g \in C} g \tag{5.14}$$

(See Proposition 3.3.1.) Thus:

Theorem 5.1.1 *The number of distinct isomorphism classes of irreducible representations of G equals the number of conjugacy classes in G.*

5.2 Representations of abelian groups

Let us look at a finite abelian group H, and assume that the field $\mathbb{F}$ is algebraically closed and has characteristic not a divisor of $|H|$. Then the group algebra $\mathbb{F}[H]$ is also abelian (i.e. multiplication is commutative), and so each endomorphism algebra $\mathrm{End}_{\mathbb{F}}(L_i)$ is abelian. This can only be if each L_i is one-dimensional.

The formula

$$|G| = \sum_{i=1}^{s} [\dim_{\mathbb{F}}(L_i)]^2,$$

where $L_1, ..., L_s$ is a maximal collection of non-isomorphic simple left ideals in $\mathbb{F}[G]$, shows that each L_i is one-dimensional if and only if the number s of distinct irreducible representations of G equals $|G|$. Thus, each irreducible representation of G is one dimensional if and only if the number of conjugacy classes in G equals $|G|$, i.e. if each conjugacy class contains just one element. But this means that G is abelian. Thus,

Theorem 5.2.1 *Assume the ground field $\mathbb{F}$ has characteristic 0 and is algebraically closed. All irreducible representations of a finite abelian group are one-dimensional. Conversely, if all irreducible representations of a finite group are one dimensional then the group is abelian.*

Exercises

1. Let G be a cyclic group, and $\mathbb{F}$ algebraically closed. Decompose $\mathbb{F}[G]$ as a direct sum of one-dimensional representations of G.

Chapter 6

Characters of Finite Groups

In this chapter we work only with finite dimensional representations of a finite group G over a field $\mathbb{F}$ which has 0 characteristic. Mostly, we will also need to assume that $\mathbb{F}$ is algebraically closed. In the later part of the chapter we take $\mathbb{F} = \mathbb{C}$.

6.1 Definition and Basic Properties

If ρ is a representation of a finite group on a finite dimensional $\mathbb{F}$-vector space E then the function

$$\chi_\rho : G \to \mathbb{F} : g \mapsto \operatorname{tr}\big(\rho(g)\big) \tag{6.1}$$

is called the *character* of the representation ρ.

Note that

$$\chi_\rho(e) = \dim \rho \tag{6.2}$$

and that the character is a *central function*, i.e. invariant under conjugation:

$$\chi_\rho(ghg^{-1}) = \chi_\rho(h) \tag{6.3}$$

for all $g, h \in G$.

We also have

$$\chi_\rho = \chi_{\rho'}$$

whenever ρ and ρ' are equivalent representations.

Sometimes it is notationally convenient to write

$$\chi_E$$

instead of χ_ρ.

It is readily seen that

$$\begin{aligned} \chi_{E\oplus F} &= \chi_E + \chi_F \qquad (6.4)\\ \chi_{E\otimes F} &= \chi_E\chi_F \qquad (6.5)\end{aligned}$$

Thus, if E decomposes as

$$E = \bigoplus_{i=1}^{m} n_i E_i,$$

where E_i are representations, then

$$\chi_E = \sum_{i=1}^{s} n_i \chi_{E_i} \qquad (6.6)$$

Note that each character function χ extends naturally to a linear function

$$\chi : \mathbb{F}[G] \to \mathbb{F}$$

which is central in the sense that

$$\chi(ab) = \chi(ba) \qquad \text{for all } a, b \in \mathbb{F}[G]. \qquad (6.7)$$

6.2 Character of the Regular Representation

We examine the character of the regular representation of G:

$$\chi^{\text{reg}} \stackrel{\text{def}}{=} \text{character of the regular representation} \qquad (6.8)$$

As usual, we may view this as a function on $\mathbb{F}[G]$. Then,

$$\chi^{\text{reg}}(a) = \text{trace of the linear map } \mathbb{F}[G] \to \mathbb{F}[G] : x \mapsto ax \qquad (6.9)$$

Consider an element

$$b = \sum_{x\in G} b_x x \in \mathbb{F}[G]$$

Then

$$by = \sum_{x\in G} b_x xy = b_e y + \sum_{z\in G, z\neq y} b_{zy^{-1}} z$$

and so, in terms of the basis of $\mathbb{F}[G]$ given by the elements of G, left multiplication by b has a matrix with b_e running down the main diagonal. Hence

$$\chi^{\text{reg}}(b) = |G| b_e \tag{6.10}$$

Put another way,

$$\frac{1}{|G|}\mathrm{Tr}\left(\rho^{\text{reg}}(b)\right) = b_e \tag{6.11}$$

Recall that the group algebra $\mathbb{F}[G]$ contains simple left ideals $L_1, ..., L_s$, where s is the number of conjugacy classes in G, each irreducible representation of G is isomorphic to some L_i, and the algebra $\mathbb{F}[G]$ is isomorphic to the product of two-sided ideals $\mathbb{F}[G]_i$, where $\mathbb{F}[G]_i$ is the sum of all left ideals isomorphic to L_i. Furthermore,

$$\mathbb{F}[G]_i \mathbb{F}[G]_j = 0 \quad \text{if } i \neq j$$

Moreover,

$$\mathbb{F}[G]_i \simeq \mathrm{End}_{\mathbb{F}}(L_i)$$

Let

$$\chi_i = \text{character of the representation on } L_i. \tag{6.12}$$

Thus, every character χ of G is a linear combination of the form

$$\chi = \sum_{i=1}^{s} n_i \chi_i, \tag{6.13}$$

where n_i is the number of copies of L_i in a direct sum decomposition of the representation for χ into irreducible representations.

In particular, for the character χ^{reg}, we have

$$\chi^{\text{reg}} = \sum_{i=1}^{s} d_i \chi_i, \tag{6.14}$$

is the number of copies of L_i in a direct sum decomposition of $\mathbb{F}[G]$ into simple left ideals. We know that

$$d_i = \dim_{D_i} L_i,$$

where D_i is the division ring

$$D_i = \mathrm{End}_{\mathbb{F}[G]_i} L_i$$

When $\mathbb{F}$ is algebraically closed, d_i equals $\dim_{\mathbb{F}} L_i$.

Recalling (6.9), and noting that

$$a_j\mathbb{F}[G]_i = 0 \quad \text{if } a_j \in \mathbb{F}[G]_j \text{ and } j \neq i,$$

we have

$$\chi_i(a_j) = 0 \qquad \text{if } a_j \in \mathbb{F}[G]_j \text{ and } j \neq i \tag{6.15}$$

Thus,

$$\chi_i\Big|\mathbb{F}[G]_j = 0 \quad \text{if } j \neq i \tag{6.16}$$

Equivalently,

$$\chi_i(u_j) = 0 \quad \text{if } j \neq i \tag{6.17}$$

where, as usual, u_j is the generating idempotent for $\mathbb{F}[G]_j$. On the other hand,

$$\chi_i(u_i) = \dim_{\mathbb{F}} L_i \tag{6.18}$$

because the central element u_i acts as the idenitity on $L_i \subset A_i$. In fact, this also implies that

$$\chi^{\text{reg}}(yu_i) = d_i\chi_i(y) \quad \text{for all } y \in G \tag{6.19}$$

Because of (6.17) and (6.18) it follows that if

$$\sum_i c_i\chi_i = 0$$

where $c_1, ..., c_s \in k$, then, on applying this to a_j,

$$c_j \dim_{\mathbb{F}} L_i = 0.$$

Thus, if $\mathbb{F}$ has characteristic 0, then the irreducible characters are linearly independent over $\mathbb{F}$.

There is one very fundamental consequence of linear independence of irreducible characters, which justifies the name 'character':

Theorem 6.2.1 *If two representations have the same character then they are equivalent.*

Proof. Let $E_1, ..., E_s$ be a maximal collection of inequivalent irreducible representations of G. If E is a representation of G then E is isomorphic to a direct sum

$$E \simeq \bigoplus_{i=1}^{s} n_iE_i \tag{6.20}$$

where n_iE_i is a direct sum of n_i copies of E_i. Then

$$\chi_E = \sum_{i=1}^{s} n_i\chi_i$$

The coefficients n_i are uniquely determined by χ_E, and hence so is the decomposition (6.20) up to isomorphism. $\boxed{\text{QED}}$

6.3 Fourier Expansion

We assume, as usual, that the character of $\mathbb{F}$ does not divide $|G|$.

Let

$$b = \sum_{x\in G} b_x x \in k[G]$$

Recall that, technically, b is a function $G \to k$.

Then

$$\chi^{\text{reg}}(bx^{-1}) = |G|b_x, \quad \text{for any } x \in G.$$

and so

$$b_x = \frac{1}{|G|}\sum_{i=1}^{s} d_i\chi_i(bx^{-1}) \tag{6.21}$$

Thus,

$$b = \sum_{x\in G}\left(\sum_{i=1}^{s}\frac{d_i}{|G|}\chi_i(bx^{-1})\right)x \tag{6.22}$$

Choose a $\mathbb{F}$-basis in each L_i, and let $\rho_i(x)^l_m$ be the matrix entries of

$$\rho_i(x) : L_i \to L_i : y \mapsto xy.$$

Then

$$\chi_i(bx^{-1}) = \sum_{l,m}\rho_i(b)^m_l\rho_i(x^{-1})^l_m$$

So we can rewrite (6.22) as:

$$b = \frac{1}{|G|}\sum_{i=1}^{s} d_i\sum_{l,m}\rho_i(b)^m_l\big(\sum_{x\in G}\rho_i(x^{-1})^l_m x\big) \tag{6.23}$$

In the case where G is a cyclic group with generator z, the irreducible representations are all one dimensional and $\rho(z^k)$ has the form $e^{i\frac{2\pi k}{n}}$. Then (6.22) is a Fourier expansion of the function b.

This shows that the 'matrix elements'

$$\sum_{x\in G}\rho_i(x^{-1})^l_m x \tag{6.24}$$

span $\mathbb{F}[G]$. Here $1 \le l, m \le \dim_{\mathbb{F}} L_i$. When $\mathbb{F}$ is algebraically closed, the total number of these elements is

$$\sum_{i=1}^{s}(\dim_{\mathbb{F}} L_i)^2 = |G| = \dim_{\mathbb{F}} k[G],$$

and so the matrix elements form a basis of $\mathbb{F}[G]$. We will return to a more detailed account of this in section 6.4.

Let us look at the expansion of the idempotents $u_i \in k[G]_i$ which form a basis of the center of $\mathbb{F}[G]$. Setting $b = u_i$ in (6.22) gives:

$$u_i = \sum_{x\in G}\frac{d_i}{|G|}\chi_i(u_i x^{-1})x + 0 = \frac{d_i}{|G|}\sum_{x\in G}\chi_i(x^{-1})x = \frac{1}{|G|}\sum_{x\in G}\chi^{\mathrm{reg}}(u_i x^{-1})$$

where we have used the fact that $\chi_i(u_j y) = 0$ for all $j \neq i$, and

$$d_i\chi_i(y) = d_i\chi_i(u_i y) = \chi^{\mathrm{reg}}(u_i y)$$

for all $y \in G$, and d_i is the multiplicity of L_i in $\mathbb{F}[G]$, equal to $\dim_{\mathbb{F}} L_i$ in case $\mathbb{F}$ is algebraically closed.

This lets us express the basis elements u_i for the center of $\mathbb{F}[G]$ in terms of the basis elements

$$b_C = \sum_{x\in C} x, \qquad C \text{ running over the set } \mathcal{C} \text{ of all conjugacy classes in } G. \tag{6.25}$$

The central idempotent corresponding to the character χ_i is:

$$u_i = \frac{d_i}{|G|}\sum_{x\in G}\chi_i(x^{-1})x = \frac{d_i}{|G|}\sum_{C\in\mathcal{C}}\chi_i(C^{-1})b_C, \tag{6.26}$$

where $\chi_i(C)$ is the constant value of χ_i on the conjugacy class C.

Note an immediate consequence:

The multiplicities d_i are not divisible by the character of $\mathbb{F}$. (6.27)

(NEED MORE DETAILS HERE: More conclusions can be drawn: for example, that each d_i is a divisor of $|G|$. Thus, when $\mathbb{F}$ is algebraically closed, the dimension of any irreducible representation is a divisor of $|G|$.)

Recall that, technically, an element $f \in k[G]$ is a function $f : x \mapsto f(x)$ on G with values in $\mathbb{F}$. The usual pointwise product on the algebra $\mathbb{F}[G]$ is thus given by

$$fh = \sum_{s\in G} f_s s \sum_{t\in G} h_t t = \sum_{x\in G}\left(\sum_{y\in G} f_y h_{y^{-1}x}\right) x$$

Keeping this in mind, let us define the normalized *convolution* of functions on G:

$$f * h(x) \stackrel{\text{def}}{=} \frac{1}{|G|}\sum_{y\in G} f(y)h(y^{-1}x) \tag{6.28}$$

The property

$$u_i u_j = \delta_{ij} u_i$$

then translates into

$$\frac{d_i d_j}{|G|}\sum_{x\in G}(\chi_i * \chi_j)(x^{-1})x = \delta_{ij}\frac{d_i}{|G|}\sum_{x\in G}\chi_i(x^{-1})x,$$

which implies

$$\boxed{\chi_i * \chi_j = \delta_{ij}\tfrac{1}{d_i}\chi_i.} \tag{6.29}$$

In particular,

$$\frac{1}{|G|}\sum_{x\in G}\chi_i(x)\chi_i(x^{-1}) = 1, \quad \text{if } \mathbb{F} \text{ is algebraically closed.} \tag{6.30}$$

The idempotent u_i corresponds to a submodule $\mathbb{F}[G]_i$ of $\mathbb{F}[G]$ containing d_i copies of the irreducible representation whose character is χ_i. For a general idempotent u, the relation between u and the corresponding character χ_u for the submodule $\mathbb{F}[G]u$ is worked out in the exercises. The result is

$$\sum_{x\in G}\chi_u(x^{-1})x = \sum_{g\in G} gug^{-1} \tag{6.31}$$

6.4 Orthogonality Relations

Recall the notion of a matrix element for a representation of G on a vector space E. It is a function on G of the form

$$G \to \mathbb{F} : g \mapsto f^*(gu)$$

where $f^* \in E^*$ and $u \in E$.

It is visually convenient to write an element $u \in E$ as a 'ket':

$$|u\rangle$$

and an element w in the dual space E^* as a 'bra'

$$\langle w\,|$$

The evaluation of w on u is then given by the 'bra-ket'

$$\langle w\,|\,u\rangle$$

Thus, a matrix element for a representation ρ is the function given by

$$g \mapsto \langle w\,|\,\rho(g)\,|u\rangle$$

for some bra $\langle w|$ and ket $|u\rangle$.

If $\{e_i\}_{i\in I}$ is a basis of E then the elements of the dual basis $\{e^i\}_{i\in I}$ may be written as

$$\langle e^i\,| \stackrel{\text{def}}{=} e^i. \tag{6.32}$$

Consider two irreducible representations E and F of the finite group G. By Schur's lemma,

$$\mathrm{Hom}_G(E,F) = 0 \quad \text{if } E \text{ and } F \text{ are not equivalent}$$

and (with the field $\mathbb{F}$ being algebraically closed),

$$\dim_{\mathbb{F}} \mathrm{Hom}_G(E,F) = 1 \qquad \text{if } E \text{ and } F \text{ are equivalent.}$$

Consider any $\mathbb{F}$-linear map

$$T : E \to F.$$

If we symmetrize it with respect to the action of G we obtain

$$T' = \sum_{g \in G} gTg^{-1}$$

The symmetrized element T' can be understood this way: there is a natural action of G on $\mathrm{Hom}_{\mathbb{F}}(E, F)$ and $\mathrm{Hom}_G(E, F)$ is the subspace on which G acts trivially; T' is $|G|$ times the projection of T onto this subspace as in Theorem 3.5.1.

If E and F are inequivalent then, of course, T' must be 0. Consider for T the operator

$$T = |f\rangle\langle e| : E \to F : |v\rangle \mapsto \langle e|v\rangle|f\rangle$$

(Such operators, of course, span $\mathrm{Hom}_{\mathbb{F}}(E, F)$.) Then

$$\sum_{g \in G} g|f\rangle\langle e|g^{-1} = T' = 0.$$

We conclude then:

Theorem 6.4.1 *If ρ_E and ρ_F are inequivalent irreducible representations of a finite group G on vector spaces E and F, respectively, then the matrix elements of ρ and ρ' are orthogonal in the sense that*

$$\sum_{g \in G} \langle f'|\rho_F(g)|f\rangle\langle e'|\rho_E(g^{-1})|e\rangle = 0 \tag{6.33}$$

for all $\langle f'| \in F^*$, $\langle e'| \in E^*$ and all $|e\rangle \in E$, $|f\rangle \in F$. In particular,

$$\sum_{g \in G} \chi_E(g)\chi_F(g^{-1}) = 0. \tag{6.34}$$

Equation (6.34) follows from (6.33) on letting e and f run over basis elements of E and F, respectively, and e' and f' over corresponding dual bases, and then summing over e and f.

Now assume that $\mathbb{F}$ is algebraically closed and has characteristic 0. Let E be a fixed irreducible representation of G. Then Schur's lemma implies that for any $T \in \mathrm{End}_{\mathbb{F}}(E)$ the symmetrized operator T' is a multiple of the identity. The value of this multiplier is easily obtained by comparing traces:

$$\sum_{g \in G} gTg^{-1} = T' = \frac{|G|}{\dim_{\mathbb{F}} E}\mathrm{tr}(T)I, \tag{6.35}$$

noting that both sides have trace equal to $|G|\mathrm{tr}(T)$.

Working with a basis $\{e_i\}_{i\in I}$ of E, we then have

$$\langle e^j|T'|e_i\rangle = \frac{|G|}{\dim_{\mathbb{F}} E}\mathrm{tr}(T)\delta_i^j \quad \text{for all } i,j \in I. \tag{6.36}$$

Taking for T the particular operator

$$T = \rho(h)|e_m\rangle\langle e^s|,$$

we obtain:

Theorem 6.4.2 *If $\mathbb{F}$ is algebraically closed and has characteristic 0, and ρ_E is an irreducible representation of the finite group G on a vector space E over $\mathbb{F}$ then, for any $h \in G$,*

$$\frac{1}{|G|}\sum_{g\in G}\langle e^s|\rho_E(g^{-1})|e_i\rangle\langle e^j|\rho_E(gh)|e_m\rangle = \frac{1}{\dim_{\mathbb{F}} E}\rho_E(h)^s_m\delta_i^j \quad \textit{for } i,j,s,m \in I. \tag{6.37}$$

for any basis $\{|e_i\rangle\}_{i\in I}$ of E. In particular,

$$\frac{1}{|G|}\sum_{g\in G}\langle e^s|\rho_E(g^{-1})|e_i\rangle\langle e^j|\rho_E(g)|e_m\rangle = \frac{1}{\dim_{\mathbb{F}} E}\delta^s_m\delta_i^j \quad \textit{for all } i,j,s,m \in I. \tag{6.38}$$

For the character χ_E we have:

$$\frac{1}{|G|}\sum_{g\in G}\chi_E(g)\chi_E(g^{-1}h) = \frac{1}{\dim_{\mathbb{F}} E}\chi_E(h), \tag{6.39}$$

and

$$\frac{1}{|G|}\sum_{g\in G}\chi_E(g)\chi_E(g^{-1}) = 1, \tag{6.40}$$

6.5 The Invariant Inner Product

In this section, $k = \mathbb{C}$, the field of complex numbers.

On the finite group G we have the normalized Haar measure μ:

$$\mu(S) = \frac{1}{|G|}|S|, \quad \text{for all } S \subset G$$

Now consider any representation ρ of G on a vector space E. Take any inner-product $\langle \cdot, \cdot \rangle'$ on E, and define:

$$\langle u, v\rangle = \frac{1}{|G|} \sum_{x \in G} \langle \rho(x)u, \rho(x)v\rangle' \quad \text{for } u, v \in E \tag{6.41}$$

This is readily checked to be an inner-product. Furthermore, we have the invariance

$$\langle \rho(x)u, \rho(x)v\rangle = \langle u, v\rangle \quad \text{for all } x \in G, \text{ and } u, v \in E$$

This property is also described as *unitarity* of the representation of G on E relative to the inner product $\langle \cdot, \cdot \rangle$, because each $\rho(x)$ is then a unitary operator on E.

Unitarity leads to the following fact:

$$\langle u, x^{-1}v\rangle = \langle xu, v\rangle = \overline{\langle v, xu\rangle} \tag{6.42}$$

Writing ρ for the representation, this property of unitarity of ρ can also be expressed as

$$\rho(x^{-1}) = \rho(x)^* \tag{6.43}$$

Thus, the representation ρ is unitary if and only if each $\rho(x)$ is unitary.

6.6 The Invariant Inner Product on Function Spaces

The group algebra

$$\mathbb{F}[G]$$

is, as a set and as a vector space, the space of all functions

$$G \to \mathbb{F}.$$

The distinction between $\mathbb{F}[G]$ and the function space lies in the multiplicative structure: the product of $f, h \in \mathbb{F}[G]$ is given in the function space by the convolution:

$$fh = \sum_{s \in G} f_s s \sum_{t \in G} h_t t = \sum_{x \in G} \left(\sum_{y \in G} f_y h_{y^{-1}x} \right) x = \sum_{x \in G} (f *' h)(x)x$$

where the convolution $f *' h$ is given by

$$f *' h(x) \stackrel{\text{def}}{=} \sum_{y \in G} f(y)h(y^{-1}x) \tag{6.44}$$

The general relation

$$\chi_\rho(ab) = \mathrm{Tr}(\rho(a)\rho(b))$$

specializes to

$$\chi^{\mathrm{reg}}(fh) = \mathrm{Tr}\left(\rho^{\mathrm{reg}}(f)\rho^{\mathrm{reg}}(h)\right) = \sum_{x \in G} f(x)h(x^{-1}) \tag{6.45}$$

This specifies a bilinear pairing on $\mathbb{F}[G]$.

We will now specialize to the case

$$\mathbb{F} = \mathbb{C}.$$

Let

$$L^2(G) = \{\text{all functions } G \to \mathbb{C}\}$$

This is a complex vector space with an inner-product:

$$\langle f, h \rangle = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{h(x)}, \qquad \text{for all } F, H \in L^2(G) \tag{6.46}$$

Recalling that every element of $\mathbb{C}[G]$ is a function on G:

$$\sum_{y \in G} a_y y = a : G \to \mathbb{C}$$

we see that

$$L^2(G) = \mathbb{C}[G] \tag{6.47}$$

as complex vector spaces.

The product on the algebra $\mathbb{C}[G]$ is given in the function notation by

$$fh = \sum_{s \in G} f_s s \sum_{t \in G} h_t t = \sum_{x \in G} \left(\sum_{y \in G} f_y h_{y^{-1}x} \right) x$$

Thus, the product corresponds to $|G|$ times the normalized *convolution* of functions on G:

$$f * h(x) \stackrel{\text{def}}{=} \frac{1}{|G|} \sum_{y \in G} f(y)h(y^{-1}x) \tag{6.48}$$

Multiplication on the left by f on $\mathbb{C}[G]$ is given by the operator M_f:

$$M_f h = |G| f * h \tag{6.49}$$

The left regular representation ρ^{reg} of G on $\mathbb{C}[G]$, corresponds in the function notation to

$$\rho^{\text{reg}}_x f(y) = f(x^{-1}y) \qquad \text{for all } f \in L^2(G) \text{ and } x, y \in G$$

This representation is *unitary*:

$$\langle \rho^{\text{reg}}_x f, \rho^{\text{reg}}_x h \rangle = \langle f, h \rangle \quad \text{for all } f, h \in L^2(G) \text{ and } x \in G \tag{6.50}$$

The results of Theorem 6.4.1 and Theorem 6.4.2 of the preceding section can be summarized thus:

Theorem 6.6.1 *For irreducible complex representations of a finite group G, the following hold:*

(i) *matrix elements for inequivalent irreducible representations are orthogonal;*

(ii) *if ρ is an irreducible representation of G on a complex vector space E then, considering the matrices $\rho(x)$ relative to a basis in E which is orthonormal relative to some invariant inner product, different matrix elements are orthogonal and each matrix element ρ^i_j has norm squared given by*

$$\|\rho^i_j\|^2_{L^2} = \frac{1}{\dim E} \tag{6.51}$$

(iii) *the convolution of matrix elements relative to an orthonormal basis of an irreducible representation is a multiple of a matrix element for the same representation, the multiplier being 0 or $1/\dim E$;*

(iv) *if χ_E is the character of an irreducible representation of G on a vector space E, then*

$$\chi_E * \chi_E = \frac{1}{\dim E}\chi_E \tag{6.52}$$

(v) *characters of inequivalent irreducible representations are orthogonal;*

(vi) *if χ_E is an irreducible representation of G then*

$$\|\chi_E\|_{L^2} = 1 \tag{6.53}$$

Note that we derived the convolution property part (iv) earlier (6.29) using idempotents.

The matrix elements for irreducible representations form not only an orthogonal system of functions, they form a basis of $L^2(G)$:

Theorem 6.6.2 *For a finite group G, let $E_1, ..., E_s$ be a maximal set of non-isomorphic irreducible representations of G. Choose an invariant inner product on each E_r, and an orthonormal basis. Then the scaled matrix elements*

$$(\dim E_r)^{-1/2}(\rho_{E_r})^i_j \tag{6.54}$$

form an orthonormal basis of $L^2(G)$.

The characters $\chi_1, ..., \chi_s$ form an orthonormal basis of the space of central functions on G.

Proof. We have seen that the functions in (6.54) are orthonormal in $L^2(G)$. The total number of these functions is

$$\sum_{r=1}^{s}(\dim E_r)^2.$$

But this is precisely the number of elements in G, i.e. it is equal to $\dim L^2(G)$. Thus, the functions (6.54) do form a basis of $L^2(G)$. For the characters, observe again that they are orthonormal, and there are s of them; but s is the number of conjugacy classes in G, and so is the dimension of the space of central functions on G. $\boxed{\text{QED}}$

We can show that the matrix entries span $L^2(G)$ by using the Fourier expansion (6.22):

$$b = \sum_{x\in G}\left(\sum_{r=1}^{s}\frac{d_r}{|G|}\chi_r(bx^{-1})\right)x$$

For $f \in L^2(G)$, let M_f be the operator

$$L^2(G) \to L^2(G) : h \mapsto M_f h$$

which corresponds to multiplication in $\mathbb{C}[G]$. Let $L_1, ..., L_r$ be a maximal set of non-isomorphic simple left ideals in $\mathbb{C}[G]$. Let M_f^r be the restriction of M_f to $L_r \subset \mathbb{C}[G] = L^2(G)$. Then

$$\begin{aligned} f(x) &= \sum_{r=1}^{s} \frac{d_r}{|G|} \mathrm{Tr}\left(M_f^r \rho_r(x^{-1})\right) &(6.55)\\ &= \sum_{r=1}^{s} \frac{d_r}{|G|} \sum_{1 \le i,j \le d_r} (M_f^r)_i^j \rho_r(x^{-1})_j^i &(6.56)\end{aligned}$$

Let us note the following result which is useful in proving irreducibility sometimes:

Proposition 6.6.1 *A character χ is irreducible if and only if $\|\chi\|_{L^2} = 1$.*

Proof. Suppose χ decomposes as

$$\chi = \sum_{i=1}^{s} n_i \chi_i,$$

where $\chi_1, ..., \chi_s$ are the irreducible characters. Then

$$\|\chi\|_{L^2(G)}^2 = \sum_{i=1}^{s} n_i^2,$$

and so the norm of χ is 1 if and only if all n_i are zero except for one which equals 1. $\boxed{\text{QED}}$

Here is an immediate application of these considerations. Consider the product group G^n. Let χ_i be the character of the irreducible representation E_i. The tensor products of the character functions produce the functions

$$\chi_{i_1} \otimes \cdots \otimes \chi_{i_n} : G^n \to \mathbb{C} : (x_1, ..., x_n) \mapsto \chi_{i_1}(x_1) \ldots \chi_{i_n}(x_n)$$

which are characters of the representations of G^n on $E_{i_1} \otimes \cdots \otimes E_{i_n}$, they orthonormal in $L^2(G^n)$, and s^n in number. Note that s^n is the number of conjugacy classes in G^n. Thus, $E_{i_1} \otimes \cdots \otimes E_{i_n}$ run over all the irreducible representations of G^n.

6.7 Orthogonality Revisited

As usual, $\chi_1, ..., \chi_s$ denote all the distinct complex irreducible characters of G.

Recall the orthogonality relations:

$$\sum_{g\in G} \chi_i(g)\chi_j(g^{-1}) = \delta_{ij}|G| \tag{6.57}$$

Since a character is constant on each conjugacy class, we can rewrite this as:

$$\sum_{C\in\mathcal{C}} |C|\chi_i(C)\overline{\chi_j(C)} = \sum_{C\in\mathcal{C}} |C|\chi_i(C)\chi_j(C^{-1}) = \delta_{ij}|G| \tag{6.58}$$

where

$$\mathcal{C} = \text{the set of all conjugacy classes in } G, \tag{6.59}$$

and $\chi(C)$ denotes the constant value of χ on the conjugacy class C.

The values of the irreducible characters form an $s \times s$ matrix

$$[\chi_r(C)]_{r\in\mathcal{R},C\in\mathcal{C}}$$

where now we use the convenient notation

$$\mathcal{R} = \text{the set of all irreducible representations of } G. \tag{6.60}$$

The entry

$$\chi_r(C)$$

is viewed as lying on 'row' r and column C. Note that we do have a square matrix, since

$$|\mathcal{C}| = |\mathcal{R}| = s.$$

The matrix has rows which are orthogonal, when each column C is weighted with $(|C|/|G|)^{1/2}$. Consequently, the columns of this weighted matrix are also orthogonal, which we may state as:

Theorem 6.7.1 *With notation as established,*

$$\sum_{r\in\mathcal{R}} \chi_r(C_i)\chi_r(C_j^{-1}) = \delta_{ij}\frac{|G|}{|C_i|^{1/2}|C_j|^{1/2}} = \frac{|G|}{|C_i|}\delta_{ij} \tag{6.61}$$

for any conjugacy classes C_i and C_j.

There is another, direct way, to see this identity (from Zagier [10]): let $(g,h) \in G \times G$ and consider the map

$$T_{(g,h)}(a) = gah^{-1} \qquad \text{for } a \in \mathbb{F}[G] \text{ and } g, h \in G.$$

Computing the trace of $T_{(g,h)}$ using the basis of of $\mathbb{F}[G]$ given by the elements of G we see readily that

$$\mathrm{Trace}(T_{(g,h)}) = \begin{cases} 0 & \text{if } g \text{ and } h \text{ are not in the same conjugacy class;} \\ \frac{|G|}{|C|} & \text{if } g \text{ and } h \text{ both belong to the same conjugacy class } C. \end{cases} \tag{6.62}$$

We can also compute the trace of $T_{(g,h)}$ from the decomposition of $\mathbb{F}[G]$ into the two-sided ideals $\mathbb{F}[G]_r$:

$$\mathrm{Trace}(T_{(g,h)}) = \sum_{r \in \mathcal{R}} \mathrm{Tr}\left(T_{(g,h)}|\mathbb{F}[G]_r\right) \tag{6.63}$$

The trace on the right may be worked out by using the isomorphism

$$\rho_r^{\mathrm{reg}} : \mathbb{F}[G]_r \to \mathrm{End}_{\mathbb{F}}(L_r)$$

Thus:

$$\mathrm{Tr}\left(T_{(g,h)}|\mathbb{F}[G]_r\right) = \mathrm{Tr}\left(\rho_r^{\mathrm{reg}} \circ T_{(g,h)}\Big|\mathbb{F}[G]_r \circ \left(\rho_r^{\mathrm{reg}}\right)^{-1}\right)$$

(See (5.9) in this context.) The trace on the right is best computed via the identification

$$\mathrm{End}_{\mathbb{F}}(L_r) \simeq L_r \otimes L_r^*$$

which leads to

$$\begin{aligned} \mathrm{Tr}\left(T_{(g,h)}|\mathbb{F}[G]_r\right) &= \mathrm{Tr}\left(\rho_r^{\mathrm{reg}}(g)\right)\mathrm{Tr}\left(\rho_r^{\mathrm{reg}}(h^{-1})\right) \\ &= \chi_r(g)\chi_r(h^{-1}) \end{aligned}$$

Combining this with (6.63) and (6.62) yields the desired orthogonality relation (6.61).

Exercises

1. Let $u = \sum_{h \in G} u(h)h$ be an idempotent in $A = \mathbb{F}[G]$, and let χ_u be the character of the regular representation of G restricted to Au:

$$\chi_u(x) = \text{Trace of } Au \to Au : y \mapsto xy.$$

(i) Show that, for any $x \in G$,

$$\chi_u(x) = \text{Trace of } A \to A : y \mapsto xyu.$$

(ii) Check that for $x, g \in G$,

$$xgu = \sum_{h \in G} u(g^{-1}x^{-1}h)h$$

(iii) Conclude that:

$$\chi_u(x) = \sum_{g \in G} u(g^{-1}x^{-1}g), \qquad \text{for all } x \in G. \tag{6.64}$$

Equivalently,

$$\sum_{x \in G} \chi_u(x^{-1})x = \sum_{g \in G} gug^{-1} \tag{6.65}$$

(iv) Show that the dimension of the representation on Au is

$$d_u = |G|u(1_G)$$

where 1_G is the unit element in G.

2. Let $y = \sum_{x \in G} y(x)x \in \mathbb{Z}[G]$, and suppose that y^2 is a multiple of y and $y(1_G) = 1$.

   (i) Show that there is a positive integer γ which is a divisor of $|G|$, and for which $\gamma^{-1}y$ is an idempotent.

   (ii) Show that the dimension of the representation space for the idempotent $\gamma^{-1}y$ is a divisor of $|G|$.

   [Sol: Let $A = \mathbb{Q}[G]$, and let A' be a complementary subspace to Ay, i.e. $A = Ay \oplus A'$. Suppose $y^2 = \gamma y$. The trace of $T_y : A \to A : x \mapsto xy$ is, on one hand, $|G|y(1_G) = |G|$, and it is also equal to $0 + \gamma \dim_{\mathbb{Q}}(Ay)$, because T_y maps A' into the complementary space Ay, and on Ay it acts as multiplication by γ.]

3. Suppose a group G is represented irreducibly on a finite-dimensional vector space V over an algebraically closed field $\mathbb{F}$. Let B be a non-zero bilinear form on V, i.e. a bilinear function $V \times V \to k$, which is G-invariant in the sense that $B(gv, gw) = B(v, w)$ for all vectors $v, w \in V$ and $g \in G$. Show that

(i) B is non-degenerate. [Hint: View B as a linear map $V \to V^*$ and use Schur's lemma.]

(ii) if B' is also a G-invariant bilinear form on V then $B' = cB$ for some $c \in B$.

(iii) If G is a finite group, and $k = \mathbb{C}$, then either B or $-B$ is positive-definite, i.e. $B(v, v) > 0$ for all non-zero $v \in V$.

Chapter 7

Some Arithmetic

In this chapter $\mathbb{F}$ will be a field, G a finite group, with $|G| \neq 0$ in $\mathbb{F}$.

7.1 Characters as Algebraic Integers

Recall the unitarity result of Proposition 1.9.1: if ρ is a representation of a finite group G over a field $\mathbb{F}$ in which $|G| \neq 0$ then for each $x \in G$ there is a basis of the representation space relative to which the matrix of $\rho(x)$ is diagonal and the diagonal entries are all $|G|$-th roots of unity. As an immediate consequence we have:

Proposition 7.1.1 *Suppose G is a finite group, $\mathbb{F}$ a field, and $|G| \neq 0$ in $\mathbb{F}$. Assume also that $\mathbb{F}$ contains all $|G|$th-roots of 1. Then, for any representation ρ of G on a finite-dimensional vector space $V_\rho \neq 0$ over $\mathbb{F}$, the character value $\chi_\rho(x)$ is a sum of $|G|$-th roots of 1, for all $x \in G$.*

For much more on arithmetic properties of characters and representations see Serre's book [18].

7.2 Dimension of Irreducible Representations

7.3 Rationality

Chapter 8

Representations of S_n

We denote by S_n the group of permutations on $\{1, ..., n\}$.

The theory of groups arose from the study of symmetry properties of polynomial functions of roots of polynomial equations. This is essentially the study of representations of the symmetric group on spaces of polynomials $P(X_1, ..., X_n)$. Young's paper [20] introduced a method of constructing polynomials with certain symmetry properties using tableaux which we describe below. Frobenius used these tableaux and much more to work out all the irreducible representations of S_n and their characters. A crucial part of the exposition in this chapter is the proof that certain elements of the group algebra $\mathbb{F}[S_n]$, generated from Young tableaux, are primitive idempotents; the proofs here are due to John von Neumann who communicated them to in a letter to Hermann Weyl [19].

We have seen that the irreducible representations of a finite group G over a field $\mathbb{F}$, whose characteristic does not divide $|G|$, correspond to left ideals in $\mathbb{F}[G]$ generated by primitive idempotents. Moreover, if $\mathbb{F}$ is algebraically closed then the number of non-isomorphic irreducible representations of G equals the number of conjugacy classes in G. We will apply this to the case $G = S_n$.

We will work out, for each conjugacy class in S_n, a primitive idempotent in $\mathbb{F}[S_n]$ and show that these generate all the irreducible representations of S_n. In more detail, for each partition of n as

$$n = \lambda_1 + \cdots + \lambda_n,$$

into positive integers λ_i, displayed in the form of a tableau T (as explained later), we will construct a primitive idempotent y_T. The simple left ideals

$\mathbb{F}[S_n]y_T$, formed from all the distinct partitions of n, will be shown to be inequivalent, and their sum is a direct sum. Now we have seen that, for $\mathbb{F}$ algebraically closed, the number of non-isomorphic irreducible representations of G equals the number of conjugacy classes in G. The latter are, as we will see, in one-to-one correspondence with the partitions of n. Thus, every irreducible representation of S_n is isomorphic to exactly one of the simple left ideals $\mathbb{F}[S_n]y_T$.

8.1 Conjugacy Classes and Young Tableaux

Recall that any element in S_n can be expressed in a unique way as a product of disjoint cycles:

$$(a_{11}, ..., a_{1\lambda_1}) \dots (a_{m1}, ..., a_{m\lambda_m})$$

where the a_{ij} are distinct and run over $\{1, ..., n\}$. This permutation thus specifies a *partition*

$$(\lambda_1, ..., \lambda_m)$$

of n into positive integers $\lambda_1, ..., \lambda_m$:

$$\lambda_1 + \cdots + \lambda_m = n.$$

By convention, we require that

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m.$$

Two permutations are conjugate if and only if they have the same cycle structure, i.e. the partition of n is the same for the permutations.

Thus, the conjugacy classes of S_n correspond one to one to partitions of n.

A Young *tableau* is a matrix of the form

$$\begin{array}{cccccc} a_{11} & ... & ... & ... & ... & a_{1\lambda_1} \\ a_{21} & ... & ... & a_{2\lambda_2} & & \\ \vdots & \vdots & \vdots & & & \\ a_{m1} & \ldots & a_{m\lambda_m} & & & \end{array} \tag{8.1}$$

We will take the entries all distinct and drawn from $\{1, ..., n\}$. If the numbers are in their natural order reading 'book style', we call it a *standard* tableau.

Thus, each tableau is associated to a partition of n, and for each partition there is a unique standard tableau.

Thus, the standard tableaux correspond one to one to the conjugacy classes in S_n.

Clearly the group S_n acts on the set of tableaux corresponding to each partition of n.

For a tableau T, Young introduced two subgroups of S_n: those which preserve each row and those which preserve each column. Let

$$R_T = \text{the subgroup of all } p \in S_n \text{ which preserve each row of } T \tag{8.2}$$

$$C_T = \text{the subgroup of all } q \in S_n \text{ which preserve each column of } T \tag{8.3}$$

Young's symmetrizer for the tableau is the element

$$y_T \stackrel{\text{def}}{=} c_T r_T = \sum_{q \in C_T, p \in R_T} (-1)^q qp \in \mathbb{Z}[S_n], \tag{8.4}$$

where

$$c_T = \sum_{q \in C_T} (-1)^q q \tag{8.5}$$

$$r_T = \sum_{p \in R_T} p \tag{8.6}$$

We have used the notation

$$(-1)^q = \operatorname{sgn}(q).$$

Observe that

$$R_T \cap C_T = \{\text{identity permutation}\}$$

Consequently, each element in the set

$$C_T R_T = \{qp : q \in C_T, p \in R_T\}$$

can be expressed in the form qp for a unique $q \in C_T$ and a unique $p \in R_T$.

8.2 Construction of Irreducible Representations of S_n

If $\mathbb{F}$ is any field, there is the natural ring homomorphism

$$\mathbb{Z} \to \mathbb{F} : m \mapsto m_{\mathbb{F}} \stackrel{\text{def}}{=} m1_{\mathbb{F}},$$

which is injective if $\mathbb{F}$ has characteristic 0, and which induces an injection of $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ onto the image of $\mathbb{Z}$ in $\mathbb{F}$ if the characteristic of $\mathbb{F}$ is $p \neq 0$. To avoid too much notational distraction, we shall often sacrifice precision and denote $m1_{\mathbb{F}}$ as simply m instead of $m_{\mathbb{F}}$, bearing in mind that this might be the 0 element in $\mathbb{F}$. An element of $\mathbb{F}$ of the form $m1_{\mathbb{F}}$, with $m \in \mathbb{Z}$, will simply be called an *integer* in $\mathbb{F}$.

Passing to the group algebras, there is naturally induced a ring homomorphism

$$\mathbb{Z}[S_n] \to \mathbb{F}[S_n] : a \mapsto a_k,$$

for any $n \in \{1, 2, ...\}$. Again, this homomorphism is an injection, with image also denoted $\mathbb{Z}[S_n]$, if $\mathbb{F}$ has characteristic 0; on the other hand, if $\mathbb{F}$ has characteristic $p \neq 0$ then there is induced an injective ring homomorphism $\mathbb{Z}_p[S_n] \to \mathbb{F}[S_n]$, and in this case we denote the image of $\mathbb{Z}[S_n]$ in $\mathbb{F}[S_n]$ by $\mathbb{Z}_p[S_n]$. Again, we will often simply write a instead of a_k. For instance, the image of the Young symmetrizer $y_T \in \mathbb{Z}[S_n]$ in $\mathbb{F}[S_n]$ is denoted simply by y_T in the statement of the following result.

Theorem 8.2.1 *Let $n \in \{2, 3, ...\}$ and $\mathbb{F}$ a field in which $n! \neq 0$. Let T be a Young tableau for n. Then there is a positive integer γ_T, dividing $n!$, such that the element $e_T = \frac{1}{\gamma_T} y_T$ is a primitive idempotent in $\mathbb{F}[S_n]$. The corresponding representation space $\mathbb{F}[S_n]y_T$ has dimension $d_{k,T}$ which satisfies*

$$\gamma_T d_{k,T} = n! \qquad \text{in } \mathbb{F}. \tag{8.7}$$

There are elements $v_1, ..., v_{d_{k,T}} \in \mathbb{Z}[S_n]y_T$ whose images in $\mathbb{F}[S_n]$ form a $\mathbb{F}$-basis of $\mathbb{F}[S_n]y_T$. If $\mathbb{F}$ has characteristic 0 then

$$d_T = d_{k,T} = \frac{n!}{\gamma_T} \tag{8.8}$$

does not depend on the field $\mathbb{F}$, and the elements $v_1, ..., v_{d_T}$ form a $\mathbb{Q}$-basis of $\mathbb{Q}[S_n]y_T$.

Proof. The primitivity criterion in Theorem 4.5.1(i) will be our key tool. Fix $g \in S_n$, and let

$$z = ygy. \tag{8.9}$$

Our first objective is to prove that z is an integer multiple of y.

Observe that

$$\begin{aligned} qy &= (-1)^q y, \qquad \text{for all } q \in C_T & (8.10)\\ yp &= y \quad \text{for all } p \in R_T. & (8.11) \end{aligned}$$

Consequently,

$$qzp = (-1)^q z \quad \text{for all } p \in R_T \text{ and } q \in C_T. \tag{8.12}$$

Note that, of course, this holds also for y:

$$qyp = (-1)^q y \quad \text{for all } p \in R_T \text{ and } q \in C_T. \tag{8.13}$$

In fact we will now show that the property (8.12) forces z to be an integer multiple of y.

Writing z as

$$z = \sum_{s \in S_n} z(s)s,$$

(note that each $z(s)$ is an integer) we see then that, for $q \in C_T$ and $p \in P_T$,

$$z(qp) = \text{coeff of 1 in } q^{-1}zp^{-1} = (-1)^q z(1)$$

and so

$$z = z(1)y + \sum_{s \notin C_T R_T} z(s)s. \tag{8.14}$$

Next we show that the second term on the right is 0. Here we shall use a crucial fact (proved below in Proposition 8.3.2) about Young tableaux which makes the whole argument work:

If $s \notin C_T R_T$ then there are transpositions $\sigma \in R_T$ and $\tau \in C_T$ such that

$$\tau s \sigma = s, \tag{8.15}$$

Then

$$\begin{aligned} \tau z \sigma &= -z \quad \text{by (8.12), but also}\\ \tau z \sigma &= -z(1)y + \sum_{s \notin C_T R_T} z(s)s, \quad \text{by (8.14) and (8.15).} \end{aligned}$$

It follows that

$$\sum_{s \notin C_T R_T} z(s)s = 0$$

and so

$$z = z(1)y, \tag{8.16}$$

i.e. ygy is an integer multiple of y.

Consequently, yxy is a $\mathbb{F}$-multiple of y for every $x \in \mathbb{F}[S_n]$.

As a special case, on taking g to be the identity element in $z = ygy$, i.e. with $z = y^2$, we have

$$yy = \gamma y, \tag{8.17}$$

where

$$\gamma = (y^2)(1) \tag{8.18}$$

is the coefficient in $y^2 \in \mathbb{Z}[S_n]$ of the identity element in S_n. In particular, the multiplier γ is an integer.

If $\gamma \neq 0$ in $\mathbb{F}$, then

$$e = \gamma^{-1}y$$

is clearly an idempotent in $\mathbb{F}[S_n]$. We will show shortly that γ is a positive integer and is indeed not 0 in the field $\mathbb{F}$. Then e is an idempotent in $\mathbb{F}[S_n]$ and, moreover, exe is a $\mathbb{F}$-multiple of e for all $x \in \mathbb{F}[S_n]$, and hence by the primitivity criterion in Theorem 4.5.1(i), e is a primitive idempotent.

Consider the right multiplication map

$$T_y : \mathbb{F}[S_n] \to \mathbb{F}[S_n] : a \mapsto ay$$

This is $\mathbb{F}$-linear, on the subspace Ay it equals multiplication by the constant γ and maps any complementary subspace into $\mathbb{F}[S_n]y$, and so has trace equal to $\gamma \dim_k\bigl(k[S_n]y\bigr)$. On the other hand, in terms of the standard basis of $k[S_n]$ given by the elements of S_n, the trace of T_y is

$$\mathrm{Trace}(T_y) = n!y(1) = n!,$$

since, from the definition of y it is clear that

$$y(1) = 1.$$

Thus,

$$\gamma \dim_{\mathbb{F}}\bigl(\mathbb{F}[S_n]y\bigr) = n! \tag{8.19}$$

in $\mathbb{F}$. By assumption, $n! \neq 0$ in $\mathbb{F}$, and so $\gamma \neq 0$ in $\mathbb{F}$. For the special case $k = \mathbb{Q}$, the relation (8.19) shows that γ is a positive integer divisor of $n!$.

Since the $\mathbb{F}$-linear span of $\mathbb{Z}[S_n]y_T$ is $\mathbb{F}[S_n]y_T$, there is a $\mathbb{F}$-basis $v_1, ..., v_{d_{k,T}}$ of $\mathbb{F}[S_n]y_T$ inside $\mathbb{Z}[S_n]y_T$. Let us check that $v_1, ..., v_{d_{k,T}} \in \mathbb{Z}[S_n]y_T$ are linearly independent over $\mathbb{Q}$. For notational simplicity let us write d for $d_{k,T}$. If a non-trivial rational linear combination of the v_j's is 0 then, by clearing denominators and common factors, there are integers $a_1, ..., a_d \in \mathbb{Z}$, not all zero, with greatest common divisor 1, such that

$$a_1v_1 + \cdots + a_dv_d = 0. \tag{8.20}$$

If $\mathbb{F}$ has characteristic 0 then $\mathbb{F}$ effectively contains $\mathbb{Z}$ and so the $\mathbb{F}$-linear independence of the v_j rules out (8.20); if $\mathbb{F}$ has finite characteristic p then $k \supset \mathbb{Z}/p\mathbb{Z}$, and then reducing (8.20) mod p and bearing in mind that p is not a factor of some a_j, the relationship (8.20) is again impossible by linear independence over $\mathbb{F}$. Hence, $v_1, ..., v_d \in \mathbb{Z}[S_n]y_T$ are linearly independent over $\mathbb{Q}$. $\boxed{\text{QED}}$

From the preceding result we can draw a remarkable conclusion: for any Young tableau T, the representation ρ_T of S_n in characteristic 0, there is a basis in the representation space relative to which the matrix for $\rho_T(x)$ has rational entries for all $x \in S_n$. Indeed, the following result gives a more refined formulation of this observation.

Theorem 8.2.2 *Let $\rho_T : S_n \to \mathrm{End}_{\mathbb{F}}(E_T)$ be the irreducible representation of S_n on a vector space E_T over a field $\mathbb{F}$ of characteristic 0, where $n \in \{2, 3, ...\}$, associated to a Young tableau T. Then there is a basis of E relative to which for each $x \in S_n$ the matrix of $\rho_T(x)$ has entries of the form r/s where $r, s \in \mathbb{Z}$ and s is coprime to $1, 2, ..., n$.*

Proof We will use notation and observations from the proof of Theorem 8.2.1. Recall, for instance, that there are elements $v_1, ..., v_d \in \mathbb{Z}[S_n]y_T$ which form a $\mathbb{F}$-basis of $\mathbb{F}[S_n]y_T$ and a $\mathbb{Q}$-basis of $\mathbb{Q}[S_n]y_T$.

Left-multiplication by $x \in S_n$ on $\mathbb{Q}[S_n]y_T$ has matrix, relative to the basis $v_1, ..., v_d$, with rational entries:

$$xv_j = \rho_T(x)v_j = \sum_{m=1}^{d} \rho_T(x)_{jm}v_m, \tag{8.21}$$

where $\rho_T(x)_{jm}$ is rational for each $j, m \in \{1, ..., d\}$. Since the characteristic of $\mathbb{F}$ is 0, $\mathbb{Q}$ is contained inside $\mathbb{F}$, and so this establishes the claim that the matrix of $\rho_T(x)$, relative to the basis $v_1, ..., v_d$ of $\mathbb{F}[S_n]y_T$, has rational entries. For fixed $x \in S_n$, let us write

$$\rho_T(x)_{jm} = \frac{r_{jm}}{s_{jm}} \tag{8.22}$$

where $r_{jm}, s_{jm} \in \mathbb{Z}$ are coprime integers (in particular, we take $s_{jm} = 1$ if $r_{jm} = 0$). Let p be a prime divisor of $n!$. We will show that none of the s_{jm} is divisible by p. Let a be the largest integer p^a is a divisor of s_{jm} for some $m \in \{1, ..., d\}$, and suppose $a \geq 1$; then multiplying both sides in (8.21) by p^a produces, in the field $\mathbb{F}$, the relation

$$0 = p^a x v_j = \sum_{m=1}^{d} \frac{p^a r_{jm}}{s_{jm}} v_m \tag{8.23}$$

where, on the right, at least one of the coefficients is not 0 in $\mathbb{F}$. But this contradicts the linear independence of $v_1, ..., v_d$ over the field $\mathbb{F}$. Hence, the matrix entries $\rho_T(x)_{jm}$ can be expressed as $\frac{r_{jm}}{s_{jm}}$, with r_{jm}, s_{jm} coprime and s_{jm} not divisible by p. $\boxed{\text{QED}}$

IS THERE CANONICAL/CONVENIENT BASIS FOR $\mathbb{Q}[S_n]y_T$?

WHAT IS THE STRUCTURE OF THE $\mathbb{Z}$-module $\mathbb{Z}[S_n]y_T$?

8.3 Some properties of Young tableaux

In this section we will prove the combinatorial fact used in establishing that Young symmetrizers are primitive idempotents. We will also prove a result that will lead to the fact that the Young symmetrizers for different partitions of n provide inequivalent irreducible representations of S_n.

Consider partitions λ and λ' of n. If $\lambda' \neq \lambda$ then there is a smallest j for which $\lambda'_j \neq \lambda_j$. If, for this j, $\lambda'_j > \lambda_j$ then we say that $\lambda' > \lambda$ in *lexicographic* order. This is an order relation on the partitions of n. The largest element is

$$(n)$$

and the smallest element is $(1, 1, ..., 1)$.

Note also that permutations act on tableaux. If $g \in S_n$, and T is a tableau, with entries T_{jk}, then gT is a tableaux whose jk entry is $g(T_{jk})$.

Proposition 8.3.1 *Let T and T' be Young tableaux, and λ and λ' the corresponding partitions of n. If $\lambda' > \lambda$ in the lexicographic order, then there are two entries in the same row of T' which are in the same column of T. Consequently, there exists a transposition σ lying in $R_{T'} \cap C_T$.*

Proof. If $\lambda'_1 > \lambda_1$ then there must exist two entries in the first row of T' which lie in the same column of T. If $\lambda'_1 = \lambda_1$, and all elements of the first row of T' lie in different columns of T, we can move these elements 'vertically' in T all to the first row, obtaining a tableau T_1 whose first row is a permutation of the first row of T'. Note that $T_1 = q_1 T$, for some $q_1 \in C_T$. Next we compare the second row of T' with that of T_1. Again, if the rows are of equal length then there is a vertical move in T_1 (which is therefore also a vertical move in T, because $C_{q_1 T} = C_T$) which produces a tableau $T_2 = q_2 q_1 T$, with $q_2 \in C_T$, whose first row is the same as that of T_1, and whose second row is a permutation of the second row of T'. Proceeding this way, we reach the first j for which the j-th row of T' has more elements that the j-th row of T. Then each of the first $j-1$ rows of T' is a permutation of the corresponding row of T_{j-1}; focusing on the tableaux made up of the remaining rows, we see that there are two elements in the j-th row of T' which lie in the same column in T_{j-1}. Since the columns of T_{j-1} are, as sets, identical to those of T, we are done. $\boxed{\text{QED}}$

Now we turn to rearrangement arguments for tableaux associated to the same partition.

Proposition 8.3.2 *Let T and T' be Young tableaux associated to a common partition λ. Let s be the element of S_n for which $T' = sT$. Then $s \notin C_T R_T$ if and only if there are two elements which are in the same row of T' and also in the column of T. Thus, $s \notin C_T R_T$ if and only if there is a transposition $\sigma \in R_T$ and a transposition $\tau \in C_T$, for which*

$$\tau s \sigma = s. \tag{8.24}$$

Proof. Suppose that $s = qp$, with $q \in C_T$ and $p \in R_T$. Consider two elements $s(i)$ and $s(j)$, with $i \neq j$, lying in the same row of T':

$$T'_{ab} = s(i), \quad T'_{ac} = s(j).$$

Thus, i, j lie in the same row of T:

$$T_{ab} = i, \quad T_{ac} = j.$$

The images $p(i)$ and $p(j)$ are also from the same row of T (hence different columns) and then $qp(i)$ and $qp(j)$ would be in different columns of T. Thus the entries $s(i)$ and $s(j)$, lying in the same row in T', lie in different columns of T.

Conversely, suppose that if two elements lie in the same row of T' then they lie in different columns of T. We will show that the permutation $s \in S_n$ for which $T' = sT$ has to be in $C_T R_T$. Bear in mind that the sequence of row lengths (i.e. the partition of n) for T' is the same as for T. Consider the elements of the first row of T'. They are distributed over distinct columns of T. Therefore, by moving these elements 'vertically' we can bring them all to the first row. This means that there is an element $q_1 \in C_T$ such that $T_1 = q_1 T$ and T' have the same *set* of elements for their first rows. Next, the elements of the second row of T' are distributed over distinct columns in T, and hence also in $T_1 = q_1 T$. Hence there is a vertical move

$$q_2 \in C_{q_1 T} = C_T,$$

for which $T_2 = q_2 T_1$ and T' have the same set of first row elements and also the same set of second row elements.

Proceeding in this way, we obtain a $q \in C_T$ such that each row of T' is equal, as a *set*, to the corresponding row of qT:

$$\{T'_{ab} : 1 \leq b \leq \lambda_a\} = \{q(T_{ab}) : 1 \leq b \leq \lambda_a\}, \qquad \text{for each } a.$$

But then we can permute horizontally, i.e. permute, for each fixed a, the numbers T_{ab} so that the $q(T_{ab})$ match the T_{ab}. Thus, there is a $p \in R_T$, such that

$$T' = qp(T).$$

Thus,

$$s = qp \in C_T R_T.$$

Finally, suppose $s \notin C_T R_T$. Then there is a row a, and two entries $i = T_{ab}$ and $j = T_{ac}$, whose images $s(i)$ and $s(j)$ lie in a common column of T. Let $\sigma = (i, j)$ and $\tau = \big(s(i), s(j)\big)$. Then $\sigma \in R_T$, $\tau \in C_T$, and

$$\tau s \sigma = s,$$

which is readily checked on i and j.

Conversely, suppose $\tau s \sigma = s$, where $\sigma = (i\, j) \in R_T$. Then i and j are in the same row of T, and so $s(i)$ and $s(j)$ are in the same row in T'. Now

$s(i) = \tau(s(j))$ and $s(j) = \tau(s(i))$. Since $\tau \in C_T$ it follows that $s(i)$ and $s(j)$ are in the same column of T. $\boxed{\text{QED}}$

A Young tableau is said to be a *standard* tableau if the entries in each row are in increasing order (left to right) and the numbers in each column are also in increasing order (top to bottom). For example:

$$\begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{6} \\ \boxed{3} & \boxed{4} & \\ \boxed{5} & & \end{array}$$

Such a tableau must, of necessity, start with 1 at the top left box, and each new row begins with the smallest number not already listed in any of the preceding rows.

It is useful to note that all numbers lying directly 'south', directly 'east', and southeast of a given entry are larger than this entry, and those to the north, west, and northwest are lower.

In general, the boxes of a tableau are ordered in 'book order', i.e. we read the boxes left to right along a row an then move down to the next row.

The standard tableaux, for a given partition, can be linearly ordered: if T and T' are standard tableaux, we declare that

$$T < T'$$

if the first entry T_{ab} of T which is different from the corresponding entry T'_{ab} of T' satisfies $T_{ab} < T'_{ab}$. The tableaux for a given partition can then be written in increasing order.

With this ordering we have:

Proposition 8.3.3 *If T and T' are tableaux with a common partition, and $T < T'$, then there are two entries in some row of T' which lie in one column of T. Consequently, there exists a transposition σ lying in $R_T \cap C_{T'}$.*

Proof. Let $x = T_{ab}$ be the first entry of T which is less than the corresponding entry $y = T'_{ab}$. The entry x appears somewhere in the tableau T'. Because ab is the *first* location where T differs from T', and $T_{ab} = x$, we see that x cannot appear prior to the location T'_{ab}. But x being $< y = T'_{ab}$, it can also not appear directly south, east, or southeast of T'_{ab}. Thus, x must appear in T' in a row below the a-th row and in a column $c < b$. Thus, the numbers T_{ac} (which equals T'_{ac}) and $T_{ab} = x$, appearing in the a-th row of T, appear in the c-th column of T'. $\boxed{\text{QED}}$

8.4 Orthogonality of Young symmetrizers

Consider Young tableaux T and T', associated with distinct partitions λ and λ' of n. We will show that the corresponding irreducible representations are not isomorphic.

Theorem 8.4.1 *If T and T' are Young tableaux associated to different partitions λ and λ', respectively, then*

$$y_{T'}y_T = 0 \qquad \textit{if } \lambda' > \lambda \textit{ in lexicographic order.} \tag{8.25}$$

If $T_1, ..., T_m$ are Young tableaux associated to distinct partitions, then the sum $\sum_{j=1}^m \mathbb{F}[S_n]y_{T_j}$ is a direct sum, if the characteristic of $\mathbb{F}$ does not divide $n!$.

Proof. Suppose that the partition associated to T' is greater, in lexicographic order, than the one associated to T. Then, by Proposition 8.3.1, there is a transposition $\sigma \in R_{T'} \cap C_T$. Then

$$y_{T'}y_T = y_{T'}\sigma\sigma y_T = (y_{T'})(-y_T) = -y_{T'}y_T$$

Thus, $y_{T'}y_T$ is 0.

Order the T_j, so that their associated partitions are in decreasing order lexicographically. Suppose $\sum_{j=1}^m \mathbb{F}[S_n]y_{T_j}$ is not a direct sum. Let r be the smallest element of $\{1, ..., n\}$ for which there exist $x_j \in \mathbb{F}[S_n]y_{T_j}$, for $j \in \{1, ..., r\}$, with $x_r \neq 0$, are such that

$$\sum_{j=1}^r x_j = 0.$$

Multiplying on the right by y_{T_r}, produces

$$\gamma_{T_r} x_r = 0$$

Now γ_{T_r} is a divisor of $n!$, and so the characteristic of $\mathbb{F}$ does not divide γ_{T_r}, and so

$$x_r = 0.$$

This contradiction proves that $\sum_{j=1}^m \mathbb{F}[S_n]y_{T_j}$ is a direct sum. $\boxed{\text{QED}}$

Moreover, we have

Theorem 8.4.2 *If T and T' are Young tableaux corresponding to different partitions of n, then the irreducible representations $\mathbb{F}[S_n]y_T$ and $\mathbb{F}[S_n]y_{T'}$ are inequivalent.*

Proof. Recall the relationship (6.65) between an idempotent u and the character χ_u of the representation $\mathbb{F}[S_n]u$:

$$\sum_{x\in S_n}\chi_u(x^{-1})x=\sum_{s\in S_n}sus^{-1} \tag{8.26}$$

We will use this to show that χ_{y_T} cannot be equal to $\chi_{y_{T'}}$.

The row and column Young subgroups behave as follows under the action of S_n on tableaux:

$$R_{sT}=sR_Ts^{-1}, \qquad \text{and} \quad C_{sT}=sC_Ts^{-1}. \tag{8.27}$$

Consequently,

$$y_{sT}=sy_Ts^{-1}. \tag{8.28}$$

The primitive idempotents corresponding to y_T and y_{sT} are obtained by scaling by a common term γ which is a divisor of $n!=|G|$, where $G=S_n$. Therefore,

$$\frac{1}{|G|}\sum_{x\in S_n}\chi_{y_T}(x^{-1})x=\frac{1}{|G|\gamma}\sum_{s\in S_n}y_{sT} \tag{8.29}$$

and

$$\frac{1}{|G|}\sum_{w\in S_n}\chi_{y_{T'}}(w^{-1})w=\frac{1}{|G|\gamma}\sum_{t\in S_n}y_{tT'} \tag{8.30}$$

(Note that since each y_T is in $\mathbb{Z}[S_n]$, this implies that *every character value $\chi_i(x)$ is rational*!) Multiplying, we obtain:

$$\frac{1}{|G|}\sum_{x\in S_n}\chi_{y_{T'}}*\chi_{y_T}(x^{-1})x=\frac{1}{|G|^2}\sum_{s,t\in S_n}y_{tT'}y_{sT} \tag{8.31}$$

We may assume that the partition of n corresponding to T' is greater, lexicographically, than the partition for T. Then the same is true for tT' and sT, for any $t,s\in S_n$, and so each term in the sum on the right side of (8.31) is 0. Consequently,

$$\frac{1}{|G|}\sum_{x\in S_n}\chi_{y_{T'}}*\chi_{y_T}(x^{-1})x=0. \tag{8.32}$$

Now if the representations for y_T and $y_{T'}$ were equivalent then

$$\chi_{y_{T'}} = \chi_{y_T}$$

and then

$$\chi_{y_{T'}} * \chi_{y_T} = \frac{1}{\tilde{d}_T}\chi_{y_T},$$

where $\tilde{d}_T$ is the multiplicity of $\mathbb{F}[S_n]y_T$ in the regular representation. Consequently, we would have

$$\chi_{y_T} = 0$$

But, evaluating the character at the identity, this would imply:

$$n! = 0 \quad \text{in } \mathbb{F},$$

which, by the hypothesis on the characteristic of $\mathbb{F}$, is false. This proves that y_T and $y_{T'}$ generate inequivalent irreducible representations. $\boxed{\text{QED}}$

As a corollary of the proof, we have the observation:

Proposition 8.4.1 *The characters of S_n (for any field of characteristic not dividing $n!$) take integer values.*

Proof. We have seen that each character is a (real) rational number; in fact, every non-zero representation of S_n, for fields with characteristic not dividing $n!$, is realizable by matrices with rational entries. But we also know that characters are sums of roots of unity and so are algebraic integers. Therefore, each character is actually an integer. $\boxed{\text{QED}}$

Now turning to tableaux for a fixed partition, we have the following result whose proof is virtually identical to that of Theorem 8.4.1 (but uses Proposition 8.3.3):

Theorem 8.4.3 *If T and T' are standard Young tableaux associated to a common partition, then*

$$y_T y_{T'} = 0 \qquad \text{if } T < T'. \tag{8.33}$$

If $T_1, ..., T_m$ are all the standard Young tableaux associated to a common partition, then the sum $\sum_{j=1}^{m} \mathbb{F}[S_n]y_{T_j}$ is a direct sum, if the characteristic of $\mathbb{F}$ does not divide $n!$.

We do not prove this here, but the two-sided ideal containing the minimal left ideal $\mathbb{F}[S_n]y_{T_1}$ is the direct sum $\sum_{j=1}^{m} \mathbb{F}[S_n]y_{T_j}$, with notation as in the preceding theorem.

Chapter 9

Commutants

The commutant C of a set S of operators is the set of all operators which commute with all operators in S. The double commutant of S is the commutant of C. The relationship between the original collection S and its double commutant is the subject of several results in algebra and analysis. In this chapter we will recall, in the light of commutants, results we have proven before.

9.1 The Commutant

Consider a module E over a ring A (with unit element 1). An endomorphism

$$f \in \mathrm{End}_A(E)$$

is, by definition, a map $f : E \to E$ which is additive

$$f(u+v) = f(u) + f(v) \qquad \text{for all } u, v \in E, \tag{9.1}$$

and *commutes* with the action of A:

$$f(au) = af(u) \qquad \text{for all } a \in A, \text{ and } u \in E. \tag{9.2}$$

The case of most interest to us is $A = \mathbb{F}[G]$, where G is a finite group and $\mathbb{F}$ a field, and E is a finite dimensional vector space over $\mathbb{F}$, with a given representation of G on E. In this case, the conditions (9.1) and (9.2) are equivalent to $f \in \mathrm{End}_{\mathbb{F}}(E)$ commuting with all the elements of G represented on E. Thus, $\mathrm{End}_{\mathbb{F}[G]}(E)$ is the commutant for the representation of G on E.

Sometimes the notation

$$\mathrm{End}_G(E)$$

is used instead of $\mathrm{End}_{\mathbb{F}[G]}(E)$. To be more technically precise, we don't need the field $\mathbb{F}$ to define $\mathrm{End}_G(E)$. If E is a module over a ring R (say the integers or, at the other extreme, a field $\mathbb{F}$) then $\mathrm{End}_G(E)$ is the set of all elements of $\mathrm{End}_R(E)$ which commute with the action of G.

Let us recall a consequence of Schur's Lemma 4.1.2:

Proposition 9.1.1 *Let G be a finite group represented on a finite dimensional vector space E over an algebraically closed field $\mathbb{F}$. Then the commutant of this representation consists of multiples of the identity operator on E if and only if the representation is irreducible.*

Suppose now that, with notation as above, the finite-dimensional representation E is reducible:

$$E = E_1^{n_1} \oplus \ldots \oplus E_r^{n_r} \tag{9.3}$$

where each E_i is irreducible, each $n_i \in \{1, 2, 3, ...\}$, and $E_i \not\simeq E_j$ as G-representations when $i \neq j$. By Schur's lemma, the only G-linear map $E_i \to E_j$, for $i \neq j$, is 0. Consequently, any element in the commutant $\mathrm{End}_G(E)$ can be displayed as a block-diagonal matrix

$$\begin{pmatrix} C_1 & 0 & 0 & \ldots & 0 \\ 0 & C_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & 0 \\ 0 & 0 & 0 & \ldots & C_r \end{pmatrix} \tag{9.4}$$

where each C_i is in $\mathrm{End}_G(E_i^{n_i})$. Moreover, any element of

$$\mathrm{End}_G(E_i^{n_i})$$

is itself an $n_i \times n_i$ matrix, with entries from

$$D_i = \mathrm{End}_G(E_i),$$

which, by Schur's lemma, is a division ring (equal to the field $\mathbb{F}$ if the latter is algebraically closed). Conversely, any such matrix clearly specifies an element of $\mathrm{End}_G(E_i^{n_i})$.

Thus, we have completely analyzed the structure of the commutant algebra $\mathrm{End}_G(E)$. It is the algebra of block diagonal matrices (9.4), where each C_i is any arbitrary $n_i \times n_i$ matrix with entries from D_i:

$$\mathrm{End}_G(E) \simeq \prod_{i=1}^{r} \mathrm{Matr}_{n_i \times n_i}(D_i) \tag{9.5}$$

Indeed, this is a special case of Theorem 4.2.3, because $\mathrm{End}_G(E)$ is the commutant of the image of the semisimple algebra $\mathbb{F}[G]$ in $\mathrm{End}_{\mathbb{F}}(E)$ and so is also semisimple.

9.2 The Double Commutant

Recall that a ring B is *simple* if any two simple left ideals in B are isomorphic as left B-modules. In this case B is the internal direct sum of a finite number of simple left ideals, all isomorphic to each other.

Consider a left ideal L in a simple ring B, viewed as a B-module. The commutant of the action of B on L is the ring

$$C = \mathrm{End}_B(L).$$

The double commutant is

$$D = \mathrm{End}_C(L).$$

Every element $b \in B$ gives a multiplication map

$$l(b) : L \to L : a \mapsto ba,$$

which, of course, commutes with every $f \in \mathrm{End}_B(L)$. Thus, each $l(b)$ is in $\mathrm{End}_C(L)$. We can now recall Rieffel's Theorem 4.3.3 in this language:

Theorem 9.2.1 *Let B be a simple ring, L a non-zero left ideal in B, and*

$$C = \mathrm{End}_B(L), \qquad D = \mathrm{End}_C(L),$$

the relevant commutant and double commutant. Then the double commutant D is essentially the original ring B, in the sense that the natural map $l : B \to D$, specified by

$$l(b) : L \to L : a \mapsto ba, \qquad \textit{for all } a \in L \textit{ and } b \in B,$$

is an isomorphism.

Let E be a semisimple module over a ring A, and C the commutant $\mathrm{End}_A(E)$. Then A is mapped into the double commutant $D = \mathrm{End}_C(E)$ by the map

$$l : A \to D : a \mapsto l(a), \quad \text{where } l(a) : E \to E : x \mapsto ax.$$

The *Jacobson density theorem* explains how big $l(A)$ is inside D:

Theorem 9.2.2 *Let E be a semisimple module over a ring A, and let C be the commutant $\mathrm{End}_A(E)$. Then for any $f \in D = \mathrm{End}_C(E)$, and any $x_1, ..., x_n \in E$, there exists an $a \in A$ such that*

$$f(x_i) = ax_i, \qquad \text{for } i = 1, ..., n.$$

In particular, if A is an algebra over a field $\mathbb{F}$, and E is finite dimensional as a vector space over $\mathbb{F}$, then $D = l(A)$, i.e. every element of D is given by multiplication by an element of A.

Proof. Let $E' = E^n$. Then any element of

$$C' \stackrel{\text{def}}{=} \mathrm{End}_A(E')$$

is given by an $n \times n$ matrix with entries in C. Note that E' is a module over the ring C'. The map

$$f' : E' \to E' : (y_1, ..., y_n) \mapsto \big(f(y_1), ..., f(y_n)\big).$$

is C'-linear, i.e.

$$f' \in \mathrm{End}_{C'}(E').$$

Now E', being semisimple, can be split as

$$E' = Ax \bigoplus F,$$

where $x = (x_1, ..., x_n)$ is any element of E', and F is an A-submodule of E'. Let

$$p : E' \to Ax \subset E'$$

be the corresponding projection. This is, of course, A-linear and can be viewed as an element of C'. Consequently, $f'p = pf'$, and so

$$f'\big(p(x)\big) = p\big(f'(x)\big) \in Ax.$$

Since $p(x) = x$, this implies the desired result. $\boxed{\text{QED}}$

Exercises

1. Suppose E is a right-module over a semisimple ring A. Then $\mathrm{Hom}_A(E, A)$ is a left A-module in the natural way via the left-multiplication in A. Show that the map

$$E \to \mathrm{Hom}_A\left(\mathrm{Hom}_A(E, A), A\right) : x \mapsto \hat{x}$$

where $\hat{x}(f) = f(x)$ for all $f \in \mathrm{Hom}_A(E, A)$, is injective.

2. Prove Burnside's theorem: If G is a group of endomorphisms of a finite dimensional vector space E over a field $\mathbb{F}$, and E is simple as a G-module, then $\mathbb{F}[G]$, the linear span of G inside $\mathrm{End}_{\mathbb{F}}(E)$, is equal to the whole of $\mathrm{End}_{\mathbb{F}}(E)$.

3. Prove Wedderburn's theorem: Let E be a simple module over a ring A, and suppose that it is faithful in the sense that if a is non-zero in A then the map $l(a) : E \to E : x \mapsto ax$ is also non-zero. If E is finite dimensional over the division ring $C = \mathrm{End}_A(E)$ then $l : A \to \mathrm{End}_C(E)$ is an isomorphism.

Chapter 10

Decomposing a Module using the Commutant

Consider a module E over a semisimple ring A. Let C be the commutant of the action of A on E:

$$C = \mathrm{End}_A(E).$$

If $E \neq 0$ then this is a ring with $1 \neq 0$, and E is a left C-module.

In this chapter we will see how the simple ideals in a semisimple algebra A specify a decomposition of an A-module when the latter is viewed as a module over the commutant C. This method is the foundation of *Schur-Weyl duality*, which we will explore in Chapter 11.

We will go over essentially the same set of ideas and results in three distinct ways, beginning with a quick, but abstract, approach. The second aproach is a more concrete one, in terms of matrices and bases. The third approach considers, again, the general setting of modules over semisimple rings, but focuses more on the relationship between simple left ideals in A and simple C-submodules of an A-module.

10.1 Joint Decomposition

Consider a semisimple ring A and a left A-module E. We have seen before in Theorem 4.6.2 that E decomposes as a direct sum

$$E \simeq \bigoplus_{i=1}^{r} L_i \otimes_{D_i} \mathrm{Hom}_A(L_i, E) \tag{10.1}$$

where $L_1, ..., L_r$ is a maximal collection of non-isomorphic simple left ideals in A, and D_i is the division ring $\mathrm{Hom}_A(L_i, L_i)$. The isomorphism is given by

$$\bigoplus_{i=1}^{r} L_i \otimes \mathrm{Hom}_A(L_i, E) \to E : \sum_{i=1}^{r} x_i \otimes f_i \mapsto \sum_{i=1}^{r} f_i(x_i) \tag{10.2}$$

The tensor product on the right in (10.1) is over D_i but then it becomes a left A-module through the left A-module structure on L_i.

Even though $\mathrm{Hom}_A(L_i, E)$ is not, naturally, an A-module, it *is* a left C-module, where

$$C = \mathrm{Hom}_A(E, E)$$

is the commutant of the action of A on E: if $c \in C$ and $f \in \mathrm{Hom}_A(L_i, E)$ then

$$cf = c \circ f$$

is also in $\mathrm{Hom}_A(L_i, E)$. This makes $\mathrm{Hom}_A(L_i, E)$ a left C-module.

Thus, the right side in (10.1) is a C-module in a natural way. It is clear that the isomorphism (10.1) is also C-linear. Thus,

> (10.1)*is an isomorphism when both sides are viewed as modules over the product ring* $A \times C$.

A striking feature now emerges:

Theorem 10.1.1 *Let E be a left module over a semisimple ring A, and let C be the ring* $\mathrm{Hom}_A(E, E)$*, the commutant of A acting on E. Let L be a simple left ideal in A, and assume that* $\mathrm{Hom}_A(L, E) \neq 0$*, i.e. that E contains some submodule isomorphic to L. Then the C-module* $\mathrm{Hom}_A(L, E)$ *is simple.*

Proof. Let $f, h \in \mathrm{Hom}_A(L, E)$, with $h \neq 0$. We will show that $f = ch$, for some $c \in C$. Consequently, any non-zero C-submodule of $\mathrm{Hom}_A(L, E)$ is all of $\mathrm{Hom}_A(L, E)$.

If u is any non-zero element in L then $L = Au$, and so it will suffice to show that $f(u) = ch(u)$.

We decompose E as the internal direct sum

$$E = F \oplus \bigoplus_{i \in S} E_i,$$

where each E_i is a submodule isomorphic with L, and F is a submodule containing no submodule isomorphic to L. For each $i \in S$ the projection $E \to E_i$, composed with the inclusion $E_i \subset E$, then gives an element

$$p_i \in C.$$

Since $h \neq 0$, there is some $j \in S$ such that $p_j h(u) \neq 0$. Then $p_j h : L \to E_j$ is an isomorphism. Moreover, for any $i \in S$, the map

$$E_j \to E_i : p_j h(au) \mapsto p_i f(au) \qquad \text{for all } a \in A,$$

is well-defined, and extends to an A-linear map

$$c_i : E \to E$$

which is 0 on F and on $E_{\mathbb{F}}$ for $k \neq j$. Note that there are only finitely many i for which $p_i\big(f(u)\big)$ is not 0, and so there are only finitely many i for which c_i is not 0. Let $S' = \{i \in S : c_i \neq 0\}$. Then, piecing together f from its components $p_i f = c_i p_j h$, we have

$$\sum_{i \in S'} c_i p_j h = f.$$

Thus

$$c = \sum_{i \in S'} c_i p_j$$

is an element of $\mathrm{End}_A(E)$ for which $f = ch$. $\boxed{\text{QED}}$

We may observe one more fact about $\mathrm{Hom}_A(L, E)$, for a simple left ideal L in A:

Proposition 10.1.1 *If $L = Ay$ is a simple left ideal in a semisimple ring A, with y an idempotent, and E is a left A-module, then the map*

$$J : \mathrm{Hom}_A(L, E) \to yE : f \mapsto f(y)$$

is an isomorphism of C-modules, where $C = \mathrm{Hom}_A(E)$.

Proof. Note that yE is a left C-module.

By semisimplicity of A, there is a projection map $p : A \to L$, i.e. p is an A-linear surjection and $p|L$ is the identity map. Then for any $f \in \mathrm{Hom}_A(L, E)$ we have

$$f(y) = fp(y1) = yf(p(1)) \in yE.$$

The map

$$J : \mathrm{Hom}_A(L, E) \to yE : f \mapsto f(y)$$

is trivially C-linear.

The kernel of J is clearly 0.

To prove that J is surjective, consider any $v \in yE$; define a map

$$f' : L \to yE : x \mapsto xv.$$

This is clearly A-linear, and $J(f') = yv = v$, because $v \in yE$ and $y^2 = y$. Thus, J is surjective. $\boxed{\text{QED}}$

Putting everything together we have the isomorphism of $A \times C$-modules

$$E \simeq \bigoplus_{i=1}^{r} L_i \otimes y_i E \tag{10.3}$$

where $L_i = y_i A$, with y_i idempotent, runs over a maximal collection of non-isomorphic simple left ideals in A.

10.2 Decomposition by the Commutant

In this section $\mathbb{F}$ is an algebraically closed field. We work with a finite dimensional vector space V over $\mathbb{F}$, and A a subalgebra of $\mathrm{End}_{\mathbb{F}}(V)$. Thus, V is an A-module. Let C be the commutant:

$$C = \mathrm{End}_A(V).$$

Our objective is to establish Schur's decomposition of V into simple C-modules $e_{ij}V$:

Theorem 10.2.1 *Let A be a subalgebra of $\mathrm{End}_{\mathbb{F}}(V)$, where $V \neq 0$ is a finite-dimensional vector space over an algebraically closed field $\mathbb{F}$. Let*

$$C = \mathrm{End}_A(V)$$

be the commutant of A. Then there exist primitive idempotents $\{e_{ij} : 1 \leq i \leq r, 1 \leq j \leq n_i\}$ in A which generate a decomposition of A into simple left ideals:

$$A = \bigoplus_{1 \leq i \leq r, 1 \leq j \leq n_i} Ae_{ij}, \tag{10.4}$$

and also decompose V, viewed as a C-module, into a direct sum

$$V = \bigoplus_{1 \leq i \leq r, 1 \leq j \leq n_i} e_{ij} V, \tag{10.5}$$

where each non-zero $e_{ij}V$ is a simple C-submodule of V.

Before proceeding to the proof of the theorem, let us make one observation. Because A is semisimple we can decompose it as a direct sum of simple left ideals Ae_j:

$$A = \bigoplus_{j=1}^{N} Ae_j$$

where the e_j are primitive idempotents with

$$e_1 + \cdots + e_N = 1, \quad \text{and} \quad e_i e_j = 0 \quad \text{for all } i \neq j.$$

Then V decomposes as a direct sum

$$V = e_1 V \oplus \ldots \oplus e_N V.$$

The commutant C maps each subspace $e_j V$ into itself. Thus, the $e_j V$ give a decomposition of V as a direct sum of C-submodules. What is, however, not clear is that each non-zero $e_j V$ is a simple C-module; the hard part of Theorem 10.2.1 provides the simplicity of the submodules in the decomposition (10.5).

Most of the remainder of this section is devoted to proving this result. We will follow Dieudonné and Carrell [4] in examining the detailed matrix structure of A, to generate the decomposition of V.

We decompose V into a direct sum

$$V = \bigoplus_{i=1}^{r} V^i, \quad \text{with} \quad V^i = V_{i1} \oplus \ldots \oplus V_{in_i} \tag{10.6}$$

where $V_{i1}, ..., V_{in_i}$ are isomorphic simple A-submodules of V, and $V_{i\alpha}$ is *not* isomorphic to $V_{j\beta}$ when $i \neq j$. By Schur's lemma, elements of C map each V^i into itself. To simplify the notation greatly, *we can then just work within a particular V^i*. Thus let us take for now

$$V = \bigoplus_{j=1}^{n} V_j,$$

where each V_j is a simple A-module and the V_j are isomorphic to each other as A-modules. Let

$$m = \dim_{\mathbb{F}} V_j$$

Fix a basis

$$u_{11}, ..., u_{1m}$$

of the $\mathbb{F}$-vector space V_1 and, using fixed A-linear isomorphisms $V_1 \to V_i$, construct a basis

$$u_{i1}, ..., u_{im}$$

in each V_i. Then the matrices of elements in A are block diagonal, with n blocks, each block being an arbitrary $m \times m$ matrix T with entries in the field $\mathbb{F}$:

$$\begin{bmatrix} T & & & 0 \\ 0 & T & & \\ & & \cdots & \\ 0 & & & T \end{bmatrix} \tag{10.7}$$

Thus, the algebra A is isomorphic to the matrix algebra $\mathrm{Matr}_{m\times m}(\mathbb{F})$ by

$$T \mapsto \begin{bmatrix} T & & & 0 \\ 0 & T & & \\ & & \cdots & \\ 0 & & & T \end{bmatrix} \tag{10.8}$$

The typical matrix in $C = \mathrm{End}_A(V)$ then has the form

$$\begin{bmatrix} s_{11}I & s_{12}I & & s_{1n}I \\ s_{21}I & s_{22}I & & \cdot \\ \cdot & \cdot & & \cdot \\ s_{n1}I & & & s_{nn}I \end{bmatrix} \tag{10.9}$$

where I is the $m \times m$ identity matrix. Reordering the basis in V as

$$u_{11}, u_{21}, ..., u_{n1}, u_{12}, u_{22}, ..., u_{n2}, ..., u_{1m}, ..., u_{nm},$$

displays the matrix (10.9) as the block diagonal matrix

$$\begin{bmatrix} [s_{ij}] & 0 & \cdot & 0 \\ 0 & [s_{ij}] & \cdot & \\ \cdot & \cdot & \cdot & \\ 0 & & & [s_{ij}] \end{bmatrix} \tag{10.10}$$

where s_{ij} are arbitrary elements of the field $\mathbb{F}$. Thus C is isomorphic to the algebra of $n \times n$ matrices $[s_{ij}]$ over $\mathbb{F}$. Now the algebra $\mathrm{Matr}_{n\times n}(\mathbb{F})$ is decomposed into a sum of n simple ideals, each consisting of the matrices which have all entries zero except possibly those in one particular column. Thus,

each simple left ideal in C is n-dimensional over $\mathbb{F}$.

Let M^i_{jh} be the matrix for the linear map $V \to V$ which takes u_{ih} to u_{ij} and is 0 on all the other basis vectors. Then, from (10.7), the matrices

$$M_{jh} = M^1_{jh} + \cdots + M^n_{jh} \tag{10.11}$$

form a basis of A, as a vector space over $\mathbb{F}$. Let

$$e_j = M_{jj},$$

for $1 \leq j \leq m$. This corresponds, in $\mathrm{Matr}_{m\times m}(\mathbb{F})$, to the matrix with 1 at the jj entry and 0 elsewhere. Then A is the direct sum of the simple left ideals Ae_j.

The subspace e_jV has the vectors

$$u_{1j}, u_{2j}, ..., u_{nj}$$

as a basis, and so e_jV is n-dimensional. Moreover, e_jV is mapped into itself by C:

$$C(e_jV) = e_jCV \subset e_jV.$$

Consequently, e_jV is a C-module. Since it has the same dimension as any simple C-module, it follows that e_jV cannot have a non-zero proper C-submodule, i.e. e_jV is a simple C-module.

We have completed the proof of Theorem 10.2.1.

Finally, let us note:

Proposition 10.2.1 *If u, u' are idempotents in a ring A which generate the same left ideal, and if E is an A-module, then uE and $u'E$ are isomorphic C-submodules of E, where $C = \mathrm{End}_A(E)$.*

Proof. Since $Au = Au'$, we have then

$$u = bu', \quad u' = b'u, \quad \text{for some } b, b' \in A.$$

Note that

$$u = bb'u, \qquad \text{and} \qquad u' = b'bu'.$$

Then the maps

$$f : uE \to u'E : y \mapsto b'y, \qquad \text{and} \qquad f' : u'E \to uE : w \mapsto bw$$

are inverses to each other and are C-linear. $\boxed{\text{QED}}$

10.3 Submodules relative to the Commutant

In this section we will go over the decomposition of a module by the commutant again, using a different approach, somewhat along the lines of Weyl [19].

For this section we will work with a *right* A-module E, which is automatically also a C-module, where C is the commutant $\mathrm{End}_A(E)$, and set up a one-to-one correspondence between the simple C-submodules of E and simple left ideals in A.

Consider first a finite group G represented on a finite-dimensional $\mathbb{F}$-vector space E. It will be convenient to view E as a *right* $\mathbb{F}[G]$-module by the action:

$$v \cdot g = g^{-1}v, \qquad \text{for } v \in E \text{ and } g \in G.$$

More generally, we work with a semisimple ring A and a right A-module E.

Let

$$\hat{E} = \mathrm{Hom}_A(E, A) \tag{10.12}$$

consisting of all additive maps $\hat{\phi} : E \to A$ satisfying

$$\hat{\phi}(xa) = \hat{\phi}(x)a \quad \text{for all } x \in E \text{ and } a \in A. \tag{10.13}$$

Then $\hat{E}$ is a left A-module:

$$\textit{if } \hat{\phi} \in \hat{E} \textit{ and } a \in A \textit{ then } a\hat{\phi} \in \hat{E}.$$

The map

$$\hat{E} \times E \to A : (\hat{\phi}, x) \mapsto \hat{\phi}(x) \tag{10.14}$$

is bilinear when all modules involved are viewed as $\mathbb{Z}$-linear, and so induces a linear map on the tensor product specified through

$$I : \hat{E} \otimes_{\mathbb{Z}} E \to A : \hat{\phi} \otimes x \mapsto \hat{\phi}(x) \tag{10.15}$$

Note that

$$aI(\hat{\phi} \otimes x) = I(a\hat{\phi} \otimes x) \quad \text{for all } a \in A,\ \hat{\phi} \in \hat{E}, \text{ and } x \in E. \tag{10.16}$$

For any additive subgroup $W \subset E$ define

$$W_{\#} \stackrel{\text{def}}{=} I(\hat{E} \otimes W) \subset A \tag{10.17}$$

The simple fact (10.16) has the following immediate consequence:

Lemma 10.3.1 *$W_{\#}$ is a left ideal in A.*

Before proceeding further let us look[1] at the case of interest for our purposes of studying representations of a finite group G. Let E then be a finite-dimensional vector space over the field $\mathbb{F}$, on which G is represented. Then E is a right $\mathbb{F}[G]$-module by the action:

$$E \times \mathbb{F}[G] \to E : (v, x) \mapsto \hat{x}v,$$

where

$$\hat{x} = \sum_{g \in G} x(g)g^{-1}.$$

An element of $\hat{\phi}$ of $\hat{E}$, expressed as

$$\hat{\phi}(v) = \sum_{g \in G} \hat{\phi}_g(v)g \qquad \text{for all } v \in E$$

is completely specified by the coefficient $\hat{\phi}_e : E \to k$, because

$$\hat{\phi}_e(v) = \hat{\phi}_e(vg^{-1}) = \hat{\phi}_e(gv) \quad \text{for all } g \in G \text{ and } v \in E.$$

Conversely, as is readily checked, every ϕ in the dual $E^* = \mathrm{Hom}(E, k)$ specifies an element $\hat{\phi} \in \hat{E}$:

$$\hat{\phi}(v) = \sum_{g \in G} \phi(gv)g \qquad \text{for all } v \in E.$$

Now we can see quite concretely how a subspace $W \subset E$ gives rise to a left ideal $W_{\#}$ in $\mathbb{F}[G]$. Evaluating $\hat{\phi} \in \hat{E}$ arising from $\phi \in E^*$, on a general element w of W we have

$$\hat{\phi}(w) = \sum_{g \in G} \langle \phi, gw \rangle g$$

---

[1] We follow Weyl [19].

The set of all such elements is $W_\#$:

$$W_\# = \left\{ \sum_{g \in G} \langle \phi, gw \rangle g \,:\, \phi \in E^*, w \in W \right\} \tag{10.18}$$

It is clear that this is a left ideal:

$$h \sum_{g \in G} \langle \phi, gw \rangle g = \sum_{g \in G} \langle (h^{-1})^* \phi, gw \rangle g$$

The explicit form of $W_\#$ given in (10.18) helps moor our discussions to a concrete base.

We return now to the general setting, with E being a right A-module, where A is a semisimple ring.

We are interested primarily in $W \subset E$ of the form

$$Ea$$

for $a \in A$. The principal fact about such W is that

Ea is a C-submodule of E,

where

$$C = \mathrm{End}_A(E) \tag{10.19}$$

is the commutant of the action of A on E.

We note that

$$(Ea)_\# = E_\# a, \qquad \text{for all } a \in A. \tag{10.20}$$

(From the definition of I we can see that $E_\#$ is the sum of all the simple right ideals in A isomorphic to the simple submodules of E.)

Before proceeding to the next observation we need an auxilliary result:

Lemma 10.3.2 *Let $x \in E$, and $\hat{\phi} \in \hat{E}$. Then the map*

$$L : E \to E : y \mapsto x\hat{\phi}(y)$$

is in the commutant $C = \mathrm{End}_A(E)$. In particular, if W is a C-submodule of E then $x\hat{\phi}(w) \in W$ for all $w \in W$.

Proof. For any $a \in A$ and $y \in E$ we have

$$L(ya) = x\hat{\phi}(ya) = x\hat{\phi}(y)a = L(y)a,$$

which shows that $L \in \mathrm{End}_A(E)$. $\boxed{\text{QED}}$

Now we can prove:

Proposition 10.3.1 *Suppose W is a C-submodule of E. Let u be any idempotent generator of the left ideal $W_\#$, i.e. $W_\# = Au$ and $u^2 = u$. Then*

$$W = Eu.$$

Thus, every C-submodule of E is of the form Eu where u is an idempotent in A.

Proof. Since $u \in W_\#$, we can write u as a sum of terms of the form $\hat{\phi}(w)$ with $\hat{\phi} \in \hat{E}$ and $w \in W$. Then, for any $x \in E$, we see that xu is a sum of terms of the form $x\hat{\phi}(w)$. The latter are all in W (by Lemma 10.3.2), and so $xu \in W$. Thus

$$Eu \subset W.$$

Now we will show that $W \subset Eu$, by using the idempotent nature of u. Let $w \in W$; we will show that wu equals w, and so then w would be in Eu. Now for any $\phi \in \hat{E}$ we have

$$\hat{\phi}(wu) = \hat{\phi}(w)u = \hat{\phi}(w), \qquad \text{because } \hat{\phi}(w) \in W_\# = Au.$$

Consequently, the element $x = wu - u \in E$ is annihilated by all elements of $\mathrm{Hom}_A(E, A)$. Decompose E as a direct sum of simple A-submodules E_i, and let x_i be the component of x in E_i. Now E_i is isomorphic as an A-module to a simple right ideal in A, and this isomorphism yields an element of $\mathrm{Hom}_A(E, A)$. Thus, x_i is 0 for each i, and so $wu = u$. $\boxed{\text{QED}}$

Going in the reverse direction we have:

Proposition 10.3.2 *Suppose W and W' are C-submodules of E with $W'_\# \subset W_\#$. Then $W' \subset W$. In particular, if $W_\# = W'_\#$ then $W = W'$.*

Proof. We can write $W'_\# = Au'$ and $W_\# = Au$, where $u, u' \in A$ are generating idempotents for these left ideals. If $W'_\# \subset W_\#$ then $u' \in Au$, and so $u' = au$, for some $a \in A$, which then implies that $W' = Eu' \subset Eu = W$. $\boxed{\text{QED}}$

We have the following useful consequence:

Proposition 10.3.3 *A C-submodule W of E is simple if the left ideal $W_\#$ is simple. Assuming that $E_\# = A$, the converse also holds: if W is a simple C-submodule of E then the left ideal $W_\#$ is simple. In particular, in the latter case, if u is a primitive idempotent in A then Eu is a simple C-submodule of E.*

Proof. Suppose $W_\#$ is a simple left ideal in A. Let $W' \subset W$ be a C-submodule of W. Then $W'_\# \subset W_\#$, and so $W'_\#$ is 0 or $W_\#$. If $W'_\# = \{0\}$ then $W' = \{0\}$ (by the argument used in the proof of Proposition 10.3.1). If $W'_\# = W_\#$ then, by the preceding proposition, $W = W'$. Thus, W is a simple C-submodule of E.

Conversely, suppose W is a simple C-submodule of E. Suppose J is a left ideal inside $W_\#$. Then $J = Au'$ and $W_\# = Au$, for some $u' \in J$ and $u \in W_\#$. Then $u' \in Au$ and so

$$Eu' \subset Eu = W.$$

Thus $Eu' = 0$ or $Eu' = Eu$. Applying $\#$ we see that $E_\# u' = 0$ or $E_\# u' = E_\# u$. Thus, if $E_\# = A$ then J is 0 or $W_\#$. Thus, $W_\#$ is simple. $\boxed{\text{QED}}$

Thus there is an almost one-to-one correspondence between ideals in A and C-submodules of E, with simple submodules arising from simple ideals. However, some care needs to be exercised; with notation as above, we have:

Proposition 10.3.4 *If u is a primitive idempotent in A* and *if the ideal Au is inside $E_\#$, then Eu is a simple C-module.*

Proof. If $Au \subset E_\#$ then

$$Au = Auu \subset E_\# u \subset Au,$$

and so

$$Au = E_\# u = (Eu)_\#, \qquad \text{by (10.20)}.$$

Since u is assumed to be a primitive idempotent, we see that $(Eu)_\#$ is simple and so Eu is a simple C-module. $\boxed{\text{QED}}$

Exercises

1. Let A be a semisimple ring. Consider a right A-module E, an element $\vec{e} \in E$, and let N be the annihilator of $\vec{e}$:

$$N = \{n \in A : \vec{e}n = 0\}$$

Then N is a right ideal in A. Let $A = N \oplus N^{\perp}$ be a decomposition of A into complementary right ideals, and let $P^{\perp}$ be the projection map onto $N^{\perp}$. Show that for any left ideal L in A:

(i) $P^{\perp}(L) \subset L$ [Hint: If $1 \in A$ decomposes as $1 = u + u^{\perp}$, with $u \in N$ and $u^{\perp} \in N^{\perp}$, then $P^{\perp}x = u^{\perp}x$ for all $x \in A$.]

(ii) the map
$$F : \vec{e}\,L \to P^{\perp}(L) : \vec{e}\,x \mapsto P^{\perp}x$$
is well-defined;

(iii) the map
$$H : P^{\perp}(L) \to \vec{e}\,L : x \mapsto \vec{e}x$$
is the inverse of F.

2. Let G be a finite group, represented on a finite-dimensional vector space E over a field $\mathbb{F}$ of characteristic 0. View E as a *right* $\mathbb{F}[G]$-module, by defining
$$vx = \hat{x}v \qquad \text{for all } v \in E \text{ and } x \in \mathbb{F}[G]$$
where
$$\hat{x} = \sum_{g \in G} x(g)g^{-1}.$$
Suppose $\vec{e} \in E$ is such that the *set* $G\vec{e}$ is a basis of E. Denote by H the isotropy subgroup $\{h \in G : h\vec{e} = \vec{e}\}$, and $N = \{n \in \mathbb{F}[G] \, : \, \vec{e}n = 0\}$.

(i) Show that
$$\mathbb{F}[G] = N \oplus \mathbb{F}[G/H],$$
where $k[G/H]$ is the right ideal in $\mathbb{F}[G]$ consisting of all x for which $hx = x$ for every $h \in H$, and that the projection map on $\mathbb{F}[G/H]$ is given by
$$x \mapsto \frac{1}{|H|} \sum_{h \in H} hx$$

(ii) Show, using Problem 1, that for any left ideal L in $\mathbb{F}[G]$,
$$\dim_{\mathbb{F}}(\vec{e}L) = \frac{1}{|H|} \sum_{h \in H} \chi_L(h),$$
where $\chi_L(a)$ is the trace of the map $L \to L : y \mapsto ay$.

Chapter 11

Schur-Weyl Duality

We turn now to a specific implementation of the dual decomposition theory developed in the preceding chapter. Consider the permutation group S_n represented on a finite-dimensional vector space V over an algebraically closed field $\mathbb{F}$. Then S_n also acts by permutations of tensor components on the tensor power $V^{\otimes n}$. This is then a module over the ring $A = \mathbb{F}[S_n]$. As we shall see, the commutant turns out to be spanned by the operators $T^{\otimes n}$ on $V^{\otimes n}$ with T running over the group $GL_{\mathbb{F}}(V)$ of all invertible linear endomorphisms of V. This then leads to an elegant relationship between characters of S_n and characters of $GL_{\mathbb{F}}(V)$.

We begin with the identification of the commutant of the action of S_n on $V^{\otimes n}$. Then we will go through a fast proof of the Weyl duality formula connecting characters of S_n and that of $GL_{\mathbb{F}}(V)$, using the more abstract approach to duality developed in section 10.1. In the last section we will prove this duality formula again, but by more explicit computation.

11.1 The Commutant for S_n acting on $V^{\otimes n}$

In this section, V is a finite dimensional vector space over a field with an infinite number of elements (thus no non-zero polynomial vanishes at all elements of the field).

The permutation group S_n has a natural left action on $V^{\otimes n}$:

$$\sigma \cdot (v_1 \otimes \ldots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}$$

Our objective in this section is to prove the following central result from Weyl [19]:

Theorem 11.1.1 *The commutant of the action of S_n on $V^{\otimes n}$ is the linear span of all endomorphisms $T^{\otimes n} : V^{\otimes n} \to V^{\otimes n}$, with T running over all invertible endomorphisms on V.*

Proof. Fix a basis $e_1, ..., e_N$ of V, and let $e^1, .., e^N$ be the dual basis in V^*:

$$\langle e^i, e_j \rangle = \delta^i_j.$$

For a linear mapping

$$X : V^{\otimes n} \to V^{\otimes n}$$

let

$$X_{i_1 j_1; \ldots, i_n j_n} = \left\langle e^{i_1} \otimes \ldots \otimes e^{i_n}, X\big(e_{j_1} \otimes \ldots \otimes e_{j_n}\big)\right\rangle \tag{11.1}$$

Then X commutes with the action of S_n if and only if (11.1) remains invariant when i and j are replaced by $i \circ \sigma$ and $j \circ \sigma$, for any $\sigma \in S_n$.

Relabel the $m = N^2$ pairs ij with numbers from $1, ..., m$.

We will show that if $F \in \mathrm{End}(V^{\otimes n})$ satisfies

$$\sum_{a_1, \ldots, a_n \in \{1, \ldots, m\}} F_{a_1 \ldots a_n} (T^{\otimes n})_{a_1 \ldots a_n} = 0 \qquad \text{for all invertible } T \in \mathrm{End} V$$

then

$$\sum_a F_{a_1 \ldots a_n} X_{a_1 \ldots a_n} = 0 \tag{11.2}$$

for all X in the commutant of S_n. This will imply the desired result.

Consider the polynomial in the $m = N^2$ indeterminates T_a given by

$$p(T) = \left(\sum_{a_1, \ldots, a_n \in \{1, \ldots, m\}} F_{a_1 \ldots a_n} T_{a_1} ... T_{a_n} \right) \det[T_{ij}]$$

The hypothesis says that this polynomial is equal to 0 for all choices of values of T_a in the field $\mathbb{F}$. This implies that the polynomial $p(T)$ is 0. Since the polynomial $\det[T_{ij}]$ is certainly not 0, it follows that

$$\sum_a F_{a_1 \ldots a_n} T_{a_1} ... T_{a_n} = 0$$

as a polynomial. This means that each $F_{a_1 \ldots . a_n}$ is 0, and hence we have the desired equality (11.2). $\boxed{\text{QED}}$

The representations of S_n and $GL_{\mathbb{F}}(V)$ (the group of invertible elements in $\mathrm{End}_{\mathbb{F}}(V)$) on $V^{\otimes n}$ clearly commute with each other. The preceding theorem says that they are in fact dual to each other, in the sense that the subalgebras of $\mathrm{End}(V^{\otimes n})$ they generate are each other's commutants.

As consequence of these observations we conclude that the representation of $GL_{\mathbb{F}}(V)$ on $V^{\otimes n}$ decomposes into a direct sum of irreducible representations, each being a subspace of the form $yV^{\otimes n}$, with y a primitive idempotent in $\mathbb{F}[S_n]$.

11.2 Schur-Weyl Character Duality I

Let $E \neq 0$ be a left module over semisimple ring A, and

$$C = \mathrm{End}_A(E)$$

the commutant. Let $L_1, ..., L_r$ be a maximal set of simple left ideals in A. Then $\mathrm{Hom}_A(L_i, E)$ is a left C-module, and the tensor product

$$L_i \otimes \mathrm{Hom}_A(L_i, E)$$

is a left A-module from the structure on L_i and a left C-module from the structure on $\mathrm{Hom}_A(L_i, E)$. Thus, it is a left module over the ring $A \times C$.

We have seen in section 10.1 that:

- each $\mathrm{Hom}_A(L_i, E)$ is a *simple* C-module,
- if $y_i \in L_i$ is non-zero idempotent in L_i then the map

  $$\mathrm{Hom}_A(L_i, E) \mapsto y_i E : f \mapsto f(y_i)$$

  is an isomorphism of C-modules, and
- the map

  $$J : \bigoplus_{i=1}^{r} L_i \otimes \mathrm{Hom}_A(L_i, E) \to E : \sum_{i=1}^{r} x_i \otimes f_i \mapsto \sum_{i=1}^{r} f_i(x_i)$$

  is an isomorphism which is both A-linear and C-linear.

Consider now any $a \in A$ and $c \in C$. Then $(a, c) \in A \times C$, acting on $L_i \otimes \mathrm{Hom}_A(L_i, E)$ goes over, via J, to the map

$$E \to E : v \mapsto acv.$$

Now we assume that A is an algebra over a field $\mathbb{F}$, and E is finite-dimensional as a vector space over $\mathbb{F}$. Then the trace of $ac \in \mathrm{End}_{\mathbb{F}}(E)$ can be computed using the isomorphism J:

$$\mathrm{Tr}(ac) = \sum_{i=1}^{r} \mathrm{Tr}(a|L_i)\mathrm{Tr}(c|y_iE), \tag{11.3}$$

where $a|L_i$ is the element in $\mathrm{End}_{\mathbb{F}}(L_i)$ given by $x \mapsto ax$.

We specialize now to

$$A = \mathbb{F}[S_n]$$

acting on $V^{\otimes n}$, where V is a finite-dimensional vector space over an infinite field $\mathbb{F}$. Then, as we know, C is spanned by elements of the form $B^{\otimes n}$, with B running over $GL_{\mathbb{F}}(V)$. The distinct simple left ideals in A correspond to inequivalent irreducible representations of S_n. Let the set $\mathcal{R}$ label these representations; thus there is a maximal set of non-isomorphic simple left ideals L_α, with α running over $\mathcal{R}$. Then we have, for any $\sigma \in S_n$ and any $B \in GL_{\mathbb{F}}(V)$, the *Schur-Weyl duality* formula

$$\boxed{\mathrm{Tr}(B^{\otimes n} \cdot \sigma) = \textstyle\sum_{\alpha \in \mathcal{R}} \chi_\alpha(\sigma)\chi^\alpha(B)} \tag{11.4}$$

where χ_α is the characteristic of the representation of S_n on $L_\alpha = y_\alpha A$, and χ^α that of $GL_{\mathbb{F}}(V)$ on $y_\alpha V^{\otimes n}$.

Recall the Schur orthogonality relation

$$\frac{1}{n!} \sum_{\sigma \in S_n} \chi_\alpha(\sigma)\chi_\beta(\sigma^{-1}) = \delta_{\alpha\beta} \quad \text{for all } \alpha, \beta \in \mathcal{R}.$$

Using this with (11.4), we have

$$\chi^\alpha(B) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_\alpha(\sigma) s^\sigma(B)$$

where

$$s^\sigma(B) = \mathrm{Tr}(B^{\otimes n} \cdot \sigma). \tag{11.5}$$

Note that s^σ *depends only on the conjugacy class of* σ, rather than on the specific choice of σ. Denoting by K a typical conjugacy class, we then have

$$\boxed{\chi^\alpha(B) = \sum_{K\in\mathcal{C}} \frac{|K|}{n!}\chi_\alpha(K)s^K(B)} \tag{11.6}$$

where $\mathcal{C}$ is the set of all conjugacy classes in S_n, $\chi_\alpha(K)$ is the value of χ_α on any element in K, and s^K is s^σ for any $\sigma \in K$.

In the following section we will prove the character duality formulas (11.4) and (11.6) again, by a more explicit method.

11.3 Schur-Weyl Character Duality II

We will now work through a proof of the Schur-Weyl duality formulas by more explicit computations. This section is entirely independent of the preceding.

The results and proofs of this section work over any algebraically closed field of zero characteristic, but it will be notationally convenient to simply work with the complex field $\mathbb{C}$.

Let $V = \mathbb{C}^N$, on which the group $GL(N,\mathbb{C})$ acts in the natural way. Let

$$e_1, ..., e_N$$

be the standard basis of $V = \mathbb{C}^N$.

We know that $V^{\otimes n}$ decomposes as a direct sum of submodules of the form

$$y_\alpha V^{\otimes n},$$

with y_α running over a set of primitive idempotents in $\mathbb{C}[S_n]$, such that the left ideals $\mathbb{C}[S_n]\hat{y}_\alpha$ form a decomposition of $\mathbb{C}[S_n]$ into simple left submodules.

Let

$$\chi^\alpha$$

be the characteristic of the irreducible representation ρ_α of $GL(N,\mathbb{C})$ on the subspace $y_\alpha V^{\otimes n}$, and

$$\chi_\alpha$$

be the characteristic of the representation of S_n on $\mathbb{C}[S_n]y_\alpha$.

Our goal is to establish the relation between these two characters.

If a matrix $g \in GL(N, \mathbb{C})$ has all eigenvalues distinct, then the corresponding eigenvectors are linearly independent and hence form a basis of V. Changing basis, g is conjugate to a diagonal matrix

$$D(\vec{\lambda}) = D(\lambda_1, ..., \lambda_N) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_N \end{bmatrix}$$

Then $\chi^\alpha(g)$ equals $\chi^\alpha\big(D(\vec{\lambda})\big)$. We will evaluate the latter.

The action of $D(\vec{\lambda})$ on the vector

$$y_\alpha(e_{i_1} \otimes \ldots \otimes e_{i_n})$$

is simply multiplication by

$$\lambda_{i_1} ... \lambda_{i_n}.$$

Fix a partition of n given by

$$\vec{f} = (f_1, ..., f_N) \in \mathbb{Z}^N_{\geq 0}$$

with

$$|\vec{f}| = f_1 + \cdots + f_N = n,$$

and let

$$\vec{\lambda}^{\vec{f}} = \prod_{j=1}^N \lambda_j^{f_j}$$

and

$$V(\vec{f}\,) = \{v \in V^{\otimes n} \,:\, D(\vec{\lambda})v = \vec{\lambda}^{\vec{f}} v \quad \text{for all } \vec{\lambda} \in U(1)^N\}$$

Thus every eigenvalue of $D(\vec{\lambda})$ is of the form $\vec{\lambda}^{\vec{f}}$.

Consequently,

$$\chi^\alpha\big(D(\vec{\lambda})\big) = \sum_{\vec{f} \in \mathbb{Z}^N_{\geq 0}} \vec{\lambda}^{\vec{f}} \dim\big(y_\alpha V(\vec{f}\,)\big) \tag{11.7}$$

The space $V(\vec{f}\,)$ has a basis given by the *set*

$$\{\sigma \cdot e_1^{\otimes f_1} \otimes \cdots \otimes e_N^{\otimes f_N} \,:\, \sigma \in S_n\}$$

Note that

$$\vec{e}^{\otimes \vec{f}} = e_1^{\otimes f_1} \otimes \cdots \otimes e_N^{\otimes f_N}$$

is indeed in $V^{\otimes n}$, because $|\vec{f}| = n$.

Then, by Exercise 8.2(ii), the dimension of $y_\alpha V(\vec{f}\,)$ is

$$\dim\bigl(y_\alpha V(\vec{f}\,)\bigr) = \frac{1}{f_1!...f_N!} \sum_{\sigma \in S_n(\vec{f})} \chi_\alpha(\sigma) \tag{11.8}$$

where

$$S_n(\vec{f})$$

is the subgroup of S_n consisting of elements which preserve the sets

$$\{1, ..., f_1\}, \{f_1 + 1, ..., f_2\}, ..., \{f_{N-1} + 1, ..., f_N\}$$

and we have used the fact (Exercise 9.1) that χ_α equals the characteristic of the representation of S_n on $\mathbb{C}[S_n]\hat{y}_\alpha$.

Thus,

$$\chi^\alpha\bigl(D(\vec{\lambda})\bigr) = \sum_{\vec{f} \in \mathbb{Z}_{\geq 0}^N} \vec{\lambda}^{\vec{f}} \frac{1}{f_1!...f_N!} \sum_{\sigma \in S_n(\vec{f})} \chi_\alpha(\sigma) \tag{11.9}$$

The character χ_α is constant on conjugacy classes. So the second sum on the right here should be reduced to a sum over conjugacy classes. Note that, with obvious notation,

$$S_n(\vec{f}) \simeq S_{f_1} \times \ldots \times S_{f_N}$$

The conjugacy class of a permutation is completely determined by its cycle structure: i_1 1-cycles, i_2 2-cycles,... . For a given sequence

$$\vec{i} = (i_1, i_2, ..., i_m) \in \mathbb{Z}_{\geq 0}^m$$

the number of such permutations in S_m is

$$\frac{m!}{(i_1!1^{i_1})(i_2!2^{i_2})(i_3!3^{i_3})...(i_m!m^{i_m})} \tag{11.10}$$

because, in distributing $1, ..., m$ among such cycles, the i_k $\mathbb{F}$-cycles can be arranged in $i_k!$ ways and each such $\mathbb{F}$-cycle can be expressed in $\mathbb{F}$ ways.

Alternatively, the denominator in (11.10) is the size of the isotropy group of any element of the conjugacy class.

The cycle structure of an element of

$$(\sigma_1, ..., \sigma_N) \in S_{f_1} \times \cdots \times S_{f_N}$$

is described by a sequence

$$[\vec{i}_1, ..., \vec{i}_N] = (\underbrace{i_{11}, i_{12}, ..., i_{1f_1}}_{\vec{i}_1}, ..., \underbrace{i_{N1}, ..., i_{Nf_N}}_{\vec{i}_N})$$

with i_{jk} being the number of $\mathbb{F}$-cycles in the permutation σ_j. Let us denote by

$$\chi_\alpha([\vec{i}_1, ..., \vec{i}_N])$$

the value of χ_α on the corresponding conjugacy class in S_n. Then

$$\sum_{\sigma \in S_n(\vec{f})} \chi_\alpha(\sigma) = \sum_{[\vec{i}_1,...,\vec{i}_N] \in [\vec{f}]} \chi_\alpha([\vec{i}_1, ..., \vec{i}_N]) \prod_{j=1}^N \frac{f_j!}{(i_{j1}!1^{i_{j1}})(i_{j2}!2^{i_{j2}}) \dots}$$

Here the sum is over the set $[\vec{f}]$ of all $[\vec{i}_1, ..., \vec{i}_N]$ for which

$$i_{j1} + 2i_{j2} + \cdots + ni_{jn} = f_j \quad \text{for all } j \in \{1, ..., N\}$$

(Of course, i_{jn} is 0 when $n > f_j$.)

Returning to the expression for χ^α in (11.9) we have:

$$\begin{aligned} \chi^\alpha\big(D(\vec{\lambda})\big) &= \sum_{\vec{f} \in \mathbb{Z}^N_{\geq 0}} \vec{\lambda}^{\vec{f}} \sum_{[\vec{i}_1,...,\vec{i}_N] \in [\vec{f}]} \chi_\alpha([\vec{i}_1, ..., \vec{i}_N]) \prod_{j=1}^N \frac{1}{(i_{j1}!1^{i_{j1}})(i_{j2}!2^{i_{j2}}) \dots (i_{jn}!n^{i_{jn}})} \\ &= \sum_{\vec{f} \in \mathbb{Z}^N_{\geq 0}} \vec{\lambda}^{\vec{f}} \sum_{[\vec{i}_1,...,\vec{i}_N] \in [\vec{f}]} \chi_\alpha([\vec{i}_1, ..., \vec{i}_N]) \prod_{1 \leq j \leq N,\, 1 \leq k \leq n} \frac{1}{i_{jk}!\, k^{i_{jk}}} \end{aligned}$$

Now χ_α is constant on conjugacy classes in S_n. The conjugacy class in $S_{f_1} \times \cdots \times S_{f_N}$ specified by the cycle structure

$$[\vec{i}_1, ..., \vec{i}_N]$$

corresponds to the conjugacy class in S_n specified by the cycle structure

$$\vec{i} = (i_1, ..., i_n)$$

with

$$\sum_{j=1}^{N} i_{jk} = i_k \quad \text{for all } k \in \{1, ..., n\} \tag{11.11}$$

Recall again that

$$\sum_{k=1}^{n} k i_{jk} = f_j \tag{11.12}$$

Note that then

$$\vec{\lambda}^{\vec{f}} = \prod_{k=1}^{n} (\lambda_1^{ki_{1k}} \dots \lambda_N^{ki_{Nk}})$$

Combining these observations we have

$$\chi^\alpha\big(D(\vec{\lambda})\big) = \sum_{\vec{i} \in \mathbb{Z}^N_{\geq 0}} \chi_\alpha(\vec{i}) \frac{1}{1^{i_1} 2^{i_2} ... n^{i_n}} \sum_{i_{jk}} \prod_{k=1}^{n} \frac{\lambda_1^{ki_{1k}} ... \lambda_N^{ki_{Nk}}}{i_{1k}! i_{2k}! ... i_{Nk}!} \tag{11.13}$$

where the inner sum on the right is over all $[\vec{i}_1, ..., \vec{i}_N]$ corresponding to the cycle structure $\vec{i} = (i_1, ..., i_n)$ in S_n, i.e. satisfying (11.11). We observe now that this sum simplifies:

$$\sum_{i_{jk}} \prod_{k=1}^{n} \frac{\lambda_1^{ki_{1k}} ... \lambda_N^{ki_{Nk}}}{i_{1k}! i_{2k}! ... i_{Nk}!} = \frac{1}{i_1! ... i_n!} \prod_{k=1}^{n} (\lambda_1^k + \cdots + \lambda_N^k)^{i_k} \tag{11.14}$$

This produces

$$\boxed{\chi^\alpha\big(D(\vec{\lambda})\big) = \textstyle\sum_{\vec{i} \in \mathbb{Z}^N_{\geq 0}} \chi_\alpha(\vec{i}) \frac{1}{(i_1! 1^{i_1})(i_2! 2^{i_2}) ... (i_n! n^{i_n})} \prod_{k=1}^{n} s_k(\vec{\lambda})^{i_k}} \tag{11.15}$$

where $s_1, ..., s_n$ are the symmetric polynomials given by

$$s_m(X_1, ..., X_n) = X_1^m + \cdots + X_n^m \tag{11.16}$$

We can also conveniently define

$$s_m(B) = \mathrm{Tr}(B^m) \tag{11.17}$$

Then

$$\boxed{\chi^\alpha(B) = \sum_{\vec{i}\in\mathbb{Z}^N_{\geq 0}} \chi_\alpha(\vec{i})\frac{1}{(i_1!1^{i_1})(i_2!2^{i_2})\dots(i_n!n^{i_n})}\prod_{m=1}^n s_m(B)^{i_m}} \tag{11.18}$$

for all $B \in GL(N, \mathbb{C})$ with distinct eigenvalues, and hence for all $B \in GL(N, \mathbb{C})$.

The sum on the right in (11.18) is over all conjugacy classes in S_n, each labelled by its cycle structure

$$\vec{i} = (i_1, \dots, i_n).$$

Note that the number of elements in this conjugacy class is exactly $n!$ divided by the denominator which appears on the right inside the sum. Thus, we can also write the *Schur-Weyl duality* formula as

$$\boxed{\chi^\alpha(B) = \sum_{K\in\mathcal{C}} \frac{|K|}{n!}\chi_\alpha(K)s^K(B)} \tag{11.19}$$

where $\mathcal{C}$ is the set of all conjugacy classes in S_n, and

$$s^K \stackrel{\text{def}}{=} \prod_{m=1}^n s_m^{i_m} \tag{11.20}$$

if K has the cycle structure $\vec{i} = (i_1, \dots, i_n)$.

Note that up to this point we have *not needed to assume that α labels an irreducible representation of S_n*. We have merely used the character χ_α corresponding to some left ideal $\mathbb{C}[S_n]y_\alpha$ in $\mathbb{C}[S_n]$, and the corresponding $GL(n, \mathbb{C})$-module $\hat{y}_\alpha V^{\otimes n}$.

We will now assume that χ_α indeed labels the irreducible characters of S_n. Then we have the Schur orthogonality relations

$$\frac{1}{n!}\sum_{\sigma\in S_n} \chi_\alpha(\sigma)\chi_\beta(\sigma^{-1}) = \delta_{\alpha\beta}$$

These can be rewritten as

$$\sum_{K\in\mathcal{C}} \chi_\alpha(K)\frac{|K|}{n!}\chi_\beta(K^{-1}) = \delta_{\alpha\beta} \tag{11.21}$$

Thus, the $|\mathcal{C}| \times |\mathcal{C}|$ square matrix $[\chi_\alpha(K^{-1})]$ has the inverse $\frac{1}{n!}\,[|K|\,\chi_\alpha(K)]$. Thus, also:

$$\sum_{\alpha\in\mathcal{R}} \chi_\alpha(K^{-1})\frac{|K'|}{n!}\chi_\alpha(K') = \delta_{KK'}, \tag{11.22}$$

where $\mathcal{R}$ labels a maximal set of inequivalent irreducible representations of S_n. Consequently, multiplying (11.19) by $\chi_\alpha(K^{-1})$ and summing over α, we obtain:

$$\boxed{\textstyle\sum_{\alpha\in\mathcal{R}} \chi^\alpha(B)\chi_\alpha(K) = s^K(B)} \tag{11.23}$$

for every conjugacy class K in S_n, where we used the fact that $K^{-1} = K$.

Observe that

$$\boxed{s^K(B) = \mathrm{Tr}(B^{\otimes n}\cdot\sigma)} \tag{11.24}$$

where σ, any element of the conjugacy class K, appears on the right here by its representation as an endomorphism of $V^{\otimes n}$. The identity (11.24) is readily checked if σ is the cycle $(12\ldots n)$, and then the general case follows by observing that

$$\mathrm{Tr}(B^{\otimes j}\otimes B^{\otimes l}\cdot\sigma\theta) = \mathrm{Tr}(B^{\otimes j})\mathrm{Tr}(B^{\otimes l})$$

if σ and θ are the disjoint cycles $(12\ldots j)$ and $(j+1\ldots n)$.

Thus the duality formula (11.23) coincides exactly with the formula (11.4) we proved in the previous section.

Exercises

1. Let $A = \mathbb{F}[G]$, where G is a finite group and $\mathbb{F}$ a field. There is, as usual, the map

$$A \to A : x \mapsto \hat{x} = \sum_{g\in G} x(g^{-1})g,$$

which is an isomorphism of the algebra A onto the opposite algebra A^{opp}. Let y be a non-zero idempotent in A. Consider the mapping

$$P : A\hat{y}\times Ay \to k : (a,b) \mapsto \chi_{\mathrm{reg}}(a\hat{b}) = \mathrm{Tr}\big(\rho_{\mathrm{reg}}(a\hat{b})\big) \tag{11.25}$$

where on the right we have the left-regular representation $\rho_{\mathrm{reg}}(x) : A \to A : v \mapsto xv$, and it character given by

$$\chi_{\mathrm{reg}}(x) = \mathrm{Tr}\big(\rho_{\mathrm{reg}}(x)\big).$$

(i) Show that if $a \in A$ is such that

$$\chi_{\mathrm{reg}}(ab) = 0$$

for all $b \in A$, then $a = 0$.

(ii) Let $(Ay)'$ be the dual vector space $\mathrm{Hom}_{\mathbb{F}}(Ay, k)$. Show that the map

$$J : A\hat{y} \to (Ay)' : v \mapsto P(v, \cdot)$$

is an isomorphism. [Hint: Check that $\mathrm{Tr}\big(\rho_{\mathrm{reg}}(v\hat{a})\big) = P(v, ay)$ for all $a \in A$ and $v \in A\hat{y}$. Use this to show that J is injective.]

(iii) Show that for any $x \in A$ and $v \in A\hat{y}$,

$$J(xv) = J(v)\rho_{\mathrm{reg}}(\hat{x})$$

Thus the left regular representation on the left ideal $A\hat{y}$ is isomorphic to the representation $x \mapsto \rho_{\mathrm{reg}}(\hat{x})^*$ on $(Ay)'$.

(iv) For any idempotent $y \in A$, let ρ_u be the representation of G on Ay given by $\rho_u(g)v = gv$, for all $v \in Ay$ (i.e. it is the left-regular representation restricted to Ay). Show that

$$J\rho_{\hat{y}}(g)J^{-1} = \rho_y(g^{-1})^*,$$

for all $g \in G$, and on the right we have the adjoint of $\rho_y(g^{-1}) : Ay \to Ay$.

(v) Show that in the case $G = S_n$, for any idempotent $y \in A[G]$, the characters of the representations $\rho_{\hat{y}}$ and ρ_y are equal.

(vi) Check that if y is a primitive idempotent then so is $\hat{y}$.

Chapter 12

Representations of Unitary Groups

In this chapter we will study the irreducible representations and characters of the group $U(N)$ of $N \times N$ complex unitary matrices. Needless to say, this is no finite group! However, because of the special link between representations of the symmetric group and representations of $U(N)$, it is worth an examination.

The unitary group $U(N)$ consists of all $N \times N$ complex matrices U which are unitary, i.e.

$$U^*U = I$$

It is indeed a group under matrix multiplication. Being a subset of the linear space of all $N \times N$ complex matrices, it is a topological space as well. Multiplication of matrices is, of course, continuous. Moreover, the inversion map $U \mapsto U^{-1} = U^*$ is also continuous.

By a *representation* ρ of $U(N)$ we will mean a continuous mapping

$$\rho : U(N) \to \mathrm{End}_{\mathbb{C}}(V),$$

for some finite dimensional complex vector space V. Thus, $U(N)$ acts on V; for each $U \in U(N)$ there is a linear map

$$\rho(U) : V \to V$$

The *character* of ρ is the function

$$\chi_\rho : U(N) \to \mathbb{C} : U \mapsto \mathrm{tr}\,(\rho(U))$$

As before, the representation ρ is *irreducible* if $V \neq 0$, and the only subspaces of V invariant under the action of $U(N)$ are 0 and V.

12.1 The Haar Integral and Orthogonality of Characters

We shall take for granted a few facts. On the space of complex-valued continuous functions on $U(N)$ there is a linear functional, the *Haar integral*

$$f \mapsto \langle f \rangle = \int_{U(N)} f(U)\, dU$$

such that

$$\langle f \rangle \geq 0 \quad \text{if } f \geq 0,$$

and $\langle f \rangle$ is 0 if and only if f equals 0. Moreover, the Haar integral is invariant under left and right translations in the sense that

$$\int_{U(N)} f(xUy)\, dU = \int_{U(N)} f(U)\, dU \quad \text{for all } x, y \in U(N)$$

and all continuous functions f on $U(N)$. Finally, the integral is normalized:

$$\langle 1 \rangle = 1.$$

Let T denote the subgroup of $U(N)$ consisting of all diagonal matrices. Thus, T consists of all matrices

$$D(\lambda_1, ..., \lambda_N) \stackrel{\text{def}}{=} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_N \end{bmatrix}$$

with $\lambda_1, ...\lambda_N$ are complex numbers of unit modulus.

Thus T is the torus, the product of N copies of the circle group $U(1)$ of unit modulus complex numbers:

$$T \simeq U(1)^N$$

There is a natural Haar integral for continuous functions over T; for any continuous function h on T:

$$\int_T h(t)\,dt = (2\pi)^{-N}\int_0^{2\pi}\dots\int_0^{2\pi} h\big(D(e^{i\theta_1},\dots,e^{i\theta_N})\big)\,d\theta_1\dots d\theta_N \tag{12.1}$$

12.1.1 The Weyl Integration Formula

Recall that a function f on a group is said to be *central* if

$$f(xyx^{-1}) = f(y)$$

for all elements x and y of the group.

For every continuous central function f on $U(N)$ the following integration formula (due to Weyl [19, Section 17]) holds:

$$\boxed{\int_{U(N)} f(U)\,dU = \frac{1}{N!}\int_T f(t)|\Delta(t)|^2\,dt} \tag{12.2}$$

where

$$\Delta\big(D(\lambda_1,\dots,\lambda_N)\big) = \det\begin{bmatrix} \lambda_1^{N-1} & \lambda_2^{N-1} & \cdots & \lambda_{N-1}^{N-1} & \lambda_N^{N-1} \\ \lambda_1^{N-2} & \lambda_2^{N-2} & \cdots & \lambda_{N-1}^{N-2} & \lambda_N^{N-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{N-1} & \lambda_N \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \tag{12.3}$$

$$= \prod_{1\le j<k\le N}(\lambda_j-\lambda_k), \tag{12.4}$$

a well-known identity. This *Vandermonde determinant*, written out as an alternating sum,is:

$$\Delta\big(D(\lambda_1,\dots,\lambda_N)\big) = \sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\lambda_1^{N-\sigma(1)}\dots\lambda_N^{N-\sigma(N)} \tag{12.5}$$

It will be useful to note that diagonal term

$$\lambda_1^{N-1}\lambda_2^{N-2}\dots\lambda_{N-1}^{1}\lambda_N^{0}$$

involves the highest power for each λ_j among all terms.

12.1.2 Schur Orthogonality

Representations ρ_1 and ρ_2 of $U(N)$, on vector space V and W, respectively, are said to be *equivalent* if there is a linear isomorphism

$$T : V_1 \to V_2$$

such that

$$T\rho_1(U)T^{-1} = \rho_2(U) \qquad \text{for all } U \in U(N).$$

If there is no such T then the representations are *inequivalent*.

Note that if ρ_1 and ρ_2 are equivalent then they have the same character.

As with finite groups, every representation is a direct sum of irreducible representations. Hence every character is a sum of irreducible representation characters with positive integer coefficients.

Just as for finite groups, the Schur orthogonality relations hold for representations of $U(N)$: If ρ and ρ' are inequivalent irreducible representations of $U(N)$ then

$$\int_{U(N)} \chi_\rho(U)\chi_{\rho'}(U^{-1})\, dU = 0 \tag{12.6}$$

and

$$\int_{U(N)} \chi_\rho(U)\chi_\rho(U^{-1})\, dU = 1 \tag{12.7}$$

Analogously to the case of finite groups, each $\rho(U)$ is diagonal in some basis, with diagonal entries being of unit modulus. (If U^n is I for some positive integer n then the diagonal entries for a diagonal-matrix form of $\rho(U)$ are roots of unity, and hence of unit modulus; a general element of $U(N)$ is a limit of such U.)

It follows then that

$$\chi_\rho(U^{-1}) = \overline{\chi_\rho(U)} \tag{12.8}$$

The Haar integral specifies a hermitian inner-product on the space of continuous functions on $U(N)$ by

$$\langle f, h\rangle = \int_{U(N)} f(U)\overline{h(U)}\, dU \tag{12.9}$$

In terms of this inner-product the Schur orthogonality relations say that the characters χ_ρ of irreducible representations form an orthonormal set of functions on $U(N)$.

If ρ is a representation of $U(N)$ on a finite-dimensional complex vector space V then, as with finite groups, there is a Hermitian inner-product on V such that $\rho(U)$ is unitary for every $U \in U(N)$ (an analogous statement holds for real vector spaces with orthogonal matrices). Using this it is, of course, clear that each $\rho(U)$ is diagonal in some basis with diagonal entries being unit-modulus complex numbers.

12.2 Characters of Irreducible Representations

We will work out Weyl's explicit formula for the irreducible characters of $U(N)$, as well as their dimensions. Amazingly, everything falls out of Schur orthogonality of characters applied to characters evaluated on diagonal matrices.

12.2.1 Weights

Consider now an irreducible representation ρ of $U(N)$ on a vector space V.

The linear maps

$$\rho(t) : V \to V$$

with t running over the abelian subgroup T, commute with each other:

$$\rho(t)\rho(t') = \rho(tt') = \rho(t't) = \rho(t')\rho(t)$$

and so there is a basis $\{v_j\}_{1\leq j\leq d_V}$ of V with respect to which the matrices of $\rho(t)$, for all $t \in T$, are diagonal:

$$\rho(t) = \begin{bmatrix} \rho_1(t) & 0 & \cdots & 0 \\ 0 & \rho_2(t) & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & \rho_D(t) \end{bmatrix}$$

where

$$\rho_r : T \to U(1)$$

are continuous homomorphisms. Thus,

$$\rho_r\big(D(\lambda_1, ..., \lambda_N)\big) = \rho_{r1}(\lambda_1)...\rho_{rN}(\lambda_N)$$

where $\rho_{rk}(\lambda)$ is ρ_r evaluated on the diagonal matrix which has λ at the $\mathbb{F}$-th diagonal entry and all other diagonal entries are 1. Since each ρ_{rk} is a continuous homomorphism

$$U(1) \to U(1)$$

it necessarily has the form

$$\rho_{rk}(\lambda) = \lambda^{w_{rk}}$$

for some integer w_{rk}. We will refer to

$$\vec{w}_r = (w_{r1}, ..., w_{rN}) \in \mathbb{Z}^N$$

as a *weight* for the representation ρ.

12.2.2 The Weyl Character Formula

Continuing with the framework as above, we have

$$\rho_r\big(D(\lambda_1, ..., \lambda_N)\big) = \lambda_1^{w_{r1}} ... \lambda_N^{w_{rN}}.$$

Thus,

$$\chi_\rho\big(D(\lambda_1, ..., \lambda_N)\big) = \sum_{r=1}^{d_V} \lambda_1^{w_{r1}} ... \lambda_N^{w_{rN}}$$

It will be convenient to write

$$\vec{\lambda} = (\lambda_1, ..., \lambda_N)$$

and analogously for $\vec{w}$.

Two diagonal matrices in $U(N)$ whose diagonal entries are permutations of each other are conjugate within $U(N)$ (permutation of the basis vectors implements the conjugation transformation). Consequently, a character will have the same value on two such diagonal matrices. Thus,

$$\chi_\rho\big(D(\lambda_1, ..., \lambda_N)\big) \text{ is invariant under permutations of the } \lambda_j.$$

Then, by gathering similar terms, we can rewrite the character as a sum of symmetric sums

$$\sum_{\sigma \in S_N} \lambda_{\sigma(1)}^{w_1} ... \lambda_{\sigma(N)}^{w_N}$$

with $\vec{w} = (w_1, ..., w_N)$ running over a certain set of elements in $\mathbb{Z}^N$.

Thus we can express each character as a Fourier sum (with only finitely many non-zero terms)

$$\chi_\rho\big(D(\vec{\lambda})\big) = \sum_{\vec{w}\in\mathbb{Z}^N_\downarrow} c_{\vec{w}} s_{\vec{w}}(\vec{\lambda})$$

where each coefficient $c_{\vec{w}}$ is a non-negative integer, and $s_{\vec{w}}$ is the symmetric function given by:

$$s_{\vec{w}}(\vec{\lambda}) = \sum_{\sigma\in S_N}\prod_{j=1}^{N}\lambda_{\sigma(j)}^{w_j}.$$

The subscript $\downarrow$ in $\mathbb{Z}^N_\downarrow$ signifies that it consists of integer strings

$$w_1 \geq w_2 \geq \ldots \geq w_N.$$

Now ρ is irreducible if and only if

$$\int_{U(N)} |\chi_\rho(U)|^2\, dU = 1.$$

Using the Weyl integration formula, and our expression for χ_ρ, this is equivalent to

$$\int_{U(1)^N} \left|\chi_\rho(\vec{\lambda})\Delta(\vec{\lambda})\right|^2 d\lambda_1 ... d\lambda_N = N! \tag{12.10}$$

Now the product

$$\chi_\rho(\vec{\lambda})\Delta(\vec{\lambda})$$

is *skew-symmetric* in $\lambda_1, ..., \lambda_N$, and is an integer linear combination of terms of the form

$$\lambda_1^{m_1}...\lambda_N^{m_N}.$$

So, collecting together appropriate terms, $\chi_\rho(\vec{\lambda})\Delta(\vec{\lambda})$ can be expressed as an integer linear combination of the elementary skew-symmetric sums

$$\begin{aligned} a_{\vec{f}}(\vec{\lambda}) &= \sum_{\sigma\in S_N} \mathrm{sgn}(\sigma)\lambda_{\sigma(1)}^{f_1}...\lambda_{\sigma(N)}^{f_N} = \sum_{\sigma\in S_N}\mathrm{sgn}(\sigma)\lambda_1^{f_{\sigma(1)}}...\lambda_N^{f_{\sigma(N)}} \\ &= \det\begin{bmatrix} \lambda_1^{f_1} & \lambda_2^{f_1} & \ldots & \lambda_N^{f_1} \\ \lambda_1^{f_2} & \lambda_2^{f_2} & \ldots & \lambda_N^{f_2} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{f_N} & \lambda_2^{f_N} & \ldots & \lambda_N^{f_N} \end{bmatrix} \end{aligned} \tag{12.11}$$

Thus,

$$\int_{U(1)^N} \left|\chi_\rho(\vec{\lambda})\Delta(\vec{\lambda})\right|^2 d\lambda_1...d\lambda_N$$

is an integer linear combination of inner-products

$$\int_{U(1)^N} a_{\vec{f}}(\vec{\lambda})\overline{a_{\vec{f}'}(\vec{\lambda})}\, d\lambda_1 \dots d\lambda_N.$$

Now we use the simple, yet crucial, fact that on $U(1)$ there is the orthogonality relation

$$\int_{U(1)} \lambda^n \overline{\lambda^m}\, d\lambda = \delta_{nm}.$$

Consequently, distinct monomials such as $\lambda_1^{a_1}...\lambda_N^{a_N}$, with $\vec{a} \in \mathbb{Z}^N$, are orthonormal. Hence, if $f_1 > f_2 > \cdots > f_N$, then the first two expressions in (12.11) for $a_{\vec{f}}(\vec{\lambda})$ *are sums of orthogonal terms, each of norm* 1.

If $\vec{f}$ and $\vec{f}'$ are *distinct* elements of $\mathbb{Z}^N_\downarrow$, each a strictly decreasing sequence, then no permutation of the entries of $\vec{w}$ could be equal to $\vec{w}'$, and so

$$\int_{U(1)^N} a_{\vec{f}}(\vec{\lambda})\overline{a_{\vec{f}'}(\vec{\lambda})}\, d\lambda_1 \dots d\lambda_N = 0 \tag{12.12}$$

On the other hand,

$$\int_{U(1)^N} a_{\vec{f}}(\vec{\lambda})\overline{a_{\vec{f}}(\vec{\lambda})}\, d\lambda_1 \dots d\lambda_N = N! \tag{12.13}$$

because $a_{\vec{f}}(\vec{\lambda})$ is a sum of $N!$ orthogonal terms each of norm 1.

Putting all these observations, especially the norms (12.10) and (12.13), together we see that an expression of $\chi_\rho(\vec{\lambda})\Delta(\vec{\lambda})$ as an integer linear combination of the elementary skew-symmetric functions $a_{\vec{f}}$ will involve exactly one of the latter, and with coefficient ± 1:

$$\chi_\rho(\vec{\lambda})\Delta(\vec{\lambda}) = \pm a_{\vec{h}}(\vec{\lambda})$$

for some $\vec{h} \in \mathbb{Z}^N_\downarrow$. To determine the sign here, it is useful to use the lexicographic ordering on $\mathbb{Z}^N$, with $v \in \mathbb{Z}^N$ being $>$ than $v' \in \mathbb{Z}^N$ if the first non-zero entry in $v - v'$ is positive. With this ordering, let $\vec{w}$ be the highest of the weights.

Then the 'highest' term in $\chi_\rho(\vec{\lambda})$ is

$$\lambda_1^{w_1}...\lambda_N^{w_N}$$

appearing with some positive integer coefficient, and the 'highest' term in $\Delta(\vec{\lambda})$ is the diagonal term

$$\lambda_1^{N-1}...\lambda_N^0$$

Thus, the highest term in the product $\chi_\rho(\vec{\lambda})\Delta(\vec{\lambda})$ is

$$\lambda_1^{w_1+N-1}...\lambda_{N-1}^{w_{N-1}+1}\lambda_N^{w_N}$$

appearing with coefficient $+1$.

We conclude that

$$\chi_\rho(\vec{\lambda})\Delta(\vec{\lambda}) = a_{(w_1+N-1,..,w_{N-1}+1,w_N)}(\vec{\lambda}) \tag{12.14}$$

and also that the highest weight term

$$\lambda_1^{w_1}...\lambda_N^{w_N}$$

appears with coefficient 1 in the expression for $\chi_\rho\big(D(\vec{\lambda})\big)$. This gives a remarkable consequence:

Theorem 12.2.1 *In the decomposition of the representation of T given by ρ on V, the representation corresponding to the highest weight appears exactly once.*

The orthogonality relations (12.12) imply that

$$\int_{U(1)^N} \chi_\rho(\vec{\lambda})\overline{\chi_{\rho'}(\vec{\lambda})}|\Delta(\vec{\lambda})|^2\, d\lambda_1...d\lambda_N = 0 \tag{12.15}$$

for irreducible representations ρ and ρ' corresponding to *distinct* highest weights $\vec{w}$ and $\vec{w'}$.

Thus:

Theorem 12.2.2 *Representations corresponding to different highest weights are inequivalent.*

Finally, we also have an explicit expression, Weyl's formula [19, Eq (16.9)], for the character χ_ρ of an irreducible representation ρ, as a ratio of determinants:

$$\boxed{\chi_\rho\big(D(\vec{\lambda})\big) = \frac{a_{(w_1+N-1,..,w_{N-1}+1,w_N)}(\vec{\lambda})}{a_{(N-1,..,1,0)}(\vec{\lambda})}} \tag{12.16}$$

where the denominator is $\Delta(\vec{\lambda})$. The division on the right should be understood as division of polynomials in the indeterminates $\lambda_j^{\pm 1}$.

12.2.3 Weyl dimensional formula

The *dimension* of the representation ρ is equal to $\chi_\rho(I)$, but (12.16) reads $0/0$ on putting $\vec{\lambda} = (1, 1, ..., 1)$ into numerator and denominator. L'Hôpital's rule may be applied, but it is simplified by a trick used by Weyl. Take an indeterminate t, and evaluate the ratio in (12.16) at

$$\vec{\lambda} = (t^{N-1}, t^{N-2}, ..., t, 1)$$

Then $a_{\vec{h}}(\vec{\lambda})$ becomes a Vandermonde determinant

$$\begin{aligned} a_{(h_1,...,h_N)}(t^{N-1}, ..., t, 1) &= \det \begin{bmatrix} t^{h_1(N-1)} & t^{h_1(N-2)} & \dots & t^{h_1} & 1 \\ t^{h_2(N-1)} & t^{h_2(N-2)} & \dots & t^{h_2} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t^{h_N(N-1)} & t^{h_N(N-2)} & \dots & t^{h_N} & 1 \end{bmatrix} \\ &= \prod_{1 \le j < k \le N} \left(t^{h_j} - t^{h_k}\right) \end{aligned}$$

Consequently,

$$\frac{a_{(h_1,...,h_N)}(t^{N-1}, ..., t, 1)}{a_{(h'_1,...,h'_N)}(t^{N-1}, ..., t, 1)} = \prod_{1 \le j < k \le N} \frac{t^{h_j} - t^{h_k}}{t^{h'_j} - t^{h'_k}}$$

Evaluation of the polynomial in t on the right at $t = 1$ yields

$$\prod_{1 \le j < k \le N} \frac{h_j - h_k}{h'_j - h'_k} = \frac{VD(h_1, ..., h_N)}{VD(h'_1, ..., h'_N)},$$

where VD denotes the Vandermonde determinant.

Applying this to the Weyl character formula yields the wonderful Weyl *dimension formula*

$$\boxed{\dim(\rho) = \prod\nolimits_{1 \le j < k \le N} \tfrac{w_j - w_k + k - j}{k - j}} \tag{12.17}$$

for the irreducible representation ρ with highest weight $(w_1, ..., w_N)$.

12.2.4 Representations with given weights

It remains to show that every $\vec{w} \in \mathbb{Z}^N_{\downarrow}$ does correspond to an irreducible representation of $U(N)$. We will produce such a representation inside a tensor product of exterior powers of $\mathbb{C}^N$.

It will be convenient to work first with a vector $\vec{f} \in \mathbb{Z}^N_{\downarrow}$ all of whose components are ≥ 0. We can take $\vec{f}$ to be simply $\vec{w}$, in case all w_j are non-negative. If, on the other hand, $w_j < 0$ then we set

$$f_j = w_j - w_N \qquad \text{for all } j \in \{1, ..., N\}$$

Now consider a tableau of boxes:

$$\begin{array}{cccccccl}
\square & \square & \square & \cdots & \cdots & \cdots & \square & \leftarrow f_1 \text{ boxes} \\
\square & \square & \square & \cdots & \cdots & \square & & \leftarrow f_2 \text{ boxes} \\
\vdots & \vdots & \vdots & \ldots & \vdots & & & \\
\square & \cdots & \square & & & & & \leftarrow f_N \text{ boxes}
\end{array}$$

The first row has f_1 boxes, and is followed beneath by a row of f_2 boxes, and so on, with the N-th row containing f_N boxes. (We ignore the trivial case where all f_j are 0.) Let f'_1 be the number of boxes in column 1, i.e. the largest i for which $f_i \geq 1$. In this way, let f'_j be the number of boxes in column j (i.e. the largest i for which $f_i \geq j$). Now consider

$$V_f = \bigwedge^{f'_1} \mathbb{C}^N \otimes \bigwedge^{f'_2} \mathbb{C}^N \otimes \ldots \otimes \bigwedge^{f'_N} \mathbb{C}^N$$

where the 0-th exterior power is, by definition, just $\mathbb{C}$, i.e. effectively dropped.

The group $U(N)$ acts on this in the obvious way through tensor powers, and we have thus a representation ρ of $U(N)$. The appropriate tensor products of the standard basis vectors $e_1, ..., e_N$ of $\mathbb{C}^N$ form an a basis of V_f, and these basis vectors are eigenvectors of the diagonal matrix

$$D(\vec{\lambda}) \in T,$$

acting on V_f. Indeed, a basis is formed by the vectors

$$e_a = \bigotimes_{j=1}^{N} (e_{a_{1,j}} \wedge \ldots \wedge e_{a_{f'_j,j}}),$$

with each string $a_{i,1}, ..., a_{i,f'_i}$ being strictly increasing and drawn from $\{1, ..., N\}$. We can visualize e_a as being obtained by placing the number $a_{i,j}$ in the box in the i-th row at the j-th column, and then taking the wedge-product of the vectors $e_{a_{i,j}}$ along each column and then taking the tensor product over all the columns:

$$\begin{array}{ccccccc}
\boxed{e_{a_{1,1}}} & \boxed{e_{a_{1,2}}} & \boxed{e_{a_{1,3}}} & \cdots & \cdots & \cdots & \boxed{e_{a_{1,f_1}}} \\
\boxed{e_{a_{2,1}}} & \square & \square & \cdots & \cdots & \square & \\
\vdots & \vdots & \vdots & \dots & \vdots & & \\
\boxed{e_{a_{f_1',1}}} & \cdots & \square & & & &
\end{array}$$

We have

$$\rho\bigl(D(\vec{\lambda})\bigr)e_a = \bigl(\prod_{i,j}\lambda_{a_{i,j}}\bigr)e_a$$

The highest weight term corresponds to precisely e_{a^*}, where a^* has the entry 1 in all boxes in row 1, then the entry 2 in all boxes in row 2, and so on. The eigenvalue corresponding to e_{a^*} is

$$\lambda_1^{f_1}...\lambda_N^{f_N}$$

Note that the corresponding subspace inside V_f is one-dimensional, spanned by e_{a^*}. Decomposing V_f into a direct sum of irreducible subspaces under the representation ρ, it follows that e_{a^*} lies inside (exactly) one of these subspaces. This subspace $V_{\vec{f}}$ must then be the irreducible representation of $U(N)$ corresponding to the highest weight $\vec{f}$.

We took $\vec{f} = \vec{w}$ if $w_N \geq 0$, and so we are done with that case. No suppose $w_N < 0$.

We have to make an adjustment to V_f to produce an irreducible representation corresponding to the original highest weight $\vec{w} \in \mathbb{Z}^N_{\downarrow}$.

Consider then

$$V_{\vec{w}} = V_f \otimes \bigl(\bigwedge^{-N}(\mathbb{C}^N)\bigr)^{\otimes |w_N|}$$

where a negative exterior power is defined through

$$\bigwedge^{-m} V = (\bigwedge^m V)^* \text{ for } m \geq 1.$$

The representation of $U(N)$ on $\bigwedge^{-N}(\mathbb{C}^N)$ is given by

$$U \cdot \phi = (\det U)^{-1}\phi \qquad \text{for all } U \in U(N) \text{ and } \phi \in \bigwedge^{-N}(\mathbb{C}^N)$$

This is a one-dimensional representation with weight $(-1, ..., -1)$, because the diagonal matrix $D(\vec{\lambda})$ acts by multiplication by $\lambda^{-1}...\lambda_N^{-1}$.

For the representation of $U(N)$ on $V_{\vec{w}}$, we have a basis of $V_{\vec{w}}$ consisting of eigenvectors of $\rho\big(D(\vec{\lambda})\big)$; the highest weight is

$$\vec{f} + (-w_N)(-1, ..., -1) = (f_1 + w_N, ..., f_N + w_N) = (w_1, ..., w_N).$$

Thus, $V_{\vec{w}}$ contains an irreducible representation with highest weight $\vec{w}$. But

$$\dim V_{\vec{w}} = \dim V_{\vec{f}},$$

and, on using Weyl's dimension formula, this is equal to the dimension of the irreducible representation of highest weight $\vec{w}$. Thus, $V_{\vec{w}}$ is the desired irreducible representation with highest weight $\vec{w}$.

12.3 Characters of S_n from characters of $U(N)$

We will now see how Schur-Weyl duality leads to a way of determining the characters of S_n from the characters of $U(N)$. As with most of the ideas we have discussed this is due to Weyl [19].

Let $N, n \geq 1$, and consider the vector space $(\mathbb{C}^N)^{\otimes n}$. The permutation group S_n acts on this by

$$\sigma \cdot (v_1 \otimes \ldots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}$$

and the group $GL(N, \mathbb{C})$ of invertible linear maps on $\mathbb{C}^N$ also acts on $(\mathbb{C}^N)^{\otimes n}$ in the natural way:

$$B \cdot (v_1 \otimes \ldots \otimes v_n) = B^{\otimes n}(v_1 \otimes \ldots \otimes v_n) = Bv_1 \otimes \ldots \otimes Bv_n.$$

As we have seen in Chapter 11, these actions are dual in the sense that the commutant of the action of $\mathbb{C}[S_n]$ on $(\mathbb{C}^N)^{\otimes n}$ is the linear span of the operators $B^{\otimes n}$ with B running over $GL(N, \mathbb{C})$.

Since the Lie algebra of $GL(N, \mathbb{C})$, i.e. all $N \times N$ complex matrices, is spanned over the complex field by the Lie algebra of $U(N)$, it follows that the action of S_n and that of $U(N)$ are also dual on $V^{\otimes n}$.

From the Schur-Weyl duality formula it follows that:

$$\mathrm{Tr}(B^{\otimes n} \cdot \sigma) = \sum_{\alpha \in \mathcal{R}} \chi_\alpha(\sigma) \chi^\alpha(B) \tag{12.18}$$

where, on the left, σ represents the action of $\sigma \in S_n$ on $(\mathbb{C}^N)^{\otimes n}$, and $B \in U(N)$, and, on the right, $\mathcal{R}$ is a maximal set of inequivalent representations of S_n. For the representation α of S_n given by the left regular representation on a simple left ideal L_α in $\mathbb{C}[S_n]$, χ^α is the characteristic of the representation of $U(N)$ on

$$y_\alpha(\mathbb{C}^N)^{\otimes n},$$

where y_α is a non-zero idempotent in L_α.

Now the simple left ideals in $\mathbb{C}[S_n]$ correspond to

$$\vec{f} = (f_1, ..., f_n) \in \mathbb{Z}^n_{\geq 0,\downarrow}$$

(the subscript $\downarrow$ signifying that $f_1 \geq \ldots \geq f_n$) which are partitions of n:

$$f_1 + f_2 + \ldots + f_n = n.$$

Recall that associated to this partition we have a Young tableau $T_{\vec{f}}$ of the numbers $1, ..., n$ in rows of boxes:

$$\begin{array}{ccccccc}
\boxed{1} & \boxed{2} & \boxed{3} & \ldots & \ldots & \ldots & \boxed{f_1} \\
\boxed{f_1+1} & \boxed{f_1+2} & \boxed{f_1+3} & \ldots & \ldots & \boxed{f_1+f_2} & \\
\vdots & \vdots & \vdots & \ldots & \vdots & & \\
\boxed{\sum_{j<n} f_j} & \ldots & \boxed{n} & & & &
\end{array}$$

and associated to this we have an idempotent

$$y_{\vec{f}} = \sum_{q \in C_{T_{\vec{f}}}, p \in R_{T_{\vec{f}}}} (-1)^{\mathrm{sgn}\,(q)} qp$$

where $C_{T_{\vec{f}}}$ is the subgroup of S_n which, acting on the tableau $T_{\vec{f}}$, maps the entries of each column into the same column, and $R_{T_{\vec{f}}}$ preserves rows. Let

$$a_{ij} \in \{1, ..., n\}$$

be the entry in row i column j in the tableau $T_{\vec{f}}$. For example,

$$a_{21} = f_1 + 2$$

Let $e_1, ..., e_N$ be the standard basis of $\mathbb{C}^N$, as usual. Place e_1 in each of the boxes in the first row, then e_2 in each of the boxes in the second row, and so on. Let

$$e^{\otimes \vec{f}} = e_1^{\otimes f_1} \otimes \ldots \otimes e^{\otimes f_n}$$

be the tensor product of these vectors. Then

$$y_{\vec{f}}e^{\otimes \vec{f}}$$

is a multiple of

$$\sum_{q \in C_{T_{\vec{f}}}} (-1)^{\operatorname{sgn}(q)} q e^{\otimes \vec{f}}$$

Let θ be the permutation that rearranges the entries in the tableau such that as one reads the new tableau book-style (row 1 left to right, then second row left to right, and so on) the numbers are as in $T_{\vec{f}}$ read down column 1 first, then down column 2, and so on:

$$\theta : a_{ij} \mapsto a_{ji}$$

Then $y_{\vec{f}}e^{\otimes \vec{f}}$ is a multiple of θ applied to

$$\otimes_{j \geq 1} \wedge_{i \geq 1} e_{a_{ij}}$$

Thus

$$y_{\vec{f}}(\mathbb{C}^N)^{\otimes n} \neq 0$$

provided the columns in the tableau $T_{\vec{f}}$ have at most N entries each.

Under the action of a diagonal matrix

$$D(\vec{\lambda}) \in U(N)$$

with diagonal entries given by

$$\vec{\lambda} = (\lambda_1, ..., \lambda_N),$$

on $(\mathbb{C}^N)^{\otimes n}$, the vector $y_{\vec{f}}e^{\otimes \vec{f}}$ is an eigenvector with eigenvalue

$$\lambda^{f_1} ... \lambda_N^{f_N}$$

Clearly, the highest weight for the representation on $y_{\vec{f}}(\mathbb{C}^N)^{\otimes n}$ is $\vec{f}$.

Returning to the Schur-Weyl character duality formula and using in it the character formula for $U(N)$ we have

$$\operatorname{Tr}\left(D(\vec{\lambda})^{\otimes n} \cdot \sigma\right) = \sum_{\vec{w}} \chi_{\vec{w}}(\sigma) \frac{a_{(w_1+N-1,..,w_{N-1}+1,w_N)}(\vec{\lambda})}{a_{(N-1,..,1,0)}(\vec{\lambda})} \tag{12.19}$$

where the sum is over all $\vec{w} \in \mathbb{Z}^N_{\geq 0,\downarrow}$ partitioning n, i.e. with $|\vec{w}| = n$.

Multiplying through in (12.19) by the Vandermonde determinant in the denominator on the right, we have

$$\mathrm{Tr}\left(D(\vec{\lambda})^{\otimes n} \cdot \sigma\right) a_{(N-1,..,1,0)}(\vec{\lambda}) = \sum_{\vec{w}\in\mathbb{Z}^N_{\geq 0,\downarrow}, |\vec{w}|=n} \chi_{\vec{w}}(\sigma) a_{(w_1+N-1,..,w_{N-1}+1,w_N)}(\vec{\lambda}) \tag{12.20}$$

To obtain the character value $\chi_{\vec{w}}(\sigma)$ we consider

$$\mathrm{Tr}\left(D(\vec{\lambda})^{\otimes n} \cdot \sigma\right) a_{(N-1,..,1,0)}(\vec{\lambda})$$

as a polynomial in $\lambda_1, ..., \lambda_N$. Examining the right side in (12.20), we see that

$$w_1 + N - 1 > w_2 + N - 2 > \ldots > w_{N-1} + 1 > w_N$$

and the coefficient of

$$\lambda_1^{w_1+N-1} ... \lambda_1^{w_N}$$

is precisely $\chi_{\vec{w}}(\sigma)$.

Chapter 13

Frobenius Induction

In this chapter we examine Frobenius' method of constructing a representation of a larger group G from a given representation of a subgroup H.

13.1 Construction of the Induced Representation

Consider a finite group G, with a subgroup H acting on G by multiplication on the right. We have then the quotient

$$G/H = \{xH : x \in G\}$$

of all left cosets of H in G. The quotient map is

$$\begin{array}{cc} G & x \\ \pi_H \downarrow & \\ G/H & xH \end{array}$$

The group H acts on G on the right:

$$G \times H \to G : (g, h) \mapsto R_h g \stackrel{\text{def}}{=} gh \tag{13.1}$$

and maps each fiber $\pi_H^{-1}(xH)$ bijectively onto itself.

The left and right multiplication actions of G on itself induce actions on the space of functions on G with values in any set E. For any $f : G \to E$ and $y, x \in G$ we have

$$L_y f : g \mapsto f(y^{-1}g), \qquad R_y f : g \mapsto f(gy) \tag{13.2}$$

They are both left actions and clearly commute with each other:

$$L_h R_k = R_k L_h.$$

Assume now that E is a vector space, and ρ a representation of H on G. We have then the set $\mathcal{E}_\rho$ of all 'fields' on G of symmetry type ρ, i.e. all functions

$$f : G \to E$$

satisfying the equivariance property

$$R_h f(g) = f(gh) = \rho(h^{-1})f(g) \quad \text{for all } g \in G \text{ and } h \in H. \tag{13.3}$$

Then $\mathcal{E}_\rho$ is a vector space. Moreover, since L_x commutes with both R_h and $\rho(h^{-1})$, we obtain a representation L_ρ of G on $\mathcal{E}_\rho$:

$$L_\rho(x) : \mathcal{E}_\rho \to \mathcal{E}_\rho : f \mapsto L_\rho(x)f \stackrel{\text{def}}{=} L_x f \tag{13.4}$$

Thus, beginning with a representation ρ of the subgroup H, we have obtained a representation L_ρ of the larger group G. This representation is called the representation *induced* from the representation ρ of the subgroup H.

13.2 Universality of the Induced Representation

The induced representation has a certain universal property, which we shall describe first in categorical language and then in more detail, and prove this property.

Consider, as before, a subgroup H of a finite group G, and a representation ρ of H on a vector space E. As explained in the previous section, there is then a vector space $\mathcal{E}_\rho$ and a representation L_ρ of G on $\mathcal{E}_\rho$. Consider now a category whose objects are H-linear maps

$$j : E \to \mathcal{E}$$

where $\mathcal{E}$ is a vector space on which G is represented. A morphism from $j : E \to \mathcal{E}$ to $j' : E \to \mathcal{E}'$ is a G-linear map

$$a : \mathcal{E} \to \mathcal{E}'$$

such that

$$a \circ j = j'$$

In this section we will construct an object $i : E \to \mathcal{E}_\rho$ in this category which is an *initial* element in the sense that there is a unique morphism from this object to any object in this category.

For each vector $v \in E$ let $i(v)$ be the function on G with value v at e and 0 off the subgroup H:

$$i(v) : y \mapsto \begin{cases} \rho(y^{-1})v & \text{if } y \in H; \\ 0 \end{cases} \tag{13.5}$$

Then $i(v)$ belongs to $\mathcal{E}_\rho$ and

$$i : E \to \mathcal{E}_\rho$$

is a linear injection.

It is readily checked that i is H-linear.

Let E_0 be the image of i in $\mathcal{E}_\rho$:

$$E_0 = i(E) \subset \mathcal{E}_\rho$$

Now consider any decomposition of G as a union of disjoint cosets:

$$G = x_1 H \cup \ldots \cup x_r H$$

If $f \in \mathcal{E}_\rho$ then cutting it off to 0 outside any $x_i H$ yields again an element of $\mathcal{E}_\rho$:

$$1_{x_i H} f \in \mathcal{E}_\rho$$

Thus, we can decompose f as

$$f = 1_{x_1 H} f + \cdots + 1_{x_r H} f \in \mathcal{E}_\rho$$

where each $1_{x_i H}$ is supported on the coset $x_i H$ in the sense that it is zero off this coset. Thus any $f \in \mathcal{E}_\rho$ can be expressed as

$$f = L_{x_1} f_1 + \cdots + L_{x_r} f_r$$

where $f_1, ..., f_r \in E_0$, with f_i being $L_{x_i^{-1}}(1_{x_i H} f)$. This gives a direct sum decomposition

$$\mathcal{E}_\rho = x_1 E_0 \oplus \ldots \oplus x_r E_0$$

where

$$x_i E_0 = L_{x_i} E_0.$$

Now consider any G-module F and an H-linear map

$$i' : E \to F$$

Define

$$\phi : \mathcal{E}_\rho \to F$$

by requiring that on E_0, ϕ is given by

$$\phi(f) = i'\big(f(e)\big) \qquad \text{for all } f \in E_1,$$

and that ϕ is G-linear, i.e.

$$\phi(L_x f) = x\phi\big(f(e)\big) \qquad \text{for all } x \in G.$$

It is readily checked that ϕ is well-defined. Moreover, ϕ is the only map with these properties.

Thus we have proved the universal property:

Theorem 13.2.1 *If H is a subgroup of a finite group G, and $\rho : H \to \mathrm{End}_{\mathbb{F}}(E)$ is a representation of H, then there is a representation $G \to \mathrm{End}_{\mathbb{F}}(\mathcal{E}_\rho)$ of G and an H-linear map*

$$i : E \to \mathcal{E}_\rho$$

such that for any H-linear map $i' : E \to E'$, where $G \to \mathrm{End}_{\mathbb{F}}(E')$ is a representation of G, there is a unique G-linear map

$$\phi : \mathcal{E}_\rho \to E'$$

such that $\phi \circ i = i'$.

13.3 Character of the Induced Representation

We continue with the notation and hypotheses of the preceding section.

Let

$$G = x_1 H \cup \ldots \cup x_r H$$

be a decomposition of G into disjoint cosets. For any $g \in G$ the map

$$L_g : \mathcal{E}_\rho \to \mathcal{E}_\rho$$

carries the subspace $x_i E_0$ bijectively onto $g x_i E_0$, which is the subspace of functions vanishing outside the coset $g x_i H$. Thus, $g x_i E_0$ equals $x_i E_0$ if and only if $x_i^{-1} g x_i$ is in H. So the map L_g has zero trace if g is not conjugate to any element in H. If g is conjugate to an element h of H then

$$\mathrm{Tr}(L_g) = n_g \mathrm{Tr}(L_h|E_0) = n_g \chi_E(h),$$

where n_g is the number of i for which $x_i^{-1} g x_i$ is in H.

We can summarize these observations in:

Theorem 13.3.1 *The characteristic of the induced representation is given by*

$$\chi_{\mathcal{E}_\rho}(g) = \frac{1}{|H|} \sum_{x \in G} \chi_E^0(x^{-1} g x) \tag{13.6}$$

where χ_E^0 is equal to the characteristic of the representation ρ (of H on E) on $H \subset G$ and is 0 outside H.

The division by $|H|$ in (13.6) is needed because each x_i for which $x_i^{-1} g x_i$ is in H is counted $|H x_i|$ (i.e. $|H|$) times in the sum on the right (13.6).

Chapter 14

Representations of Clifford Algebras

In this chapter we will apply the results on representations of semisimple algebras we studied in Chapter 4 to an important special algebra called a Clifford algebra. This algebra arises in the study of rotation groups, but for our purposes we will focus just on the algebra itself. Since we are primarily interested in representations on vector spaces over algebraically closed fields, we will define the Clifford algebra also simply in the case of fields which allow square-roots of -1.

We will work with a fixed integer $d \geq 1$, and a field $\mathbb{F}$ which is of characteristic $\neq 2$, and contains contains $i = \sqrt{-1}$. We use the notation

$$[d] = \{1, ..., d\}.$$

14.1 Clifford Algebra

The Clifford algebra with d generators $e_1, ..., e_d$ over a field $\mathbb{F}$ is the associative algebra, with unit element generated by these elements, subject to the relations

$$\{e_r, e_s\} \stackrel{\text{def}}{=} e_r e_s + e_s e_r = 2\delta_{rs}1 \qquad \text{for all } r, s \in \{1, ..., d\}. \tag{14.1}$$

A basis of the algebra is given by all products of the form

$$e_{s_1} ... e_{s_m},$$

where $m \geq 0$, and $1 \leq s_1 < s_2 < \cdots \leq d$. Writing S for such a set $\{s_1, ..., s_m\} \subset \{1, ..., d\}$, with the elements s_i always in increasing order, we see that the algebra has a basis consisting of one element e_S for each subset S of $\{1, ..., d\}$. This leads to the formal construction of the algebra discussed below in subsection 14.1.1. Notice also that the condition (14.1) implies that every time a term $e_s e_t$, with $s > t$, is replaced by $e_t e_s$, one picks up a minus sign:

$$e_t e_s = -e_s e_t \qquad \text{if } s \neq t.$$

Keeping in mind also the condition $e_s^2 = 1$ for all $s \in [d]$, we have

$$e_S e_T = \epsilon_{ST} e_{S\Delta T}, \tag{14.2}$$

where $S\Delta T$ is the symmetric difference of the sets S and T, and

$$\epsilon_{ST} = \prod_{s\in S, t\in T} \epsilon_{st},$$

$$\epsilon_{st} = \begin{cases} +1 & \text{if } s < t; \\ +1 & \text{if } s = t; \\ -1 & \text{if } s > t, \end{cases} \tag{14.3}$$

that the empty product (which occurs if S or T is $\emptyset$) is taken to be 1.

14.1.1 Formal Construction

Now we can construct the algebra $\mathcal{C}_d$ officially. We take $\mathcal{C}_d$ to be the free vector space, with scalars in $\mathbb{F}$, over the set of all subsets of $[d]$. Denote the basis element of $\mathcal{C}_d$ corresponding to $S \subset [d]$ by e_S. Thus, every $x \in \mathcal{C}_d$ is expressed uniquely as a linear combination

$$x = \sum_{S\subset[d]} x_S e_S \qquad \text{with all } x_S \text{ in } \mathbb{F}.$$

Define a bilinear multiplication operation on $\mathcal{C}_d$ by (14.2):

$$e_S e_T = \epsilon_{ST} e_{S\Delta T}, \tag{14.4}$$

Clearly, the product is symmetric. It is also associative, as is seen from

$$e_S(e_T e_R) = \epsilon_{ST}\epsilon_{TR}\epsilon_{SR} e_{S\Delta T\Delta R} = (e_S e_T)e_R \tag{14.5}$$

Note that the dimension of $\mathcal{C}_d$ is the number of subsets of $\{1, ..., d\}$, i.e.

$$\dim \mathcal{C}_d = 2^d.$$

The element $e_\emptyset$ is the multiplicative identity element in the algebra and will usually just be denoted 1. A coefficient $x_\emptyset$ will often be written simply as x_0:

$$x_0 = x_\emptyset.$$

Observe that if $S = \{s_1, ..., s_n\}$, where $s_1 < ... < s_n$, then

$$e_{s_1} ... e_{s_n} = e_S,$$

a relation which motivated the formal construction. We will often write this element as

$$e_{s_1 ... s_n}.$$

Thus, $e_{1...d}$ means $e_{[d]}$.

14.1.2 The Center of $\mathcal{C}_d$

The nature of the center $Z(\mathcal{C}_d)$ of $\mathcal{C}_d$ depends on whether d is even or odd.

An element z of $\mathcal{C}_d$ lies in the center if and only if it commutes with each e_r, i.e. if and only if

$$e_r z e_r = z$$

holds for every $r \in [d]$. Let us analyze this relation, using the expression of z in terms of the standard basis in $\mathcal{C}_d$:

$$z = \sum_{S \subset [d]} z_S e_S.$$

For any $r \in [d]$, the product $e_r e_S e_r$ equals $\pm e_S$, and so the mapping

$$x \mapsto e_r x e_r$$

has the effect of replacing some of the coefficients x_S with their negatives. In particular, if $|S|$ is odd and $r \notin S$, or if $|S|$ is even and $r \in S$, then

$$e_r e_S e_r = -e_S,$$

and so

$$(e_r x e_r)_S = -x_S.$$

Thus if z is in the center of $\mathcal{C}_d$ then, and any $r \notin S$ with $|S|$ odd, or $r \in S$ with $|S|$ even, we have z_S equal to its own negative. Since the characteristic of the field $\mathbb{F}$ is not 2, we see that if z is in the center of the algebra then

$$z_S = 0 \qquad \text{if } |S| \neq 0 \text{ is even or if it is odd and } |S| < d.$$

In particular, if d is a positive even integer then the coefficient z_S is 0 for all non-empty S. Thus, if d is even then the center of $\mathcal{C}_d$ consists of just the scalar multiples of the identity element 1.

Now suppose d is odd. The arguments in the preceding paragraph work for all non-empty $S \subset [d]$ *except* for $S = [d]$. Moreover, it is readily checked that, for d odd, $e_{[d]}$ commutes with every e_j and so is in the center of $\mathcal{C}_d$. Thus, the center, in this case, consists of all linear combinations of $1 = e_\emptyset$ and $e_{[d]}$.

We can now summarize the results for the center of the Clifford algebra. Note that we have only needed to use, in the preceding arguments, that $1 \neq -1$ in the field $\mathbb{F}$. If $1 + 1 = 0$ in $\mathbb{F}$ then, going back to the defining relations for the Clifford algebra, we see that the algebra is then commutative.

Proposition 14.1.1 *Let $\mathcal{C}_d$ be the Clifford algebra with d generators, over any field $\mathbb{F}$ of scalars. If the characteristic of $\mathbb{F}$ is not 2 then the center of $\mathcal{C}_d$ is*

$$Z(\mathcal{C}_d) = \begin{cases} k1 & \text{if } d \text{ is even;} \\ k1 + ke_{[d]} & \text{if } d \text{ is odd.} \end{cases} \tag{14.6}$$

If $\mathbb{F}$ has characteristic 2 then $\mathcal{C}_d$ is abelian.

14.2 Semisimple Structure of the Clifford Algebra

Let us recall some structure theory for semisimple algebras. If A is a semisimple algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0, then the center $Z(A)$, as a $\mathbb{F}$-vector-space has a basis $u_1, ..., u_C$, where each u_j is a non-zero idempotent, with $u_j u_m = 0$ if $j \neq m$, and

$$1 = u_1 + \cdots + u_C.$$

The algebra A is the product of the 2-sided ideals $A_j = Au_j$, viewed as $\mathbb{F}$-algebras. The 2-sided ideal A_j is a direct sum of simple left ideals, all isomorphic to one simple left ideal L_j. If $j \neq m$ then the simple left ideals L_j and L_m are not isomorphic; every simple left A-module is isomorphic to exactly one of the L_j.

In view of the results about the center of $\mathcal{C}_d$ we can see that, if $\mathcal{C}_d$ is semisimple then the algebra $\mathcal{C}_d$ is simple if d is even, and is the direct sum of two 2-sided ideals if d is odd. In the latter case the ideals are generated by the two idempotents

$$\frac{1}{2}(1 \pm i^{d(d-1)/2} e_{[d]}),$$

where the right side arises from $e_{[d]}^2$. Note that $d(d-1)$ being even, the power $i^{d(d-1)/2}$ is an integer power of i and is therefore in the field $\mathbb{F}$.

For even d we know that, if $\mathcal{C}_d$ is semisimple then it is, in fact, simple, and hence would be the direct sum of p semisimple left ideals, each of which is p-dimensional. Thus, p must be $2^{d/2}$. We shall construct these simple left ideals directly using idempotents, and thereby prove semisimplicity of $\mathcal{C}_d$ as by-product. Semisimplicity of $\mathcal{C}_d$ can, of course, readily be checked directly.

14.2.1 Structure of $\mathcal{C}_d$ for $d \leq 2$

We shall work out the structure of Clifford algebras generated by one and two elements. We work with a field of characteristic $\neq 2$. For the case of two generator algebras, we shall also need to use the square root $i = \sqrt{-1}$ in $\mathbb{F}$.

First consider the Clifford algebra E_y generated by one element y. Then

$$E_y = k1 + ky.$$

Since

$$y^2 = 1,$$

the elements

$$u_y = \frac{1+y}{2} \qquad \text{and} \qquad u_{y,-} = \frac{1-y}{2}$$

are idempotents. Moreover,

$$yu_{y,+} = u_{y,+} \qquad \text{and} \qquad yu_{y,-} = -u_{y,-},$$

and

$$u_{y,+}u_{y,-} = 0 = u_{y,-}u_{y,+}.$$

Then E_y decomposes as the internal direct sum of left ideals:

$$E_y = E_y u_{y,+} \oplus E_y u_{y,-}.$$

This being a decomposition of the two-dimensional vector space E_y into non-zero subspaces, each is a one-dimensional subspace

$$E_y u_{y,\pm} = k u_{y,\pm}.$$

Being one-dimensional, these are necessarily simple left ideals in E_y. These left ideals are non-isomorphic as E_y-modules, because if

$$f : E_y u_{u,+} \to E_y u_{u,-}$$

were an E_y-linear mapping then $f(u_{y,+})$ would be $cu_{y,-}$ for some $c \in k$, and then

$$f(u_{u,+}) = f(u_{y,+}u_{y,+}) = u_{y,+}f(u_{y,+}) = cu_{y,+}u_{y,-} = 0.$$

In other words,

$$\mathrm{End}_{E_y}(E_y u_{y,+}, E_y u_{y,-}) = 0.$$

Now we move on to the Clifford algebra E_α generated by a pair

$$\alpha = \{r, s\}$$

of elements r, s. Then

$$E_\alpha = k1 + ke_\alpha + ke_r + ke_s.$$

We have

$$(ie_\alpha)^2 = 1,$$

and so

$$u_{\alpha,+} = \frac{1 + ie_\alpha}{2} \qquad \text{and} \qquad u_{\alpha,-} = \frac{1 - ie_\alpha}{2}$$

are idempotents.

In terms of the two idempotents $u_{\alpha,\pm}$ the algebra decomposes as the internal direct sum of left ideals $E_\alpha u_{\alpha,\pm}$:

$$E_\alpha = E_\alpha u_{\alpha,+} + E_\alpha u_{\alpha,-}$$

We observe also that, for $j \in \alpha$,

$$e_j u_{\alpha,+} = u_{\alpha,-} e_j \qquad \text{and} \qquad u_{\alpha,+} e_j = e_j u_{\alpha,-},$$

and

$$ie_\alpha u_{\alpha,+} = u_{\alpha,+} \qquad \text{and} \qquad ie_\alpha u_{\alpha,-} = u_{\alpha,-}.$$

Consequently, for any $x \in E_\alpha$,

$$xu_{\alpha,\pm}x$$

is a $\mathbb{F}$-multiple of $u_{\alpha,\pm}$. Therefore, by the simple result in Theorem 4.5.1 (i), the element $u_{\alpha,\pm}$ is a *primitive idempotent* in E_α, and the left ideals $E_\alpha u_{\alpha,\pm}$ are simple.

For the pair $\alpha = \{r, s\}$, with $r < s$, an ordered basis of $E_\alpha u_{\alpha,+}$ as a $\mathbb{F}$-vector-space is given by

$$e_r u_{\alpha,+}, \qquad u_{\alpha,+}. \tag{14.7}$$

Observing that

$$\begin{aligned} e_s e_r u_{\alpha,+} &= i(ie_r e_s)u_{\alpha,+} = iu_{\alpha,+} \\ e_s u_{\alpha,+} &= (-ie_r)(ie_r e_s)u_{\alpha,+} = -ie_r u_{\alpha,+}, \end{aligned} \tag{14.8}$$

we see that the matrices for multiplication on the left by e_r and by e_s on $E_\alpha u_{\alpha,+}$, relative to the basis (14.7) are

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \tag{14.9}$$

respectively. Similarly, the matrices for multiplication on the left by e_r and by e_s on $E_\alpha u_{\alpha,-}$, relative to the ordered basis

$$u_{\alpha,-}, \qquad e_r u_{\alpha,-} \tag{14.10}$$

are

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \tag{14.11}$$

respectively.

The mapping

$$f : E_{\alpha,+} \to E_{\alpha,-} : x \mapsto xe_r u_{\alpha,-}$$

is clearly E_α-linear, and has the following action on the basis in $E_{\alpha,+}$:

$$\begin{aligned} f(u_{\alpha,+}) &= e_r u_{\alpha,-} \\ f(e_r u_{\alpha,-}) &= u_{\alpha,+}. \end{aligned}$$

Thus, f is an isomorphism of the simple left modules $E_{\alpha,\pm}$.

14.2.2 Structure of $\mathcal{C}_d$ for even d

Assume that d is even. Fix a partition of $[d]$ into pairs, say

$$P_d = \{\{1,2\},\{3,4\},...,\{d-1,d\}\} \qquad \text{for even } d. \tag{14.12}$$

For each $\alpha = \{r, r+1\} \in P_d$, the two elements e_r and e_{r+1} generate the subalgebra

$$E_\alpha = k1 + ke_{r,r+1} \qquad + \qquad ke_r + ke_{r+1},$$

in which there are the two primitive idempotents

$$u_{\alpha,\pm} = \frac{1}{2}(1 \pm ie_\alpha).$$

We will construct primitive idempotents in $\mathcal{C}_d$ by taking products of the $u_{\alpha,\pm}$.

To this end let us first observe

Lemma 14.2.1 *The mapping*

$$E_{12} \times \cdots \times E_{d-1,d} \to \mathcal{C}_d : (a_{12},...,a_{d-1,d}) \mapsto a_{12}...a_{d-1,d} \tag{14.13}$$

induces a $\mathbb{F}$-linear isomorphism

$$f : E_{12} \otimes \cdots \otimes E_{d-1,d} \to \mathcal{C}_d \tag{14.14}$$

Proof The mapping in (14.13) is multilinear over the scalars in $\mathbb{F}$, and so induces a linear map f in (14.14). A basis of the tensor product $E_{12} \otimes \cdots \otimes E_{d-1,d}$ is given by all the elements of the form $e_{S_{12}} \otimes \cdots \otimes e_{S_{d-1,d}}$, where S_α runs over subsets of α for each $\alpha \in P_d$. The map f carries this basis element to $e_{S_{12}\cup...\cup S_{d-1,d}}$; thus f maps a basis to a basis, and so is an isomorphism. $\boxed{\text{QED}}$

Let ϵ be any mapping

$$\epsilon : P_d \to \{+,-\} : \{r,s\} \mapsto \epsilon_{rs} = \epsilon_{sr}.$$

Associate to this the idempotent

$$u_\epsilon = \prod_{\alpha \in P_d} u_{\alpha,\epsilon_\alpha} = \prod_{\{r,r+1\}\in P_d} \frac{1 + \epsilon_{r,r+1} ie_r e_{r+1}}{2}. \tag{14.15}$$

Note first that this is not 0. The following relations are readily checked:

$$\begin{aligned}\sum_{\epsilon\in P_d} u_\epsilon &= 1\\ u_\epsilon^2 &= u_\epsilon\\ u_{\epsilon'}u_\epsilon &= 0 \quad \text{if } \epsilon\neq\epsilon'.\end{aligned} \tag{14.16}$$

We also observe that

$$e_r u_\epsilon = u_{\epsilon_r} e_r, \tag{14.17}$$

where ϵ_r agrees with ϵ on all pairs except on the pair which contains r. Moreover, for any $\{r, r+1\}\in P_d$, we have

$$ie_r e_{r+1} u_\epsilon = \epsilon_{r,r+1} u_\epsilon \tag{14.18}$$

Lemma 14.2.2 *The idempotents u_ϵ are primitive.*

Proof Suppose A_1 and A_2 are subalgebras of a $\mathbb{F}$-algebra A such that the mapping

$$f : A_1\otimes A_2 \to A : a_1\otimes a_2 \mapsto a_1a_2$$

is a $\mathbb{F}$-linear isomorphism, and suppose $u_i \in A_i$ is a primitive idempotent with the property that $u_i x u_i$ is a $\mathbb{F}$-multiple of u_i for every $x \in A_i$, for $i\in\{1,2\}$. Assume, moreover, that u_1 commutes with every element of A_2, and u_2 commutes with every element of A_1. Then, for any $a_1 \in A_1$ and $a_2 \in A_2$, we have

$$u_1u_2a_1a_2u_1u_2 = u_1a_1(u_2a_2u_2)u_1 \in ku_1a_1u_1u_2 \subset ku_1u_2.$$

Then, by Theorem 4.5.1 (i) (whose proof is simple), it follows that u_1u_2 is a primitive idempotent in A. Applying this inductively to the subalgebras $E_{12}, E_{34}, .., E_{d-1,d}$, and the idempotents $u_{\alpha,\pm}$ (primitive inside E_α), we see that the products u_ϵ are primitive, for every $\epsilon\in\{+,-\}^{P_d}$. $\boxed{\text{QED}}$

The maps ϵ run over the set

$$\{+,-\}^{P_d},$$

which contains $2^{d/2}$ elements.

To summarize:

Proposition 14.2.1 *Suppose $d \geq 2$ is an even number, and P_d the set of pairs $\{\{1,2\},\{3,4\},...,\{d-1,d\}\}$. Then for every $\epsilon \in \{+,-\}^{P_d}$, the element u_ϵ is a primitive idempotent. Moreover,*

$$u_{\epsilon'}u_\epsilon = 0 \qquad \text{if } \epsilon \neq \epsilon',$$

and

$$\sum_{\epsilon \in \{+,-\}^{P_d}} u_\epsilon = 1.$$

Next we show that the u_ϵ generate isomorphic left ideals:

Lemma 14.2.3 *Let $d \geq 2$ be an even integer. For any $\epsilon, \epsilon' \in \{+,-\}^{P_d}$, the left ideals $\mathcal{C}_d u_\epsilon$ and $\mathcal{C}_d u_{\epsilon'}$ are isomorphic as left $\mathcal{C}_d$-modules.*

Proof The key to this is the observation that if $r < s$ in $[d]$, then

$$u_{rs,+}e_r = e_r u_{rs,-}, \qquad \text{and} \qquad u_{rs,-}e_r = e_r u_{rs,+}.$$

Thus to convert u_ϵ into $u_{\epsilon'}$ we can multiply by a suitable product y of the e_r's. Let $D = \{\alpha \in P_d : \epsilon_\alpha \neq \epsilon'_\alpha\}$. Let us write each pair as $\alpha = \{\alpha_1, \alpha_2\}$, with $\alpha_1 < \alpha_2$. Let

$$y = \prod_{\alpha \in D} e_{\alpha_1},$$

where the product is, say, in increasing order of the subscripts α_1. Then

$$u_\epsilon y = y u_{\epsilon'}.$$

The map

$$h : \mathcal{C}_d u_\epsilon \to \mathcal{C}_d u_{\epsilon'} : a u_\epsilon \mapsto a u_\epsilon y = a y u_{\epsilon'}$$

is $\mathcal{C}_d$-linear, maps the generator u_ϵ to the non-zero element $y u_{\epsilon'} \in \mathcal{C}_d u_{\epsilon'}$ (it is non-zero because in the expansion of $y u_{\epsilon'}$ in the standard basis, the coefficient of y is 1). Therefore, h is an isomorphism of the simple left ideals.

We can now summarize all our results for even d. For this, recall that an algebra is said to be simple if it is a direct sum of isomorphic simple left ideals.

Theorem 14.2.1 *For any even positive integer d, the Clifford algebra $\mathcal{C}_d$ generated by elements $e_1, ..., e_d$, over a field $\mathbb{F}$ which contains $i = \sqrt{-1}$ and has characteristic $\neq 2$, is a simple algebra which decomposes as the direct sum of $2^{d/2}$ simple left-ideals:*

$$\mathcal{C}_d = \bigoplus_{\epsilon \in \{+,-\}^{P_d}} \mathcal{C}_d u_\epsilon,$$

where $P_d = \{\{1,2\}, ..., \{d-1,d\}\}$, and

$$u_\epsilon = \prod_{\{r,r+1\} \in P_d} \frac{1 + \epsilon_{r,r+1} e_r e_{r+1}}{2}.$$

Each left ideal $\mathcal{C}_d u_\epsilon$ is a $2^{d/2}$-dimensional vector space over $\mathbb{F}$, and the simple left ideals $\mathcal{C}_d u_\epsilon$ are isomorphic to each other as $\mathcal{C}_d$-modules, for all $\epsilon \in \{+,-\}^{P_d}$. A basis of the vector space $\mathcal{C}_d u_\epsilon$ is given by the vectors $e_R u_\epsilon$, with R running over all subsets of $[d]_{\mathrm{odd}}$, the set of odd integers in $[d] = \{1, ..., d\}$.

Let us also make another observation.

Lemma 14.2.4 *For any $\epsilon \in \{+,-\}^{P_d}$, the simple left ideal in $\mathcal{C}_d$ generated by u_ϵ can be expressed as*

$$\mathcal{C}_d u_\epsilon = f(E_{12,\epsilon_{12}} \otimes \cdots \otimes E_{\{d-1,d\},\epsilon_{d-1,d}}), \tag{14.19}$$

where $f : E_{12} \otimes \cdots \otimes E_{d-1,d} \to \mathcal{C}_d$ is specified by $f(a_{12} \otimes \cdots \otimes a_{d-1,d}) = a_{12} ... a_{d-1,d}$.

Proof. Let $a_\alpha \in E_\alpha$, for each $\alpha \in P_d$. Then

$$a_\alpha u_{\beta,\epsilon_\beta} = u_{\beta,\epsilon_\beta} a_\alpha \qquad \text{for all } \beta \in P_d \text{ with } \beta \neq \alpha.$$

Then, since $E_{\alpha,\pm}$ has a basis, as $\mathbb{F}$-vector-space, given by the elements $u_{\beta,\pm}$ and $e_r u_{\beta,\pm}$, with $r \in \beta$, we have the following equality of ordered products

$$\prod_{\alpha \in P_d} a_\alpha u_{\alpha,\epsilon_\alpha} = (\prod_{\alpha \in P_d} a_\alpha) u_\epsilon.$$

Hence, $\prod_{\alpha \in P_d} E_{\alpha,\epsilon_\alpha}$ is a subset of the simple left ideal $\mathcal{C}_d u_\epsilon$. On the other hand, by Lemma 14.2.1, every element of $\mathcal{C}_d$ is a sum of products of the form $\prod_{\alpha \in P_d} a_\alpha$. It follows then that every element of $\mathcal{C}_d u_\epsilon$ is in the image in $f(E_{12,\epsilon_{12}} \otimes \cdots \otimes E_{\{d-1,d\},\epsilon_{d-1,d}})$. $\boxed{\text{QED}}$

14.2.3 Structure of $\mathcal{C}_d$ for odd d

Now suppose d is odd. We then also have to consider the idempotents

$$\frac{1 \pm e_d}{2}.$$

So, for odd d, define

$$P_d = \{\{1,2\}, \{3,4\}, ..., \{d-2, d-1\}, \{d\}\} \qquad \text{for odd } d, \tag{14.20}$$

and, for $\epsilon \in \{+,-\}^{P_d}$, we set

$$u_\epsilon = \left(\frac{1+\epsilon_d e_d}{2}\right) \prod_{\alpha \in P_d \setminus \{d\}} \left(\frac{1+\epsilon_\alpha i e_\alpha}{2}\right) \tag{14.21}$$

Note that this is not 0, and that this is a product of terms which commute with each other. (There is no point in making a distinction between d and $\{d\}$ as subscript.)

The only difference with the case of even d is that there is now an extra term from the idempotents corresponding to e_d. We still have the relations

$$\begin{aligned} \sum_{\epsilon \in \{+,-\}^{P_d}} u_\epsilon &= 1 \\ u_\epsilon^2 &= u_\epsilon \\ u_{\epsilon'} u_\epsilon &= 0 \quad \text{if } \epsilon \neq \epsilon'. \end{aligned} \tag{14.22}$$

Arguing analogously to the case of even d we have:

Proposition 14.2.2 *Suppose $d \geq 1$ is an odd number, and let P_d be the set $\{\{1,2\}, \{3,4\}, ..., \{d-2, d-1\}, \{d\}\}$. Then for every $\epsilon \in \{+,-\}^{P_d}$, the element u_ϵ is a primitive idempotent. Moreover,*

$$u_{\epsilon'} u_\epsilon = 0 \qquad \textit{if } \epsilon \neq \epsilon',$$

and

$$\sum_{\epsilon \in \{+,-\}^{P_d}} u_\epsilon = 1.$$

Thus, $\mathcal{C}_d$ decomposes into the internal direct sum of the simple left ideals $\mathcal{C}_d u_\epsilon$. We need now only determine which of these are isomorphic. To this end, for each $\epsilon \in \{+,-\}^{P_d}$, we introduce

$$\sigma(\epsilon) = \sum_{\alpha \in P_d} \epsilon_\alpha \qquad \mod \mathbb{Z}_2, \tag{14.23}$$

where we have identified $+$ with $0 \in \mathbb{Z}_2$ and $-$ with $1 \in \mathbb{Z}_2$.

Lemma 14.2.5 *Let $d \geq 1$ be an odd integer. For any $\epsilon, \epsilon' \in \{+,-\}^{P_d}$, the primitive left ideals $\mathcal{C}_d u_\epsilon$ and $\mathcal{C}_d u_{\epsilon'}$ are isomorphic if and only if $\sigma(\epsilon) = \sigma(\epsilon')$.*

<u>Proof</u> If

$$f : \mathcal{C}_d u_\epsilon \to \mathcal{C}_d u_{\epsilon'}$$

is $\mathcal{C}_d$-linear, then $f(u_\epsilon)$ equals $xu_{\epsilon'}$ for some $x \in \mathcal{C}_d$, and then f is given by

$$f(au_\epsilon) = f(au_\epsilon u_\epsilon) = au_\epsilon x u_{\epsilon'}.$$

We will show that if $\sigma(u) = \sigma(u')$ then there is an $x \in \mathcal{C}_d$ for which $u_\epsilon x$ equals $u_{\epsilon'}$, while if $\sigma(u) \neq \sigma(u')$ then $au_\epsilon x u_{\epsilon'}$ is 0 for all $x \in \mathcal{C}_d$.

For $\alpha \in P_{d-1} = \{\{1,2\}, ..., \{d-2, d-1\}\}$ we have

$$u_\epsilon e_{\alpha_1} = e_{\alpha_1} u_{\epsilon(1)}, \tag{14.24}$$

where $\alpha = \{\alpha_1, \alpha_2\}$, and $\epsilon(1)$ disagrees with ϵ on α and on $\{d\}$. Since ϵ and $\epsilon(1)$ differ on exactly two elements in P_d it follows that

$$\sigma(\epsilon) = \sigma(\epsilon(1)). \tag{14.25}$$

From (14.24) we see that, for any $S \subset [d-1]$,

$$u_\epsilon e_S = e_S u_{\epsilon_S},$$

where ϵ_S disagrees with ϵ on $\alpha \in P_{d-1}$ if and only if $|S \cap \alpha| = 1$, and ϵ_S disagrees with ϵ on $\{d\}$ if and only if $|S|$ is odd. Moreover, from (14.25) it follows that

$$\sigma(\epsilon) = \sigma(\epsilon_S).$$

Consider now $\epsilon, \epsilon' \in \{+,-\}^{P_d}$. Suppose $\sigma(\epsilon) \neq \sigma(\epsilon')$. Then for every $S \subset [d-1]$ we have $\epsilon_S \neq \epsilon'$ and so

$$u_\epsilon e_S u_{\epsilon'} = e_S u_{\epsilon_S} u_\epsilon = 0,$$

which then also implies

$$u_\epsilon e_S e_d u_{\epsilon'} = \pm u_\epsilon e_S u_{\epsilon'} = 0.$$

Therefore,

$$u_\epsilon x e_d u_{\epsilon'} = 0 \qquad \text{for all } x \in \mathcal{C}_d.$$

This implies that any $\mathcal{C}_d$ linear map $\mathcal{C}_d u_\epsilon \to \mathcal{C}_d u_{\epsilon'}$ is 0, and so these simple left ideals are not isomorphic as $\mathcal{C}_d$-modules.

Suppose now that $\sigma(\epsilon) = \sigma(\epsilon')$. Let

$$D = \{\alpha \in P_{d-1} : \epsilon_\alpha = \epsilon'_\alpha\},$$

and D_1 be the subset of $[d]$ consisting of one element exactly from each $\alpha \in D$ (in particular, $|D_1|$ equals $|D|$). Then $|\{\alpha \in P_d : \epsilon_\alpha \neq \epsilon'_\alpha\}|$ is even. Then either $|D|$ is even and $\epsilon'_d = \epsilon_d$ or $|D|$ is odd and $\epsilon'_d \neq \epsilon_d$. In either case, we have

$$u_\epsilon e_{D_1} = e_{D_1} u_{\epsilon'}.$$

Therefore the $\mathcal{C}_d$-linear mapping

$$\mathcal{C}_d u_\epsilon \to \mathcal{C}_d u_{\epsilon'} : a u_\epsilon \mapsto a u_\epsilon e_{D_1}$$

is non-zero, carrying u_ϵ to $e_{D_1} u_{\epsilon'} \neq 0$, and hence an isomorphism of the simple left ideals. $\boxed{\text{QED}}$

We can also verify the effect of the central element $e_{[d]}$ on the simple left ideals.

Lemma 14.2.6 *Let $d \geq 1$ be odd. Then*

$$i^{(d-1)/2} e_{[d]} x = \begin{cases} x & \text{if } x \in \mathcal{C}_d u_\epsilon \text{ where } \sigma(\epsilon) = 0; \\ -x & \text{if } x \in \mathcal{C}_d u_\epsilon \text{ where } \sigma(\epsilon) = 1. \end{cases} \tag{14.26}$$

In particular, multiplication by the central idempotent

$$z_+ = \frac{1 + i^{d(d-1)/2} e_{[d]}}{2}$$

is given by

$$z_+ x = \begin{cases} x & \text{if } x \in \mathcal{C}_d u_\epsilon \text{ where } \sigma(\epsilon) = 0; \\ 0 & \text{if } x \in \mathcal{C}_d u_\epsilon \text{ where } \sigma(\epsilon) = 1. \end{cases} \tag{14.27}$$

Proof Consider the central element

$$y = e_d \prod_{\alpha \in P_{d-1}} (ie_\alpha) = i^{(d-1)/2} e_{[d]}.$$

Multiplying with this, we have, for any $\epsilon \in \{+,-\}^{P_d} \simeq \mathbb{Z}_2^{P_d}$,

$$yu_\epsilon = (-1)^{\sigma(\epsilon)} u_\epsilon$$

Then

$$i^{d(d-1)/2} e_{[d]} u_\epsilon = y^d u_\epsilon = (-1)^{\sigma(\epsilon)d} u_\epsilon = (-1)^{\sigma(\epsilon)} u_\epsilon,$$

because d is odd. This yields (14.27). $\boxed{\text{QED}}$

Now we can summarize our results for odd d. Recall that a two-sided ideal is said to be simple if it is a direct sum of isomorphic simple left ideals.

Theorem 14.2.2 *For any odd positive integer d, the Clifford algebra $\mathcal{C}_d$ generated by elements $e_1, ..., e_d$, over a field $\mathbb{F}$ which contains $i = \sqrt{-1}$ and has characteristic $\neq 2$, decomposes as the direct sum of 2 simple two-sided ideals*

$$\mathcal{C}_d = \mathcal{C}_d^+ \oplus \mathcal{C}_d^-,$$

where the central idempotent $\frac{1+i^{d(d-1)/2}e_{[d]}}{2}$ acts as identity on $\mathcal{C}_d^+$, and on $\mathcal{C}_d^-$ as 0. Each of these two-sided ideals is the direct sum of $2^{(d-1)/2}$ simple left-ideals:

$$\mathcal{C}_d^+ = \bigoplus_{\epsilon \in \{+,-\}^{P_d}, \sigma(\epsilon)=0} \mathcal{C}_d u_\epsilon,$$

and

$$\mathcal{C}_d^- = \bigoplus_{\epsilon \in \{+,-\}^{P_d}, \sigma(\epsilon)=1} \mathcal{C}_d u_\epsilon,$$

where $P_d = \{\{1,2\}, ..., \{d-1,d\}, \{d\}\}$, and

$$u_\epsilon = \left(\frac{1+\epsilon_d e_d}{2}\right) \prod_{\{r,r+1\} \in P_d} \frac{1+\epsilon_{r,r+1} e_r e_{r+1}}{2}.$$

Each left ideal $\mathcal{C}_d u_\epsilon$ is a $2^{(d-1)/2}$-dimensional vector space over $\mathbb{F}$, and the simple left ideals $\mathcal{C}_d u_\epsilon$ within each of the two-sided ideals $\mathcal{C}_d^{\pm}$ are isomorphic to each other as $\mathcal{C}_d$-modules. A basis of the vector space $\mathcal{C}_d u_\epsilon$ is given by the vectors $e_R u_\epsilon$, with R running over all subsets of $[d-1]_{\text{odd}}$, the set of odd integers in $[d-1] = \{1, ..., d-1\}$.

Again we have an observation on tensor product decompositions.

Lemma 14.2.7 *Let $d \geq 1$ be an odd integer. Then there is an isomorphism of vector spaces given by*

$$f : E_{12} \otimes \cdots \otimes E_{d-2,d-1} \otimes E_d \to \mathcal{C}_d : a_{12} \otimes \cdots \otimes a_{d-2,d-1} \otimes a_d \mapsto a_{12} \ldots a_{d-2,d-1} a_d.$$

For any $\epsilon \in \{+,-\}^{P_d}$, the simple left ideal in $\mathcal{C}_d$ generated by u_ϵ can be expressed as

$$\mathcal{C}_d u_\epsilon = f(E_{12,\epsilon_{12}} \otimes \cdots \otimes E_{\{d-1,d\},\epsilon_{d-2,d-1}} \otimes E_d), \tag{14.28}$$

where

<u>Proof</u>. The proof that f is an isomorphism is entirely analogous to the case of even d in Lemma 14.2.1: the map f carries basis elements to basis elements.

Let $a_\alpha \in E_\alpha$, for each $\alpha \in P_d$. Then

$$u_{\beta,\epsilon_\beta} a_\alpha = a_\alpha u_{\beta,\epsilon_\beta} \qquad \text{for all } \beta \in P_{d-1} \text{ with } \beta \neq \alpha.$$

Then, since $E_{\alpha,\pm}$ has a basis, as $\mathbb{F}$-vector-space, given by the elements $u_{\beta,\pm}$ and $e_r u_{\beta,\pm}$, with $r \in \beta$, we have the following equality of ordered products

$$\prod_{\alpha \in P_d} a_\alpha u_{\alpha,\epsilon_\alpha} = (\prod_{\alpha \in P_d} a_\alpha) u_\epsilon.$$

Thus, $\prod_{\alpha \in P_d} E_{\alpha,\epsilon_\alpha}$ is a subset of the simple left ideal $\mathcal{C}_d u_\epsilon$. On the other hand, since f is an isomorphism, every element of $\mathcal{C}_d$ is a sum of products of the form $\prod_{\alpha \in P_d} a_\alpha$. It follows then that every element of $\mathcal{C}_d u_\epsilon$ is in the image in $f(E_{12,\epsilon_{12}} \otimes \cdots \otimes E_{\{d-1,d\},\epsilon_{d-2,d-1}} \otimes E_d)$. $\boxed{\text{QED}}$

14.3 Representations

In this section $\mathbb{F}$ is, as before, a field of characteristic $\neq 2$ which contains $i = \sqrt{-1}$. We will use notation from the previous section.

14.3.1 Representations with Endomorphisms

For any even positive integer d, we have seen that the Clifford algebra $\mathcal{C}_d$, with d generators $e_1, ..., e_d$, over the field $\mathbb{F}$, is a simple algebra. Recall that its basis, as a vector space is given by the elements e_S, with S running over $[d]$. All simple $\mathcal{C}_d$ modules are isomorphic to the simple left ideal

$$\mathcal{C}_d u,$$

where

$$u = \prod_{\alpha \in P_d} \frac{1 + ie_\alpha}{2},$$

where $P_d = \{\{1,2\}, ..., \{d-1, d\}\}$. A vector space basis of this left module is given by the elements $e_R u$, with R running over all subsets of $[d]_{\text{odd}}$, the set off odd numbers in $[d]$. If

$$f : \mathcal{C}_d u \to \mathcal{C}_d u$$

is $\mathcal{C}_d$-linear then $f(u)$ is xu for some $x \in \mathcal{C}_d$, and then, as seen in the proof of Lemma 14.2.2, uxu equals cu, for some $c \in k$, and so

$$f(au) = f(auu) = auf(u) = auxu = cau, \qquad \text{for all } a \in \mathcal{C}_d.$$

Thus,

$$\mathrm{End}_A(\mathcal{C}_d u) = k1,$$

where 1 here is the identity map on $\mathcal{C}_d u$. Next, for the 'double commutant', we have:

Proposition 14.3.1 *Let $\mathcal{C}_d$ be the Clifford algebra on d generators over a field of characteristic $\neq 2$ and containing $i = \sqrt{-1}$. If d is even then, for any simple $\mathcal{C}_d$-module L, the mapping*

$$\mu : \mathcal{C}_d \to \mathrm{End}_{\mathbb{F}}(L),$$

which associates to each $a \in \mathcal{C}_d$ the $\mathbb{F}$-linear map $L \to L : y \mapsto ay$, is an isomorphism of $\mathbb{F}$-algebras. If d is odd and $L^{\pm}$ a simple $\mathcal{C}_d$-module on which the central element $i^{(d-1)/2}e_{[d]}$ acts as ± 1, then

$$\mu : \mathcal{C}_d \to \mathrm{End}_{\mathbb{F}}(L^+) \oplus \mathrm{End}_{\mathbb{F}}(L^-)$$

is an isomorphism of $\mathbb{F}$-algebras, where, for each $a \in \mathcal{C}_d$, the element $\mu(a) \in \mathrm{End}_{\mathbb{F}}(L^{\pm})$ maps any $y \in L^{\pm}$ to ay.

Proof It is clear that μ is $\mathbb{F}$-linear, and maps the identity in $\mathcal{C}_d$ to the identity in $\mathrm{End}_{\mathbb{F}}(L)$.

Suppose d is even. If $a \in \mathcal{C}_d$ is such that $\mu(a)$ is 0 then, since all simple left modules of $\mathcal{C}_d$ are isomorphic, and $\mathcal{C}_d$ is itself a direct sum of such modules appearing as simple left ideals, it follows that multiplication in $\mathcal{C}_d$ by a on the left is 0, which implies that $a = a1$ is 0. Thus μ is injective. We can check directly that μ is surjective, or observe that $\dim_{\mathbb{F}} \mathcal{C}_d = 2^{d/2}$ and $\dim_{\mathbb{F}}\big(\mathrm{End}_{\mathbb{F}}(L)\big) = (\dim_{\mathbb{F}} L)^2 = (2^{d/2})^2 = 2^d$, and therefore μ must also be surjective.

The argument for odd d is very similar. $\boxed{\text{QED}}$

14.3.2 Representations on Exterior Algebras

In the Clifford algebra $\mathcal{C}_d$, with all notation and hypotheses as usual, there is the simple left ideal L_+ generated by

$$u_+ = \prod_{\alpha \in P_s} u_{\alpha,+}.$$

A basis of this as a vector space is given by the elements $e_R u_+$, with R running over all subsets of $[d]_{\mathrm{odd}}$.

The *exterior algebra* Λ_D, over the field $\mathbb{F}$, on an ordered set $D \neq \emptyset$ of generators is the free $\mathbb{F}$-vector-space with basis y_R, with R running over all finite subsets of D; the bilinear product structure on Λ_D is specified by

$$y_R y_S = \epsilon^0_{RS} y_{R\Delta S}, \tag{14.29}$$

where

$$\epsilon^0_{RS} = \prod_{r \in R, s \in S} \epsilon_{rs} \tag{14.30}$$

and

$$\epsilon_{rs} = \begin{cases} 1 & \text{if } r < s; \\ 0 & \text{if } r = s; \\ -1 & \text{if } r > s, \end{cases}$$

and the empty product in (14.30), if R or S is empty, understood to be 1. Note that

$$y_r^2 = 0 \qquad \text{and} \qquad y_r y_s = -y_s y_r$$

for all $r, s \in D$. The multiplicative identity $y_\emptyset$ will be written simply as 1. Clearly, Λ_D has dimension $2^{|D|}$.

Let V be the linear span of the elements y_r for $r \in D$:

$$V = \sum_{r \in D} k y_r \subset \Lambda_D.$$

The m-th exterior power $\Lambda^m V$ of V is the linear span in Λ_D of the basis elements y_S with $|S| = m$. Thus, Λ_D is the direct sum of these subspaces:

$$\Lambda_D = \bigoplus_{m=0}^{|D|} \Lambda^m V.$$

For any $v \in V$ the *creation operator* c_v on Λ_D is is the linear mapping on Λ_D specified by

$$c_v(y_S) = v y_S \qquad \text{for all } S \subset D,$$

where vy_S is the product in Λ_D. We write c_r for c_{y_r}; thus,

$$c_r(y_S) = y_r y_S = \begin{cases} \epsilon_{\{r\}S} y_{\{r\} \cup S} & \text{if } r \notin S; \\ 0 & \text{if } r \notin S. \end{cases} \tag{14.31}$$

Define the *annihilation operator* a_v by

$$a_v(y_S) = \sum_{r \in D} v_r a_r(y_S),$$

for $v = \sum_{r \in D} v_r y_r \in V$, and where

$$a_r(y_S) = \begin{cases} 0 & \text{if } r \notin S; \\ \epsilon_{\{r\}S} y_{S \setminus \{r\}} & \text{if } r \in S. \end{cases} \tag{14.32}$$

The annihilation operator has a convenient algebraic property:

$$a_v(xy) = (a_v x)y + (-1)^p x(a_v y), \tag{14.33}$$

if $x \in \Lambda^p V$.

Returning to the Clifford algebra, let Λ_D be the exterior algebra with generating set being $[d-1]_{\text{odd}}$. Consider the linear mapping

$$C : L_+ \to \Lambda_D : e_R u_+ \mapsto y_R \qquad \text{for all } R \subset [d]_{\text{odd}}.$$

Since, by definition, this carries a basis to a basis, it is an isomorphism of vector spaces. Let

$$\mathcal{C}_d^+$$

be $\mathcal{C}_d$ is d if even, and for d odd be the two-sided ideal in $\mathcal{C}_d$ on which $i^{(d-1)/2}e_{[d]}$ acts as multiplication by $+1$. Then, by Proposition 14.3.1,

$$\mathcal{C}_d^+ \simeq \mathrm{End}_{\mathbb{F}}(L_+),$$

and therefore

$$B : \mathcal{C}_d^+ \simeq \mathrm{End}_{\mathbb{F}}(\Lambda_{[d-1]_{\mathrm{odd}}}) \tag{14.34}$$

We have made no use at all of the exterior product structure in Λ_D as yet. This structure plays a role only when we express the isomorphism B explicitly in terms of that structure.

The following can be verified by straightforward computation:

Proposition 14.3.2 *With notation as above, and, as always, for $\mathbb{F}$ a field of charateristic $\neq 2$ and containing $i = \sqrt{-1}$,*

$$B(e_r) = c_r + a_r \qquad \textit{and} \qquad B(e_{r+1}) = i(c_r - a_r)$$

for all $r \in [d-1]_{\mathrm{odd}}$, where c_r and a_r are the creation and annihilation operators on $\Lambda_{[d-1]_{\mathrm{odd}}}$.

14.4 Superalgebra structure

In this section, we will discuss a graded structure on Clifford algebras, and review some of the results of this chapter in the context of this graded structure.

14.4.1 Superalgebras

A $\mathbb{Z}_2$-graded algebra or *superalgebra* B over a commutative unital ring R is an expression of the algebra B as a direct sum

$$B = B^0 \oplus B^1$$

where each B^p is a sub-R-module, and

$$B^p B^q \subset B^{p+q},$$

where $p+q$ is the sum in $\mathbb{Z}_2$, i.e. modulo 2. Elements of B^0 are called *even*, and elements of B^1 are called *odd*. Since $B^0B^0 \subset B^0$, we see that B^0 is a subalgebra of B. If B has a multiplicative identity 1 then 1^2 equals 1, and so $1 \in B^0$.

Elements which are even or odd are called *homogeneous*. A general element of the algebra is, of course, expressed uniquely as a sum of an even part and an odd part.

In a superalgebra B, there is the *superbracket* or *supercommutator* $\{\cdot,\cdot\}$ which is a bilinear product defined on homogeneous elements by

$$\{x,y\} \stackrel{\text{def}}{=} xy - (-1)^{pq}yx, \tag{14.35}$$

where $x \in B^p$ and $y \in B^q$. This bracket is super-skew-symmetric:

$$\{y,x\} = (-1)^{pq}yx, \qquad \text{for all } x \in B^p \text{ and } y \in B^q.$$

Moreover, the superbracket satisfies the super-Jacobi identity:

$$\{x,\{y,z\}\} = \{\{x,y\},z\} + (-1)^{pq}\{x,\{y,z\}\}, \tag{14.36}$$

for all $x \in B^p$, $y \in B^q$, and $z \in B$.

Suppose

$$A = A^0 \oplus A^1 \qquad \text{and} \qquad B = B^0 \oplus B^1$$

are $\mathbb{Z}_2$-graded associative algebras over some commutative unital ring R. We will work with the tensor product R-module

$$A \otimes B.$$

It will be convenient to write the element $a \otimes b$ simply as ab. We define a bilinear product on $A \otimes B$ by

$$(ab)(a'b') = (-1)^{qp'}(aa')(bb'),$$

if a, b, a', b' are all homogeneous, and $b \in B^q$ and $a' \in A^{p'}$. Then, as may be verified by explicit checking, $A \otimes B$ is an associative algebra over R; it has multiplicative identity 1 given by $1_A 1_B$ if A and B have identities 1_A and 1_B, respectively. In fact, it is a superalgebra, on setting

$$(A \otimes B)^p = \bigoplus_{r,s \in \mathbb{Z}_2, r+s=p} A^r \otimes B^s \tag{14.37}$$

14.4.2 Clifford algebras as superlagebras

We turn now to Clifford algebras over some fixed field $\mathbb{F}$.

The Clifford algebra $\mathcal{C}_d$ has a superalgebra structure:

$$\mathcal{C}_d = \mathcal{C}_d^0 + \mathcal{C}_d^1, \tag{14.38}$$

where $\mathcal{C}_d^0$ is the subspace spanned by all e_S with $|S|$ even, and $\mathcal{C}_d^1$ is spanned by all e_S with $|S|$ odd.

14.4.3 Tensor product decomposition

Let us, for the moment, denote the Clifford algebra with generating set $E = \{e_1, ..., e_d\}$, by $\mathcal{C}_E$. Then, if E' and E'' partition E into disjoint non-empty subsets then

$$\mathcal{C}_{E'} \otimes \mathcal{C}_{E''} \to \mathcal{C}_E : a \otimes b \mapsto ab \tag{14.39}$$

is an isomorphism of superalgebras, i.e. is is an isomorphism of algebras which carries even elements to even elements and odd to odd.

The preceding tensor product isomorphism naturally carries over to the case where $[d]$ is partitioned into multiple subsets.

In particular, if d is even, we can partition $[d]$ into pairs

$$P_d = \{\{1, 2\}, ..., \{d-1, d\}\}$$

and have the superalgebra isomorphism

$$f : E_{12} \otimes E_{34} \otimes \cdots \otimes E_{d-1,d} \to \mathcal{C}_d : a_{12} \otimes ... \otimes a_{d-1,d} \mapsto a_{12} ... a_{d-1,d} \tag{14.40}$$

where E_{rs} is the Clifford algebra over $\{e_r, e_s\}$.

If d is odd then we have the partition

$$P_d = \{\{1, 2\}, ..., \{d-1, d\}, \{d\}\}$$

and the corresponding isomorphism

$$f : E_{12} \otimes E_{34} \otimes \cdots \otimes E_{d-1,d} \otimes E_d \to \mathcal{C}_d : a_{12} \otimes ... \otimes a_{d-1,d} \mapsto a_{12} ... a_{d-1,d}, \tag{14.41}$$

where E_d is the Clifford algebra over the one generator e_d.

14.4.4 Semisimple structure

Assume, as usual, that the field $\mathbb{F}$ has character $\neq 2$.

As seen before, the Clifford algebra E_α decomposes as a direct sum of simple left ideals

$$E_\alpha = E_{\alpha,+} \oplus E_{\alpha,-},$$

where each $E_{\alpha,\pm}$ is 2-dimensional as a $\mathbb{F}$-vector-space if α is a pair, and is 1-dimensional if α is a singleton. Explicitly,

$$E_{\alpha,+} = E_\alpha u_{\alpha,+}, \qquad \text{and} \qquad E_{\alpha,-} = E_\alpha u_{\alpha,-},$$

where the primitive idempotents $u_{\alpha,\pm}$ are given by

$$\begin{aligned} u_{\{r,s\},+} &= \frac{1 + ie_r e_s}{2} \qquad \text{and} \qquad u_{\{r,s\},-} = \frac{1 - ie_r e_s}{2} \\ u_{r,+} &= \frac{1 + e_r}{2} \qquad \text{and} \qquad u_{r,-} = \frac{1 - e_r}{2}, \end{aligned} \tag{14.42}$$

for any $r, s \in [d]$ with $r \neq s$.

Let ϵ be any element of $\{+,-\}^{P_d}$. The algebra isomorphism f then maps

$$\bigotimes_{\alpha \in P_d} E_{\alpha,\epsilon_\alpha}$$

onto the simple left ideal

$$\mathcal{C}_d u_\epsilon,$$

where

$$u_\epsilon \stackrel{\text{def}}{=} \begin{cases} u_{12,\epsilon_{12}} \dots u_{\{d-1,d\},\epsilon_{\{d-1,d\}}} & \text{if } d \text{ is even;} \\ u_{12,\epsilon_{12}} \dots u_{\{d-1,d\},\epsilon_{\{d-1,d\}}} u_{d,\epsilon_d} & \text{if } d \text{ is odd.} \end{cases}$$

Thus the decomposition

$$\bigotimes_{\alpha \in P_d} E_\alpha = \bigoplus_{\epsilon \in \{+,-\}^{P_d}} \left(\bigotimes_{\alpha \in P_d} E_{\alpha,\epsilon_\alpha} \right)$$

carries over, via f, to the decomposition of the Clifford algebra $\mathcal{C}_d$ into simple left ideals:

$$\mathcal{C}_d = \bigoplus_{\epsilon \in \{+,-\}^{P_d}} \mathcal{C}_d u_\epsilon.$$

Exercises

1. Prove the super-Jacobi identity (14.36).

2. The *super-center* of $\mathcal{C}_d$ is the set of all elements a for which $\{a, x\} = 0$ for all $x \in \mathcal{C}_d$. Work out the super-center of $\mathcal{C}_d$.

3. For a basis element e_S in the Clifford algebra, show that
$$e_S^2 = (-1)^{\pi_S} 1,$$
where $\pi_S = |S|(|S| - 1)/2$ is the number of unordered pairs of elements in S.

4. Show that the Clifford algebra generated by one element e_1 contains exactly two idempotents other than 0 and 1, these being $(1 \pm e_1)/2$. Conclude that these are primitive idempotents.

5. For a Clifford algebra $\mathcal{C}_d$ over a field $\mathbb{F}$, determine all linear maps $T : \mathcal{C}_d \to k$ which have the trace-property that $T(ab) = T(ba)$ for all $a, b \in \mathcal{C}_d$.

6. Fix $d \geq 1$, and let $\mathbb{F}$ be a field of characteristic $\neq 2$. Suppose E is a left ideal in $\mathcal{C}_d$, the Clifford algebra with d generators over the field $\mathbb{F}$. Treating it as a vector space over $\mathbb{F}$, we have a linear surjection $P : \mathcal{C}_d \to E$, which is equal to the identity map on E. Construct from P a surjection $P' : \mathcal{C}_d \to E$ which is linear as a map of $\mathcal{C}_d$-modules and is, again, the identity on E.

7. Prove directly from the definition that the Clifford algebra $\mathcal{C}_d$, over a field of characteristic $\neq 2$, is semisimple.

8. Fix $d \geq 1$, and let $e_{-1}, e_0, e_1, ..., e_d$ generate a group G subject to the relations :(i) e_0 is the identity element; (ii) e_{-1} commutes with all elements; (iii) $e_r^2 = e_0$ for all $r \in \{-1, 0, ..., d\}$; (iv) $e_r e_s = e_{-1} e_s e_r$ if $r \neq s \in \{1, ..., d\}$. Work out the structure of the group algebra $\mathbb{F}[G]$. Show that $\mathcal{C}_d$ is the quotient of $\mathbb{F}[G]$ by the 2-sided ideal generated by $e_0 + e_{-1}$.

9. Suppose A and B are semisimple algebras over a field $\mathbb{F}$, and L and M simple left ideals in A and B, respectively. Suppose, moreover, that

$\mathrm{End}_A(L, L)$ and $\mathrm{End}_B(M, M)$ are one-dimensional, as vector spaces over $\mathbb{F}$. Show that $L \otimes M$ is a simple left ideal in the algebra $A \otimes B$.

10. Let d be a positive integer, and $\mathcal{C}_d$ the Clifford algebra with generators $e_1, ..., e_d$ over a field of characteristic $\neq 2$ which contains $i = \sqrt{-1}$. For any $S \subset [d]$, and $\epsilon \in \{+, -\}$, let

$$u_{S,\epsilon} = \frac{1 + \epsilon i^{|S|(|S|-1)/2} e_S}{2}.$$

Show that $u_{S,\epsilon}$ is an idempotent, and prove the following relations:

$$\begin{aligned} u_{S,+} + u_{S,-} &= 1 \\ u_{S,+}u_{S,-} &= 0 = u_{S,-}u_{S,+} \\ u_{S,+} - u_{S,-} &= i^{|S|(|S|-1)/2} e_S. \end{aligned}$$

Work out all relations between $u_{S,\epsilon}u_{T,\epsilon'}$ and $u_{T,\epsilon'}u_{S,\epsilon}$.

11. A trace τ on an algebra A, with identity 1, over a field $\mathbb{F}$ is a linear map $\tau : A \to k$, satisfying $\tau(1) = 1$ and $\tau(xy) = \tau(yx)$ for all $x, y \in A$. Let d be a positive integer and consider the Clifford algebra $\mathcal{C}_d$ with d generators $e_1, ..., e_d$, over a field $\mathbb{F}$ of characteristic $\neq 2$.

    (i) Let $\tau_0 : \mathcal{C}_d \to k$ be the linear map for which $\tau(e_\emptyset) = 1$ and $\tau(e_S) = 0$ for $S \neq \emptyset$; show that τ_0 is a trace.

    (ii) If $\tau : \mathcal{C}_d \to k$ is a trace which is 0 on all e_S for $|S|$ odd, show that $\tau = \tau_0$.

12. We work with the notation and hypotheses of the preceding problem; thus, τ_0 is the unique trace on $\mathcal{C}_d$ which vanishes on all e_S with $S \neq \emptyset$. Let $k \to k : \lambda \mapsto \overline{\lambda}$ be an automorphism of the field $\mathbb{F}$, and let

$$\mathcal{C}_d \to \mathcal{C}_d : x \mapsto x^*$$

be the unique mapping specified by requiring that $(xy)^* = y^*x^*$ for all $x, y \in \mathcal{C}_d$ and $(\lambda x) = \overline{\lambda} x^*$ for all $x \in \mathcal{C}_d$ and all $\lambda \in k$. Consider the pairing

$$\mathcal{C}_d \times \mathcal{C}_d \to k : (x, y) \mapsto \langle x, y \rangle \stackrel{\text{def}}{=} \tau(xy^*).$$

This is clearly linear in the first variable, conjugate linear in the second.

(i) Show that $\langle \cdot, \cdot \rangle$ is a metric, i.e.

$$\langle y, x \rangle = \overline{\langle x, y \rangle} \qquad \text{for all } x, y \in \mathcal{C}_d,$$

and that if $\langle x, y \rangle$ is 0 for all y then x is 0.

(ii) Show that $\{e_S : S \subset \{1, ..., d\}\}$ is an orthonormal basis of $\mathcal{C}_d$ with respect to the metric $\langle \cdot, \cdot \rangle$.

(iii) Check that left multiplication by e_j on $\mathcal{C}_d$ is hermitian operator, i.e.

$$\langle x, e_j y \rangle = \langle e_j x, y \rangle \qquad \text{for all } x, y \in \mathcal{C}_d \text{ and all } j \in \{1, ..., d\}.$$

Check that this also holds for right multiplication by e_j.

Some algebraic Background

The purpose of this Appendix is to present summary definitions and some basic results for concepts used in the book.

14.5 Some basic algebraic structures

A *group* is a set G along with an operation

$$G \times G \to G : (a, b) \mapsto a \cdot b$$

for which the following hold:

(G1) the operation is associative:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \qquad \text{for all } a, b, c \in G.$$

(G2) there is an element $e \in G$, called the *identity* element for which

$$a \cdot e = e \cdot a = a \qquad \text{for all } a \in G. \tag{14.43}$$

(G3) for each element $a \in G$ there is an element $a^{-1} \in G$, called the inverse of a, for which

$$a \cdot a^{-1} = a^{-1} \cdot a = e \tag{14.44}$$

If $e' \in G$ is an element with the same property (14.43) as e then

$$e' = e \cdot e' = e',$$

and so the identity element is unique. Similarly, if $a, a_L \in G$ are such that $a_L \cdot a$ is e, then

$$a_L = a_L \cdot e = a_L \cdot (a \cdot a^{-1}) = (a_L \cdot a) \cdot a^{-1} = e \cdot a^{-1} = a^{-1},$$

and, similarly, if $a \cdot a_R$ is e then a_R is equal to a^{-1}. Thus, the inverse of an element is unique.

Usually, we drop the $\cdot$ in the operation and simply write ab for $a \cdot b$:

$$ab = a \cdot b$$

A group is *abelian* or *commutative* if

$$ab = ba \qquad \text{for all } a, b \in G.$$

For many abelian groups, the group operation is written additively:

$$G \times G \to G : (a, b) \mapsto a + b,$$

the identity element denoted 0, and the inverse of a then denoted $-a$.

A *ring* R is a set with two operations

$$\begin{aligned} \text{addition} : \mathbb{F} \times \mathbb{F} \to \mathbb{F} : (a, b) \mapsto a + b \\ \text{multiplication} : \mathbb{F} \times \mathbb{F} \to \mathbb{F} : (a, b) \mapsto ab \end{aligned}$$

such that under addition R is an abelian group, the operation of multiplication is associative, and multiplication distributes over addition:

$$\begin{aligned} a(b + c) = ab + ac \\ (b + c)a = ba + ca \end{aligned} \tag{14.45}$$

The set $\mathbb{Z}$ of all integers is a ring under the usual arithmetic operations.

An *left ideal* I in a ring R is a non-empty subset of R with the property that $RI \subset I$, i.e.

$$xa \in I \text{ for all } x \in R \text{ and } a \in I.$$

A *right ideal* J is a nonempty subset of R for which $JR \subset R$. A subset of R is a *two-sided ideal* if it is both a left ideal and a right ideal.

In $\mathbb{Z}$ an ideal is a subset of the form $m\mathbb{Z}$, for some $m \in \mathbb{Z}$.

A *commutative ring* is a ring whose multiplication operation is commutative.

An element a in a commutative ring R is a *divisor* of $b \in R$ if $b = ac$, for some $c \in R$.

An ideal I in a commutative ring R is a *prime ideal* if it is not R and has the property that if $a, b \in R$ have their product ab in I then a or b is in I. In the ring $\mathbb{Z}$ an ideal is prime if and only if it consists of all multiples of some prime number.

Suppose R is a commutative ring with $1 \neq 0$. Then there is the ring $R[[X]]$ of *power series* in the variable X, with coefficients in R; a typical element of $R[[X]]$ is of the form

$$p(X) = \sum_{j \in \{0,1,2,\ldots\}} a_j X^j,$$

where each a_n is an element of R. Addition and multiplication are specified in the natural way

$$\sum_j a_j X^j + \sum_j b_j X^j = \sum_j (a_j + b_j) X^j$$

and

$$\left(\sum_j a_j X^j\right)\left(\sum_j b_j X^j\right) = \sum_j c_j X^j,$$

where

$$c_j = \sum_{k=0}^{j} a_k b_{j-k} \qquad \text{for all } j \in \{0,1,2,\ldots\}.$$

Technically, the power series $\sum_j a_j X^j$ is simply the sequence $(a_0, a_1, \ldots)$, written in a visually convenient way, with X standing for $(0,1,0,0,\ldots)$, and addition and multiplication given as above. By a constant we shall mean a power series $\sum_j a_j X^j$ for which $a_j = 0$ for all $j \in \{1,2,\ldots\}$.

Inside the ring of power series is the *polynomial ring* $R[X]$ which consists of all elements $\sum_j a_j X^j$ for which the set $\{j : a_j \neq 0\}$ is finite. For a non-zero polynomial, the largest j for which the coefficient of X^j is not zero is called the *degree* of the polynomial.

Consider now a field $\mathbb{F}$. If I is a non-zero ideal in $\mathbb{F}[X]$, and $q(X)$ is an element in I of smallest degree, then I consists of all the multiples of $q(X)$, i.e. I is $\mathbb{F}[X]q(X)$. This ideal is prime if and only if $q(X)$ is *irreducible*, i.e. any divisor of $q(X)$ in $\mathbb{F}[X]$ is either a constant or a constant multiple of $q(X)$. CHECK.

If R is a commutative ring, and I an ideal in R, then the quotient

$$R/I \stackrel{\text{def}}{=} \{x + I : x \in R\} \tag{14.46}$$

is a ring under the operatons

$$(x+I)+(y+I)=(x+y)+I, \qquad (x+I)(y+I)=xy+I.$$

If R is a multiplicative identity 1 then $1+I$ is the multiplicative identity in R/I.

If $a \in R$ and $m \in \{1,2,3,..\}$ the sum of m copies of a is denoted ma; more officially, define inductively:

$$1a = a \text{ and } (m+1)a = ma + a.$$

Further, setting

$$0a = 0,$$

wherein 0 on the left is the integer 0, and for $m \in \{1,2,..\}$, setting

$$(-m)a = m(-a),$$

gives a map

$$\mathbb{Z} \times R \to R : (n,a) \mapsto na$$

which is additive in n and in a, and also satisfies

$$m(na) = (mn)a \qquad \text{for all } m,n \in \mathbb{Z} \text{ and } a \in R.$$

14.6 Fields

A field is a ring, with a unit element $1 \neq 0$, in which the operation of multiplication is commutative and the non-zero elements form a group under multiplication.

In more detail, a *field* is a set $\mathbb{F}$, along with two operations

$$\begin{aligned} \text{addition} : \mathbb{F} \times \mathbb{F} \to \mathbb{F} : (a,b) \mapsto a+b \\ \text{multiplication} : \mathbb{F} \times \mathbb{F} \to \mathbb{F} : (a,b) \mapsto ab \end{aligned}$$

such that

(F1) $\mathbb{F}$ is an abelian group under addition,

(F2) $\mathbb{F} - \{0\}$ is a group under multiplication

(F3) multiplication is *distributive* over addtion:

$$\begin{aligned} a(b+c) &= ab+ac \\ (b+c)a &= ba+ca \end{aligned} \tag{14.47}$$

(F4) $\mathbb{F}$ contains at least two elements, or, equivalently, the multiplicative identity 1 is not equal to the additive identity 0

(F5) multiplication is commutative:

$$ab = ba \qquad \text{for all } a, b \in \mathbb{F} \tag{14.48}$$

Of course, in view of commutativity of multiplication, the second distributive law in (14.47) is superfluous. The distributive property implies

$$a \cdot 0 = 0 \qquad \text{for all } a \in \mathbb{F}$$

The existence of multiplicative inverses of non-zero elements implies that if the product of two elements in a field is 0 then at least one of them must be 0.

Suppose R is a commutative ring with a multiplicative identity element $1 \neq 0$, and suppose I is a prime ideal in R. Then the quotient ring R/I is a field. Applying this to the ring $\mathbb{Z}$, and a prime number p, produces the finite field

$$\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} \tag{14.49}$$

If $p(X)$ is a polynomial which has no polynomial divisors other than constants, then we have the quotient

$$\mathbb{F}[X]/I_{p(X)},$$

where

$$I_{p(X)} = p(X)\mathbb{F}[X],$$

is the ideal consisting of all multiples of $p(X)$. If $p(X)$ is irreducible then $\mathbb{F}[X]/I_{p(X)}$ is a field.

If $p(X) = \sum_{j=1}^{d} a_j X^j \in \mathbb{F}[X]$ and $\alpha \in \mathbb{F}$ then

$$p(\alpha) = \sum_{j=1}^{d} a_j \alpha^j.$$

The element α is called a *root* of $p(X)$ of $p(\alpha)$ is 0.

A field $\mathbb{F}$ is *algebraically closed* of each polynomial $p(X) \in \mathbb{F}$ of degree ≥ 1, has a root in $\mathbb{F}$. In this case, a polynomial $p(X)$ of degree $d \geq 1$, splits into a product of terms each of the form $X - \alpha$, for $\alpha \in \mathbb{F}$, and a constant.

Basic examples of fields include:

(i) the rational numbers $\mathbb{Q}$

(ii) for any prime number p, the integers modulo p form the field $\mathbb{Z}_p$

both with the usual arithmetic addition and multiplication operations.

At another extreme are the field $\mathbb{R}$ of real numbers, and the field $\mathbb{C}$ of complex numbers. The field of complex numbers is algebraically closed.

Suppose $\mathbb{F}$ is a field, and $\mathbb{F}' \subset \mathbb{F}$ is a subset which is a field under the operations inherited from $\mathbb{F}$. Then $\mathbb{F}$ is called an *extension* of $\mathbb{F}'$. Often we work with an extension that is given by adjoining certain elements to the subfield.

Consider a field $\mathbb{F}$, with multiplicative unit $1_{\mathbb{F}}$. Then

$$Z_{\mathbb{F}} = \{m \in \mathbb{Z} : m1_{\mathbb{F}} = 0\}$$

is an ideal in $\mathbb{Z}$ and so is of the form

$$Z_{\mathbb{F}} = c\mathbb{Z},$$

where c is the smallest non-negative element of $Z_{\mathbb{F}}$, and is called the *characteristic* of $\mathbb{F}$. If m and n are integers such that $mn1_{\mathbb{F}}$ is 0 then this means that $m1_{\mathbb{F}}n1_{\mathbb{F}}$ is 0 and so $m1_{\mathbb{F}}$ or $n1_{\mathbb{F}}$ is 0; thus, the ideal $Z_{\mathbb{F}}$ is a prime ideal, and so the characteristic of a field, if non-zero, is a prime number. The field $\mathbb{Z}_p$ has characteristic p.

14.7 Vector Spaces over Division Rings

A *division ring* is an algebraic structure which has all the properties of a field except for commutativity of multiplication. Thus, a division ring is a ring with a multiplicative unit element $1 \neq 0$, in which all non-zero elements have multiplicative inverses.

A *vector space* V over a division ring D is a set V equipped an operation of addition under which it is an abelian group, and a further operation of multiplication by scalars

$$D \times V \to V : (a, v) \mapsto av,$$

such that

$$\begin{aligned} (a+b)v &= av + bv \\ a(v+w) &= av + aw \\ a(bv) &= (ab)v \\ 1v &= v \end{aligned} \tag{14.50}$$

for all $v, w \in V$, and $a, b \in D$. Elements of V are called *vectors*, and elements of D are, in this context, called *scalars*.

If $n \in \{1, 2, 3, ..\}$, then D^n has a natural vector space structure over D:

$$\begin{aligned} (a_1, ..., a_n) + (b_1, ..., b_n) &= (a_1 + b_1, ..., a_n + b_n) \\ k(a_1, ..., a_n) &= (ka_1, ..., ka_n) \end{aligned} \tag{14.51}$$

for all $a_1, ..., a_n, b_1, ..., b_n, k \in D$.

A *linear combination* of elements $v_1, ..., v_n \in V$ is an element of the form $a_1v_1 + \cdots a_nv_n$, where $a_1, ..., a_n \in D$. The set of all linear combinations of elements in a set $S \subset V$ is called the *span* of S.

A subset S of V is said to be *linearly independent* if it is nonempty and for any $n \in \{1, 2, ...\}$, and $v_1, ..., v_n \in V$, a relation

$$a_1v_1 + \cdots + a_nv_n = 0$$

can hold with $a_1, ..., a_n \in D$ only if each of $a_1, ..., a_n$ is 0. Note that, in particular, no set which is linearly independent can contain the vector 0.

A *basis* of a vector space V is a subset of V which is linearly independent and whose span is V.

Theorem 14.7.1 *Every vector space, other than $\{0\}$, over a division ring has a basis. More generally, if $V \neq \{0\}$ is a vector space over a division ring D, and S is a subset of V whose span is V, and I a subset of V which is linearly independent, then there is a basis B of V of the form $B = I \cup S'$, where $S' \subset S$ is a subset of S disjoint from I. Any two bases of V have the same cardinality, and this common value is called the dimension of V and denoted* $\dim_D V$.

14.8 Modules over Rings

In this section R is a ring with a multiplicative identity element 1_R. A *left R-module* M is a set M which is an abelian group under an addition operation $+$, and there is an operation of scalar multiplication

$$R \times M \to M : (r, v) \mapsto rv$$

for which the following hold:

$$\begin{aligned} (r+s)v &= rv + sv \\ r(v+w) &= rv + rw \\ r(sv) &= (rs)v \\ 1_R v &= v \end{aligned}$$

for all $v, w \in M$, and $r, s \in R$. As for vector spaces,

$$0v = 0 \qquad \text{for all } v \in M,$$

where 0 on the left is the zero in R, and 0 on the right is 0 in M.

A *right R-module* is defined analogously, except that the multiplication by scalars is on the right:

$$M \times R \to R : (v, r) \mapsto vr$$

and so the 'associative law' reads

$$(vr)s = v(rs).$$

By convention/bias, an R-module means a left R-module.

Any abelian group A is automatically a $\mathbb{Z}$-module, because of the multiplication

$$\mathbb{Z} \times A \to A : (n, a) \mapsto na.$$

If M and N are left R-modules, a map $f : M \to N$ is *linear* if

$$f(rv + w) = rf(v) + f(w) \qquad \text{for all } v, w \in M \text{ and all } r \in R.$$

The set of all linear maps $M \to N$ is denoted

$$\mathrm{Hom}_R(M, N)$$

and is an abelian group under addition. If R is commutative, then $\mathrm{Hom}_R(M,N)$ is an R-module, with multiplication of an element $f \in \mathrm{Hom}_R(M,N)$ by a scalar $r \in R$ defined to be the map

$$rf : M \to N : v \mapsto rf(v).$$

Note that rf is linear only on using the commutativity of R.

A subset $N \subset M$ of a left R-module M is a *submodule* of M is it a module under the restrictions of addition and scalar multiplication, i.e. if $N+N \subset N$ and $RN \subset N$. In this case, the quotient

$$M/N = \{v + N \,:\, v \in M\}$$

becomes a left R-module with the natural operations

$$(v+N)+(w+N) = (v+w)+N, \qquad \text{and} \qquad r(v+N) = rv+N$$

for all $v, w \in M$ and $r \in R$. Thus, it is the unique R-module structure on M/N which makes the quotient map

$$M \to M/N : v \mapsto v+N$$

linear.

The *span* of a subset T of an R-module is the set of all elements of M which are linear combinations of elements of T. A set $I \subset M$ is *linearly independent* if I is nonempty and or any $n \in \{1,2,...\}$, $v_1,...,v_n \in M$ and $r_1,...,r_n \in R$ with $r_1v_1 + \cdots r_nv_n = 0$ the elements $r_1,...,r_n$ are all 0. A subset of M which is linearly independent and whose span is M is called a *basis* of M. If M has a basis it is said to be a *free* module.

If S is a non-empty set, and R a ring with identity 1_R, then the set $R[S]$, of all maps $f : S \to R$ for which $f^{-1}(R - \{0\})$ is finite, is a left R-module with the natural operations of addition and multiplication induced from R:

$$(f+g)(s) = f(s)+g(s), \qquad (rf)(s) = rf(s),$$

for all $s \in S$, $r \in R$, and $f, g \in R[S]$. The R-module $R[S]$ is called the *free R-module over S*. It is convenient to write an element $f \in R[S]$ in the form

$$f = \sum_{x \in S} f(x)x.$$

For $x \in S$, let $j(x)$ be the element of $R[S]$ equal to 1_R on x and 0 elsewhere. Then $j : S \to R[S]$ is an injection which is uded to identify S with the subset $j(S)$ of $R[S]$. Note that $j(S)$ is a *basis* of $R[S]$, i.e. every element of $R[S]$ can be expressed in a unique way as a linear combination of the elements of $j(S)$:

$$f = \sum_{x \in S} f(x) j(x)$$

wherein all but finitely many elements are 0 and so it is, in fact, a finite sum. If M is a left R-module and $\phi : S \to M$ a map then $\phi = \phi' \circ j$, where $\phi' : R[S] \to M$ is uniquely specified by requiring that it be linear and equal to $\phi(x)$ on $j(x)$.

14.9 Tensor Products

In this section R is a commutative ring with multiplicative identity element 1_R. We will also use, later in the section, a possibly non-commutative ring D.

Consider left R-modules $M_1, ..., M_n$. If N is also an R-module, a map

$$f : M_1 \times \cdots M_n \to N : (v_1, ..., v_n) \mapsto f(v_1, ..., v_n)$$

is called *multilinear* if it is linear in each v_j, with the other v_i held fixed; i.e. if

$$f(v_1, ..., av_k + bv_k', ..., v_n) = af(v_1, ..., v_n) + bf(v_1, ..., v_k', ..., v_n)$$

for all $v_1 \in M_1, ..., v_k, v_k' \in M_k, ..., v_n \in M_n$ and $a, b \in R$.

Consider the set $S = M_1 \times \ldots \times M_n$, and the free R-module $R[S]$, with the injection $j : S \to R[S]$. Inside $R[S]$ consider the submodule J spanned by all elements of the form

$$(v_1, ..., av_k + bv_k', ..., v_n) - a(v_1, ..., v_n) - b(v_1, ..., v_k', ...v_n)$$

with $v_1 \in M_1, ..., v_k, v_k' \in M_k, ..., v_n \in M_n$ and $a, b \in R$. The quotient R-module

$$M_1 \otimes \ldots \otimes M_n = R[S]/J \tag{14.52}$$

is called the *tensor product* of the modules $M_1, ..., M_n$. The image of $(v_1, ..., v_n) \in M_1 \times \cdots \times M_n$ under j is denoted $v_1 \otimes \cdots \otimes v_n$:

$$v_1 \otimes \cdots \otimes v_n = j(v_1, ..., v_n). \tag{14.53}$$

The tensor product construction has the following 'universal property': if $f : M_1 \times \cdots \times M_n \to N$ is a multilinear map then there is a unique linear map $f' : M_1 \otimes \cdots \otimes M_n \to N$ such that $f = f' \circ j$, specified simply by requiring that

$$f(v_1, ..., v_n) = f'j(v_1, ..., v_n),$$

for all $v_1, ..., v_n \in M$. Occasionally, the ring R needs to be stressed, and we then write the tensor product as

$$M_1 \otimes_R \cdots \otimes_R M_n.$$

There is a tensor product construction for two modules over a possibly non-commutative ring. Let D be a ring with multiplicative identity element 1_D, and suppose M_r is a right D-module and N_r a left D-module. There is then the injection

$$j : S_{rl} \to \mathbb{Z}[S_{rl}],$$

where $\mathbb{Z}[S_{rl}]$ is the free $\mathbb{Z}$-module over the set $M_r \times N_l$. Inside $\mathbb{Z}[S_{rl}]$ consider the $\mathbb{Z}$-submodule J_{rl} spanned by all elements of the form

$$(m_r d, n_l) - (m_r, dn_l)$$

with $m_r \in M_r$, $n_l \in N_l$, and $d \in D$. The quotient is the $\mathbb{Z}$-module

$$M_r \otimes_{D_i} N_l = \mathbb{Z}[S_{rl}]/J_{rl} \tag{14.54}$$

The image of $(m_r, n_l) \in M_r \in N_l$ under j is denoted

$$m_r \otimes n_l = j(m_r, n_l).$$

The key feature now is that

$$(m_r d) \otimes n_l = m_r \otimes (dn_l),$$

for all $(m_r, n_l) \in M_r \in N_l$ and $d \in D$. Indeed, we could also think of $M_r \otimes_{D_i} N_l$ as the $\mathbb{Z}$-module tensor product $M_r \otimes_{\mathbb{Z}} N_l$ quotiented by the submodule generated by all elements of the form $m_r d \otimes n_l - m_r \otimes dn_l$.

End Notes

In preparing these notes I have consulted several sources. The most influence has been from Hermann Weyl [19]. I have also consulted Lang [11] extensively. Other works that have influenced the notes include Dieudonné and Carrell [4], Don Zagier's appendix to the book of Lando and Zvonkin [10], and Serre [18]. For Clifford algebras, in addition to the Artin's book [1], the recent book by Gracia-Bondía et al. [6] is a very convenient reference.

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