Notes on the representation theory of finite groups

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Chapter 1

Introduction

1.1 Caveat

I have not had the time to proofread these Notes thoroughly. Please inform me of any typographic errors that you find. If there are any points at which the commentary is too terse, I would like to know about these as well.

1.2 Preamble

The purpose of these Notes is to give the background to the representation theory of finite groups that is necessary for deriving an explicit expression for the generating series for maps in orientable surfaces. In this sense these Notes are therefore intended for combinatorialists. The material requires Schur's Lemma, Maschke's Theorem, induced representations, the irreducible characters of the symmetric group, the central orthogonal idempotents and symmetric functions. It has not been necessary to use the Wedderburn Structure Theorem. I have taken the quickest path to this goal, and the usual applications to group theory have not been given. Results from the theory of symmetric functions that are needed have been stated explicitly for completeness, but without proof.

It is necessary to state the perspective adopted for the definition of the Schur function since there are several choices which might have been made. The Schur function is defined in these Notes as the generating series for column strict plane partitions with respect to its filling, and is then shown to be equal to a determinant of complete symmetric functions through the Gessel-Viennot construction for enumerating non-intersecting ordered n-paths. The Jacobi-Trudi Identity and the Jacobi Identity can be established algebraically in the ring of symmetric functions. The Schur function is then identified as the generating series for the evaluations of an irreducible character on the conjugacy classes of the symmetric group. This is the fundamental result of Frobenius.

The genus series for maps in locally orientable surfaces can be obtained in a similar way, but additional material is required. This involves the double-coset algebra of the hyperoctahedral group embedded in the symmetric group as the stabilizer of a specific fixed point free involution. The functions that correspond to the Schur functions in this theory are the zonal polynomials.

From the point of view of algebraic combinatorics, representation theory is a very natural area of mathematics to learn about after linear algebra, group theory and ring theory. Indeed, it is the next "engine" that we need to approach deeper enumerative problems. It is a beautiful area of mathematics which is used extensively in other areas of mathematics (for example, group theory, number theory, harmonic analysis) and its applications. Applications include mathematical physics, crystallography and probability theory.

1.3 Combinatorial motivation

The main purpose of these Notes is to give a self-contained account of the representation theory that is needed to prove the *rooted map version* of the following theorem that gives the generating series for the number of *hypermaps* in orientable surfaces with respect to vertex-, face-, and hyperedge-distribution. From this information the genus of the surface is deducible. Recall that a hypermap is a 2-face-colourable map, and that restriction to maps is by digon conflation on the hyperedges (the hyperedges are forced to have degree 2, and the digons are then conflated to edges

Theorem 1.1 *Let $h_{\mathbf{i},\mathbf{j},\mathbf{k}}$ be the number of rooted hypermaps in orientable surfaces with vertex-distribution* $\mathbf{i}$, *face-distribution* $\mathbf{j}$ *and hyperedge-distribution* $\mathbf{k}$, *and let*

$$H(\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{\mathbf{i},\mathbf{j},\mathbf{k}\geq\mathbf{0}} h_{\mathbf{i},\mathbf{j},\mathbf{k}} \mathbf{x}^i \mathbf{y}^j \mathbf{z}^k.$$

Then

$$H(\mathbf{p}(\mathbf{x}),\mathbf{p}(\mathbf{y}),\mathbf{p}(\mathbf{z})) = t\frac{\partial}{\partial t}\log\left(\sum_{\theta\in\mathcal{P}} \frac{|\theta|!}{f^\theta} t^{|\theta|} s_\theta(\mathbf{x})\, s_\theta(\mathbf{y})\, s_\theta(\mathbf{z})\right)\Bigg|_{t=1}.$$

Here, $p_i(\mathbf{x})$ is the power sum of degree i in the ground indeterminates $\mathbf{x} = (x_1, x_2, \ldots)$, $\mathbf{p}(\mathbf{x}) = (p_1(\mathbf{x}), p_2(\mathbf{x}), \ldots)$, $\mathcal{P}$ is the set of all partitions (with the null partition adjoined), s_θ is the Schur function indexed by a partition θ, and f^θ is the degree of the irreducible representation of $\mathfrak{S}_{|\theta|}$ indexed by the partition θ. Some of these may not be familiar to you, but they will be defined in time. It is important to note that, to find $h_{\mathbf{i},\mathbf{j},\mathbf{k}}$ from this theorem, it is necessary to express the Schur functions in terms of the power sum symmetric functions.

The combinatorial aspects of maps and their axiomatization has been discussed elsewhere, and this will not be included here. However, some brief comments may be useful to explain the connexion between the enumeration of maps and representation theory. Let ε_n be a fixed-point free involution in $\mathfrak{S}_{2n}$. By the Embedding Theorem of graphs, a permutation $\nu \in \mathfrak{S}_{2n}$ such that $\langle \nu, \varepsilon_n \rangle$, that acts transitively on the $2n$ edge-end labels of a graph with n edges, uniquely defines a 2-cell embedding of a graph in an orientable surface Σ. The configuration consisting of the graph and its embedding is called a *map.* If the edges of the map are deleted then Σ decomposed into a union of regions homeomorphic to open discs called the *faces* of the map. The permutation ν is called a *rotation system.* Then the faces of the map whose rotation system is ν correspond to the cycles of $\nu\varepsilon$. The genus of Σ is deducible from ν by the Euler-Poincaré Formula.

From an enumerative point of view the task is to consider any permutation $\nu \in \mathfrak{S}_{2n}$, which therefore corresponds to a pre-map (a union of maps), to filter from these the connected objects, namely maps, and then to record, in a generating series, their vertex and face distributions. The question of counting maps algebraically is therefore reduced by combinatorial means to determining the cycle-type of $\nu\varepsilon_n$ in a way that enables us to retain these distributions. Equivalently, we wish to find the number of ways $c^{\gamma}_{\alpha,\beta}$ of expressing $z \in \mathcal{C}_\gamma$ as $z = xy$ where $x \in \mathcal{C}_\alpha$ and $y \in \mathcal{C}_\beta$, where $\alpha, \beta, \gamma \vdash 2n$ and $\mathcal{C}_\alpha$ is the conjugacy class of $\mathfrak{S}_{2n}$ naturally indexed by α.

All of the representation theory that is presented here will be required to answer this apparently simple question. Theorem 1.1 is proved in Section 8.2, where fuller combinatorial details are given.

1.4 Notation

The following notation will be used throughout these Notes. There are a few notational conventions to help the reader. Linear operators are sans serif capitals $(\mathsf{P}, \mathsf{T}, \ldots)$, matrices and vectors are bold face $(\mathbf{v}, \mathbf{M}, \ldots)$, vector spaces and sets are italic capitals $(\mathcal{P}, \mathcal{S}, \mathcal{U}, \mathcal{V}, \mathcal{X}, \ldots)$.

$\leq$, "subspace of", "subgroup of".

$\sim$, equivalence of representations.

$\vdash$, "is a partition of".

$^\perp$, orthogonal complement.

$\langle \cdot, \cdot \rangle_\Lambda$, $\langle \cdot, \cdot \rangle_{\mathbb{C}^G}$, $\langle \cdot, \cdot \rangle_\rho$, various inner products.

$\langle x_1, x_1, \ldots \rangle$, the group generated by $x_1, x_2, \ldots$.

$|\theta|$, the sum of the parts of a partition θ.

$[G\colon H]$, the index of H in G, where $H < G$.

$[\mathsf{T}]_\mathcal{A}$, the matrix representing $\mathsf{T} \in \mathrm{End}\,(\mathcal{V})$ with respect to a basis $\mathcal{A}$ of $\mathcal{V}$.

1_G, the trivial representation of G.

ε_n, a prescribed fixed-point free involution in $\mathfrak{S}_n$.

ι, the identity element of G.

Λ, the ring of symmetric functions.

ν, a rotation system,

$o\left(x\right)$, the order of $x \in G$.

ρ_G, a representation of G.

$\rho \uparrow_H^G$, the induced representation of G from $H < G$, where ρ is a representation of H.

$\sigma \downarrow_H^G$, the restriction of the representation σ of G to $H < G$.

χ^ρ, the character of the representation ρ.

a_δ, the Vandermonde determinant.

$\mathrm{alt}_{\mathfrak{S}_n}$, the alternating representation of $\mathfrak{S}_n$.

$\mathcal{C}_i$, the i-th conjugacy class in an arbitrary indexing scheme.

$c_{p,q}^r$, connexion coefficient for the class algebra of $\mathbb{C}G$.

$\mathbb{C}$, the field of complex numbers.

$\mathbb{C}G$, the group algebra of G over $\mathbb{C}$.

$\deg(\rho)$, degree of the representation ρ.

$\mathbf{e}_i$, a element in the standard basis of $\mathbb{C}^n$.

$\mathrm{End}\left(\mathcal{V}\right) = \{\mathsf{T}\colon \mathcal{V} \to \mathcal{V}\colon \mathsf{T} \text{ is a linear operator}\}$, the set of endomorphisms.

$f^{(i)}$, the degree of the irreducible representation of G associated with $\mathcal{C}_i$.

$\mathrm{fix}\left(x\right)$, the set of all elements of X that are fixed by $x \in G$.

F_i, an orthogonal idempotent in $Z_{\mathbb{C}G}$.

g, the order $|G|$ of G.

G_x the stabilizer subgroup of $x \in \mathcal{X}$, where G acts on the set $\mathcal{X}$.

Gx, the G-orbit containing $x \in \mathcal{X}$, where G acts on $x \in \mathcal{X}$.

$\mathrm{GL}\left(\mathcal{V}\right) = \{\mathsf{T} \in \mathrm{End}\left(\mathcal{V}\right) : T \text{ invertible}\}$, the general linear group.

G, H, groups; usually finite; $H < G$, H is a subgroup of G.

$h^{(i)}$, the size of $\mathcal{C}_i$.

h_θ, the complete symmetric function indexed by $\theta \vdash n$.

$H(\mathbf{x}, \mathbf{y}, \mathbf{z})$, the genus series for hypermaps.

$\mathbf{I}_n$, the $n \times n$ identity matrix.

$\mathsf{I}_{\mathcal{V}}$, the identity operator in $\mathrm{End}(\mathcal{V})$.

k, the number of conjugacy classes of G.

K_i, the formal sum in $\mathbb{C}G$ of the elements of $\mathcal{C}_i$.

$\ker \mathsf{T}$, the kernel (null space) of the linear transformation T.

$l(\theta)$, the length of a partition θ.

$m_{\theta,}$ the monomial symmetric function indexed by $\theta \vdash n$.

$\mathfrak{m}$, a map.

$M(\mathbf{x}, \mathbf{y}, z)$, the genus series for maps.

$\mathcal{M}_n^\star(\mathbb{C})$, the set of all $n \times n$ invertible matrices over $\mathbb{C}$.

$\mathcal{N}_n$, $= \{1, \ldots, n\}$.

nat_G, the natural representation of G.

$\mathsf{0}_{\mathcal{V}}$, the zero operator in $\mathrm{End}(\mathcal{V})$.

$p(n)$, the partition number.

p_θ, the power sum symmetric function indexed by $\theta \vdash n$.

$\mathfrak{p}$, a pre-map.

$\mathcal{P}$, the set of all partitions, with the empty partition adjoined.

$\mathbb{Q}$, the field of rational numbers.

$\mathbb{R}$, the field of real numbers.

reg_G, the regular representation of G.

$R(G)$, the ring of class functions in $\mathbb{C}^G$.

s_θ, the Schur function indexed by $\theta \vdash n$.

$\mathrm{sgn}(\pi)$, the signum of $\pi \in \mathfrak{S}_n$.

$\mathrm{spec}(\mathsf{T})$, the spectrum of T; that is, the multiset of eigenvalues (counted with respect to multiplicity) of the operator T.

$\mathfrak{S}_n$, the symmetric group on $\{1, \ldots, n\}$.

tr, the trace function.

$\mathsf{T}|_{\mathcal{U}}$, the restriction of $\mathsf{T} \in \mathrm{End}(\mathcal{V})$ to $\mathcal{U} \leq \mathcal{V}$.

$\mathsf{T}\mathcal{V}$, the image space (range space) of the linear transformation $\mathsf{T}: \mathcal{V} \to \mathcal{W}$.

$\mathcal{U}^{\perp}$, the orthogonal complement of the vector space $\mathcal{U}$.

$\mathcal{V}$, an arbitrary finite dimensional vector space over $\mathbb{C}$; $\mathcal{U} < \mathcal{V}$, $\mathcal{U}$ is a subspace of $\mathcal{V}$.

X_ρ, Y_ρ, matrix representations of the representation ρ.

$z(\theta)$, the size of the stabilizer of $x \in \mathcal{C}_\theta$ in the symmetric group.

$Z_{\mathbb{C}G}$, the centre of the group algebra $\mathbb{C}G$.

1.5 Representations

Let $\mathcal{V}$ be a vector space over C, and let $\mathrm{GL}(\mathcal{V}) = \{\mathsf{T} \in \mathrm{End}(\mathcal{V}) : \mathsf{T} \text{ invertible}\}$. This set forms a group with composition of linear operators as product, and is called the *General Linear Group.*

Definition 1.2 *Let $\mathcal{V}$ be a vector space over $\mathbb{C}$. A* representation *of a group G is a homomorphism*

$$\rho_G : G \longrightarrow \mathrm{GL}(\mathcal{V}).$$

The degree *of ρ_G is* $\dim \mathcal{V}$ *and is denoted by* $\deg(\rho_G)$. *$\mathcal{V}$ is called the* representation space *(or sometimes the* carrier space*).*

Since ρ_G is a homomorphism, recall that $\rho_G(xy) = \rho_G(x)\rho_G(y)$ for all $x, y \in G$, and $\rho_G(\iota) = \mathsf{I}_{\mathcal{V}}$ where ι is the identity element of G. It follows that $\rho_G(x^{-1}) = \rho_G(x)^{-1}$. Often the subscript of ρ_G will be omitted when the group is clear from the context. The idea is that, under ρ_G, x is mapped to an invertible linear transformation T_x, where, of course, the action of the linear operator T_x on $\mathcal{V}$ has to be supplied in a particular instance. The vector space $\mathcal{V}$ can be thought of as carrying the action of G, which accounts for the term "carrier space."

1.5.1 Examples - 1_G, nat_G and reg_G

The following are important representations that will be used later. In each case we define $\rho_G(x)$, for $x \in G$, by specifying its action on a basis of $\mathcal{V}$.

Example 1.3 $\rho_G : G \longrightarrow \mathrm{GL}(\mathcal{V}) : x \longmapsto \mathsf{I}_{\mathcal{V}}$. *When* $\dim(\mathcal{V}) = 1$ *this is called the* trivial representation, *and it is then denoted by* 1_G.

Let $x \in \mathfrak{S}_n$. Then $\mathrm{sgn}(x)$ is equal to 1 if x is an even permutation and is equal to -1 if x is an odd permutation. This function is called the *signum* of a permutation. One of its properties is that $\mathrm{sgn}(x) = \det(\mathbf{P}_x)$, where $\mathbf{P}_x$ is the $n \times n$ matrix whose (i,j)-element is equal to 1 if $xi = j$ and is 0 otherwise. The following is an example of a non-trivial representation of degree one.

Example 1.4 $\mathrm{alt}_{\mathfrak{S}_n}: \mathfrak{S}_n \longrightarrow \mathbb{C} - \{0\} : x \longmapsto \mathrm{sgn}(x)$. *This is called the* alternating *representation.*

To see that it is a representation, let $\mathcal{A} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of $\mathcal{V}$ and, for $x \in G$, let $\mathsf{T}_x \in \mathrm{GL}(\mathcal{V})$ be such that $\mathsf{T}_x \mathbf{v}_i = \mathbf{v}_{xi}$, for $i = 1, \ldots, n$ where $x \in \mathfrak{S}_n$. Then $[\mathsf{T}_x]_{\mathcal{A}} = \mathbf{P}_x$. Let $y \in \mathfrak{S}_n$. Then $\mathsf{T}_{xy}\mathbf{v}_i = \mathbf{v}_{(xy)i} = \mathbf{v}_{x(yi)} = \mathsf{T}_x \mathbf{v}_{yi} = (\mathsf{T}_x \mathsf{T}_y)\mathbf{v}_i$ for $i = 1, \ldots, n$ so the actions of T_{xy} and $\mathsf{T}_x\mathsf{T}_y$ on a basis are the same so $\mathsf{T}_{xy} = \mathsf{T}_x\mathsf{T}_y$. Then $\mathrm{sgn}(xy) = \det(\mathsf{T}_{xy}) = \det(\mathsf{T}_x\mathsf{T}_y) = \det(\mathsf{T}_x)\det(\mathsf{T}_y) = \mathrm{sgn}(x)\mathrm{sgn}(y)$. Thus $\mathrm{alt}_{\mathfrak{S}_n}(xy) = \mathrm{alt}_{\mathfrak{S}_n}(x)\,\mathrm{alt}_{\mathfrak{S}_n}(y)$. Trivially, $\mathrm{sgn}(\iota) = 1$ so $\mathrm{alt}_{\mathfrak{S}_n}(\iota) = \mathsf{I}_{\mathcal{V}}$.

The next example involves the action of G on a set $\{\mathbf{v}_x : x \in G\}$ of generic vectors, which we regard therefore as linearly independent, whose span is $\mathcal{V}$.

Example 1.5 *Let* $\mathrm{reg}_G: G \longrightarrow \mathrm{GL}(\mathcal{V}) : z \longmapsto \mathsf{T}_z$ *where* $\mathsf{T}_z(\mathbf{v}_x) = \mathbf{v}_{zx}$, *and where* $\mathcal{V} = \mathrm{span}_{\mathbb{C}}(\{\mathbf{v}_x : x \in G\})$. *Then* reg_G *is called the* (left) regular representation, *and* $\deg(\mathrm{reg}_G) = |G|$.

To see that this is a representation, let $y, z \in G$. Then $(\mathsf{T}_y\mathsf{T}_z)\mathbf{v}_x = \mathsf{T}_y\mathbf{v}_{zx} = \mathbf{v}_{yzx} = \mathbf{v}_{(yz)x} = \mathsf{T}_{yz}\mathbf{v}_x$ for all $x \in G$ so $(\mathrm{reg}_G(y)\,\mathrm{reg}_G(z))\mathbf{v}_x = \mathrm{reg}_G(yz)\mathbf{v}_x$ for all $x \in G$. Thus $\mathrm{reg}_G(y)\,\mathrm{reg}_G(z) = \mathrm{reg}_G(yz)$. Finally, $\mathrm{reg}_G(\iota) = \mathrm{reg}_G\left(xx^{-1}\right) = \mathrm{reg}_G(x)\,\mathrm{reg}_G\left(x^{-1}\right) = \mathsf{T}_x\mathsf{T}_{x^{-1}} = \mathsf{I}_{\mathcal{V}}$.

In the next example we consider a group G of degree n so, as a subgroup of $\mathfrak{S}_n$, it acts on $\mathbf{v}_1, \ldots, \mathbf{v}_n$ where these are generic vectors whose span is $\mathcal{V}$.

Example 1.6 *Let* $\mathrm{nat}_G: G \longrightarrow \mathrm{GL}(\mathcal{V}) : x \longmapsto \mathsf{T}_x$ *where* $\mathsf{T}_z: \mathcal{V} \longrightarrow \mathcal{V}: \mathbf{v}_i \longmapsto \mathbf{v}_{zi}$, *extended linearly to* $\mathcal{V}$, *where* $\mathcal{V} = \mathrm{span}_{\mathbb{C}}(\{\mathbf{v}_1, \ldots, \mathbf{v}_n\})$. *Then* nat_G *is called the* natural representation *of* G, *and* $\deg(\mathrm{nat}_G) = n$.

To see that it is a representation we repeat the idea in Example 1.4. Let $x, y \in G$. Then $(\mathrm{nat}_G(xy))\mathbf{v}_i = \mathsf{T}_{xy}\mathbf{v}_i = \mathbf{v}_{(xy)i} = \mathbf{v}_{x(yi)} = \mathsf{T}_x\mathbf{v}_{yi} = (\mathsf{T}_x\mathsf{T}_y)\mathbf{v}_i = (\mathrm{nat}_G(x)\,\mathrm{nat}_G(y))\mathbf{v}_i$ for $i = 1, \ldots, n$ so $\mathrm{nat}_G(xy) = \mathrm{nat}_G(x)\,\mathrm{nat}_G(y)$. Also $\mathrm{nat}_G(\iota)\mathbf{v}_i = \mathsf{T}_\iota\mathbf{v}_i = \mathbf{v}_i = \mathsf{I}_{\mathcal{V}}\mathbf{v}_i$ so $\mathrm{nat}_G(\iota) = \mathsf{I}_{\mathcal{V}}$. Thus nat_G is a representation.

The final example, which is a general one, deals with the action of G on a set $\mathcal{S}$. The proof is very similar to the earlier ones.

Example 1.7 *Let* $\mathcal{S} = \{s_1, \ldots, s_m\}$ *be a set and let* G *act on* $\mathcal{S}$. *Let* $\mathcal{V} = \mathrm{span}\{v_{s_1}, \ldots, v_{s_m}\}$. *Let* $\rho: G \longrightarrow \mathrm{GL}(\mathcal{V}) : x \longmapsto \mathsf{T}_x$ *where* $\mathsf{T}_z: \mathcal{V} \longrightarrow \mathcal{V}: \mathbf{v}_{s_i} \longmapsto \mathbf{v}_{zs_i}$, *extended linearly to* $\mathcal{V}$. *Then* ρ *is called a* permutation representation *of* G, *and the degree of this representation is* m.

1.5.2 Maps and connexion coefficients

In later chapters it will be shown that the formal sums K_i of x over a conjugacy class $\mathcal{C}_i$ of G, for $i = 1, \ldots, k$, where k is the number of classes, generate the centre $Z_{\mathbb{C}\mathfrak{S}_n}$ of $\mathbb{C}\mathfrak{S}_n$ so, for $1 \leq p, q \leq k$,

$$K_p K_q = \sum_{r=1}^{k} c_{p,q}^r K_r.$$

The numbers $c_{p,q}^r$ are non-negative integers, called *connexion coefficients* for the centre. Clearly, from this expression, $c_{p,q}^r$ is the number of ways $z \in \mathcal{C}_r$ can be expressed as $z = xy$ for some $x \in \mathcal{C}_p$ and some $y \in \mathcal{C}_q$, identifying it precisely as the *combinatorial* number that we need for counting maps. Then, applying ρ, and recalling that it is a homomorphism, we have

$$\rho(K_p)\,\rho(K_q) = \sum_{r=1}^{k} c_{p,q}^r \rho(K_r).$$

The plan is to obtain and explicit evaluation of $\rho(K_p)$ so that this relation, for a sufficient number of representations ρ (they will be the irreducible representations of $\mathfrak{S}_{2n}$), will provide sufficient information for determining the $c_{p,q}^r$. This will be the final reference to the enumeration of maps until Theorem 1.1 is proved. The rest of these Notes will be concerned with the development of the algebraic material required for this purpose. In the course of doing this, a good deal of the basic material on the representation theory of finite groups will be developed.

1.6 Matrix representations

For some purposes it will be more convenient to work with matrix representations of linear operators and, to accomplish this, we introduce the idea of a matrix representation of G. Let $\rho: G \to \mathrm{GL}(\mathcal{V})$ be a representation of G of degree n and let X_ρ be the mapping defined by $X_\rho: G \to \mathcal{M}_n^\star: x \mapsto [\rho(x)]_\mathcal{A}$ where $\mathcal{A}$ is a basis of $\mathcal{V}$ and $\mathcal{M}_n^\star(\mathbb{C})$ is the set of all invertible $n \times n$ matrices over $\mathbb{C}$, regarded as a group under the usual matrix product. To show that X is a representation, let $x, y \in G$. Clearly $X_\rho(x)$ is invertible since both $\rho(x)$ and $[\cdot]_\mathcal{A}$ are invertible. Now $X_\rho(xy) = [\rho(xy)]_\mathcal{A} = [\rho(x)\rho(y)]_\mathcal{A} = [\rho(x)]_\mathcal{A}[\rho(y)]_\mathcal{A} = X_\rho(x)X_\rho(y)$. Also, $X_\rho(\iota) = [\rho(\iota)]_\mathcal{A} = [\mathsf{I}_\mathcal{V}]_\mathcal{A} = \mathbf{I}_n$. We therefore conclude that X_ρ is a representation of G, which motivates the following definition.

Definition 1.8 *Let $\rho: G \to \mathrm{GL}(\mathcal{V})$ be a representation of G of degree n, and let*

$$X_\rho: G \longrightarrow \mathcal{M}_n^\star: x \longmapsto [\rho(x)]_A.$$

Then X_ρ is called a matrix representation *of G associated with ρ.*

Example 1.9 *The following is a matrix representation X of*

$$\mathfrak{S}_3 = \{\iota, (1,2), (1,3), (2,3), (1,2,3), (1,2,3)\}$$

of degree 2 :

$$X(\iota) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad X(1,2) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$X(1,3) = \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ -1 & -\frac{1}{2} \end{bmatrix}, \qquad X(2,3) = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ 1 & -\frac{1}{2} \end{bmatrix},$$
$$X(1,2,3) = \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ -1 & -\frac{1}{2} \end{bmatrix}, \quad X(1,3,2) = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{4} \\ 1 & -\frac{1}{2} \end{bmatrix}.$$

To check this note that $(1,3)(1,2,3) = (2,3)$, where the multiplication is from left to right. Then, as confirmation,

$$X(1,3)\,X(1,2,3) = \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ -1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ 1 & -\frac{1}{2} \end{bmatrix} = X(2,3).$$

Example 1.10 *A matrix representation associated with the permutation representation ρ. Let $\mathcal{A} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of $\mathcal{V}$. From Example 1.6 the action is given by $\mathsf{T}_x\mathbf{v}_i = \mathbf{v}_{xi}$, so $[\mathsf{T}_x\mathbf{v}_i]_A = [\mathbf{v}_{xi}]_A = \mathbf{e}_i^T$ where $\mathbf{e}_i$ is the i-th standard basis vector of $\mathbb{C}^n$. Thus, for example, in $\mathfrak{S}_4$,*

$$X_\rho(1,4,2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

1.6.1 Equivalent representations

Let $\mathcal{V}$ and $\mathcal{V}'$ be vector spaces and let $\mathsf{P}\colon \mathcal{V} \to \mathcal{V}'$ be an invertible linear transformation. Let $\rho\colon G \to \mathrm{GL}(\mathcal{V})\colon x \mapsto \mathsf{T}_x$ be a representation of G and let σ be a mapping defined by $\sigma\colon G \to \mathrm{GL}(\mathcal{V}')\colon x \mapsto \mathsf{P}\mathsf{T}_x\mathsf{P}^{-1}$. It is easily seen that σ is a representation of G, since for $x, y \in G$ we have $\sigma(xy) = \mathsf{P}\rho(xy)\,\mathsf{P}^{-1} = \mathsf{P}\rho(x)\,\rho(y)\,\mathsf{P}^{-1} = \left(\mathsf{P}\rho(x)\,\mathsf{P}^{-1}\right)\left(\mathsf{P}\rho(y)\,\mathsf{P}^{-1}\right) = \sigma(x)\,\sigma(y)$. Also $\sigma(\iota) = \mathsf{P}\rho(\iota)\,\mathsf{P}^{-1} = \mathsf{P}\mathsf{P}^{-1} = \mathsf{I}_{\mathcal{V}'}$. We therefore have the following definition.

Definition 1.11 *Two representations $\rho\colon G \to \mathrm{GL}(\mathcal{V})$ and $\rho'\colon G \to \mathrm{GL}(\mathcal{V}')$ of G are said to be* equivalent *if there exists an isomorphism $\mathsf{P}\colon \mathcal{V} \to \mathcal{V}'$ such that $\rho'(x) = \mathsf{P}\rho(x)\,\mathsf{P}^{-1}$ for all $x \in G$.*

We write $\rho \sim \rho'$ to indicate that ρ and ρ' are equivalent representations. Note that in this case $\dim \mathcal{V} = \dim \mathcal{V}'$.

For the equivalence of matrix representations we have the following definition.

Definition 1.12 *Two matrix representations* $X\colon G \to \mathcal{M}_n^\star$ *and* $X'\colon G \to \mathcal{M}_m^\star$ *of* G *are said to be* equivalent *if there exists an invertible matrix* $\mathbf{M}$ *such that* $X'(x) = \mathbf{M}X(x)\,\mathbf{M}^{-1}$ *for all* $x \in G$.

Note that in this case, $m = n$.

Chapter 2

Irreducible representations

Let X be a matrix representation of G, of degree n, and let $\mathbf{M}_z$ for each $z \in G$ be the matrices of the representation. Thus $\mathbf{M}_x\mathbf{M}_y = \mathbf{M}_{xy}$ and $\mathbf{M}_\iota = \mathbf{I}_n$. We now show that, in effect, there is a matrix $\mathbf{P}$, independent of x, such that $\mathbf{N}_x = \mathbf{P}\mathbf{M}_x\mathbf{P}^{-1}$, for all $x \in G$, are block diagonal matrices with identical block structure. The block structure is therefore independent of x. In other words, the matrices $(\mathbf{M}_x : x \in G)$ can be simultaneously block diagonalized. Moreover, the set of i-th blocks, one from each $\mathbf{N}_x$, constitutes a subrepresentation of G that contains no non-trivial subrepresentations and is therefore irreducible. These irreducible representations are the fundamental building blocks of the theory and we now examine them.

2.1 Invariant subspaces

Let $\mathsf{T} \in \mathrm{GL}(\mathcal{V})$ and let $\mathcal{U} \leq \mathcal{V}$ ($\mathcal{U}$ is a subspace of $\mathcal{V}$). Then $\mathcal{U}$ is said to be a T-*invariant* subspace of $\mathcal{V}$ if $\mathsf{T}u \in \mathcal{U}$ for all $u \in \mathcal{U}$. Also $\mathsf{T}|_{\mathcal{U}}: \mathcal{U} \to \mathcal{V}: \mathbf{u} \mapsto \mathsf{T}\mathbf{u}$ is the *restriction* of T to $\mathcal{U}$. It is clearly a linear transformation since T is. Moreover, if $\mathcal{U}$ is T-invariant then $\mathsf{T}|_{\mathcal{U}}$ is a linear operator on $\mathcal{U}$.

Definition 2.1 *If $\rho: G \to \mathrm{GL}(\mathcal{V})$ is a representation of G and $\mathcal{U} \leq \mathcal{V}$, then $\mathcal{U}$ is said to be* ρ-invariant *if $\rho(x)\,\mathbf{u} \in \mathcal{U}$ for all $\mathbf{u} \in \mathcal{U}$ and for all $x \in G$. Also, $\{0\}$ and $\mathcal{V}$ are the* trivial *ρ-invariant subspaces of $\mathcal{V}$.*

This is the crucial idea. If $\mathcal{U}$ is a ρ-invariant subspace of $\mathcal{V}$ of dimension i, then it is readily seen that the restriction of ρ to $\mathcal{U}$ contributes an $i \times i$ diagonal block to the matrix representation of G for each $x \in G$.

Let $\rho: G \to \mathrm{GL}(\mathcal{V})$, and suppose that $\mathcal{U}$ is a non-trivial ρ-invariant subspace of $\mathcal{V}$. Then $\rho(x)|_{\mathcal{U}} \in \mathrm{GL}(\mathcal{U})$for all $x \in G$. Let $\rho|_{\mathcal{U}}$ be defined by $\rho|_{\mathcal{U}}: G \to \mathrm{GL}(\mathcal{U}) : x \mapsto \rho(x)|_{\mathcal{U}}$. Then, checking that $\rho|_{\mathcal{U}}$ is a representation, we have, $\rho|_{\mathcal{U}}(xy) = \rho(xy)|_{\mathcal{U}} = \rho(x)\rho(y)|_{\mathcal{U}} = \rho(x)|_{\mathcal{U}}\rho(y)|_{\mathcal{U}} = \rho|_{\mathcal{U}}(x)\,\rho|_{\mathcal{U}}(y)$ for all $x, y \in G$, and $\rho|_{\mathcal{U}}(\iota) = \rho(\iota)|_{\mathcal{U}} = \mathsf{I}_{\mathcal{U}}$. Thus $\rho|_{\mathcal{U}}$ is a representation, called the *restriction* of ρ to $\mathcal{U}$. If a representation ρ' of G can be realized as the restriction of a representation ρ of G, then ρ' is called a *subrepresentation* of ρ.

Definition 2.2 *A representation* $\rho: G \to \mathrm{GL}(\mathcal{V})$ *is said to be* irreducible *if* $\mathcal{V}$ *has no non-trivial* ρ*-invariant subspaces; that is,* ρ *does not admit any non-trivial subrepresentations.*

Example 2.3 *If* $\deg(\rho) = 1$ *then* $\rho: G \to \mathrm{GL}(\mathcal{V})$ *is irreducible since the only subspaces are* $\{0\}$ *and* $\mathcal{V}$*. The trivial representation* 1_G *and* alt_G *are therefore irreducible.*

Example 2.4 reg_G *is reducible.*

Consider $\mathrm{reg}_G: G \to \mathrm{GL}(\mathcal{V})$. Let $\mathcal{V} = \mathrm{span}\{v_z: z \in G\}$, let $\mathbf{u} = \sum_{z\in G} \mathbf{v}_z$ and let $\mathcal{U} = \mathrm{span}\{\mathbf{u}\}$. Then $\mathcal{U} \leq \mathcal{V}$ and $\dim(\mathcal{U}) = 1$. Let $\mathbf{v} \in \mathcal{U}$. Then $\mathbf{v} = \lambda\mathbf{u}$ for some $\lambda \in \mathbb{C}$ so, from Example 1.5,

$$\begin{aligned} \mathrm{reg}_G|_{\mathcal{U}}(x)(\mathbf{v}) &= \mathrm{reg}_G(x)(\lambda\mathbf{u}) = \lambda \mathsf{T}_x \sum_{z\in G} \mathbf{v}_z \\ &= \lambda \sum_{z\in G} \mathbf{v}_{xz} = \lambda \sum_{y\in G} \mathbf{v}_y = \lambda\mathbf{u} = \mathbf{v} \in \mathcal{U}. \end{aligned}$$

Thus $\mathcal{U}$ is a reg_G-invariant subspace of $\mathcal{V}$, so $\mathrm{reg}_G|_{\mathcal{U}}$ is a subrepresentation of reg_G. But $(\mathrm{reg}_G|_{\mathcal{U}}(x))(\mathbf{v}) = \mathrm{reg}_G(x)|_{\mathcal{U}}(\lambda\mathbf{u}) = \lambda\mathrm{reg}_G|_{\mathcal{U}}(\mathbf{u}) = \lambda\mathbf{u} = \mathbf{v} = 1_G(x)\mathbf{v}$ for all $v \in U$, so $\mathrm{reg}_G|_{\mathcal{U}}$ is identified as the trivial representation. It follows that the trivial representation is a subrepresentation of reg_G, so reg_G is reducible.

In fact, we will prove in Corollary 4.8 that reg_G contains every irreducible representation of G.

There are two points to note in the above example. The first is the use of the *averaging* a quantity (namely $\mathbf{v}_x$) over the whole group (although in the above instance where was no normalization by $|G|$ since it was not needed). This is a useful and frequently encountered algebraic construction. The second point is the transformation of the sum $\sum_{z\in G} \mathbf{v}_{xz}$ into $\sum_{y\in G} \mathbf{v}_y$. This is merely commutativity of addition, since x is invertible (it is in G). This property of summing over a group will be used very extensively, and further attention will not be drawn to it.

2.2 Maschke's theorem

The next lemma begins the decomposition of ρ into irreducible subrepresentations.

Lemma 2.5 *Let* $\rho: G \to \mathrm{GL}(\mathcal{V})$ *be a representation of* G*. Let* $\mathcal{U}$ *be a* ρ*-invariant subspace of* $\mathcal{V}$*. Then there exists a* ρ*-invariant subspace* $\mathcal{W}$ *of* $\mathcal{V}$ *such that* $\mathcal{V} = \mathcal{U} \oplus \mathcal{W}$*.*

Proof: Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of $\mathcal{V}$, and let $\langle \cdot, \cdot \rangle_1 : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ be a form defined by $\langle \mathbf{v}_i, \mathbf{v}_j \rangle_1 = \delta_{i,j}$ for $1 \le i, j \le n$, extended sesquilinearly (linearly in the first argument, and conjugate-linearly in the second) to $\mathcal{V} \times \mathcal{V}$. Then $\langle \cdot, \cdot \rangle_1$ is an inner product on $\mathcal{V}$ (the subscript "1" is merely to distinguish this inner product from the next one). Let $\langle \cdot, \cdot \rangle_\rho : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ be a form defined in terms of $\langle \cdot, \cdot \rangle_1$ by

$$\langle \mathbf{u}, \mathbf{v} \rangle_\rho = \sum_{x \in G} \langle \rho(x)\, \mathbf{u}, \rho(x)\, \mathbf{v} \rangle_1$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$. Then $\langle \cdot, \cdot \rangle_\rho$ is an inner product on $\mathcal{V}$. Thus $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^\perp$, where $\mathcal{U}^\perp$ is the orthogonal complement of $\mathcal{U}$.

We now show that $\mathcal{U}^\perp$ is ρ-invariant by first showing that $\langle \cdot, \cdot \rangle_\rho$ is ρ -invariant. Let $\mathbf{u}, \mathbf{v} \in \mathcal{V}$, and let $x \in G$. Then

$$\begin{aligned} \langle \rho(x)\, \mathbf{u}, \rho(x)\, \mathbf{v} \rangle_\rho &= \sum_{y \in G} \langle \rho(y)\, \rho(x)\, \mathbf{u}, \rho(y)\, \rho(x)\, \mathbf{v} \rangle_1 \\ &= \sum_{y \in G} \langle \rho(yx)\, \mathbf{u}, \rho(yx)\, \mathbf{v} \rangle_1 \\ &= \sum_{z \in G} \langle \rho(z)\, \mathbf{u}, \rho(z)\, \mathbf{v} \rangle_1 = \langle \mathbf{u}, \mathbf{v} \rangle_\rho \end{aligned}$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ and $x \in G$. Thus $\langle \cdot, \cdot \rangle_\rho$ is ρ-invariant. Now let $\mathbf{w} \in \mathcal{U}^\perp$ and consider $\rho(x)\, \mathbf{w}$ where $x \in G$. Then for $\mathbf{a} \in \mathcal{U}$,

$$\langle \mathbf{a}, \rho(x)\, \mathbf{w} \rangle_\rho = \left\langle \rho\left(x^{-1}\right) \mathbf{a}, \rho\left(x^{-1}\right) \rho(x)\, \mathbf{w} \right\rangle_\rho = \left\langle \rho\left(x^{-1}\right) \mathbf{a}, \mathbf{w} \right\rangle_\rho$$

since $\langle \cdot, \cdot \rangle_\rho$ is ρ -invariant. But $\mathcal{U}$ is ρ-invariant so $\rho\left(x^{-1}\right) \mathbf{a} \in \mathcal{U}$, whence $\left\langle \rho\left(x^{-1}\right) \mathbf{a}, \mathbf{w} \right\rangle_\rho = 0$. Thus $\langle \mathbf{a}, \rho(x)\, \mathbf{w} \rangle_\rho = 0$ for all $\mathbf{a} \in \mathcal{U}$, so $\rho(x)\, \mathbf{w} \in \mathcal{U}^\perp$. Thus $\mathcal{U}^\perp$ is ρ-invariant, and the result follows. □

Note that this proof contains another common algebraic construction. The statement of the result does not refer to an inner product. This was introduced as a convenient way of constructing a suitable subspace $\mathcal{W}$ explicitly. Recall that every finite dimensional vector space has an inner product, by transportation from the coordinatizing space.

There is a word of caution is to be given here here, since care must be taken with infinite groups. Let $\mathbb{R}^+$ be the additive group of the reals. Let

$$X_{\mathbb{R}^+} : \mathbb{R}^+ \longrightarrow \mathcal{M}_2^\star(\mathbb{R}) : x \longmapsto \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}.$$

Then,

$$X_{\mathbb{R}^+}(x)\, X_{\mathbb{R}^+}(y) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+y \\ 0 & 1 \end{bmatrix} = X_{\mathbb{R}^+}(x+y)$$

and $X_{\mathbb{R}^+}(0) = \mathbf{I}_2$ (0 is the neutral element for $\mathbb{R}^+$) Then $X_{\mathbb{R}^+}$ is a representation of $\mathbb{R}^+$. Let $(a,b)^T$ be an $X_{\mathbb{R}^+}$-invariant vector. Then

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}$$

for all $x \in \mathbb{R}$. Then $a + bx = \lambda x$ and $b = \lambda b$ for all $x \in \mathbb{R}$, so $b = 0$ and $\lambda = 1$. Thus $\mathcal{U} = \operatorname{span}\left((a,b)^T\right)$ is the only $X_{\mathbb{R}^+}$ -invariant subspace, and this has dimension 1. In particular, $\mathcal{U}$ does *not* have an $X_{\mathbb{R}^+}$-invariant complement.

To make use of the above lemma we need to introduce direct sums of linear operators and representations.

2.2.1 Direct sums of representations

Let $\mathcal{V} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_r$ and let $\mathcal{A}_i$ be a basis of $\mathcal{V}_i$ for $i = 1, \ldots, r$, so $\mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_r$ is a basis of $\mathcal{V}$. Let $\mathsf{T}_i\colon \mathcal{V}_i \to \mathcal{V}_i$ be linear operators for $i = 1, \ldots, r$. Let the mapping $\mathsf{T}_1 \oplus \cdots \oplus \mathsf{T}_r\colon \mathcal{V} \to \mathcal{V}$ be defined by

$$(\mathsf{T}_1 \oplus \cdots \oplus \mathsf{T}_r)\,\mathbf{v} = \mathsf{T}_1\mathbf{v}_1 + \cdots + \mathsf{T}_r\mathbf{v}_r$$

where $\mathbf{v} = \mathbf{v}_1 + \cdots + \mathbf{v}_r$ and $\mathbf{v}_i \in \mathcal{V}_i$ for $i = 1, \ldots, r$. Then $\mathsf{T}_1 \oplus \cdots \oplus \mathsf{T}_r$ is a linear operator on $\mathcal{V}$. Moreover,

$$[\mathsf{T}_1 \oplus \cdots \oplus \mathsf{T}_r]_{\mathcal{A}} = \begin{bmatrix} [\mathsf{T}_1]_{\mathcal{A}_1} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & [\mathsf{T}_r]_{\mathcal{A}_r} \end{bmatrix}$$

Let $\rho_i\colon G \to \mathrm{GL}(\mathcal{V}_i)$ be a representation of G for $i = 1, \ldots, r$. Let

$$\rho_1 \oplus \cdots \oplus \rho_r\colon G \to \mathrm{GL}(\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_r)$$

be defined by

$$(\rho_1 \oplus \cdots \oplus \rho_r)(x) = \rho_1(x) \oplus \cdots \oplus \rho_r(x)$$

for all $x \in G$. Then $\rho_1 \oplus \cdots \oplus \rho_r$ is a representation of G and

$$[(\rho_1 \oplus \cdots \oplus \rho_r)(x)]_{\mathcal{A}} = \begin{bmatrix} [\rho_1(x)]_{\mathcal{A}_1} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & [\rho_r(x)]_{\mathcal{A}_r} \end{bmatrix}$$

for all $x \in G$.

If $r > 1$, $\rho = \rho_1 \oplus \cdots \oplus \rho_r$ is reducible, since ρ_i is a subrepresentation of ρ. If $\rho_1, \ldots, \rho_r$ are irreducible then ρ is said to be *completely reducible*.

Thus $X_{\mathbb{R}^+}$, defined above, is not completely reducible.

Theorem 2.6 *[Maschke] Let G be a finite group over $\mathbb{C}$ (or any field of characteristic 0 or prime to g). Then every representation $\rho: G \to \mathrm{GL}(\mathcal{V})$ is completely reducible.*

Proof: We use induction over $\dim \mathcal{V}$. If $\dim \mathcal{V} = 1$, then ρ is irreducible and hence completely reducible.

Now assume that the result holds for all spaces of dimension less than n, where $\deg(\rho) = n$. If ρ is irreducible, then ρ is completely reducible and the proof is complete. If ρ is reducible then $\mathcal{V}$ has a ρ-invariant subspace $\mathcal{U}$, so $\mathcal{V} = \mathcal{U} \oplus \mathcal{W}$ for some ρ-invariant subspace $\mathcal{W}$ of $\mathcal{V}$, by Lemma 2.5. Thus $\rho = \rho|_U \oplus \rho|_W$. But $\dim \mathcal{U} < n$ and $\dim \mathcal{V} < n$. The result follows by the Principle of Mathematical Induction. □

The matrix form of Maschke's Theorem is the following.

Theorem 2.7 *Let $X: G \to \mathcal{M}_n^\star$ be a matrix representation of G. Then there exist irreducible matrix representations $X_1, \ldots, X_r$ of G and an invertible matrix $\mathbf{M}$ such that*

$$\mathbf{M}X(x)\mathbf{M}^{-1} = \begin{bmatrix} X_1(x) & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & X_r(x) \end{bmatrix}$$

for all $x \in G$.

The remarkable aspect of this apparently simple proof is that the matrices $X(x)$, $x \in G$, are simultaneously block diagonalizable. It is easy to overlook this consequence of the proof.

Chapter 3

The character of a representation

The character of a representation ρ is the function χ^ρ that assigns to $x \in G$ the trace of $\rho(x)$. The remarkable fact about the character function is that such a simple similarity invariant of a matrix should have such fundamental and powerful properties, and should preserve so much information about G. Very often it allows us to avoid using the representing matrices themselves.

3.1 Properties of the trace

We will need the following properties of the trace.

Definition 3.1 *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. Then the* trace *of* $\mathbf{A}$ *is* $\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{i,i}$ *where* $a_{i,j}$ *is the* (i,j)*-element of* $\mathbf{A}$.

Lemma 3.2 *Let* $\mathbf{A}$ *be an* $n \times m$ *matrix and let* $\mathbf{B}$ *be an* $m \times n$ *matrix. Then* $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$.

Proof:

$$\begin{aligned}
\operatorname{tr}(\mathbf{AB}) &= \sum_{j=1}^{n} [\mathbf{AB}]_{j,j} = \sum_{j=1}^{n}\sum_{k=1}^{m} a_{j,k}b_{k,j} = \sum_{k=1}^{m}\sum_{j=1}^{n} b_{k,j}a_{j,k} \\
&= \sum_{k=1}^{m} [\mathbf{BA}]_{k,k} = \operatorname{tr}(\mathbf{BA}).
\end{aligned}$$

□

It follows that if $\mathbf{A}$ and $\mathbf{B}$ are similar matrices then $\operatorname{tr}(\mathbf{A}) = \operatorname{tr}(\mathbf{B})$, and this supports the following definition.

Definition 3.3 *Let* $\mathsf{T}: \mathcal{V} \rightarrow \mathcal{V}$ *be a linear operator. Then the* trace *of* T *is* $\operatorname{tr}(\mathsf{T}) = \operatorname{tr}([\mathsf{T}]_{\mathcal{A}})$ *where* $\mathcal{A}$ *is a basis of* $\mathcal{V}$.

For the next result we need some notation. Let $\mathcal{N}_n = \{1, \ldots, n\}$. If $\alpha, \beta \subseteq \mathcal{N}_n$ and $\mathbf{A}$ is an $n \times n$ matrix, let $\mathbf{A}[\alpha|\beta]$ denote the submatrix of $\mathbf{A}$ with rows elements of $\mathbf{A}$ selected from positions in $\alpha \times \beta$. Let $\bar{\alpha} = \mathcal{N}_n - \alpha$.

Theorem 3.4 *Let* $\mathbf{A}, \mathbf{B}$ *be* $n \times n$ *matrices and let* $\mathcal{N}_n = \{1, \ldots, n\}$. *Let* $\sigma(\alpha) = \sum_{i \in \alpha} i$. *Then*

$$\det(\mathbf{A} + \mathbf{B}) = \sum_{r=0}^{n} \sum_{\alpha, \beta \subseteq \mathcal{N}_n} (-1)^{\sigma(\alpha) + \sigma(\beta)} (\det \mathbf{A}[\alpha|\beta]) \left(\det \mathbf{B}\left[\bar{\alpha}|\bar{\beta}\right]\right)$$

where the sum is over all α, β *such that* $|\alpha| = |\beta| = r$.

The proof of this result is directly from determinant theory.

Corollary 3.5 *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix over* $\mathbb{C}$ *with eigenvalues* $\lambda_1, \ldots, \lambda_n$. *Then* $\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$.

Proof: Let $f(x) = \prod_{i=1}^{n} (x - \lambda_i)$ be the characteristic polynomial of $\mathbf{A}$ is so $\left[x^{n-1}\right] f(x) = -\sum_{i=1}^{n} \lambda_i$. On the other hand, from Theorem 3.4,

$$\begin{aligned} f(x) &= \det(x\mathbf{I} - \mathbf{A}) = \sum_{r=0}^{n} \sum_{\alpha \subseteq \mathcal{N}_n, |\alpha| = r} (\det(-\mathbf{A})[\alpha|\alpha]) \left(\det(x\mathbf{I})\left[\bar{\alpha}|\bar{\beta}\right]\right) \\ &= \sum_{r=0}^{n} \sum_{\alpha \subseteq \mathcal{N}_n, |\alpha| = r} (-1)^r x^{n-r} \det \mathbf{A}[\alpha|\alpha] \end{aligned}$$

so

$$\left[x^{n-1}\right] f(x) = -\sum_{\alpha \subseteq \mathcal{N}_n, |\alpha| = 1} \det \mathbf{A}[\alpha|\alpha] = -\operatorname{tr}(\mathbf{A}),$$

and the result follows. □

This result can of course be proved directly from the Jordan normal form, or from the Triangular Form Theorem. The above proof uses only properties of the determinant.

3.2 Characters

Definition 3.6 *Let* $\rho: G \rightarrow \mathrm{GL}(\mathcal{V})$ *be a representation of* G. *Let* $\mathcal{A}$ *be any basis of* $\mathcal{V}$. *Then the* character *of* ρ *is the function* $\chi^\rho: G \rightarrow \mathbb{C}$ *such that*

$$\chi^\rho(x) = \operatorname{tr}\left([\rho(x)]_A\right).$$

χ^ρ *is said to be* irreducible *if* ρ *is irreducible. The* degree *of* χ^ρ *is the degree of* ρ $(= \dim \mathcal{V})$.

When ρ is understood from the context we may abbreviate χ^{ρ} to χ.

Definition 3.7 *Let $X: G \to \mathcal{M}_n^{\star}$ be a matrix representation of G. The* character *of X is the function $\chi: G \to \mathbb{C}$ such that*

$$\chi(x) = \operatorname{tr}(X(x))$$

for all $x \in G$.

Lemma 3.8 *Let ρ' and ρ be representations of G. If $\rho' \sim \rho$ then $\chi^{\rho'} = \chi^{\rho}$.*

Proof: From Definition 1.11, there exists an invertible matrix $\mathbf{P}$ such that $\rho'(x) = \mathbf{P}\rho(x)\mathbf{P}^{-1}$ for all $x \in G$. Then, from Lemma 3.2, $\chi^{\rho'}(x) = \operatorname{tr}(\rho'(x)) = \operatorname{tr}\left(\mathbf{P}\rho(x)\mathbf{P}^{-1}\right) = \operatorname{tr}(\rho(x)) = \chi^{\rho}(x)$. □

In fact, the converse of this result is also true, but we shall require a considerable amount of additional material to prove it. It is proved in Theorem 4.9. This strengthening of the lemma indicates the importance of the characters.

3.2.1 Characters of nat_G and reg_G

Example 3.9 *Let $\rho = \mathrm{alt}_{\mathfrak{S}_n}$. Then $\chi^{\rho}(x) = \operatorname{sgn}(x)$.*

This is simply the observation that $\chi^{\rho}(x) = \operatorname{tr}(\operatorname{sgn}(x)) = \operatorname{sgn}(x)$. This is the *alternating character*. It appears in the determinant function. Indeed, if the alternating character is replaced by χ^{ρ} in the definition of the determinant, then the resulting function is called an *immanant*. It is an interesting matrix function that has been studied extensively.

Let $x \in \mathfrak{S}_n$ and let $\operatorname{fix}(x)$ be the set of all i, $1 \le i \le n$, such that $xi = i$. These are the points of $\{1, \ldots, n\}$ that are fixed by x.

Example 3.10 *Let $\rho = \mathrm{nat}_{\mathfrak{S}_n}$. Then $\chi^{\rho}(x) = |\operatorname{fix}(x)|$.*

To see this, let X be a matrix representation associated with ρ with respect to the basis $\mathcal{A} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Then, from Example 1.6,

$$[X(x)]_{i,j} = \begin{cases} 1 & \text{if } x(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\chi^{\rho}(x) = |\{i: xi = i, 1 \le i \le n\}| = |\operatorname{fix}(x)|$.

Example 3.11

$$\chi^{\mathrm{reg}_G}(x) = \begin{cases} |G| & \textit{if } x = \iota, \\ 0 & \textit{otherwise.} \end{cases}$$

Let $G = \{x_1, \ldots, x_n\}$, let $\mathcal{A} = \{v_{x_1}, \ldots, v_{x_n}\}$ and let $\mathcal{V} = \mathrm{span}(\mathcal{A})$. Then, from Example 1.5, $\mathrm{reg}_G(x)\,\mathbf{v}_{x_i} = \mathbf{v}_{xx_i}$ for $i = 1, \ldots, n$, so

$$[[\mathrm{reg}_G(x)]_{\mathcal{A}}]_{i,j} = \begin{cases} 1 & \text{if } xx_i = x_j, \\ 0 & \text{otherwise.} \end{cases}$$

For contributions to the trace we require that $i = j$. But $xx_i = x_i$ if and only if $x = \iota$. Thus $\chi^{\mathrm{reg}_G}(x) = \mathrm{tr}\,[\mathrm{reg}_G(x)] = n$ if $x = \iota$ and is 0 otherwise.

Example 3.12 *Let* $\rho_i \colon G \to \mathrm{reg}_G \mathrm{GL}(\mathcal{V}_i)$, $i = 1, \ldots, r$, *be representations of* G *and let* $\rho = \rho_1 \oplus \cdots \oplus \rho_r$. *Then* $\chi^\rho = \chi^{\rho_1} + \cdots + \chi^{\rho_r}$.

Let $\mathcal{V} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_r$. Then ρ is a representation of G with $\mathcal{V}$ is its representation space. Let $\mathcal{A}_i$ be a basis of $\mathcal{V}_i$ for $i = 1, \ldots, r$. Then

$$\begin{aligned} \chi^\rho(x) &= \mathrm{tr}\,[(\rho_1 \oplus \cdots \oplus \rho_r)(x)]_{\mathcal{A}} \\ &= \mathrm{tr}\,[\rho_1(x)]_{\mathcal{A}_1} + \cdots + \mathrm{tr}\,[\rho_1(x)]_{\mathcal{A}_r} \\ &= \chi^{\rho_1}(x) + \cdots + \chi^{\rho_r}(x) \end{aligned}$$

for all $x \in G$ so $\chi^\rho = \chi^{\rho_1} + \cdots + \chi^{\rho_r}$.

3.2.2 Some notation

Let ρ_i be a representation of G. As a notational device, let

$$m_i \rho_i = \underbrace{\rho_i \oplus \cdots \oplus \rho_i}_{m_i}.$$

By Maschke's Theorem (Theorem2.6), any representation ρ of a finite group G can be expressed as a decomposition

$$\rho = m_1 \rho_1 \oplus \cdots \oplus m_r \rho_r$$

into irreducibles. The integer m_i is called the *multiplicity* of ρ_i in ρ. Moreover, $\rho(x) = m_1 \rho_1(x) \oplus \cdots \oplus m_r \rho_r(x)$ for $x \in G$, so

$$\chi^\rho(x) = m_1 \chi^{\rho_1}(x) + \cdots + m_r \chi^{\rho_r}(x).$$

3.2.3 Preliminaries from group theory.

We will need some results from group theory for developing some of the properties of the characters.

Theorem 3.13 *[Lagrange] Let* G *be a group and let* $H < G$. *Then* $|H|$ *divides* $|G|$.

The number of left (right) cosets of H in G is the *index* of H in G, and is denoted by $[G \colon H]$. If G is finite, then $[G \colon H] = |G| / |H|$.

Definition 3.14 *Let G be a group and let $x \in G$. The* order *of x is $o(x) = \min\{i: x^i = \iota\}$.*

Proposition 3.15 *If G is a finite group and $x \in G$ then $o(x) < \infty$.*

Proof: Consider the subgroup $\langle x \rangle = \{\iota, x, x^2, \ldots\}$ of G generated by x. Since $|G| < \infty$, there exists j, i with $j > i$ such that $x^j = x^i$. Then $x^{j-i} = \iota$, so $\{l: x^l = \iota\}$ is non-empty and has a least member. The result follows. □

Corollary 3.16 *Let G be a group and let $x \in G$. Then $o(x)$ divides $|G|$.*

Proof: $x^{o(x)} = \iota$ so $\langle x \rangle$ has at least $o(x)$ elements. Suppose that it has fewer that $o(x)$ elements. Then some pair of elements in $\{\iota, x, \ldots, x^{o(x)-1}\}$ must be equal. Thus $x^p = x^q$ for some $1 \leq p < q < o(x)$, whence $x^{q-p} = \iota$. But $1 < q - p < o(x)$. This is a contradiction with the minimality of $o(x)$. Thus $|\langle x \rangle| = o(x)$ so, by Lagrange's Theorem (Theorem 3.13), $o(x)$ divides $|G|$. □

Corollary 3.17 *Let G be a finite group, and let $x \in G$. Then $x^{|G|} = \iota$.*

Proof: From Corollary 3.16, $o(x)$ divides $|G|$ so $|G| = l o(x)$, for some $l \geq 1$. Thus $x^{|G|} = \left(x^{o(x)}\right)^l = \iota^l = \iota$. □

3.2.4 Character of a conjugate and the inverse

We may now prove a useful result about characters.

Theorem 3.18 *Let G be a finite group and let ρ be a representation of G. Then*

1. $\chi^\rho(\iota) = \deg(\rho)$,
2. $\chi^\rho\left(y^{-1}xy\right) = \chi^\rho(x)$ for all $x, y \in G$,
3. $\chi^\rho\left(x^{-1}\right) = \overline{\chi^\rho(x)}$.

Proof: 1) $\rho: G \to \mathrm{GL}(\mathcal{V})$ where $\dim \mathcal{V} = n$. Let $\mathcal{A}$ be a basis of $\mathcal{V}$. Then $\chi^\rho(\iota) = \mathrm{tr}(\rho(\iota)) = \mathrm{tr}([\rho(\iota)]_\mathcal{A}) = \mathrm{tr}(\mathbf{I}_n) = n = \dim \mathcal{V} = \deg(\rho)$.

2) $\chi^\rho\left(y^{-1}xy\right) = \mathrm{tr}\left(\rho\left(y^{-1}xy\right)\right) = \mathrm{tr}\left(\rho\left(y^{-1}\right)\rho(x)\rho(y)\right) = \mathrm{tr}(\rho(x)) = \chi^\rho(x)$.

3) From Corollary 3.5, $\chi(x) = \mathrm{tr}(\rho(x)) = \sum_{\lambda \in \mathrm{spec}(\rho(x))} \lambda$, where $\mathrm{spec}(\rho(x))$ is the multiset of all eigenvalues of $\rho(x)$ (so repeated eigenvalues are included). Since $\lambda \in \mathrm{spec}(\rho(x))$, there exists a non-zero vector $\mathbf{v} \in \mathcal{V}$ such that $\rho(x)\mathbf{v} = \lambda\mathbf{v}$. Now $\lambda \neq 0$ since $\rho(x)$ is invertible. Thus λ^{-1} is an eigenvalue of $\rho\left(x^{-1}\right)$. Thus

$$\chi^\rho\left(x^{-1}\right) = \sum_{\lambda \in \mathrm{spec}(\rho(x))} \lambda^{-1}.$$

Let $g = |G|$. Then, by Corollary 3.17, $\lambda^g \mathbf{v} = \rho(x)^g \mathbf{v} = \rho(x^g)\mathbf{v} = \rho(\iota)\mathbf{v} = \mathsf{I}_{\mathcal{V}}\mathbf{v} = \mathbf{v}$ so $\lambda^g = 1$. Then $|\lambda| = 1$ so $\lambda\overline{\lambda} = 1$. Thus $\lambda^{-1} = \overline{\lambda}$. Then

$$\chi^\rho\left(x^{-1}\right) = \sum_{\lambda \in \operatorname{spec}(\rho(x))} \lambda^{-1} = \sum_{\lambda \in \operatorname{spec}(\rho(x))} \overline{\lambda} = \overline{\sum_{\lambda \in \operatorname{spec}(\rho(x))} \lambda} = \overline{\chi^\rho(x)}.$$

This completes the proof. □

Part (2) of Theorem 3.18 states that χ^ρ is constant on the conjugacy classes of G. Such a function is called a *class function* of G, and will be considered again in Chapter 5

We note the following result concerning real characters.

Corollary 3.19 *Let G be a finite group and let χ be a character of G. If x and x^{-1} are conjugates in G then $\chi(x) \in \mathbb{R}$.*

Proof: x and x^{-1} are conjugates in G so there exist $y \in G$ such that $x^{-1} = yxy^{-1}$. Then, from Theorem 3.18, $\overline{\chi(x)} = \chi\left(x^{-1}\right) = \chi\left(yxy^{-1}\right) = \chi(x)$, and the result follows. □

Corollary 3.20 *Let χ be a character of $\mathfrak{S}_n$. Then $\chi(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$.*

Proof: x and x^{-1} are conjugates in $\mathfrak{S}_n$. The result follows from Corollary 3.19. □

This corollary will be of importance in Chapter 8, where the character theory of the symmetric group is studied in greater detail.

Chapter 4

The orthogonality of the characters

4.1 Schur's Lemma

We begin with a fundamental lemma that will be used repeatedly.

Lemma 4.1 *[Schur] Let* $\rho: G \to \mathrm{GL}(\mathcal{V})$ *and* $\rho': G \to \mathrm{GL}(\mathcal{V}')$ *be irreducible representations of* G. *If* $\mathsf{T}: \mathcal{V} \to \mathcal{V}'$ *is a linear transformation such that* $\mathsf{T}\rho(x) = \rho'(x)\mathsf{T}$ *for all* $x \in G$, *then*

1. Then
   (a) either $\mathsf{T} = 0_{\mathcal{V}}$,
   (b) or T is an isomorphism.
2. Moreover,
   (a) if $\rho \nsim \rho'$ then $\mathsf{T} = 0_{\mathcal{V}}$,
   (b) if $\rho = \rho'$ then $\mathsf{T} = \alpha \mathsf{I}_{\mathcal{V}}$ for some $\alpha \in \mathbb{C}$.

Proof: 1) We consider $\ker(\mathsf{T})$. Let $\mathbf{v} \in \ker(\mathsf{T})$. Then $\mathsf{T}\rho(x)\mathbf{v} = \rho'\mathsf{T}\mathbf{v} = \mathbf{0}$ so $\rho(x)\mathbf{v} \in \ker(\mathsf{T})$. Thus $\ker(\mathsf{T})$ is ρ-invariant. But ρ is irreducible so $\ker(\mathsf{T}) = \{\mathbf{0}\}$ or $\mathcal{V}$. There are therefore two case.

Case (1a): Let $\ker(\mathsf{T}) = \mathcal{V}$. Then $\mathsf{T} = 0_{\mathcal{V}}$ and the proof is complete.

Case (1b): Let $\ker(\mathsf{T}) = \{\mathbf{0}\}$. Then $\mathsf{T} \neq 0_{\mathcal{V}}$. We now show that $\mathsf{T}\mathcal{V}$ is a ρ'-invariant subspace of $\mathcal{V}'$. Let $\mathbf{v}' \in \mathsf{T}\mathcal{V}$. Then $\mathbf{v}' = \mathsf{T}\mathbf{v}$ for some $\mathbf{v} \in \mathcal{V}$. Let $x \in G$. Then $\rho'(x)\mathbf{v}' = \rho'(x)\mathsf{T}\mathbf{v} = \mathsf{T}\rho(x)\mathbf{v}$ so $\rho'(x)\mathbf{v}' \in \mathsf{T}\mathcal{V}$. Thus $\mathsf{T}\mathcal{V}$ is a ρ'-invariant subspace of $\mathcal{V}'$. But ρ' is irreducible so $\mathsf{T}\mathcal{V} = \{\mathbf{0}\}$ or $\mathcal{V}'$. But $\mathsf{T}\mathcal{V} \neq \{\mathbf{0}\}$ since $\mathsf{T} \neq 0_{\mathcal{V}}$. Thus $\mathsf{T}\mathcal{V} = \mathcal{V}'$, so T is surjective. But $\ker(\mathsf{T}) = \{\mathbf{0}\}$ so T is injective. Thus T is an isomorphism.

2) (a) We assume that $\rho \nsim \rho'$. Suppose $\mathsf{T} \neq 0_{\mathcal{V}}$. Then, from (1b), T is an isomorphism so $\rho \sim \rho'$, which is a contradiction. Thus $\mathsf{T} = 0_{\mathcal{V}}$.

2)(b) We assume that $\rho \sim \rho'$. Then $\mathsf{T}\rho(x) = \rho(x)\mathsf{T}$ for all $x \in G$. Let $\alpha \in C$. Then $(\mathsf{T} - \alpha\mathsf{I})\rho(x) = \rho(x)(\mathsf{T} - \alpha\mathsf{I})$ so, from (1), $\mathsf{T} - \alpha\mathsf{I} = 0_{\mathcal{V}}$ or $\mathsf{T} - \alpha\mathsf{I}$ is an isomorphism. Now $\operatorname{spec}(\mathsf{T}) \neq \emptyset$, so select $\alpha \in \operatorname{spec}(\mathsf{T})$. Thus there exists a non-zero $\mathbf{v} \in \mathcal{V}$ such that $\mathsf{T}\mathbf{v} = \alpha\mathbf{v}$, whence $(\mathsf{T} - \alpha\mathsf{I})\mathbf{v} = \mathbf{0}$. Since $\mathbf{v} \neq \mathbf{0}$, then $\ker(\mathsf{T}) \neq \{\mathbf{0}\}$, so $\mathsf{T} - \alpha\mathsf{I}$ is not an isomorphism. Then $\mathsf{T} - \alpha\mathsf{I} = 0_{\mathcal{V}}$, and the result follows. □

There is a matrix formulation of this result.

Lemma 4.2 *[Schur] Let $X\colon G \to \mathcal{M}_n^\star$ and $X'\colon G \to \mathcal{M}_m^\star$ be irreducible representations of G. Let $\mathbf{M}$ be a matrix such that $\mathbf{M}X(x) = X'(x)\mathbf{M}$ for all $x \in G$.*

1. Then
   (a) either $\mathbf{M} = \mathbf{0}_n$,
   (b) or $\mathbf{M}$ is invertible.
2. Moreover,
   (a) if $X \nsim X'$ then $\mathbf{M} = \mathbf{0}_n$,
   (b) if $X = X'$ then $\mathbf{M} = \alpha\mathbf{I}_n$ for some $\alpha \in \mathbb{C}$.

The next result is a technical one which will be used in the next section.

Corollary 4.3 *Let Let $X\colon G \to \mathcal{M}_n^\star$ and $X'\colon G \to \mathcal{M}_m^\star$ be irreducible representations of G. Let $\mathbf{M}$ be an $n \times m$ matrix over $\mathbb{C}$, and let $\mathbf{N}$ be the $n \times m$ matrix*

$$\mathbf{N} = \frac{1}{|G|}\sum_{x \in G} X(x)\,\mathbf{M}X'\left(x^{-1}\right).$$

Then

$$\mathbf{N} = \begin{cases} \frac{1}{n}(\operatorname{tr}(\mathbf{M}))\,\mathbf{I}_n & \text{if } X = X', \\ \mathbf{0}_n & \text{if } X \neq X'. \end{cases}$$

Proof: We check that $X(y)\mathbf{N} = \mathbf{N}X'(y)$ for all $y \in G$, and then apply Schur's Lemma. Now

$$\begin{aligned} X(y)\mathbf{N} &= \frac{1}{|G|}\sum_{x \in G} X(yx)\,\mathbf{M}X'\left(x^{-1}\right), \text{ since } X \text{ is a representation} \\ &= \frac{1}{|G|}\sum_{x \in G} X(yx)\,\mathbf{M}X'\left(x^{-1}y^{-1}\right)X'(y), \text{ since } X'\left(y^{-1}\right)X'(y) = \mathbf{I}_n \\ &= \left(\frac{1}{|G|}\sum_{z \in G} X(z)\,\mathbf{M}X'\left(z^{-1}\right)\right)X'(y) = \mathbf{N}X'(y). \end{aligned}$$

From Schur's Lemma, if $X \nsim X'$ then $\mathbf{N} = \mathbf{0}_n$. On the other hand, if $X = X'$ then $\mathbf{N} = \alpha \mathbf{I}_n$ for some $\alpha \in \mathbb{C}$, so $\operatorname{tr}(\mathbf{N}) = n\alpha$. But

$$\begin{aligned}\operatorname{tr}(\mathbf{N}) &= \frac{1}{|G|}\sum_{x\in G}\operatorname{tr}\left(X(x)\,\mathbf{M}X\left(x^{-1}\right)\right)\\ &= \frac{1}{|G|}\sum_{x\in G}\operatorname{tr}\left(X(x)^{-1}X(x)\,\mathbf{M}\right)\text{ from Lemma 3.2}\\ &= \frac{1}{|G|}\sum_{x\in G}\operatorname{tr}(\mathbf{M}) = \operatorname{tr}(\mathbf{M}),\end{aligned}$$

so $n = \alpha \operatorname{tr}(\mathbf{M})$ and the result follows. □

4.2 The vector space $\mathbb{C}^G$

We now construct a vector space in which the characters reside. Let G be a finite group. Then $\mathbb{C}^G = \{f : G \to \mathbb{C}\}$, equipped with the operations

$$\begin{aligned}(f+g)(x) &= f(x)+g(x),\text{ for all } x\in G,\\ (\lambda f)(x) &= \lambda f(x),\text{ for all } x\in G, \lambda\in\mathbb{C}\end{aligned}$$

is a vector space. If χ is a character of G then $\chi \in \mathbb{C}^G$. Let $\langle\cdot,\cdot\rangle_{\mathbb{C}^G} : \mathbb{C}^G \times \mathbb{C}^G \to \mathbb{C}$ be a form defined by

$$\langle f,g\rangle_{\mathbb{C}^G} = \frac{1}{|G|}\sum_{x\in G} f(x)\overline{g(x)}$$

for all $f, g \in \mathbb{C}^G$. Then $\langle\cdot,\cdot\rangle_{\mathbb{C}^G}$ is an inner product on $\mathbb{C}^G$. Where the context makes it clear, we will abbreviate $\langle\cdot,\cdot\rangle_{\mathbb{C}^G}$ to $\langle\cdot,\cdot\rangle$. We can use this inner product to distinguish irreducible representations.

4.2.1 Orthogonality of the characters

Theorem 4.4 *[Orthogonality relations for the first kind] Let χ, χ' be (irreducible) characters corresponding, respectively, to the irreducible representations ρ, ρ' of G. Then*

$$\langle\chi,\chi'\rangle = \begin{cases}1 & \text{if } \rho\sim\rho',\\ 0 & \text{if } \rho\nsim\rho'.\end{cases}$$

Proof: Let X, X' be matrix representations corresponding to ρ, ρ', respectively. Then

$$\begin{aligned}\langle\chi,\chi'\rangle &= \frac{1}{|G|}\sum_{x\in G}\chi(x)\overline{\chi(x)}\\ &= \frac{1}{|G|}\sum_{x\in G}\chi(x)\chi\left(x^{-1}\right),\text{ from Thm. 3.18 (3)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|G|}\sum_{x\in G}\operatorname{tr}(X(x))\operatorname{tr}\left(X'\left(x^{-1}\right)\right)\\
&= \frac{1}{|G|}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{x\in G}[X(x)]_{i,i}\left[X'\left(x^{-1}\right)\right]_{j,j}.
\end{aligned}
$$

Let $\mathbf{P}^{i,j}$ be the $n\times m$ matrix with (k,l) -element equal to $\delta_{i,k}\delta_{l,j}$. Let

$$\mathbf{S}^{i,j}=\frac{1}{|G|}\sum_{x\in G}X(x)\,\mathbf{P}^{i,j}X'\left(x^{-1}\right).$$

Then

$$
\begin{aligned}
\left[\mathbf{S}^{i,j}\right]_{i,j} &= \frac{1}{|G|}\sum_{x\in G}\sum_{k,l}[X(x)]_{i,k}\left[\mathbf{P}^{i,j}\right]_{k,l}\left[X'\left(x^{-1}\right)\right]_{l,j}\\
&= \frac{1}{|G|}\sum_{x\in G}\sum_{k,l}[X(x)]_{i,k}\,\delta_{i,k}\delta_{l,j}\left[X'\left(x^{-1}\right)\right]_{l,j}\\
&= \frac{1}{|G|}\sum_{x\in G}[X(x)]_{i,i}\left[X'\left(x^{-1}\right)\right]_{j,j}
\end{aligned}
$$

so

$$\langle\chi,\chi'\rangle=\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\mathbf{S}^{i,j}\right]_{i,j}.$$

Then, from Corollary 4.3,

$$\mathbf{S}^{i,j}=\begin{cases}\mathbf{0}_{n\times m} & \text{if } X\nsim X',\\ \frac{1}{n}\left(\operatorname{tr}\left(\mathbf{P}^{i,j}\right)\right)\mathbf{I}_n & \text{if } X\sim X'.\end{cases}$$

But $\operatorname{tr}\left(\mathbf{P}^{i,j}\right)=0$ unless $i=j$, in which case $\operatorname{tr}\left(\mathbf{P}^{i,j}\right)=1$. Thus

$$\left[\mathbf{S}^{i,j}\right]_{i,j}=\begin{cases}0 & \text{if } X\nsim X',\\ 0 & \text{if } i\neq j,\\ \frac{1}{n} & \text{if } i=j \text{ and } X\sim X'.\end{cases}$$

Now, if $X=X'$, then $\chi=\chi'$, from Lemma 3.8 , whence

$$\langle\chi,\chi'\rangle=\begin{cases}1 & \text{if } X\sim X',\\ 0 & \text{if } X\nsim X',\end{cases}$$

and the result follows. □

4.2.2 A test for the irreducibility of characters

We may use this result to test whether a representation is irreducible.

Lemma 4.5 *Let G be a finite group and let $\rho = m_1\rho_1 \oplus \cdots \oplus m_r\rho_r$, where $\rho_1, \ldots, \rho_r$ are irreducible representations of G. Then*

$$\langle \chi^\rho, \chi^\rho \rangle = \sum_{i=1}^{r} m_i^2.$$

Proof: From Theorem 4.4

$$\langle \chi^\rho, \chi^\rho \rangle = \sum_{i,j=1}^{r} m_i m_j \langle \chi^{\rho_i}, \chi^{\rho_j} \rangle = \sum_{i,j=1}^{r} m_i m_j \delta_{i,j}$$

and the result follows. □

Corollary 4.6 *Let G be a finite group and let ρ be a representation of G. Then ρ is irreducible if and only if $\langle \chi^\rho, \chi^\rho \rangle = 1$.*

Proof: Immediate, from Lemma 4.5. □

Example 4.7 *The matrix representation given in Example 1.9 is irreducible.*

Let χ denote the character of this representation. Then, by direct calculation with the representing matrices,

$$\begin{aligned} \langle \chi, \chi \rangle &= \frac{1}{|G|} \sum_{x \in \mathfrak{S}_3} \chi(x) \overline{\chi(x)} \\ &= \frac{1}{6} (2 \cdot 2 + 0 \cdot 0 + 0 \cdot 0 + 0 \cdot 0 + (-1) \cdot (-1) + (-1) \cdot (-1)) = 1. \end{aligned}$$

Thus, from Corollary 4.6, it follows that the representation is irreducible.

4.3 Some properties of irreducible representations

We are now in a position to complete the connexion between equivalence of representations and equivalence of characters, and to show that every irreducible representation occurs at least once in the regular representation reg_G of G.

Corollary 4.8 *Let G be a finite group. Let $\rho_1, \ldots, \rho_r$ be the set of all irreducible representations of G. Then*

1. the multiplicity of ρ_i in reg_G is $\deg(\rho_i)$.
2. $|G| = \sum_{i=1}^{r} (\deg(\rho_i))^2$.
3. $1 \leq r \leq |G|$.

Proof: 1) Let reg denote reg_G. Now

$$\langle \chi^{\mathrm{reg}_G}, \chi^{\rho_i} \rangle = \frac{1}{|G|} \sum_{x \in G} \chi^{\mathrm{reg}_G}(x) \overline{\chi^{\rho_i}(x)}.$$

But, from Example 3.11, $\chi^{\mathrm{reg}_G}(x) = |G|$ if $x = \iota$, and is 0 otherwise, so

$$\langle \chi^{\mathrm{reg}_G}, \chi^{\rho_i} \rangle = \frac{1}{|G|} \chi^{\mathrm{reg}_G}(\iota) \overline{\chi^{\rho_i}(\iota)} = \overline{\chi^{\rho_i}(\iota)} = \deg(\rho_i).$$

By Maschke's Theorem (Theorem 2.6) reg $= \oplus_{j=1}^r m_j \rho_j$ where $m_1, \ldots, m_j$ are non-negative integers. Then

$$\langle \chi^{\mathrm{reg}_G}, \chi^{\rho_i} \rangle = \left\langle \sum_{j=1}^r m_j \chi^{\rho_j}, \chi^{\rho_i} \right\rangle = \sum_{j=1}^r m_j \langle \chi^{\rho_j}, \chi^{\rho_i} \rangle = m_i$$

by the orthogonality of the irreducible characters (Theorem 4.4) whence

$$\mathrm{reg}_G = \oplus_{j=1}^r (\deg(\rho_j)) \rho_j.$$

This establishes (1).

2) Let $\mathcal{V}$ be the representation space for reg_G. Now, from (1)

$$|G| = \dim \mathcal{V} = \chi^{\mathrm{reg}_G}(\iota) = \sum_{j=1}^r (\deg \rho_j) \chi^{\rho_j}(\iota) = \sum_{j=1}^r (\deg \rho_j)^2,$$

giving the result.

3) $\deg(\rho_i) \geq 1$ so, from (2), $1 \leq r \leq |G|$. □

Thus, in reg_G, every irreducible representation of G occurs with multiplicity equal to its degree. This enables us to prove the converse of Lemma 3.8, which is contained in the following result.

Theorem 4.9 *Let ρ, ρ' be representations of a finite group G. Then $\rho \sim \rho'$ if and only if $\chi^\rho \sim \chi^{\rho'}$.*

Proof: If $\rho \sim \rho'$ then $\chi^\rho \sim \chi^{\rho'}$ from Lemma 3.8. We now prove the reverse implication. Since G is finite, ρ and ρ' are completely reducible, by Maschke's Theorem (Theorem 2.6) so $\rho = \oplus_{i=1}^r m_i \rho_i$ and $\rho' = \oplus_{j=1}^r n_i \rho_i$ where $\rho_1, \ldots, \rho_r$ are the irreducible representations of G. Then $\chi = \sum_{i=1}^r m_i \chi^{\rho_i}$ and $\chi' = \sum_{j=1}^r n_i \chi^{\rho_i}$. Then, by the orthogonality of the irreducible characters (Theorem 4.4),

$$m_i = \langle \chi^\rho, \chi^{\rho_i} \rangle = \left\langle \chi^{\rho'}, \chi^{\rho_i} \right\rangle = n_i$$

for $i = 1, \ldots, r$, so $\rho \sim \rho'$, and the result follows. □

There is an expression for $\langle \chi, \chi' \rangle$ that is convenient when χ and χ' are irreducible characters of G. From Theorem 3.18(2) we know that χ is constant on the conjugacy classes of G. Let $\mathcal{C}$ denote a conjugacy class of G and let $x \in \mathcal{C}$. We denote the value of χ at any element in $\mathcal{C}$ by $\chi(\mathcal{C})$. Thus $\chi(x) = \chi(\mathcal{C})$.

Corollary 4.10 *Let G be a finite group and let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be the conjugacy classes of G. Let χ, χ' be irreducible characters of G. Then*

1. $\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum_{i=1}^{k} |\mathcal{C}_i| \chi(\mathcal{C}_i) \overline{\chi'(\mathcal{C}_i)}$.
2. $\langle \chi, \chi' \rangle = \delta_{\chi,\chi'}$.

Proof: 1)

$$\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum_{x \in G} \chi(x) \overline{\chi'(x)} = \frac{1}{|G|} \sum_{i=1}^{k} \sum_{x \in \mathcal{C}_i} \chi(x) \overline{\chi'(x)}$$

and the result follows since, from Theorem 3.18(2), χ, χ' are constant on conjugacy classes.

2) Let ρ and ρ' be the representations corresponding to χ and χ'. From the orthogonality of the characters (Theorem 4.4), $\langle \chi, \chi' \rangle = 1$ if $\rho \sim \rho'$ and is 0 otherwise. But, from Theorem 4.9, $\rho \sim \rho'$ if and only if $\chi = \chi'$. The result follows. □

Chapter 5

Class functions

The ring of class functions is natural ring in which to consider further properties of the characters.

5.1 An inner product

The set $\mathbb{C}^G = \{f\colon G \to \mathbb{C}\}$, with pointwise sum and product, is a ring, and

$$\langle f, g\rangle_{\mathbb{C}^G} = \frac{1}{|G|}\sum_{x\in G} f(x)\,\overline{g(x)}$$

defines an inner product for all $f, g \in \mathbb{C}^G$.

Definition 5.1 *$f \in \mathbb{C}^G$ is said to be a* class function *if $f(x) = f\left(yxy^{-1}\right)$ for all $x, y \in G$.*

The set of all class functions of G is denoted by $R(G)$. Let ρ be a representation of G. Then clearly $\chi^\rho \in \mathbb{C}^G$. In fact, we know, from Theorem 3.18(2), that the stronger containment $\chi^\rho \in R(G)$ holds.

Lemma 5.2 $\dim(R(G)) = k$, *where k is the number of conjugacy classes of G.*

Proof: To determine $\dim(R(G))$ we construct a basis of $R(G)$. For $i = 1, \ldots, k$, let $e_i \in \mathbb{C}^G$ be defined by $e_i(x) = 1$ if $x \in C_i$ and 0 otherwise. Then $e_i \in R(G)$, for $i = 1, \ldots, k$. Let $f(\mathcal{C}_i)$ denote the value of $f \in R(G)$ and any $x \in G$. Then $f(x) = f(\mathcal{C}_1)\,e_1(x) + \cdots + f(\mathcal{C}_k)\,e_k(x)$ for all $x \in G$, so $f = f(\mathcal{C}_1)\,e_1 + \cdots + f(\mathcal{C}_k)\,e_k$. Thus $R(G) \le \operatorname{span}\{e_1, \ldots, e_k\}$. The reverse inclusion is trivial, and $e_1, \ldots, e_k$ are clearly so $\{e_1, \ldots, e_k\}$ is a basis of $R(G)$, whence $\dim(R(G)) = k$. $\square$

We show that the irreducible characters of G form another basis of $R(G)$ and, for this, the following preliminary lemma is needed.

Lemma 5.3 *Let ρ be an irreducible representation of G and let $f \in R(G)$. Then*

$$\sum_{x\in G} \overline{f(x)}\rho(x) = \frac{|G|}{\deg \rho} \langle \chi^\rho, f\rangle_{\mathbb{C}^G} \mathsf{I}_\mathcal{V}$$

where $\mathcal{V}$ is the representation space.

Proof: Let $\mathsf{T} = \sum_{x\in G} \overline{f(x)}\rho(x)$. Now, for $y \in G$,

$$\begin{aligned}
\rho(y)\mathsf{T} &= \sum_{x\in G} \overline{f(x)}\rho(yx) \\
&= \sum_{x\in G} \overline{f(x)}\rho\left(yxy^{-1}\right)\rho(y) \\
&= \sum_{x\in G} \overline{f(yxy^{-1})}\rho\left(yxy^{-1}\right)\rho(y) \text{ since } f \in R(G) \\
&= \left(\sum_{x\in G} \overline{f(z)}\rho(z)\right)\rho(y) = \mathsf{T}\rho(y).
\end{aligned}$$

Thus $\rho(y)\mathsf{T} = \mathsf{T}\rho(y)$ for all $y \in G$ so, by Schur's Lemma (Lemma 4.1(2b)), $\mathsf{T} = \alpha \mathsf{I}_\mathcal{V}$ for some $\alpha \in \mathbb{C}$. Then $\sum_{x\in G} \overline{f(x)}\rho(x) = \alpha\mathsf{I}_\mathcal{V}$ and, taking the trace, we have

$$\operatorname{tr}\left(\sum_{x\in G} \overline{f(x)}\rho(x)\right) = \sum_{x\in G} \overline{f(x)}\operatorname{tr}(\rho(x)) = \sum_{x\in G} \overline{f(x)}\chi^\rho(x) = |G| \langle \chi^\rho, f\rangle_{\mathbb{C}^G},$$

and $\operatorname{tr}(\alpha\mathsf{I}_\mathcal{V}) = \alpha \operatorname{tr}(\mathsf{I}_\mathcal{V}) = \alpha \dim \mathcal{V} = \alpha \deg \rho$, whence $\alpha \deg \rho = |G| \langle \chi^\rho, f\rangle_{\mathbb{C}^G}$. The result follows. □

5.2 The character basis

The next result gives the character basis for $R(G)$.

Theorem 5.4 *Let G be a finite group. Then set of all irreducible characters of G is an orthonormal basis of $R(G)$.*

Proof: Let $\mathcal{V}$ be the representation space for reg_G so $\mathcal{V} = \operatorname{span}\{v_x : x \in G\}$. Then $\dim(\mathcal{V}) = |G|$. Since G is finite we have, by Maschke's Theorem (Thm. 2.6), $\mathrm{reg}_G = m_1\rho_1 \oplus \cdots \oplus m_r\rho_r$ where $\mathcal{V} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_r$ and $\rho_i : G \to \mathrm{GL}(\mathcal{V}_i)$, for $i = 1, \ldots, r$ are the irreducible representations of G. Let k be the number of conjugacy classes of G so, from Lemma 5.2, $\dim R(G) = k$. But $\chi^{\rho_1}, \ldots, \chi^{\rho_r} \in R(G)$ so $r \le k$. Suppose that $r \ne k$. Then there exists a non-zero function $f \in R(G)$ that is orthogonal to $\chi^{\rho_1}, \ldots, \chi^{\rho_r}$, so $\langle \chi^{\rho_i}, f\rangle_{\mathbb{C}^G} = 0$ for $i = 1, \ldots, r$. Then, from Lemma 5.3, $\sum_{x\in G} \overline{f(x)}\mathrm{reg}(x) = 0_\mathcal{V}$. Let $y \in G$. Then

$$\mathbf{0} = 0_\mathcal{V}\mathbf{v}_y = \sum_{x\in G} \overline{f(x)}\mathrm{reg}(x)\,\mathbf{v}_y = \sum_{x\in G} \overline{f(x)}\mathbf{v}_{xy} = \sum_{z\in G} \overline{f(zy^{-1})}\mathbf{v}_z.$$

But $\{v_x : x \in G\}$ is a basis of $\mathcal{V}$, so $f\left(zy^{-1}\right) = 0$ for all $y, z \in G$, whence $f(x) = 0$ for all $x \in G$. Thus $f = 0$. But this is a contradiction, so $r = k$. Then $\operatorname{span}\{\chi^{\rho_1}, \ldots, \chi^{\rho_k}\} \leq R(G)$. But $\chi^{\rho_1}, \ldots, \chi^{\rho_r}$ are linearly independent since, by Corollary 4.10, there are pairwise mutually orthogonal. Thus $\{\chi^{\rho_1}, \ldots, \chi^{\rho_k}\}$ is an orthonormal basis of $R(G)$, which completes the proof. □

The following result now clarifies the question of the number of distinct irreducible representations of G.

Corollary 5.5 *Let G be a group with precisely k conjugacy classes. Then G has precisely k distinct characters (and therefore k distinct irreducible representations).*

Proof: The first part is from Theorem 5.4 and the fact that $\dim R(G) = k$. The second part follows from Theorem 4.9 that asserts that $\chi^{\rho_i} \sim \chi^{\rho_j}$ if and only if $\rho_i \sim \rho_j$. □

It is worthwhile recording the following result.

Corollary 5.6 *Let G be a group with precisely k conjugacy classes, and let $\rho_1, \ldots, \rho_k$ be irreducible representations of G. Then, for $x \in G$,*

$$\frac{1}{|G|} \sum_{i=1}^{k} (\deg \rho_i)\, \chi^{\rho_i}(x) = \delta_{\iota,x}.$$

Proof: From Corollary 5.5, there are k distinct irreducible representations, so $\mathrm{reg}_G = \oplus_{i=1}^{k} (\deg \rho_i)\, \rho_{i.}$ Then $\chi^{\mathrm{reg}_G} = \sum_{i=1}^{k} (\deg \rho_i)\, \chi^{\rho_{i\cdot}}$, so $\chi^{\mathrm{reg}_G}(x) = \sum_{i=1}^{k} (\deg \rho_i)\, \chi^{\rho_{i\cdot}}(x)$ for all $x \in G$. But from Example 3.11, $\chi^{\mathrm{reg}_G}(x) = |G|\, \delta_{\iota,x}$, and from Corollary 4.8(2), $|G| = \sum_{i=0}^{k} (\deg \rho_i)^2$. The result follows immediately. □

Chapter 6

The group algebra

In this chapter we develop the orthogonal idempotents and it is these that will lead to the determination of the $c_{\alpha,\beta}^{\gamma}$ that are needed for the proof of Theorem 1.1.

6.1 The centre $Z_{\mathbb{C}G}$

Let $G = \{x_1, \ldots, x_g\}$ be a finite group and let

$$\mathbb{C}G = \{a_1 x_1 + \cdots + a_g x_g : a_1, \ldots, a_g \in \mathbb{C}\}.$$

Let $x, y \in \mathbb{C}G$, so $x = \sum_{i=1}^{g} a_i x_i$ and $y = \sum_{j=1}^{g} b_j x_j$. The product xy is defined to be $xy = \sum_{i,j=1}^{g} a_i a_j (x_i x_j)$. Then $\mathbb{C}G$ is called the *group algebra* of G over $\mathbb{C}$. Its dimension is g, and $\{x_1, \ldots, x_g\}$ is a basis. The *centre* of $\mathbb{C}G$ is

$$Z_{\mathbb{C}G} = \{z \in \mathbb{C}G : xz = zx \text{ for all } x \in \mathbb{C}G\}.$$

The centre $Z_{\mathbb{C}G}$ is a subalgebra of $\mathbb{C}G$, and its elements are called *central* elements. It is easy to construct a basis for the centre.

Lemma 6.1 *Let $K_i = \sum_{a \in C_i} a$, for $i = 1, \ldots, k$. Then $\{K_1, \ldots, K_k\}$ is a basis for $Z_{\mathbb{C}G}$.*

Proof: For an arbitrary $y \in G$

$$yK_i = \sum_{x \in C_i} yx = \sum_{x \in C_i} \left(yxy^{-1}\right) y = K_i y.$$

Then $yK_i = K_i y$ for all $y \in G$ so $K_i \in Z_{\mathbb{C}G}$. Moreover, summing y over C_j we have $K_i K_j = K_j K_i$ for all $1 \leq i, j \leq k$, so the K_i's commute.

Let $x \in Z_{\mathbb{C}G}$. Then $x \in \mathbb{C}G$ so $x = \sum_{u \in G} \alpha_u u$ where $\alpha_u \in \mathbb{C}$, and $xy = yx$ for all $y \in G$. Thus

$$\sum_{u \in G} \alpha_u u = x = yxy^{-1} = \sum_{u \in G} \alpha_u yuy^{-1} = \sum_{u \in G} \alpha_{y^{-1}uy} u,$$

so, equating coefficients of these two expressions we have $\alpha_u = \alpha_{y^{-1}uy}$ for all $y \in G$, since $\{u: u \in G\}$ is a basis of $\mathbb{C}G$. Then $x = \sum_{i=1}^{k} \beta_i K_i$ where $\beta_i \in \mathbb{C}$ for $i = 1, \ldots, k$. It follows that $Z_{\mathbb{C}G} \leq \text{span}\{K_i: i = 1, \ldots, k\}$. The reverse inclusion is immediate so $Z_{\mathbb{C}G} = \text{span}\{K_i: i = 1, \ldots, k\}$.

To test for linear independence, let $\sum_{i=1}^{k} \gamma_i K_i = 0$. Consider $x \in G$. Then $x \in C_j$ for a unique j, where $1 \leq j \leq k$. Equating the coefficients of x on both sides of this expression we have $\gamma_j = 0$. Thus $\gamma_1 = \cdots = \gamma_k = 0$ so $\{K_1, \ldots, K_k\}$ is linearly independent. We conclude that $\{K_1, \ldots, K_k\}$ is a basis for $Z_{\mathbb{C}G}$. $\square$

It follows that, for $1 \leq p, q \leq k$,

$$K_p K_q = \sum_{i=1}^{k} c_{p,q}^{r} K_r$$

for unique $c_{p,q}^{r} \in C$, for $r = 1, \ldots, k$. These are the *connexion coefficients* of $Z_{\mathbb{C}G}$, and they are the coefficients that have combinatorial significance in the study of maps.

6.2 The orthogonal idempotents

Let G be a finite group. Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be its conjugacy classes, and let $h^{(i)} = |\mathcal{C}_i|$ for $i = 1, \ldots, k$. By Corollary 5.5, the irreducible representations of G are indexed by the numbers $1, \ldots, k$ (although this is not, in general, a natural index). Let ρ_i denote an irreducible representation, for $i = 1, \ldots, k$. Let $\chi^{(i)}$ denote the (irreducible) character associated with ρ_i, and let $\chi_j^{(i)}$ denote the value of $\chi^{(i)}$ at any $x \in \mathcal{C}_j$, for $j = 1, \ldots, k$, (recalling that $\chi^{(i)} \in R(G)$). Let $f^{(i)} = \deg \rho_i$.

Theorem 6.2 *[Orthogonality relations for the irreducible characters]*

1. $\frac{1}{|G|} \sum_{i=1}^{k} h^{(i)} \chi_i^{(p)} \overline{\chi_i^{(q)}} = \delta_{p,q}$ for $1 \leq p, q \leq k$ (relation of the first kind),

2. $\sum_{i=1}^{k} \chi_p^{(i)} \overline{\chi_q^{(i)}} = \frac{h^{(p)}}{|G|} \delta_{p,q}$ for $1 \leq p, q \leq k$ (relation of the second kind).

Proof: 1) This is a restatement of Corollary 4.10.

2) Let P be a $k \times k$ matrix such that

$$[\mathbf{P}]_{j,i} = \chi_i^{(j)} \sqrt{\frac{h^{(i)}}{|G|}}.$$

Then, from (1), $\sum_{i=1}^{k}[\mathbf{P}]_{p,i}[\mathbf{P}^{\star}]_{i,q} = \delta_{p,q}$, so $\mathbf{P}\mathbf{P}^{\star} = \mathbf{I}$. Then $\mathbf{P}^{\star}\mathbf{P} = \mathbf{I}$ so $\mathbf{P}^{T}\overline{\mathbf{P}} = \mathbf{I}$. Thus $\sum_{i=1}^{k}\left[\mathbf{P}^{T}\right]_{p,i}\left[\overline{\mathbf{P}}\right]_{i,q} = \delta_{p,q}$ for $1 \leq p,q \leq k$, so

$$\sum_{i=1}^{k}\chi_p^{(i)}\sqrt{\frac{h^{(p)}}{|G|}}\cdot\overline{\chi_q^{(i)}}\sqrt{\frac{h^{(q)}}{|G|}} = \delta_{p,q},$$

and the result follows. □

We may now give an expression for the connexion coefficients.

Lemma 6.3 *For* $1 \leq p,q,r \leq k$,

$$c_{p,q}^{r} = \frac{1}{g}h^{(p)}h^{(q)}\sum_{i=1}^{k}\frac{1}{f^{(i)}}\chi_p^{(i)}\chi_q^{(i)}\overline{\chi_r^{(i)}}.$$

Proof: Since $c_{p,q}^{r}$ is the connexion coefficient of the class algebra of G,

$$K_pK_q = \sum_{i=1}^{k}c_{p,q}^{i}K_i.$$

Then, upon taking the representation,

$$\rho_j\left(K_pK_q\right) = \rho_j\left(K_p\right)\rho_j\left(K_q\right) = \sum_{i=1}^{k}c_{p,q}^{i}\rho_j\left(K_i\right).$$

To determine $\rho_j\left(K_p\right)$ we note that $xK_p = K_px$ for all $x \in G$ since K_p is central. Taking the representation, we have $\rho_j\left(x\right)\rho_j\left(K_p\right) = \rho_j\left(K_p\right)\rho_j\left(x\right)$ for all $x \in G$ so, by Schur's Lemma (Lemma 4.1), $\rho_j\left(K_p\right) = \alpha\mathsf{I}_{\mathcal{V}}$ for some $\alpha \in \mathbb{C}$. Taking the trace gives $\chi^{(j)}\left(K_p\right) = \alpha\deg\rho_j = \alpha f^{(j)}$. Then $\alpha = h^{(p)}\chi_p^{(j)}/f^{(j)}$, whence

$$\rho_j\left(K_p\right) = h^{(p)}\frac{\chi_p^{(j)}}{f^{(j)}}\mathsf{I}_{\mathcal{V}}.$$

Substituting this into the expansion of $\rho_j\left(K_pK_q\right)$ gives

$$\left(h^{(p)}\frac{\chi_p^{(j)}}{f^{(j)}}\cdot h^{(q)}\frac{\chi_q^{(j)}}{f^{(j)}}\right)\mathsf{I}_{\mathcal{V}} = \left(\sum_{i=1}^{k}c_{p,q}^{i}h^{(i)}\frac{\chi_i^{(j)}}{f^{(j)}}\right)\mathsf{I}_{\mathcal{V}}$$

whence

$$\frac{1}{f^{(j)}}h^{(p)}h^{(q)}\chi_p^{(j)}\chi_q^{(j)} = \sum_{i=1}^{k}c_{p,q}^{i}h^{(i)}\chi_i^{(j)}.$$

Now multiply throughout by $\overline{\chi_r^{(j)}}$ and sum over j to obtain

$$h^{(p)}h^{(q)}\sum_{j=1}^{k}\frac{1}{f^{(j)}}\chi_p^{(j)}\chi_q^{(j)}\overline{\chi_r^{(j)}} \quad = \quad \sum_{i=1}^{k}c_{p,q}^{i}h^{(i)}\sum_{j=1}^{k}\chi_i^{(j)}\overline{\chi_r^{(j)}}$$

$$\begin{aligned} &= \sum_{i=1}^{k} c_{p,q}^{i} h^{(i)} \frac{g}{h^{(i)}} \delta_{i,j} \text{ from Thm. 6.2(2)} \\ &= g c_{p,q}^{r} \end{aligned}$$

and the result follows. □

In fact, we can construct the orthogonal idempotents to span the centre. These are given by the following theorem.

Theorem 6.4 *Let*

$$F_i = \frac{f^{(i)}}{g} \sum_{j=1}^{k} \overline{\chi_j^{(i)}} K_j,$$

for $i = 1, \ldots, k$. *Then*

1. $K_j = h^{(j)} \sum_{i=1}^{k} \frac{1}{f^{(i)}} \overline{\chi_j^{(i)}} F_i$, *for* $j = 1, \ldots, k$,
2. $F_i F_j = \delta_{i,j} F_i$ *for* $1 \le i, j \le k$.

Proof: 1) Substituting the expression for F_j into the right hand side of (1), we have

$$\begin{aligned} h^{(j)} \sum_{i=1}^{k} \frac{1}{f^{(i)}} \overline{\chi_j^{(i)}} F_i &= \frac{h^{(j)}}{g} \sum_{m=1}^{k} K_m \sum_{l=1}^{k} \chi_j^{(l)} \overline{\chi_m^{(l)}} \\ &= \frac{h^{(j)}}{g} \sum_{m=1}^{k} K_m \frac{g}{h^{(j)}} \delta_{m,j} \text{ from Thm. 6.2(2)} \\ &= K_j, \end{aligned}$$

which is the result.

2) We use Lemma 6.3 and the orthogonality of the characters. For $1 \le i, j \le k$ we have

$$\begin{aligned} F_i F_j &= \frac{1}{g^2} f^{(i)} f^{(j)} \sum_{l,m=1}^{k} \overline{\chi_l^{(i)} \chi_m^{(i)}} K_l K_m \\ &= \frac{1}{g^2} f^{(i)} f^{(j)} \sum_{l,m=1}^{k} \overline{\chi_l^{(i)} \chi_m^{(i)}} \sum_{r=1}^{k} K_r \left(\frac{1}{g} h^{(l)} h^{(m)} \sum_{s=1}^{k} \frac{1}{f^{(s)}} \chi_l^{(s)} \chi_m^{(s)} \overline{\chi_r^{(s)}} \right) \end{aligned}$$

from Lemma 6.3. Then

$$\begin{aligned} F_i F_j &= \frac{1}{g^3} f^{(i)} f^{(j)} \sum_{r=1}^{k} K_r \sum_{s=1}^{k} \frac{1}{f^{(s)}} \overline{\chi_r^{(s)}} \left(\sum_{l=1}^{k} h^{(l)} \overline{\chi_l^{(i)}} \chi_l^{(s)} \right) \left(\sum_{m=1}^{k} h^{(m)} \overline{\chi_m^{(i)}} \chi_m^{(s)} \right) \\ &= \frac{1}{g} f^{(i)} f^{(j)} \sum_{r=1}^{k} K_r \sum_{s=1}^{k} \frac{1}{f^{(s)}} \overline{\chi_r^{(s)}} \delta_{i,s} \delta_{j,s} \text{ from Thm. 6.2(1)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{g} f^{(i)} f^{(j)} \sum_{r=1}^{k} K_r \frac{1}{f^{(i)}} \overline{\chi_r^{(i)}} \delta_{i,j} \\
&= \delta_{i,j} \frac{1}{g} f^{(j)} \sum_{r=1}^{k} \overline{\chi_r^{(i)}} K_r = F_i \delta_{i,j}.
\end{aligned}
$$

This completes the proof. □

Note that $\{F_1, \ldots, F_k\}$ is therefore a basis of $Z_{\mathbb{C}G}$ consisting of orthogonal idempotents. Theorem 6.4 gives an explicit expression for the orthogonal idempotents of an arbitrary finite group. This includes the hyperoctahedral group, that is needed for treating maps in locally orientable surfaces.

Chapter 7

Induced representations

Let $H < G$. We now consider the idea of obtaining a representation of H from a representation of G, and a representation of G from a representation of H.

7.1 Restricted and induced characters

Definition 7.1 *Let G be a group, and let $H \leq G$. Let X be a matrix representation of G. Then the* restriction *of X to H is $X \downarrow_H^G$ where $X \downarrow_H^G (x) = X (x)$ for all $x \in X$.*

Clearly, this is a representation. To go in the other direction we need the following lemma.

Lemma 7.2 *Let G be a group and let $H \leq G$. Let $(t_1, \ldots, t_l)$ be a transversal for the left cosets of H in G. Let Y be a matrix representation of H. Let $X : G \to \mathcal{M}_n^\star$ be such that, for all $x \in G$,*

$$X (x) = \left[Y \left(t_i^{-1} x t_j\right)\right]_{l\times l}$$

(a block matrix), where $Y (y) = \mathbf{0}$ if $y \notin H$. Then X is a representation of G.

Proof: Let $x, y \in G$. Then, by block matrix multiplication,

$$\begin{aligned}
\left[X (x) X (y)\right]_{p,q} &= \left[Y \left(t_i^{-1} x t_j\right)\right]_{l\times l} \left[Y \left(t_i^{-1} y t_j\right)\right]_{l\times l} \\
&= \sum_{j=1}^{l} Y \left(t_p^{-1} x t_j\right) Y \left(t_j^{-1} y t_q\right).
\end{aligned}$$

There are two cases to be considered.

Case 1: Let $t_p^{-1} x t_q \notin H$. But $t_p^{-1} x t_q = \left(t_p^{-1} x t_j\right) \left(t_j^{-1} y t_q\right)$, so either $t_p^{-1} x t_j \notin H$ or $t_j^{-1} y t_q \notin H$. Then $Y \left(t_p^{-1} x t_j\right) Y \left(t_j^{-1} y t_q\right)$ for $i = 1, \ldots, l$, so $\left[X (x) X (y)\right]_{p,q} = \mathbf{0}$. But $Y \left(t_p^{-1} x t_q\right) = 0$ since $t_p^{-1} x t_q \notin H$, so $\left[X (xy)\right]_{p,q} = \left[X (x) X (y)\right]_{p,q}$.

Case 2: Let $z = t_p^{-1} x t_q \notin H$. Since $y t_q \in G$, it is in precisely one coset, $t_r H$, of H in G. Thus $y t_q = t_r u$ for some $u \in H$, so $u = t_r^{-1} u t_q \in H$ and r is uniquely defined. Thus the expression for $[X(x) X(y)]_{p,q}$ simplifies, and gives $[X(x) X(y)]_{p,q} = Y\left(t_p^{-1} x t_r\right) Y\left(t_r^{-1} u t_q\right)$. But $t_p^{-1} x t_r = \left(t_p^{-1} x y t_q\right)\left(t_q^{-1} y^{-1} t_r\right) = z u^{-1} \in H$ since $z, u \in H$. Thus $[X(x) X(y)]_{p,q} = Y\left(t_p^{-1} x y t_q\right) = [X(xy)]_{p,q}$.

We conclude from these two cases that $X(xy) = X(x) X(y)$.

We now show that $X(\iota) = \mathbf{I}$. Now $X(\iota) = \left[Y\left(t_i^{-1} t_j\right)\right]_{l \times l}$. Suppose that $t_i^{-1} t_j \in H$ for $i \neq j$. Then $t_j = t_i v$ for some $v \in H$, whence $t_j = t_i v H = t_i H$. But $t_j H \cap t_i H = \emptyset$. This is a contradiction so $t_i^{-1} t_j \notin H$ for $i \neq j$. If $i = j$, then $Y\left(t_i^{-1} t_j\right) = Y(\iota) = I$, so $X(\iota) = \mathbf{I} \oplus \cdots \oplus \mathbf{I} = \mathbf{I}$ (the final identity matrix has a different size from the constituents of the direct sum).

Thus X is a matrix representation of G. □

This lemma supports the following definition.

Definition 7.3 *Let G be a group and let $H \leq G$. Let $(t_1, \ldots, t_l)$ be a transversal for the left cosets of H in G. Let Y be a matrix representation of H. Then the* induced representation *of G by H is $Y \uparrow_H^G$ where, for all $x \in G$,*

$$Y \uparrow_H^G (x) = \left[Y\left(t_i^{-1} x t_j\right)\right]_{l \times l}$$

and $Y(y) = \mathbf{0}$ if $y \notin H$.

The following gives an example of the construction.

Example 7.4 *Let $G = S_3$ and $H = \{\iota, (23)\}$. The left cosets of H in G are*

$$\begin{aligned} D_1 &= \iota H = (23) H = \{\iota, (23)\}, \\ D_2 &= (12) H = (132) H = \{(12), (132)\}, \\ D_2 &= (13) H = (123) H = \{(13), (123)\}. \end{aligned}$$

Then $\{\iota, (12), (13)\}$ is a coset transversal of H in G. If Y is a matrix representation of H, then

$$X \uparrow_H^G (\iota) = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad X \uparrow_H^G (12) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & Y(23) \end{bmatrix},$$

$$X \uparrow_H^G (13) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & Y(23) & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad X \uparrow_H^G (23) = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & Y(23) \\ \mathbf{0} & Y(23) & \mathbf{0} \end{bmatrix},$$

$$X \uparrow_H^G (123) = \begin{bmatrix} \mathbf{I} & Y(23) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ Y(23) & \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad X \uparrow_H^G (132) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & Y(23) \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}.$$

As a check, note that $(12)(23) = (132)$. Then $X\uparrow_H^G(12)\,X\uparrow_H^G(23) = X\uparrow_H^G(132)$. This is confirmed by multiplying out the corresponding matrices.

We now examine the dependency of the characters of $X\uparrow_H^G$ on the selection of transversal.

Lemma 7.5 *Let G be a group and let $H \le G$. Let $(s_1, \ldots, s_l)$ and $(t_1, \ldots, t_l)$ be a transversals for the left cosets of H in G. Let Y be a matrix representation of H. Then*

$$\operatorname{tr}\left[Y\left(s_i^{-1}xs_j\right)\right]_{l\times l} = \operatorname{tr}\left[Y\left(t_i^{-1}xt_j\right)\right]_{l\times l}$$

for all $x \in G$.

Proof: Since $(s_1, \ldots, s_l)$ and $(t_1, \ldots, t_l)$ are transversals for the left cosets of H in G, we may assume, without loss of generality, relabelling if necessary, that $t_iH = s_iH$ for $i = 1, \ldots, l$. Thus $t_ia_i = s_ib_i$ for $\mathrm{a}_i.b_i \in H$, so $s_i = t_ic_i$ where $c_i = a_ib_i^{-1} \in H$. Then $s_i^{-1}xs_i = c_i^{-1}\left(t_i^{-1}xt_i\right)c_i$ so $s_i^{-1}xs_i$ and $t_i^{-1}xt_i$ are conjugates. Thus, if $t_i^{-1}xt_i \in H$ then $s_i^{-1}xs_i \in H$, so $\operatorname{tr}\left[Y\left(s_i^{-1}xs_j\right)\right]_{l\times l} = \operatorname{tr}\left[Y\left(t_i^{-1}xt_j\right)\right]_{l\times l}$. On the other hand, if $t_i^{-1}xt_i \notin H$ then $s_i^{-1}xs_i \notin H$, so $Y\left(s_i^{-1}xs_j\right) = \mathbf{0} = Y\left(t_i^{-1}xt_j\right)$. Thus

$$\begin{aligned}\operatorname{tr}\left[Y\left(t_i^{-1}xt_j\right)\right]_{l\times l} &= \sum_{i=0}^{l}\operatorname{tr}\left[Y\left(t_i^{-1}xt_i\right)\right]_{l\times l} = \sum_{i=0}^{l}\operatorname{tr}\left[Y\left(s_i^{-1}xs_i\right)\right]_{l\times l}\\ &= \operatorname{tr}\left[Y\left(s_i^{-1}xs_j\right)\right]_{l\times l},\end{aligned}$$

completing the proof. □

The next result is now immediate.

Corollary 7.6 *Let G be a group and let $H \le G$. Let $(t_1, \ldots, t_l)$ be a transversals for the left cosets of H in G. Let ρ be a representation of H and let χ be its character. Then $\chi\uparrow_H^G$ is independent of the choice of transversal.*

7.2 An explicit expression for $\chi\uparrow_H^G$

There is a convenient formula for determining $\chi\uparrow_H^G$.

Lemma 7.7 *Let G be a finite group, let Y be a representation of H where $H \le G$, and let χ be the character of Y. Then*

$$\chi\uparrow_H^G(x) = \frac{1}{h^{(r)}}\frac{|H|}{|G|}\sum_{z\in\mathcal{C}_r\cap H}\chi(z)$$

where $x \in \mathcal{C}_r$.

Proof: Let ϕ denote $\chi \uparrow_H^G$, and let $\{t_1, \ldots, t_l\}$ be a transversal of H in G. Then by Definition 7.3 ,$\phi(x) = \sum_{i=1}^{l} \chi\left(t_i^{-1} x t_i\right)$. Now, for any $a \in H$, $\{t_1 a, \ldots, t_l a\}$ is also a transversal of H in G so, by Lemma 7.5, $\phi(x) = \sum_{i=1}^{l} \chi\left(a^{-1} t_i^{-1} x t_i a\right)$. Thus, summing over all $a \in H$,

$$|H|\,\phi(x) = \sum_{a \in H} \sum_{i=1}^{l} \chi\left(a^{-1} t_i^{-1} x t_i a\right) = \sum_{b \in G} \chi\left(b^{-1} x b\right)$$

since $G = t_1 H \cup \cdots \cup t_k H$.

We consider a conjugate $b_1^{-1} x b_1$ of $x \in \mathcal{C}_r$ for fixed $b_1 \in G$. Let $b_2^{-1} x b_2$ be a conjugate of $b_1^{-1} x b_1$. Now $b_1^{-1} x b_1 = b_2^{-1} x b_2 \Leftrightarrow x = x\left(b_2 b^{-1}\right) \Leftrightarrow b_2 b^{-1} \in Z_x$ where Z_x is the centralizer of $x \in G$. Thus each conjugate $b_1^{-1} x b_1$ of x is constructed $|Z_x|$ times in the sum over $b \in G$, so

$$\sum_{b \in G} \chi\left(b^{-1} x b\right) = |Z_x| \sum_{z \in \mathcal{C}_r} \chi(z) = |Z_x| \sum_{z \in \mathcal{C}_r \cap H} \chi(z)$$

since, from Definition 7.3, $\chi(z) = 0$ if $z \notin H$. Now

$$\begin{aligned} b_1^{-1} x b_1 = b_2^{-1} x b_2 \quad &\Leftrightarrow \quad b_2 b_1^{-1} \in Z_x \\ &\Leftrightarrow \quad b_2 b_1^{-1} Z_x = Z_x \\ &\Leftrightarrow \quad b_1^{-1} Z_x = b_2^{-1} Z_x \end{aligned}$$

so the number of distinct conjugates of x is equal to the number of cosets of Z_x in G. But $Z_x G$, and $x \in C_r$, so $h^{(r)} = [G \colon Z_x] = |G| / |H|$ since G is finite. Thus $|Z_x| = |G| / h^{(r)}$. Combining these we have

$$|H|\,\phi(x) = \sum_{b \in G} \chi\left(b^{-1} x b\right) = |Z_x| \sum_{z \in \mathcal{C}_r \cap H} \chi(z) = \frac{|G|}{h^{(r)}} \sum_{z \in \mathcal{C}_r \cap H} \chi(z)$$

and the result follows. □

Chapter 8

The characters of the symmetric group

We now apply this theory to derive expressions for the irreducible characters of $\mathfrak{S}_n$. From Corollary 3.20 we know that the characters of $\mathfrak{S}_n$ are real. It will be shown that certain symmetric functions are the generating series for the evaluation of irreducible characters of the symmetric group at conjugacy classes, and that computations with characters of the symmetric group can therefore be conducted through the ring Λ of symmetric functions. This is a well studied ring and its properties are well understood.

8.1 Frobenius's construction

Some notation is needed. Let $\mathcal{A} \subset \{1, 2, \ldots\}$ where $|\mathcal{A}| < \infty$. Let $\mathfrak{S}_{\mathcal{A}}$ denote the set of all permutations of the elements of $\mathcal{A}$. For $\theta \vdash n$, with $l(\theta) = r$, let

$$\mathfrak{S}_\theta = \mathfrak{S}_{\{1,\ldots,\theta_1\}} \times \mathfrak{S}_{\{\theta_1+1,\ldots,\theta_1+\theta_2\}} \times \cdots \times \mathfrak{S}_{\{\theta_1+\cdots+\theta_{r-1}+1,\ldots,\theta_1+\cdots+\theta_r\}}.$$

$\mathfrak{S}_\theta$ is called a *Young subgroup* of $\mathfrak{S}_n$. Let $z(\alpha) = 1^{i_1} 2^{i_2} \cdots i_1! i_2! \cdots$ with $\alpha = \left[1^{i_1}, 2^{i_2}, \cdots\right]$, $\theta! = \theta_1! \theta_2! \cdots$ where $\theta = (\theta_1, \theta_2, \ldots)$ and $h^\alpha = |\mathcal{C}_\alpha|$.

Let m_θ be a monomial symmetric function, p_θ be a power sum symmetric function, h_θ a complete symmetric function and s_θ a Schur function.

We begin by constructing a character that contains the information that we want.

Lemma 8.1 *Let $\alpha, \theta \vdash n$ and let $x \in C_\alpha$. Then*

$$1 \uparrow_{\mathfrak{S}_\theta}^{\mathfrak{S}_n} (x) = [m_\theta]\, p_\alpha.$$

Proof: Let $\phi_\theta(x) = 1\uparrow_{\mathfrak{S}_\theta}^{\mathfrak{S}_n}$. From Lemma 7.7, for $x \in C_\alpha$,

$$\phi_\theta(x) = \frac{|\mathfrak{S}_n|}{|\mathfrak{S}_\theta|\, h^\alpha} \sum_{z \in \mathcal{C}_\alpha \cap \mathfrak{S}_\theta} 1 = \frac{z(\alpha)}{\theta!} |\ \mathcal{C}_\alpha \cap \mathfrak{S}_\theta| \,.$$

We now determine $|\mathcal{C}_\alpha \cap \mathfrak{S}_\theta|$. Let $\pi \in \mathcal{C}_\alpha \cap \mathfrak{S}_\theta$, and let π have $a_{j,k}$ cycles of length k on the set $\{\theta_1+\cdots+\theta_{j-1}+1,\ldots,\theta_1+\cdots+\theta_j\}$. Then $\mathbf{A} = [a_{j,k}]_{r\times n}$, where $r = l(\theta)$, satisfies the two conditions

I) $\sum_{k=1}^n k a_{j,k} = \theta_j$, for $j = 1,\ldots,r$,

II) $\sum_{j=1}^r a_{j,k} = i_k$ for $k = 1,\ldots,n$.

In the sums that arise the attachment of the symbols I and II to the summation conditions indicates which of these conditions is to be applied. The cycles of π can be selected in

$$\prod_{j=1}^r \frac{\theta_j!}{1^{a_{j,1}} \ldots n^{a_{j,n}} a_{j,1}! \cdots a_{j,n}!}$$

ways. Thus

$$|\mathcal{C}_\alpha \cap \mathfrak{S}_\theta| = \sum_{\mathbf{A}\geq \mathbf{0};(I,II)} \prod_{j=1}^r \frac{\theta_j!}{1^{a_{j,1}} \ldots n^{a_{j,n}} a_{j,1}! \cdots a_{j,n}!}.$$

Then, after some routine simplification,

$$\begin{aligned}
\phi_\theta(x) &= \sum_{\mathbf{A}\geq\mathbf{0};(I,II)} \prod_{l=1}^n \left(\frac{i_l!}{\prod_{j=1}^r a_{j,l}!} \right) \\
&= \sum_{\mathbf{A}\geq\mathbf{0};(I)} \prod_{l=1}^n \binom{i_l}{a_{1,l},\ldots,a_{m,l}} \\
&= \left[x_1^{\theta_1}\cdots x_r^{\theta_r}\right] \sum_{\mathbf{A}\geq\mathbf{0}} \prod_{l=1}^n \binom{i_l}{a_{1,l},\ldots,a_{m,l}} x^{la_{1,l}} \cdots x_r^{la_{r,l}} \\
&= \left[x_1^{\theta_1}\cdots x_r^{\theta_r}\right] \prod_{l=1}^n \sum_{a_{1,l}+\cdots a_{r,l}=i_l} \binom{i_l}{a_{1,l},\ldots,a_{m,l}} x^{la_{1,l}} \cdots x_r^{la_{r,l}} \\
&= \left[x_1^{\theta_1}\cdots x_r^{\theta_r}\right] \prod_{l=1}^n \left(x_1^l+\cdots+x_n^l\right)^{i_l} = \left[x_1^{\theta_1}\cdots x_r^{\theta_r}\right] p_\alpha = [m_\theta]\, p_\alpha,
\end{aligned}$$

which completes the proof. □

To obtain the irreducible characters, we will use the following facts about the ring Λ of symmetric functions in the ground indeterminates $x_1, x_2, \ldots$. For $\alpha \in \mathbb{N}^n$, let

$$a_\alpha = \det\left[x_i^{\alpha_j}\right]_{n\times n}.$$

Let $\delta = (n-1, n-2, \ldots, 1, 0)$. Then

$$a_\delta = \prod_{1 \le ij \le n} (x_j - x_i),$$

the Vandermonde determinant. The Jacobi-Trudi Identity states that

$$s_\lambda (x_1, \ldots, x_n) = \frac{a_{\delta+\lambda}}{a_\delta}.$$

A form $\langle \cdot, \cdot \rangle_\Lambda$ on Λ defined by

$$\langle p_\alpha, p_\beta \rangle_\Lambda = z(\alpha)\, \delta_{\alpha,\beta}$$

for $|\alpha| = |\beta|$, extended linearly, is an inner product on Λ. Moreover, $\langle s_\alpha, s_\beta \rangle_\Lambda = \delta_{\alpha,\beta}$, so $\{s_\theta : \theta \vdash n\}$ is an orthonormal basis of $\Lambda^{(n)}$, the ring of symmetric functions of degree n.

Recall that $R(G)$ is the set of all class functions in $\mathbb{C}^G$. Let $\xi \in R(G)$. Then, from Theorem 5.4, $\xi = c_1 \chi^{\rho_1} + \cdots + c_k \chi^{\rho_k}$ where $\rho_1, \ldots, \rho_k$ are the irreducible representations of G and $c_1, \ldots, c_k \in C$, and k is the number of conjugacy classes of G. We call ξ an *arbitrary character* of G. If $c_1, \ldots, c_k$ are integers, then ξ is called a *generalized character* of G. If $c_1, \ldots, c_k$ are non-negative integers, then there is a representation of G for which ξ is the character.

Lemma 8.2 *Let ξ be a generalized character of G. If $\langle \xi, \xi \rangle_{\mathbb{C}^G} = 1$ and $\xi(\iota) > 0$ then ξ is an irreducible character of G.*

Proof: ξ is a generalized character of G so $\xi = n_1 \chi^{\rho_1} + \cdots + n_k \chi^{\rho_k}$ where $n_1, \ldots, n_k$ are integers. Now $\langle \xi, \xi \rangle_{\mathbb{C}^G} = 1$, so $n_1^2 + \cdots + n_k^2 = 1$, by the orthogonality of the irreducible characters. Thus $k = 1$, whence $n_1^2 = 1$ so $n_1 = \pm 1$. Then $\xi = \pm \chi^{\rho_1}$. But $\xi(\iota) > 0$, and $\chi^{\rho_1}(\iota) = \deg \rho_1 > 0$ so $\xi = \chi^{\rho_1}$, and the result follows. □

Since we are now considering only $\mathfrak{S}_n$, we will adapt the notation for irreducible characters. For an arbitrary finite group G there are k irreducible characters, where k is the number of conjugacy classes of G. For $\mathfrak{S}_n$, $k = p(n)$, the *partition number* (the number of partitions of n). Thus $\{\theta : \theta \vdash n\}$ is an index set for the irreducible characters of $\mathfrak{S}_n$. In fact, it is a *natural index* set for these characters, in the sense that a θ is a natural index for the conjugacy classes of $\mathfrak{S}_n$ and the irreducible representations can be constructed from the conjugacy class. We do not need to go so far as constructing the irreducible representations for the purposes of these Notes. Thus $\left\{\chi^\theta : \theta \vdash n\right\}$ is a complete set of irreducible representations for $\mathfrak{S}_n$. If $x \in \mathfrak{S}_n$, the value of χ^θ at $x \in C_\alpha$ is denoted by χ^θ_α. From Corollary 3.20, $\chi^\theta_\alpha \in \mathbb{R}$ for all $\alpha \vdash n$.

It will be useful to have the orthogonality relations for the irreducible characters of $\mathfrak{S}_n$ rewritten in this notation. The result is simply a rewriting of Theorem 6.2.

Theorem 8.3 *[Orthogonality relations]*

1. $\frac{1}{n!}\sum_{\alpha\vdash n} h^\alpha \chi^\theta_\alpha \chi^\phi_\alpha = \delta_{\theta,\phi}$ *where* $\theta, \phi \vdash n$,
2. $\frac{1}{n!}\sum_{\theta\vdash n} \chi^\theta_\alpha \chi^\theta_\beta = h^\alpha \delta_{\alpha,\beta}$ *where* $\alpha, \beta \vdash n$.

The next result gives the celebrated expression of Frobenius for the generating series for the values of an irreducible character of the symmetric group in terms of a Schur function.

Theorem 8.4 *Let* $\lambda \vdash n$. *Then*

$$s_\lambda = \frac{1}{n!}\sum_{\alpha\vdash n} h^\alpha \chi^\lambda_\alpha p_\alpha.$$

Proof: From Lemma 8.1, $[m_\lambda]\, p_\alpha = \phi^\lambda(x)$, for $x \in \mathcal{C}_\alpha$, where $\phi^\lambda = 1\uparrow^{\mathfrak{S}_n}_{\mathfrak{S}_\lambda}$. Let $P(\mathbf{x})$ be a symmetric function of homogeneous degree r over $\mathbb{C}$, and let $\beta \vdash n + r$. Then $\left[\mathbf{x}^\beta\right] P(\mathbf{x})\, p_\alpha$ is the evaluation of an arbitrary character of $\mathfrak{S}_n$ at $x \in \mathcal{C}_\alpha$. Now select $P(\mathbf{x}) = a_\delta$. Then $\psi^\lambda\colon \mathfrak{S}_n \to \mathbb{C}\colon x \mapsto \left[\mathbf{x}^{\lambda+\delta}\right] a_\delta p_\alpha$ is a generalized character of $\mathfrak{S}_n$. Let ψ^λ_α denote $\psi^\lambda(x)$ where $x \in \mathcal{C}_\alpha$. Now $\{s_\theta\colon \theta \vdash n\}$ is an orthonormal basis of $\Lambda^{(n)}$ so there exist unique $c_{\alpha,\theta}$ such that

$$p_\theta = \sum_{\theta\vdash n} c_{\alpha,\theta} s_\theta.$$

Thus, by the Jacobi-Trudi Identity,

$$p_\alpha a_\delta = \sum_{\theta\vdash n} c_{\alpha,\theta} a_{\delta+\theta}.$$

Then, $\psi^\lambda_\alpha = \left[\mathbf{x}^{\lambda+\delta}\right] a_\delta p_\alpha = c_{\alpha,\lambda}$, whence

$$p_\alpha = \sum_{\theta\vdash n} \psi^\theta_\alpha s_\theta.$$

From the orthogonality of the s_θ we have $\langle p_\alpha, s_\lambda\rangle_\Lambda = \psi^\lambda_\alpha$, so

$$s_\lambda = \sum_{\theta\vdash n} \frac{1}{z(\theta)} \psi^\lambda_\theta p_\theta$$

by the orthogonality of the p_θ, with $\langle p_\theta, p_\theta\rangle_\Lambda = z(\theta)$.

We complete the argument by showing that ψ^λ is an irreducible character of $\mathfrak{S}_n$. Now

$$\begin{aligned} 1 &= \langle s_\lambda, s_\lambda\rangle_\Lambda \text{ (orthonormality of the } s_\theta) \\ &= \sum_{\alpha,\beta\vdash n} \frac{1}{z(\alpha)} \frac{1}{z(\beta)} \langle p_\alpha, p_\beta\rangle_\Lambda \psi^\lambda_\alpha \psi^\lambda_\beta \end{aligned}$$

$$\begin{aligned} &= \sum_{\alpha\vdash n} \frac{1}{z(\alpha)} \psi^\lambda_\alpha \psi^\lambda_\alpha \text{ (orthonormality of the } p_\theta) \\ &= \frac{1}{n!} \sum_{\alpha\vdash n} \sum_{x\in\mathcal{C}_\alpha} \psi^\lambda(x)\,\psi^\lambda(x) \\ &= \frac{1}{n!} \sum_{\alpha\vdash n} \sum_{x\in\mathcal{C}_\alpha} \psi^\lambda(x)\,\psi^\lambda\left(x^{-1}\right) \\ &= \left\langle \psi^\lambda, \psi^\lambda \right\rangle_{\mathbb{C}\mathfrak{S}_n}. \end{aligned}$$

Thus $\left\langle \psi^\lambda, \psi^\lambda \right\rangle_{\mathbb{C}\mathfrak{S}_n} = 1$. Moreover, $\psi^\lambda(\iota) = [m_\lambda]\, p_1^n > 1$. Then, from Lemma 8.2, ψ^λ is an irreducible character of $\mathfrak{S}_n$. Thus $\psi^\lambda = \chi^\lambda$, and the result follows. □

This completes the tasks of expressing the connexion coefficient $[K_\gamma]\, K_\alpha K_\beta$ for the class algebra of $\mathfrak{S}_n$ in terms of the irreducible characters χ^λ_θ, and deriving the generating series for $\left(\chi^\lambda_\theta \colon \theta \vdash n\right)$, for fixed λ, as a symmetric function.

8.2 The genus series for maps

The proof of Theorem 1.1 follows, after some enumerative work has been done. Let $\mathfrak{p}$ be a pre-map. Let i_t be the number of vertices of $\mathfrak{p}$ of degree t, let j_t be the number of faces of $\mathfrak{p}$ of degree t, for $t = 1, 2, \ldots,$ and let n be the number of edges of $\mathfrak{p}$. Let x_t mark vertices of degree t, let y_t mark faces of degree t, and let z mark edges. Let $R(\mathbf{x}, \mathbf{y}|z)$ be the generating series for the number, $r_{\mathbf{i},\mathbf{j},n}$, for rotation systems $\nu \in \mathfrak{S}_{2n}$ such that $\nu\varepsilon_n$ has cycle-type $\mathbf{j}$, where ε_n is a fixed fixed-point free involution in $\mathfrak{S}_{2n}$. We adopt the convention for generating series that the arguments before the "|" are ordinary indeterminates, while those following it are exponential. Let Ω be an operator on $\mathbb{Q}[[\mathbf{x}, \mathbf{y}, z]]$with the action $\Omega f(\mathbf{x}, \mathbf{y}, z) = 2zf\left(\mathbf{x}, \mathbf{y}, \frac{1}{2}z\right).$

Let $m_{\mathbf{i},\mathbf{j},n}$ be the number of rooted maps in orientable surfaces with vertex-distribution $\mathbf{i}$,face-distribution $\mathbf{j}$ and n edges. Let M be the genus series for rooted maps in orientable surfaces, so

$$M(\mathbf{x}, \mathbf{y}, z) = \sum_{\mathbf{i},\mathbf{j}\geq\mathbf{0}, n\geq 0} m_{\mathbf{i},\mathbf{j},n} \mathbf{x}^{\mathbf{i}} \mathbf{y}^{\mathbf{j}} z^n.$$

Lemma 8.5 *Let $M(\mathbf{x}, \mathbf{y}, z)$ be the genus series for rooted maps in orientable surfaces. Then*

1. $M(\mathbf{x}, \mathbf{y}, z) = \Omega \log R(\mathbf{x}, \mathbf{y}|z)$, *where*
2. $R(\mathbf{x}, \mathbf{y}|z) = \sum_{n\geq 0} \frac{1}{n!} z^n \sum_{\nu,\phi\vdash 2n} \frac{h^\phi}{h^{[2^n]}} \left([K_\phi]\, K_\nu K_{[2^n]}\right) \mathbf{x}^\nu \mathbf{y}^\phi.$

Proof: 1) Let $\mathfrak{m}$ be a rooted map with n edges. Let $\mathcal{X} = \left\{1^{-1}, 1^+, \ldots, n^-, n^+\right\}$, be the set of labels for the edge-end positions (of which there are n). Let the edges of $\mathfrak{m}$ correspond to $\left\{1^{-1},1^{+}\right\},\ldots,\left\{n^{-},n^{+}\right\}$. Then $\mathfrak{m}$ is uniquely encoded, up to diffeomorphisms of the surface, by a permutation $\nu \in \mathfrak{S}_{2n}$ specifying the vertex cycles, and the fixed fixed-point free involution $\varepsilon_n = \left(1^{-1}1^{+}\right)\cdots\left(n^{-}n^{+}\right) \in \mathfrak{S}_{2n}$ specifying the edges. Since $\mathfrak{m}$ is connected, the group $\langle \nu, \varepsilon_n \rangle$ acts transitively on $\mathcal{X}$. For an arbitrary $\nu \in \mathfrak{S}_{2n}$ the corresponding structure is a union of maps, namely a pre-map. The set of maps is recoverable from the set of pre-maps as the primes (the maximal connected premaps.

Let $c_{\mathbf{i},\mathbf{j},n}$ be the number of transitive rotation systems among those counted by $r_{\mathbf{i},\mathbf{j},n}$. Let $C\left(\mathbf{x},\mathbf{y}|z\right)$ be the generating series for $\left(c_{\mathbf{i},\mathbf{j},n}\colon \mathbf{i},\mathbf{j} \geq \mathbf{0}, n \geq 0\right)$. But maps are the primes among all maps, so $R = \exp C$. Now each rooted map with n edges has 1 assigned by convention to the root edge, and the root edge has a prescribed orientation. The remaining edges can be labelled in $(n-1)!$ ways and can be oriented in 2^{n-1} ways. Thus $c_{\mathbf{i},\mathbf{j},n} = t\left(n\right) m_{\mathbf{i},\mathbf{j},n}$,where $t\left(n\right) = 2^{n-1}\left(n-1\right)!$, so

$$C\left(\mathbf{x},\mathbf{y}|z\right) = \sum_{n\geq 0} t\left(n\right) m_{\mathbf{i},\mathbf{j},n} \mathbf{x}^{\mathbf{i}} \mathbf{y}^{\mathbf{j}} \frac{z^n}{n!},$$

whence

$$\Omega \log R\left(\mathbf{x},\mathbf{y}|z\right) = \Omega C\left(\mathbf{x},\mathbf{y}|z\right) = M\left(\mathbf{x},\mathbf{y},z\right).$$

2) Now $r_{\mathbf{i},\mathbf{j},n} = \left|\mathcal{C}_{\nu\varepsilon_n} \cap \mathcal{C}_{\phi}\right|$ where $\nu = \left[1^{i_1}, 2^{i_2}, \ldots\right]$ and $\phi = \left[1^{j_1}, 2^{j_2}, \ldots\right]$. Then

$$\begin{aligned}
r_{\mathbf{i},\mathbf{j},n} &= \sum_{(a,b)\in\mathcal{C}_\nu\times\{\varepsilon_n\}, ab\in\mathcal{C}_\phi} 1 \\
&= \frac{1}{h^{[2^n]}} \sum_{(a,b)\in\mathcal{C}_\nu\times\mathcal{C}_{[2^n]}, ab\in\mathcal{C}_\phi} 1 \\
&= \frac{h^{\phi}}{h^{[2^n]}} \sum_{(a,b)\in\mathcal{C}_\nu\times\mathcal{C}_{[2^n]}, ab=c, c \text{ fixed}} 1 \\
&= \frac{h^{\phi}}{h^{[2^n]}} \left[K_\phi\right] K_\nu K_{[2^n]}.
\end{aligned}$$

This completes the proof. □

The final proof may come as something of an anticlimax, now that all of the necessary representation theory has been completed! However, if this is what you feel, return to the Introduction and review just how much was needed to reach this point.

Proof: [Theorem 1.1] The proof follows immediately from Lemma 6.3, that gives the connexion coefficients if the class algebra of $\mathbb{C}\mathfrak{S}_{2n}$ in terms of the irreducible characters of $\mathfrak{S}_{2n}$, and from Theorem 8.4 that gives the generating series for the values of these characters at the conjugacy classes in terms of a Schur function. □

Properties of the ring $\Lambda_{(n)}$ can be used to proceed further in the investigation of maps. Moreover, the approach can be generalised to deal with the enumeration of rooted maps in all surfaces, both orientable and non-orientable. In this case it is necessary to use the *double coset algebra* the hyperoctohedral group embedded in $\mathfrak{S}_{4n}$ as the stabiliser of a single matching. This algebra replaces the class algebra of the symmetric group.

8.3 Additional topics

At some point I intend to include additional topics. These will be motivated partly by combinatorial questions, and will include some more of the classical material on representation theory.

1. The commutant algebra.
2. Tensor products of representations, the characteristic map.
3. Frobenius reciprocity, Fourier analysis on groups, Gel'fand pairs.
4. The Gel'fand pair $(\mathfrak{S}_{2n}, \mathfrak{B}_n)$; combinatorial aspects of the Hecke algebra; orthogonal idempotents.
5. Maps in locally orientable surfaces, embedding theorem, genus series.