Double orthodontia formulas and Lascoux positivity

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Outline

- Schubert polynomials and flagged Weyl modules
- Orthodontia formula for flagged Weyl modules
  - and key positivity of their dual characters
- Orthodontia formula for double Grothendieck polynomials
  - and a curious Lascoux positivity result

Goal: Analogue of flagged Weyl module for Grothendieck polynomials.

Schubert varieties

Flag variety $\mathcal{F}\ell_n$ is $\{(V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n) \colon V_i$ i-dim subspace of $\mathbb{C}^n\}$.

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- $\mathcal{F}\ell_n$ is paved by affines C_w $(w \in S_n)$
- $\rightsquigarrow H^{\mathrm{odd}}(\mathcal{F}\ell_n) = 0$ and $[\overline{C_w}]$ form basis for $H^*(\mathcal{F}\ell_n)$
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Theorem (Borel '53)

$$H^*(\mathcal{F}\ell_n) \cong \frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle \mathbb{C}[x_1, \ldots, x_n]_+^{S_n} \rangle}.$$

Want to lift $[X_w]$ to $\mathbb{C}[x_1, \ldots, x_n]$.

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$$\partial_i(f) := \frac{f - s_i \cdot f}{x_i - x_{i+1}},$$

for $i \in [n-1]$. $(s_i \cdot f := f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n))$

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For $w \in S_n$, recursively define *Schubert polynomials*:

$$\mathfrak{S}_w(\mathbf{x}) = \begin{cases} x_1^{n-1} x_2^{n-2} \ldots x_{n-1} & \text{if } w = w_0 \\ \partial_i(\mathfrak{S}_{ws_i}(\mathbf{x})) & \text{if } \ell(w) < \ell(ws_i). \end{cases}$$

Schur polynomials

Example

Schur polynomials $s_\lambda := \mathrm{ch}(V_\lambda)$ are $\mathfrak{S}_w$ for "Grassmannian w".

(The GL_n-irreps V_λ are "representation-theoretic avatars" of Grassmannian $\mathfrak{S}_w$.)

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$$[X_u] \cdot [X_v] = \sum_w c_{uv}^w [X_w] \quad \leftrightsquigarrow \quad V_\lambda \otimes V_\mu = \bigoplus_\nu V_\nu^{\oplus c_{\lambda\mu}^\nu}$$

intersection nos. $\leftrightsquigarrow$ multiplicities of irreps

c_{uv}^w: "Littlewood–Richardson coefficients"

Central problem: Combinatorial formula for c_{uv}^w?

Rothe diagrams

(Towards representation-theoretic avatars of general $\mathfrak{S}_w$)

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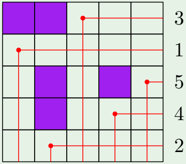
Definition

- Draw $n \times n$ grid with dots in i-th row and $w(i)$-th column
- Draw “death rays” emanating east and south of each dot
- Remaining squares are $D(w)$.

Running Example

$w = 31542$ (one line notation) $\rightsquigarrow D(31542)$:



Flagged Weyl modules

$$D \rightsquigarrow \mathcal{M}_D \qquad \text{“\textit{flagged Weyl module}”}$$

(representation of $B := \{\text{upper triangular matrices}\} \subseteq \mathrm{GL}_n$)

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Theorem (Kraśkiewicz–Pragacz '87)

The dual character $\mathrm{ch}^*(\mathcal{M}_{D(w)})$ *is the Schubert polynomial* $\mathfrak{S}_w$.

(*Dual character* of V is $\mathrm{ch}^*(V)(x_1, \ldots, x_n) = \mathrm{tr}(\mathrm{diag}(x_1^{-1}, \ldots, x_n^{-1}) \colon V \to V)$.)

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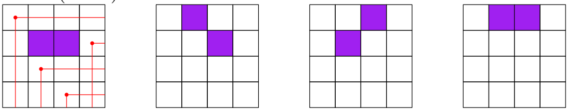
$D = D(1423)$



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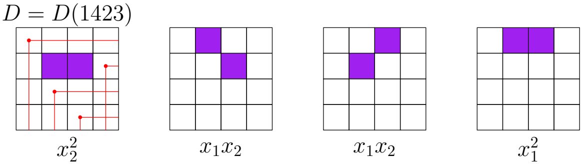
- $\mathcal{S}(D) := \{$"diagrams obtained by bubbling boxes of D upwards"$\}$
- Rep theory: monomials appearing in $\mathrm{ch}^*(\mathcal{M}_D)$ is $\{\mathbf{x}^{\mathrm{wt}(C)} : C \in \mathcal{S}(D)\}$
  ($\rightsquigarrow$ can check "in one go" if $\mathbf{x}^\alpha$ appears.)



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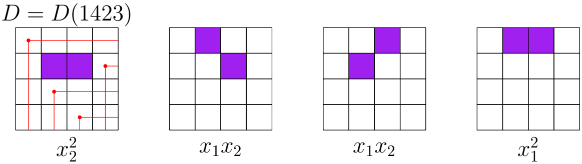

Other ways: pipe dreams (Fomin–Kirillov), bumpless pipe dreams (Lam–Lee–Shimozono)

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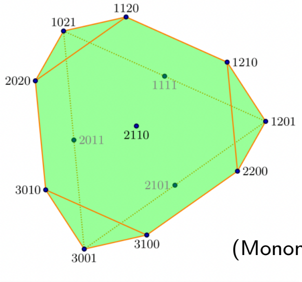
$\rightsquigarrow$ polyhedral geometry of $\mathcal{N}(D) := \{\mathrm{wt}(C) \colon \mathbf{x}^{\mathrm{wt}(C)} \text{ appears in } \mathrm{ch}^*(\mathcal{M}_D)\}$



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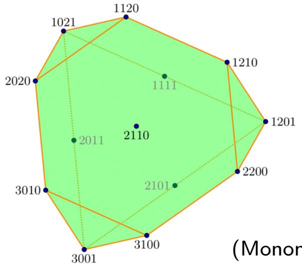

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Theorem (Fink–Mészáros–St. Dizier, '18)

$\mathcal{N}(D)$ is saturated.

(*Saturated*: $S = \mathrm{conv}(S) \cap \mathbb{Z}^n$.)

([FMS]: $\mathrm{conv}(\mathcal{N}(D))$ is *generalized permutahedron* – good combinatorics)

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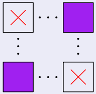
D is *%-avoiding* if it does not have any instance of:



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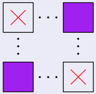
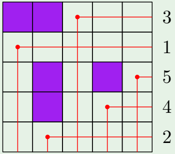


Proposition

The Rothe diagram $D(w)$ is %-avoiding for all $w \in S_n$.

Running Example



Orthodontic sequence

$$D_j := j\text{-th column of a diagram } D$$

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If D is %-avoiding, it can be reduced to the empty diagram via:

- *Remove columns:* $D \mapsto D \setminus D_j$ *when* $D_j = [i]$
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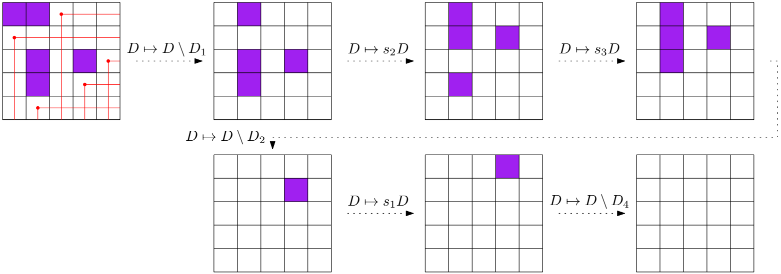

Orthodontia for flagged Weyl modules

$$\pi_i(f) := \partial_i(x_i f).$$

Theorem (Magyar '98, "orthodontia formula")

Let D be a %-avoiding diagram. Then:

- $\mathrm{ch}^*(\mathcal{M}_D) = x_1 \ldots x_i \cdot \mathrm{ch}^*(\mathcal{M}_{D \setminus D_j})$ *if* $D_j = [i]$.
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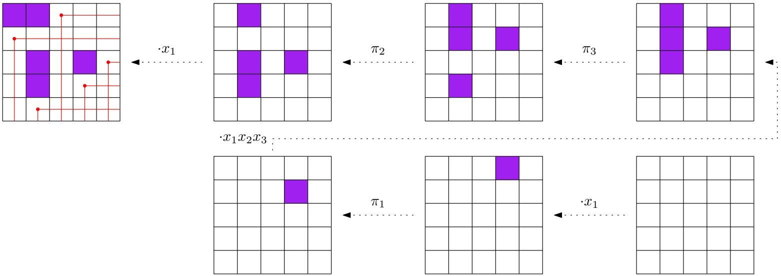
Proof involves: $\mathcal{M}_D \cong \{$sections of a line bundle on a Bott–Samelson variety$\}$.

Uses comb. of chamber sets (Leclerc–Zelevinsky), geom. of Frobenius splitting (Van der Kallen).



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Corollary (Magyar '98)

For any %-avoiding diagram D, the dual character $\mathrm{ch}^*(\mathcal{M}_D)$ can be obtained from $1 \in \mathbb{C}[\mathbf{x}]$ by applying various $\cdot x_1 \dots x_i$ and π_i.

Remark

This formula is "increasing in degree"(!).

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Definition

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Lemma (Reiner–Shimozono '98)

For any k and α, the polynomial $x_1 \dots x_k \cdot \kappa_\alpha$ is a $\mathbb{Z}_{\geq 0}$-linear combination of key polynomials.

What does orthodontia buy us?

Proposition

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(Result is false for general D.)

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$\rightsquigarrow$ Lowest degree part of $\mathfrak{G}_w(\mathbf{x};\mathbf{0})$ is $\mathfrak{S}_w$.

($\mathfrak{G}_w(\mathbf{x};\mathbf{0})$ is the *ordinary Grothendieck polynomial*.)

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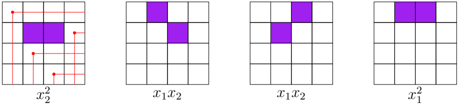

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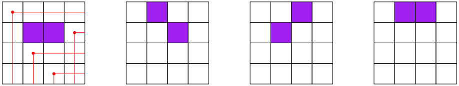

x_2^2 $x_1 x_2$ $x_1 x_2$ x_1^2

Pechenik–Speyer–Weigandt '24:

- $\deg(\mathfrak{G}_w) = \mathrm{raj}(w)$
- $\mathfrak{G}_w^{\mathrm{top}}(\mathbf{x}; \mathbf{y}) = f(\mathbf{x})g(\mathbf{y})$

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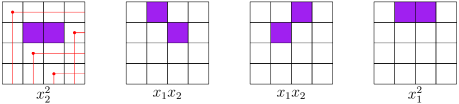
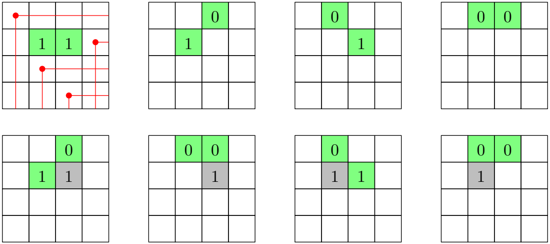


Hafner–Mészáros–S.–St. Dizier '24: $\{$monomials in vexillary $\mathfrak{G}_w(\mathbf{x};\mathbf{0})\}$.



(What is the rep-theoretic meaning of this?)

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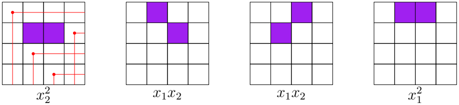

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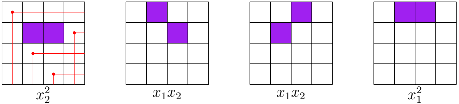


- Want to "access" $\mathfrak{G}_D$ for %-avoiding D (e.g. induction)

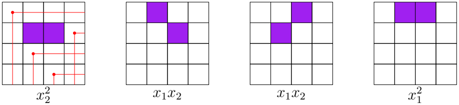
Flagged Weyl modules in K-theory

Combinatorics of $\mathfrak{S}_w$ often extends to $\mathfrak{G}_w(\mathbf{x};\mathbf{y})$.

Goal

What is the analogue of $\mathcal{M}_D$ for $\mathfrak{G}_w$?

- Want {monomials in $\mathfrak{G}_w$}:
  $\mathfrak{G}_w^{\mathrm{top}}$: Pechenik–Speyer–Weigandt '24
  Vexillary $\mathfrak{G}_w(\mathbf{x};\mathbf{0})$: HMSS '24



- Want to "access" $\mathfrak{G}_D$ for %-avoiding D (e.g. induction)
- Guess: $\mathcal{M}_D$ is the right framework for $\mathfrak{G}_w$-to-$\mathfrak{S}_v$ expansion.

Observation

For Grassmannian w: $\mathfrak{G}_w$-to-$\mathfrak{S}_v$ has only Grassmannian v, i.e.

$$\mathfrak{G}_w(\mathbf{x};\mathbf{0}) = \mathrm{ch}^*(H^*(\mathcal{F}\ell(n),\mathcal{E}))$$

for some vector bundle $\mathcal{E}$.

Orthodontia for double Grothendieck polynomials

Schubert story:

Orthodontia for double Grothendieck polynomials

Schubert story:

Theorem (Magyar '98, "orthodontia formula")

Let D be a %-avoiding diagram. Then:

- $\mathrm{ch}^*(\mathcal{M}_D) = x_1 \dots x_i \cdot \mathrm{ch}^*(\mathcal{M}_{D \setminus D_j})$ *if* $D_j = [i]$.
- $\mathrm{ch}^*(\mathcal{M}_D) = \pi_i(\mathrm{ch}^*(\mathcal{M}_{s_i D}))$ *when* $i \in D_k$ *implies* $i + 1 \in D_k$ *for all* k.

Theorem (Kraśkiewicz–Pragacz '87)

The dual character $\mathrm{ch}^*(\mathcal{M}_{D(w)})$ *is the Schubert polynomial* $\mathfrak{S}_w$.

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Goal

For %-avoiding D, *define* $\mathscr{G}_D \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ *so that* $\mathscr{G}_{D(w)} = \mathfrak{G}_w(\mathbf{x}; \mathbf{y})$.

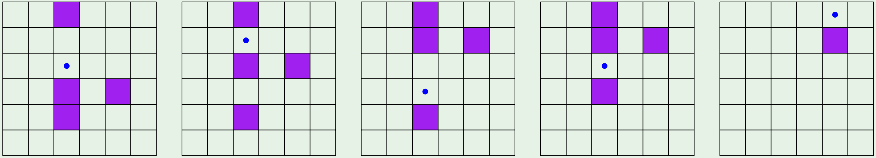
Easier goal: Define $\mathcal{G}_D \in \mathbb{C}[\mathbf{x}]$ so that $\mathcal{G}_{D(w)} = \mathfrak{G}_w(\mathbf{x}; \mathbf{0})$.

Orthodontia algorithm

Definition

Let C be the leftmost nonempty column of D. The *first missing tooth* is the minimal i so that $i \notin C$ and $i+1 \in C$.

Example



$i := 3$ $i := 2$ $i := 4$ $i := 3$ $i := 1$

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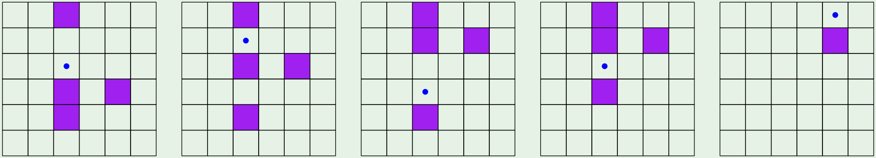

$i := 3$ $\quad$ $i := 2$ $\quad$ $i := 4$ $\quad$ $i := 3$ $\quad$ $i := 1$

An algorithm (Magyar '98) to reduce %-avoiding D:

1. Remove any columns $D_j = [i]$
2. Swap rows i and $i+1$, for $i :=$ first missing tooth.
3. Repeat step 1 & 2 until empty

Orthodontia for ordinary Grothendieck polynomials

Definition (Mészáros–S.–St. Dizier '22)

For %-avoiding D, define $\mathcal{G}_D \in \mathbb{C}[\mathbf{x}]$ recursively:

- If some $D_j = [i]$, then $\mathcal{G}_D = x_1 \dots x_i \cdot \mathcal{G}_{D \setminus D_j}$
- Otherwise, $\mathcal{G}_D = \overline{\pi}_i(\mathcal{G}_{s_i D})$ where i is the first missing tooth,

where $\overline{\pi}_i := \pi_i((1 - x_{i+1})f)$.

Theorem (Mészáros–S.–St. Dizier '22)

When $D = D(w)$ is a Rothe diagram, $\mathcal{G}_D = \mathfrak{G}_w(\mathbf{x}; \mathbf{0})$.

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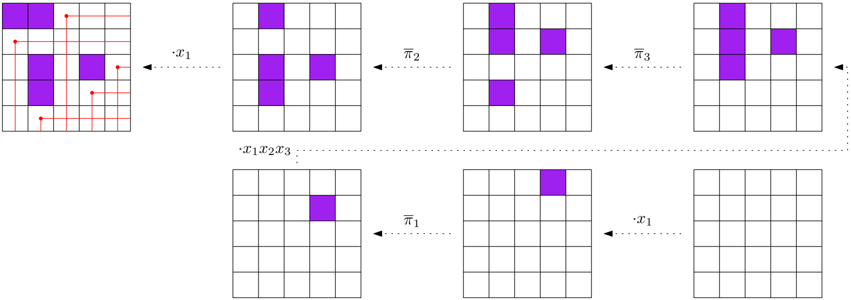
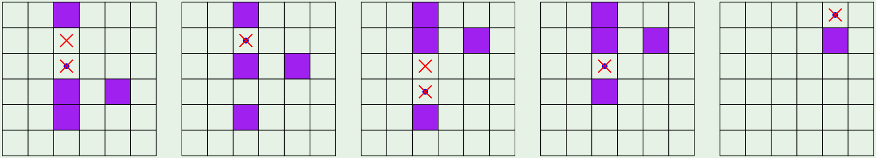


Orthodontia algorithm, II

Definition

Let D_k be the leftmost nonempty column of D. Let i be the first missing tooth and $j := k - \#\{a \leq i \colon a \notin D_k\}$. The *first missing double-tooth* is (i, j).

Example



$i := 3$, $j := 1$ — $i := 2$, $j := 2$ — $i := 4$, $j := 1$ — $i := 3$, $j := 2$ — $i := 1$, $j := 4$

Double orthodontic polynomials

Goal

For %-avoiding D, define $\mathscr{G}_D \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ *so that* $\mathscr{G}_{D(w)} = \mathfrak{S}_w(\mathbf{x}; \mathbf{y})$.

Double orthodontic polynomials

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For %-avoiding D, define $\mathscr{G}_D \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ *so that* $\mathscr{G}_{D(w)} = \mathfrak{G}_w(\mathbf{x}; \mathbf{y})$.

$$\overline{\omega}_i^{\{j\}} := \prod_{k=1}^{i} (x_k + y_j - x_k y_j)$$

$$\overline{\pi}_{i,j} := \overline{\partial}_i((x_i + y_j - x_i y_j) f)$$

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For %-avoiding D, define $\mathscr{G}_D \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ recursively:

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- Otherwise, $\mathscr{G}_D = \overline{\pi}_{i,j}(\mathscr{G}_{s_i D})$ for (i, j) the first missing double-tooth

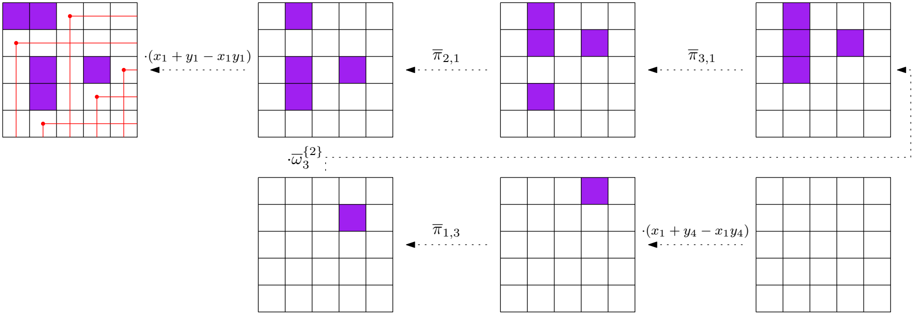
Theorem (S.–St. Dizier)

When $D = D(w)$ is a Rothe diagram, $\mathscr{G}_D = \mathfrak{G}_w(\mathbf{x}; \mathbf{y})$.

Orthodontia for double Grothendieck polynomials

Theorem (S.–St. Dizier)

When $D = D(w)$ is a Rothe diagram, $\mathcal{G}_D = \mathfrak{G}_w(\mathbf{x};\mathbf{y})$.



$$\left(\overline{\omega}_3^{\{2\}} := (x_1 + y_2 - x_1y_2)(x_2 + y_2 - x_2y_2)(x_3 + y_2 - x_3y_2)\right)$$

Orthodontia for double Grothendieck polynomials, II

Theorem (S.–St. Dizier)

When $D = D(w)$ is a Rothe diagram, $\mathscr{G}_D = \mathfrak{G}_w(\mathbf{x};\mathbf{y})$.

$\mathrm{ch}^*(\mathcal{M}_D)$ is invariant under reordering columns, but $\mathscr{G}_D$ is not.

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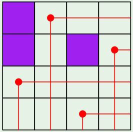
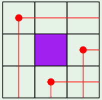
Example

$$\mathfrak{S}_{2413}(\mathbf{x}) = x_1x_2\mathfrak{S}_{132}(\mathbf{x})$$
$$\mathfrak{G}_{2413}(\mathbf{x};\mathbf{0}) = x_1x_2\mathfrak{G}_{132}(\mathbf{x};\mathbf{0})$$
$$\mathfrak{G}_{2413}(\mathbf{x};\mathbf{y}) \neq g(\mathbf{x},\mathbf{y}) \cdot \mathfrak{G}_{132}(\mathbf{x};\mathbf{y}) \quad \text{for any } g$$

$D(2413)$



$D(132)$

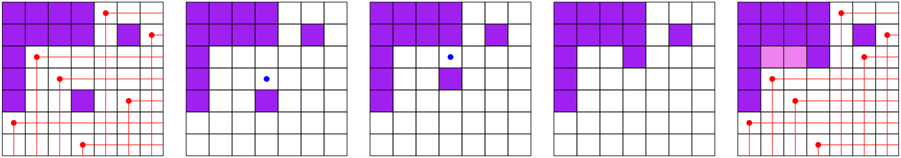


Orthodontia for double Grothendieck polynomials, III

Theorem (S.–St. Dizier)

When $D = D(w)$ is a Rothe diagram, $\mathscr{G}_D = \mathfrak{G}_w(\mathbf{x};\mathbf{y})$.

Proof idea: "Find almost-Rothe-diagrams in reduction sequence for $D(w)$"



Orthodontia for double Grothendieck polynomials, III

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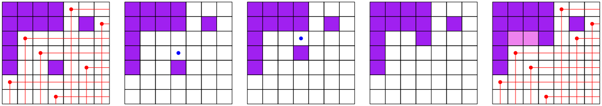
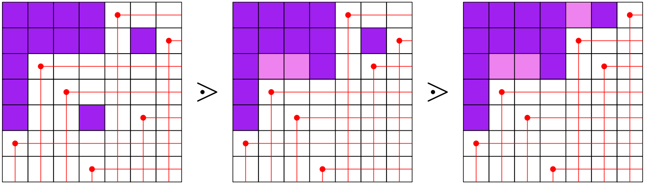


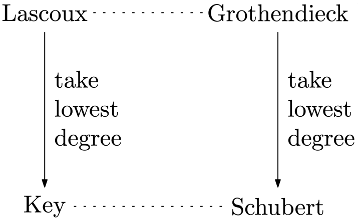
“orthodontic sort”:



(what’s the geometric meaning of this?)

Lascoux polynomials

Lascoux polynomials are "K-theoretic analogues" of key polynomials:



Definition

For $\alpha \in \mathbb{Z}^n_{\geq 0}$, recursively define *Lascoux polynomials*:

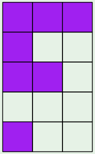
$$\mathfrak{L}_\alpha(\mathbf{x}) = \begin{cases} x_1^{\alpha_1} \dots x_n^{\alpha_n} & \text{if } \alpha_1 \geq \cdots \geq \alpha_n \\ \overline{\pi}_i(\mathfrak{L}_{s_i\alpha}(\mathbf{x})) & \text{if } \alpha_i < \alpha_{i+1}, \end{cases}$$

where $\overline{\pi}_i(f) := \pi_i((1 - x_{i+1})f)$.

Double Lascoux polynomials...?

$$\alpha \rightsquigarrow D(\alpha) \qquad \textit{“skyline diagram”}$$

Example

$\alpha = (3,1,2,0,1) \quad \rightsquigarrow \quad D(\alpha) =$ 

Observation (Mészáros–S.–St. Dizier, '22)

For all α, $\mathscr{G}_{D(\alpha)}(\mathbf{x};\mathbf{0}) = \mathfrak{L}_\alpha(\mathbf{x})$.

Who is $\mathscr{G}_{D(\alpha)}(\mathbf{x};\mathbf{y})$? And what about reordered-column $D(\alpha)$'s?

A curious Lascoux positivity conjecture

$$\mathscr{G}_D^{\mathrm{bot}} := \text{lowest degree part of } \mathscr{G}_D.$$

($\mathscr{G}_{D(w)}^{\mathrm{bot}}(\mathbf{x}; -\mathbf{y})$ is the *double Schubert polynomial*.)

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Conjecture (S.–St. Dizier)

The polynomial $x_1^n \ldots x_n^n \mathscr{G}_D^{\mathrm{bot}}(x_n^{-1}, \ldots, x_1^{-1}; -1, \ldots, -1)$ *is a graded nonnegative sum of Lascoux polynomials.*

Example

The polynomial $x_1^4 x_2^4 x_3^4 x_4^4 \mathscr{G}_{D(2143)}^{\mathrm{bot}}(x_4^{-1}, x_3^{-1}, x_2^{-2}, x_1^{-1}; -1, -1, -1, -1)$ is

$$x_1^4x_2^3x_3^4x_4^3 + x_1^4x_2^4x_3^4x_4^2 + x_1^4x_2^4x_3^3x_4^3 - x_1^4x_2^3x_3^4x_4^4 - x_1^4x_2^4x_3^3x_4^4 - 4x_1^4x_2^4x_3^4x_4^3 + 3x_1^4x_2^4x_3^4x_4^4$$

which is

$$(\mathfrak{L}_{(4,3,4,3)} + \mathfrak{L}_{(4,4,4,2)}) - (\mathfrak{L}_{(4,3,4,4)} + 2\mathfrak{L}_{(4,4,4,3)}) + \mathfrak{L}_{(4,4,4,4)}$$

A curious Lascoux positivity conjecture, II

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Orthodontia: $x_1^n \dots x_n^n \mathscr{G}_D^{\mathrm{bot}}(x_n^{-1}, \dots, x_1^{-1}; -1, \dots, -1)$ is obtained from

- $f \mapsto \overline{\pi}_i(f)$,
- $f \mapsto x_1 \dots x_i (1 - x_{i+1}) \dots (1 - x_n) f$.

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Since $\overline{\pi}_i(\mathfrak{L}_\alpha) = \mathfrak{L}_{\alpha'}$, $\overline{\pi}_i$ preserves graded Lascoux positivity.

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Conjecture: The product $\mathfrak{L}_\alpha \cdot x_1 \dots x_i (1 - x_{i+1}) \dots (1 - x_n)$ is graded Lascoux positive. (cf. key positivity of $\kappa_\alpha \cdot x_1 \dots x_i$.) □

A curious Lascoux positivity result

Corollary (S.–St. Dizier)

When columns of D can be ordered by inclusion, the polynomial $x_1^n \dots x_n^n \mathscr{G}_D^{\mathrm{bot}}(x_n^{-1}, \dots, x_1^{-1}; -1, \dots, -1)$ is a graded nonnegative sum of Lascoux polynomials.

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Sketch.

In this case, $x_1^n \dots x_n^n \mathscr{G}_D^{\mathrm{bot}}(x_n^{-1}, \dots, x_1^{-1}; -1, \dots, -1)$ can be obtained from $f \mapsto x_1 \dots x_i(1 - x_{i+1}) \dots (1 - x_n)f$, followed by $f \mapsto \overline{\pi}_i(f)$.

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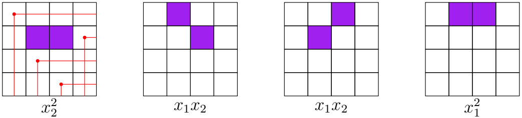
Follows from Orelowitz–Yu '23: $G_w \cdot \mathfrak{L}_\alpha$ is graded Lascoux positive.

($G_w :=$ stable Grothendieck) □

Thank you!

Goal

Find analogue of $\mathcal{M}_D$ for Grothendieck polynomials.



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