Cluster algebras and recurrence relations

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math.uchicago.edu/∼linus/austin-apr-24.pdf

Outline

- Somos sequences,
    - and their surprising properties
- Cluster algebras and the Laurent phenomenon,
    - and how to use it to prove the properties above
- Coxeter friezes,
    - and their surprising properties

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Somos-4 sequence

- Define (x_n) by
$$x_n = \frac{x_{n-1}x_{n-3} + x_{n-2}^2}{x_{n-4}},$$
initial terms $x_1 = x_2 = x_3 = x_4 = 1$.

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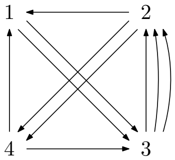

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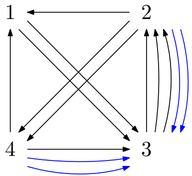

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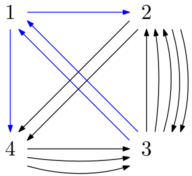

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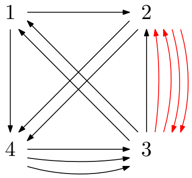

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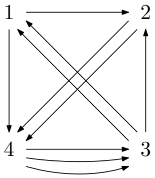

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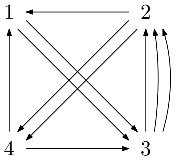

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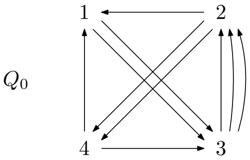

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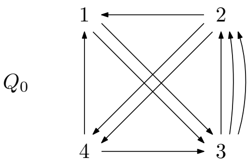

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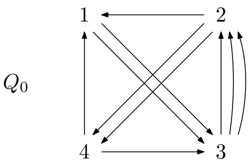

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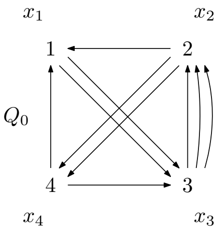

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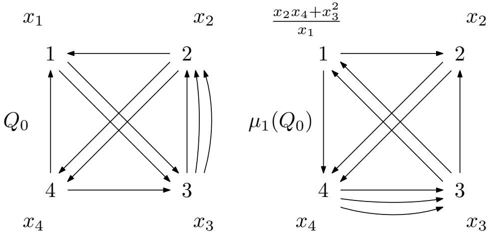

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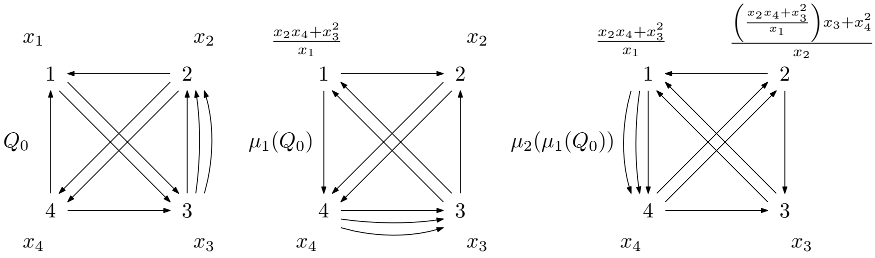

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The *cluster algebra* $\mathcal{A}(Q_0)$ is the subalgebra of $\mathbb{C}(x_1, \ldots, x_n)$ generated by all φ_i appearing in some $\mu_{s_\ell}(\ldots(\mu_{s_1}((Q_0, \mathbf{x})))\ldots)$. (The φ_i are called *cluster variables*).

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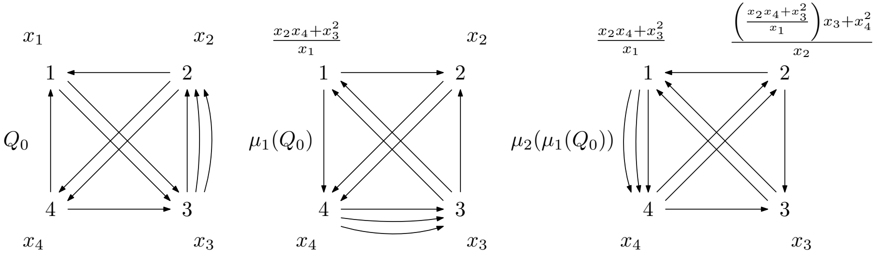

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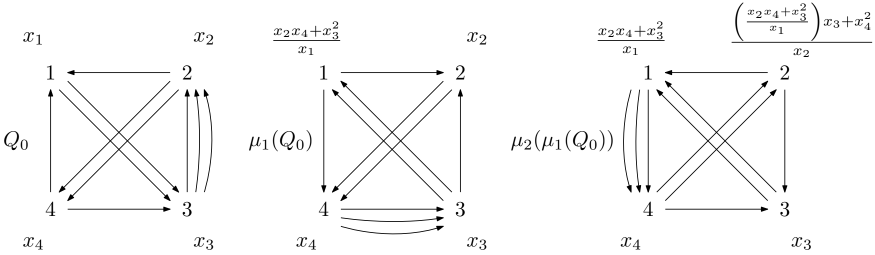

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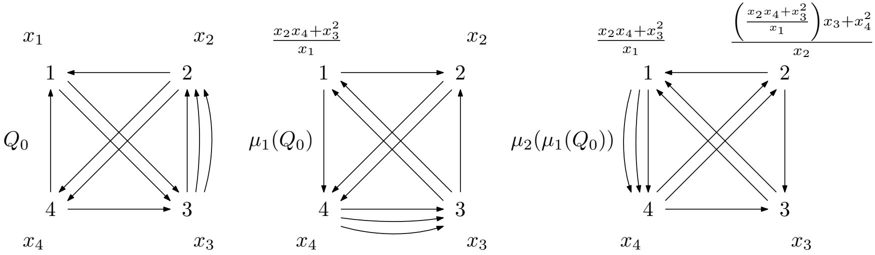

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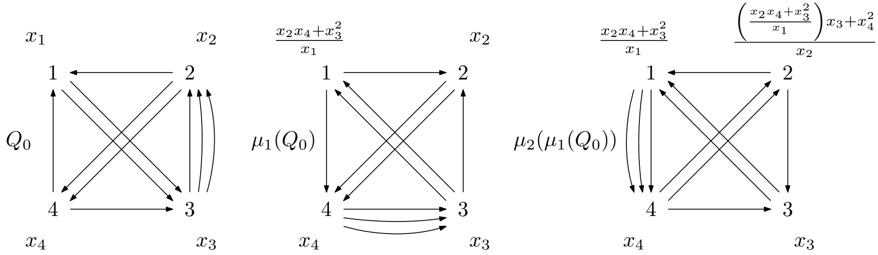

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Theorem ([FZ], Laurent phenomenon)

Let $\mathcal{A}(Q_0)$ *be a cluster algebra and fix* $(Q, \varphi) = \mu_{s_\ell}(\dots(\mu_{s_1}((Q_0, \mathbf{x}))) \dots)$. *Let* $F \in \mathcal{A}$ *be a cluster variable. Then* $F \in \mathbb{Z}[\varphi_1^{\pm}, \dots, \varphi_n^{\pm}]$.

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$$\frac{\frac{x_2+1}{x_1} + \frac{x_1+x_2+1}{x_1x_2} + 1}{\frac{x_2+1}{x_1} \cdot \frac{x_1+x_2+1}{x_1x_2}} = \frac{x_2^2 + x_2 + x_1 + x_2 + 1 + x_1x_2}{(x_2+1)(x_1+x_2+1)\cdot\frac{1}{x_1}}$$

Laurent phenomenon, again

Theorem ([FZ], Laurent phenomenon)

Let $\mathcal{A}(Q_0)$ be a cluster algebra and fix $(Q, \varphi) = \mu_{s_\ell}(\ldots(\mu_{s_1}((Q_0, \mathbf{x})))\ldots)$. Let $F \in \mathcal{A}$ be a cluster variable. Then $F \in \mathbb{Z}[\varphi_1^{\pm}, \ldots, \varphi_n^{\pm}]$.

Most often we will use it in the case $(Q, \varphi) = (Q_0, \mathbf{x})$, but...

Example

Take $Q_0 := 1 \to 2$. We saw earlier that

$$\mathbf{s} := \left(Q_0, \left(\frac{x_2 + 1}{x_1}, \frac{x_1 + x_2 + 1}{x_1 x_2}\right)\right)$$

was a seed in $\mathcal{A}(Q_0)$. Let's verify Laurent phenomenon for $\mathbf{s}$ and $F = x_1$:

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Somos-4, again

Theorem ([FZ], Laurent phenomenon)

Let $\mathcal{A}(Q_0)$ be a cluster algebra and fix $(Q, \varphi) = \mu_{s_\ell}(\ldots(\mu_{s_1}((Q_0, \mathbf{x})))\ldots)$. Let $F \in \mathcal{A}$ be a cluster variable. Then $F \in \mathbb{Z}[\varphi_1^{\pm}, \ldots, \varphi_n^{\pm}]$.

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- Somos-4 sequence: $x_n = \frac{x_{n-1}x_{n-3}+x_{n-2}^2}{x_{n-4}}$. (Ignore initial terms for now)

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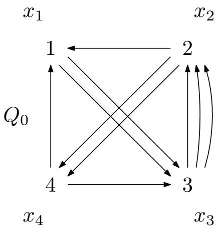
- Take Q_0 as before.



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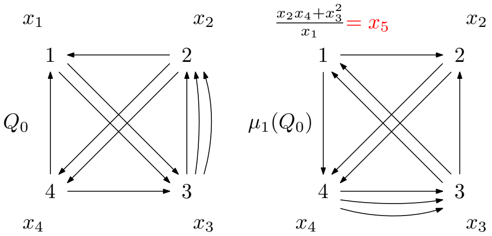
- Take Q_0 as before.
- Then $\mu_1(Q_0)$ is "Q_0 but rotated"; x_5 is "the new cluster variable".



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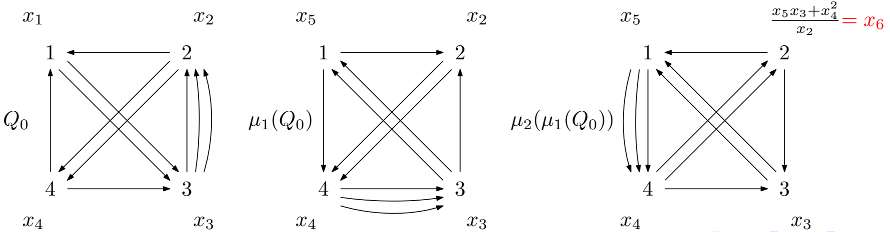
- Take Q_0 as before.
- Then $\mu_1(Q_0)$ is "Q_0 but rotated"; x_5 is "the new cluster variable".
- Similarly for $\mu_2(\mu_1(Q_0))$. Now x_6 is "the new cluster variable".



Somos-4, again again

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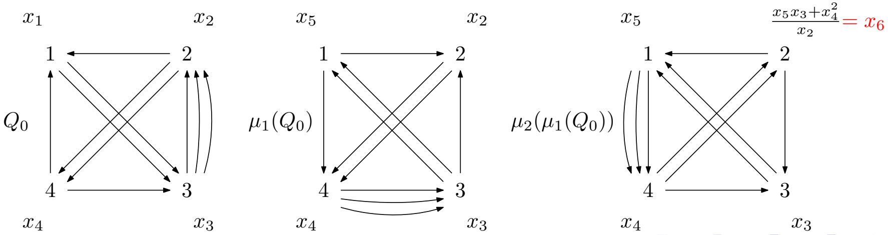
Let $\mathcal{A}(Q_0)$ be a cluster algebra and fix $(Q, \varphi) = \mu_{s_\ell}(\dots(\mu_{s_1}((Q_0, \mathbf{x})))\dots)$. Let $F \in \mathcal{A}$ be a cluster variable. Then $F \in \mathbb{Z}[\varphi_1^{\pm}, \dots, \varphi_n^{\pm}]$.

- Each x_n is a cluster variable in $\mathcal{A}(Q_0)$:



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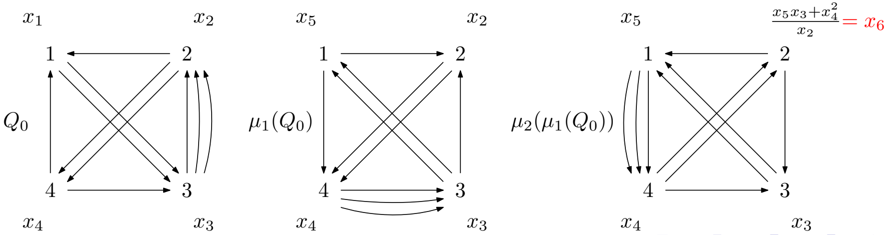
- Each x_n is a cluster variable in $\mathcal{A}(Q_0)$:
  - (e.g., $x_5 \in \mu_1(Q_0, \mathbf{x})$, and $x_6 \in \mu_2\mu_1(Q_0, \mathbf{x})$, and $x_7 \in \mu_3\mu_2\mu_1(Q_0, \mathbf{x})$, etc.)



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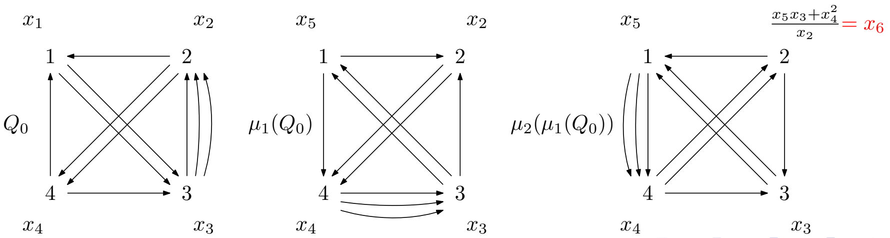
- Laurent phenomenon: $x_n \in \mathbb{Z}[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}, x_4^{\pm}]$ for all n.



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Let $\mathcal{A}(Q_0)$ be a cluster algebra and fix $(Q, \varphi) = \mu_{s_\ell}(\ldots(\mu_{s_1}((Q_0, \mathbf{x})))\ldots)$. Let $F \in \mathcal{A}$ be a cluster variable. Then $F \in \mathbb{Z}[\varphi_1^{\pm}, \ldots, \varphi_n^{\pm}]$.

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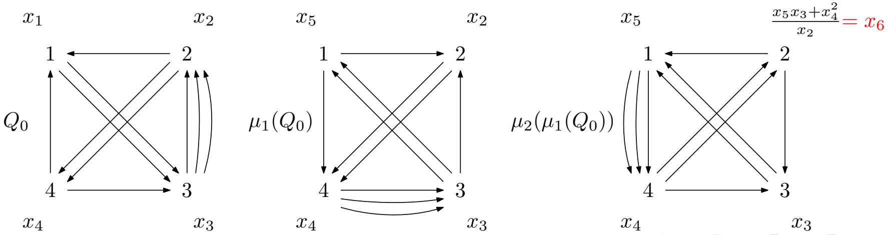
- Laurent phenomenon: $x_n \in \mathbb{Z}[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}, x_4^{\pm}]$ for all n.
- In particular if $x_1 = x_2 = x_3 = x_4 = 1$ then $x_n \in \mathbb{Z}$.



Somos-5, again

Theorem ([FZ], Laurent phenomenon)

Let $\mathcal{A}(Q_0)$ be a cluster algebra and fix $(Q, \varphi) = \mu_{s_\ell}(\dots(\mu_{s_1}((Q_0, \mathbf{x})))\dots)$. Let $F \in \mathcal{A}$ be a cluster variable. Then $F \in \mathbb{Z}[\varphi_1^{\pm}, \dots, \varphi_n^{\pm}]$.

- Somos-5 sequence: $x_n = \frac{x_{n-1}x_{n-4}+x_{n-2}x_{n-3}}{x_{n-5}}$

Somos-5, again

Theorem ([FZ], Laurent phenomenon)

Let $\mathcal{A}(Q_0)$ be a cluster algebra and fix $(Q, \varphi) = \mu_{s_\ell}(\dots(\mu_{s_1}((Q_0, \mathbf{x})))\dots)$. Let $F \in \mathcal{A}$ be a cluster variable. Then $F \in \mathbb{Z}[\varphi_1^{\pm}, \dots, \varphi_n^{\pm}]$.

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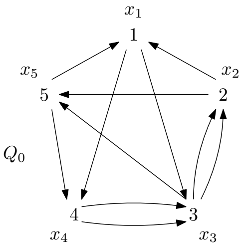

Somos-5, again

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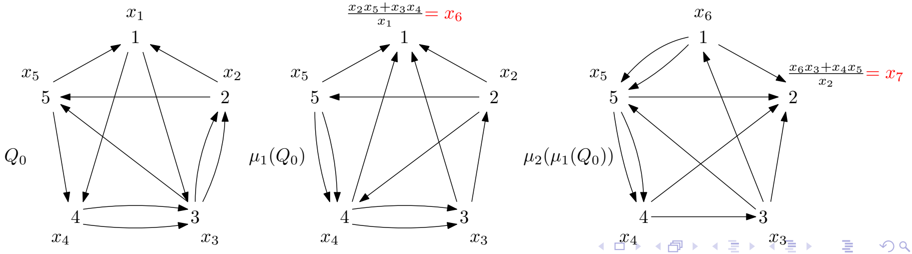

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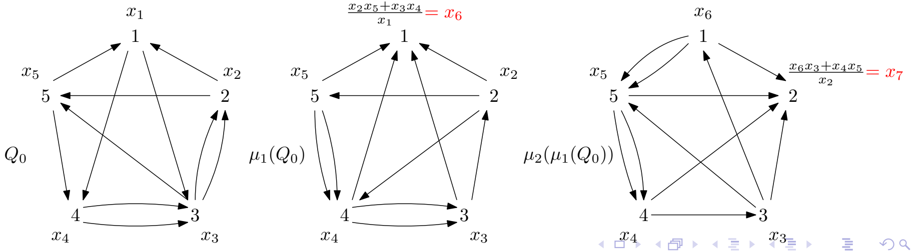
- Laurent phenomenon: $x_n \in \mathbb{Z}[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}, x_4^{\pm}, x_5^{\pm}]$ for all n.



Other sequences

- Key ingredient: quiver Q_0 and vertex v so that $\mu_v(Q_0)$ is "Q_0 but rotated".

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Some examples from their classification:

- "Somos": $x_n = \frac{x_{n-1}^r x_{n-3}^r + x_{n-2}^s}{x_{n-4}}$ and $x_n = \frac{x_{n-1}^r x_{n-4}^r + x_{n-2}^s x_{n-3}^s}{x_{n-5}}$
- "Gale-Robinson": $x_n = \frac{x_{n-r}x_{n-N+r} + x_{n-s}x_{n-N+s}}{x_{n-N}}$

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Fordy+Hone, "Discrete integrable systems and Poisson algebras from cluster maps" (2012): Given a sequence as in [FM], when is $(x_n, x_{n+1}, \ldots, x_{n+N}) \mapsto (x_{n+1}, x_{n+2}, \ldots, x_{n+N+1})$ an integrable system?

Coxeter friezes

- A *Coxeter frieze* is an array of numbers arranged as below such that any array like $\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$ satisfies $ad - bc = 1$.

$$
\begin{array}{ccccccccccccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & 2 & & 1 & & 4 & & 3 & & 1 & & 2 & & 3 & & 2 & & 2 & & \\
 & 3 & & 1 & & 3 & & 11 & & 2 & & 1 & & 5 & & 5 & & 3 & & 1 & \\
\cdots & & 1 & & 2 & & 8 & & 7 & & 1 & & 2 & & 8 & & 7 & & 1 & & \cdots \\
 & 2 & & 1 & & 5 & & 5 & & 3 & & 1 & & 3 & & 11 & & 2 & & 1 & \\
 & & 1 & & 2 & & 3 & & 2 & & 2 & & 1 & & 4 & & 3 & & 1 & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$

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  - (This one has all integer entries, and satisfies a glide symmetry...)

	1		1		1		1		1		1		1		1		1		1	
		2		1		4		3		1		2		3		2		2		
	3		1		3		11		2		1		5		5		3		1	
...		1		2		8		7		1		2		8		7		1		...
	2		1		5		5		3		1		3		11		2		1	
		1		2		3		2		2		1		4		3		1		
	1		1		1		1		1		1		1		1		1		1	

Coxeter friezes

Claim

- *Every Coxeter frieze with boundary $= 1$ satisfies a glide symmetry.*
- *If a Coxeter frieze has boundary $= 1$ and a "path of 1's", then every entry in the frieze is an integer.*

	1		1		1		1		1		1		1		1		1		1	
		2		1		4		3		1		2		3		2		2		
	3		1		3		11		2		1		5		5		3		1	
...		1		2		8		7		1		2		8		7		1		...
	2		1		5		5		3		1		3		11		2		1	
		1		2		3		2		2		1		4		3		1		
	1		1		1		1		1		1		1		1		1		1	

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```
     1     1     1     1     1     1     1     1     1     1
        2     1     4     3     1     2     3     2     2
     3     1     3    11     2     1     5     5     3     1
...     1     2     8     7     1     2     8     7     1      ...
     2     1     5     5     3     1     3    11     2     1
        1     2     3     2     2     1     4     3     1
     1     1     1     1     1     1     1     1     1     1
```

Coxeter friezes as recurrences

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$$\begin{array}{ccc} & b & \\ a & & * \\ & c & \end{array} \quad \rightsquigarrow * = \frac{bc+1}{a}$$

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$$\begin{array}{ccccccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & * & & a_1 & & * & & * & & * & & * & & \\
 & * & & a_2 & & * & & * & & * & & * & & * & \\
\cdots & & a_3 & & * & & * & & * & & * & & * & & \cdots \\
 & * & & a_4 & & * & & * & & * & & * & & * & \\
 & & a_5 & & * & & * & & * & & * & & * & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
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 & & * & & a_1 & & * & & * & & * & & * & & \\
 & * & & a_2 & & * & & * & & * & & * & & * & \\
\dots & & a_3 & & \frac{a_2 a_4 + 1}{a_3} & & * & & * & & * & & * & & \dots \\
 & * & & a_4 & & * & & * & & * & & * & & * & \\
 & & a_5 & & * & & * & & * & & * & & * & & \\
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 & & * & & a_1 & & * & & * & & * & & * & & \\
 & * & & a_2 & & F & & * & & * & & * & & * & \\
\cdots & & a_3 & & \frac{a_2 a_4+1}{a_3} & & * & & * & & * & & * & & \cdots \\
 & * & & a_4 & & * & & * & & * & & * & & * & \\
 & & a_5 & & * & & * & & * & & * & & * & & \\
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where $F := \dfrac{\frac{a_2 a_4+1}{a_3} a_1 + 1}{a_2}$

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$$\begin{array}{ccccccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & * & & \color{blue}{a_1} & & * & & * & & * & & * & & \\
 & * & & a_2 & & \color{blue}{F} & & * & & * & & * & & * & \\
\cdots & & a_3 & & \color{blue}{\frac{a_2 a_4+1}{a_3}} & & * & & * & & * & & * & & \cdots \\
 & * & & \color{blue}{a_4} & & * & & * & & * & & * & & * & \\
 & & a_5 & & \color{blue}{\frac{a_4+1}{a_5}} & & * & & * & & * & & * & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}$$

where $F := \dfrac{\frac{a_2 a_4+1}{a_3} a_1 + 1}{a_2}$

Wavefronts as quivers

Let me index the entries like this:

$$
\begin{array}{ccccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & x_{06} & & x_{17} & & x_{28} & & x_{39} & & x_{4,10} & & \\
 & x_{05} & & x_{16} & & x_{27} & & x_{38} & & x_{49} & & x_{5,10} & \\
\cdots & & x_{15} & & x_{26} & & x_{37} & & x_{48} & & x_{59} & & \cdots \\
 & x_{14} & & x_{25} & & x_{36} & & x_{47} & & x_{58} & & x_{69} & \\
 & & x_{24} & & x_{35} & & x_{46} & & x_{57} & & x_{68} & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$

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$$\begin{array}{ccccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & x_{06} & & {\color{blue}x_{17}} & & x_{28} & & x_{39} & & x_{4,10} & & \\
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 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}$$

Dictionary

"wavefront" $\leftrightsquigarrow$ triangulation of $(n+1)$-gon $\leftrightsquigarrow$ quiver

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 & & x_{06} & & {\color{blue}x_{17}} & & x_{28} & & x_{39} & & x_{4,10} & & \\
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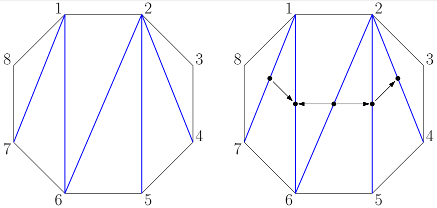
 & x_{14} & & {\color{blue}x_{25}} & & x_{36} & & x_{47} & & x_{58} & & x_{69} & \\
 & & {\color{blue}x_{24}} & & x_{35} & & x_{46} & & x_{57} & & x_{68} & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}$$

Dictionary

"wavefront" $\leftrightsquigarrow$ triangulation of $(n+1)$-gon $\leftrightsquigarrow$ quiver



Wavefronts as quivers

Let me index the entries like this:

$$
\begin{array}{ccccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & x_{06} & & {\color{blue}x_{17}} & & x_{28} & & x_{39} & & x_{4,10} & & \\
 & x_{05} & & {\color{blue}x_{16}} & & x_{27} & & x_{38} & & x_{49} & & x_{5,10} & \\
\cdots & & x_{15} & & {\color{blue}x_{26}} & & x_{37} & & x_{48} & & x_{59} & & \cdots \\
 & x_{14} & & {\color{blue}x_{25}} & & x_{36} & & x_{47} & & x_{58} & & x_{69} & \\
 & & {\color{blue}x_{24}} & & x_{35} & & x_{46} & & x_{57} & & x_{68} & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$

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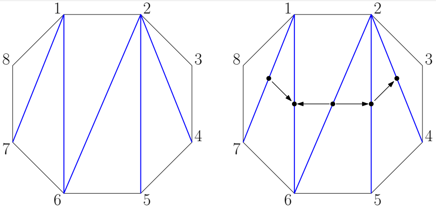


(On each diagonal, add a vertex.
If two diagonals meet, draw arrow
oriented “wrt clockwise order”)

Frieze propagation as diagonal flipping

Dictionary

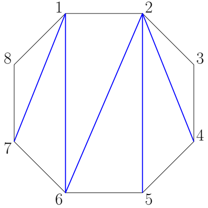
"wavefront"	$\leftrightsquigarrow$	triangulation	$\leftrightsquigarrow$	quiver
propagate wavefront	$\leftrightsquigarrow$	"flip diagonal"	$\leftrightsquigarrow$	mutate vertex

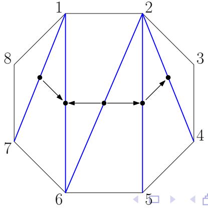
Frieze propagation as diagonal flipping

Dictionary

"wavefront"	⇝	triangulation	⇝	quiver
propagate wavefront	⇝	"flip diagonal"	⇝	mutate vertex

	1		1		1		1		1		1	
		x_{06}		x_{17}		x_{28}		x_{39}		$x_{4,10}$		
	x_{05}		x_{16}		x_{27}		x_{38}		x_{49}		$x_{5,10}$	
$\cdots$		x_{15}		x_{26}		x_{37}		x_{48}		x_{59}		$\cdots$
	x_{14}		x_{25}		x_{36}		x_{47}		x_{58}		x_{69}	
		x_{24}		x_{35}		x_{46}		x_{57}		x_{68}		
	1		1		1		1		1		1	





Frieze propagation as diagonal flipping

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propagate wavefront	$\leftrightsquigarrow$	"flip diagonal"	$\leftrightsquigarrow$	mutate vertex

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		x_{06}		x_{17}		x_{28}		x_{39}		$x_{4,10}$		
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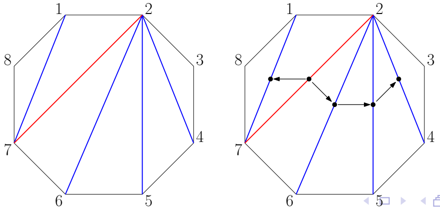
	x_{14}		x_{25}		x_{36}		x_{47}		x_{58}		x_{69}	
		x_{24}		x_{35}		x_{46}		x_{57}		x_{68}		
	1		1		1		1		1		1	



Frieze entries as cluster variables

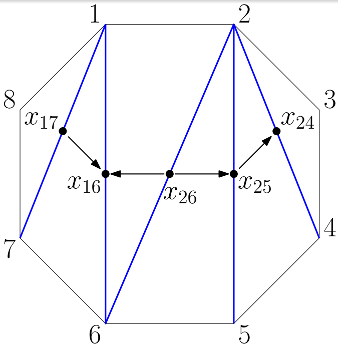
Dictionary

“wavefront”	$\leftrightsquigarrow$	triangulation	$\leftrightsquigarrow$	quiver
propagate wavefront	$\leftrightsquigarrow$	“flip diagonal”	$\leftrightsquigarrow$	mutate vertex
new frieze entry	$\leftrightsquigarrow$		$\leftrightsquigarrow$	new cluster variable

Frieze entries as cluster variables

Dictionary

“wavefront”	⇝	triangulation	⇝	quiver
propagate wavefront	⇝	“flip diagonal”	⇝	mutate vertex
new frieze entry	⇝		⇝	new cluster variable



Frieze entries as cluster variables

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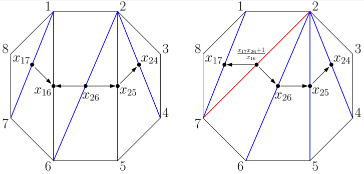

Every frieze entry is a cluster variable in $\mathcal{A}(Q_0)$.

Corollary

Every frieze entry is a Laurent polynomial in "initial wavefront quantities". (In particular, if there is a "wavefront of 1*'s" then every entry is in* $\mathbb{Z}$*.)*

Glide symmetry

Theorem (Penner)

Given any "initial wavefront quantities" x_{ij}, there exists a hyperbolic metric on $\mathbb{H}^2$ along with points $p_1, \ldots, p_n \in \partial\mathbb{H}^2$ and horocycles around each p_i so that $x_{ij} = \lambda(p_i, p_j)$.

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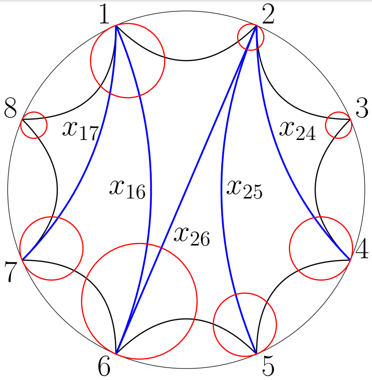

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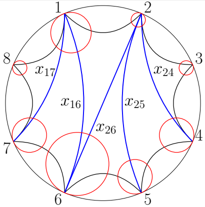

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propagate wavefront	$\leftrightsquigarrow$	"flip diagonal"	$\leftrightsquigarrow$	mutate vertex
initial frieze entry	$\leftrightsquigarrow$	λ-lengths	$\leftrightsquigarrow$	initial cluster variable
new frieze entry	$\leftrightsquigarrow$		$\leftrightsquigarrow$	new cluster variable

Glide symmetry

Corollary

New frieze entries are equal to λ-lengths of the "new diagonal".

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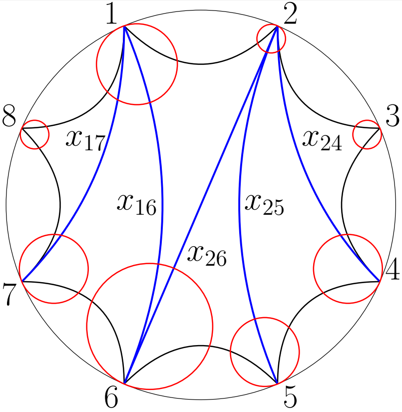

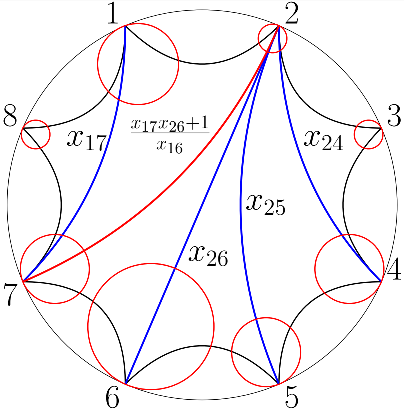

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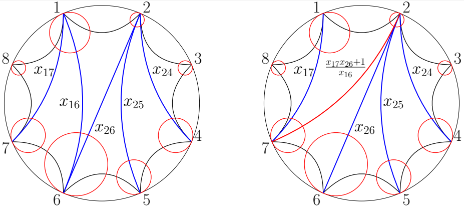

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new frieze entry	$\leftrightsquigarrow$	new λ-length	$\leftrightsquigarrow$	new cluster variable

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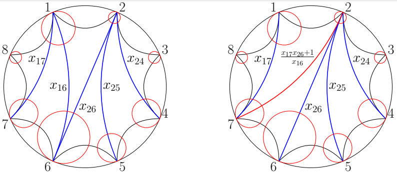

Dictionary

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initial frieze entry	⇜⇝	λ-lengths	⇜⇝	initial cluster variable
new frieze entry	⇜⇝	new λ-length	⇜⇝	new cluster variable

Corollary

We have $x_{ij} = \lambda(p_{i \bmod n}, p_{j \bmod n})$; thus every frieze with $\partial = 1$ is periodic.

Remarks on Laurent phenomenon

Theorem ([FZ], Laurent phenomenon)

Let $\mathcal{A}(Q_0)$ *be a cluster algebra and fix* $(Q, \varphi) = \mu_{s_\ell}(\dots(\mu_{s_1}((Q_0, \mathbf{x})))\dots)$. *Let* $F \in \mathcal{A}$ *be a cluster variable. Then* $F \in \mathbb{Z}[\varphi_1^{\pm}, \dots, \varphi_n^{\pm}]$.

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Inclusion $\mathcal{A}(Q_0) \hookrightarrow \mathbb{C}[\varphi_1^{\pm},\dots,\varphi_n^{\pm}]$ induces embedding of "cluster torus" $\mathrm{Spec}(\mathbb{C}[\varphi_1^{\pm},\dots,\varphi_n^{\pm}]) \to \mathrm{Spec}(\mathcal{A}(Q_0))$. *They are geometrically interesting.*

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Remark

Laurent phenomenon implies

$$\mathcal{A}(Q_0) \subseteq \bigcap_{(Q,\varphi)} \mathbb{C}[\varphi_1^{\pm},\dots,\varphi_n^{\pm}].$$

The RHS is "upper cluster algebra". They are algebraically interesting.

Thank you!

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