

# DEFINING SLE IN MULTIPLY CONNECTED DOMAINS WITH THE BROWNIAN LOOP MEASURE

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ABSTRACT. We define the Schramm-Loewner evolution ( $SLE_\kappa$ ) in multiply connected domains for  $\kappa \leq 4$  using the loop measure. We show that in the case of the annulus, this is the same as the measure obtained recently by Dapeng Zhan. We use the loop formation to give a different derivation of the partial differential equation for the partition function in the annulus.

## 1. INTRODUCTION

The Schramm-Loewner evolution ( $SLE$ ) is a conformally invariant or conformally covariant family of measures on curves in the plane. It was proposed by Schramm [25] as a candidate for the scaling limit of loop-erased walk and percolation interfaces, and it has turned out to be the crucial tool in the rigorous development of two-dimensional critical phenomenon. Before  $SLE$ , there had been much theoretical, but mathematically nonrigorous, development using conformal field theory.

In conformal field theory, the standard parameter to characterize a field is the central charge  $\mathbf{c}$ . There is a major difference between  $\mathbf{c} \leq 1$  and  $\mathbf{c} > 1$ , and  $SLE$  appears in the former case which is all we consider in this paper. The parameter for  $SLE$  is denoted  $\kappa > 0$ . For each  $\mathbf{c} < 1$ , there are two values of  $\kappa$ , one less than four and one greater than four, given by

$$\mathbf{c} = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa}.$$

The smaller value corresponds to the simple curve case, and we concentrate on this in this paper. For  $\mathbf{c} = 1$ ,  $\kappa = 4$  is a double root which also corresponds to simple curves. Important examples are  $\kappa = 2, \mathbf{c} = -2$  (loop-erased walks),  $\kappa = 8/3, \mathbf{c} = 0$  (self-avoiding walks),  $\kappa = 3, \mathbf{c} = 1/2$  (interfaces of Ising clusters),  $\kappa = 4, \mathbf{c} = 1$  (interfaces of free fields). In all cases, but for self-avoiding walk,  $SLE$  has been proved to be the scaling limits of the models [21, 29, 27]

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♣ The letter  $c$  is standard in the physics literature for central charge. Since we use  $c$  for arbitrary constants, it is not a good choice for a parameter. Our compromise is to use a bold-face  $\mathbf{c}$ .

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Another conformally invariant measure on (in this case, nonsimple) curves in the plane is given by Brownian motion. A variant of this measure, called the Brownian loop measure, arises in the study of  $SLE$  [14, 24]. This is a  $\sigma$ -finite measure on nonsimple curves that arises as a scaling limit of a random walk loop measure, see [23] and [20, Chapter 9]. It is closely related to the determinant of the Laplacian

and the Gaussian free field, see, e.g., [6], but we will only need to view it as a measure on paths. The key properties of the measure are conformal invariance and the restriction property.

In this paper we view  $SLE_\kappa$  as a (positive) measure  $\mu_D(z, w)$  on curves (modulo increasing reparametrization) in a domain  $D$  of total mass  $\Psi_D(z, w)$  connecting two distinct points  $z, w$ . Here  $z, w$  can be interior points or boundary points but in the latter case we make some smoothness assumptions on the boundary. We expect these curves to arise as normalized limits of measures on lattice curves. If  $0 < \Psi_D(z, w) < \infty$ , we can normalize the measure to produce a probability measure that we denote by  $\mu_D^\#(z, w)$ . There are various assumptions we can make on the measures. We will be more precise later, but let us discuss them now. The first is conformal covariance:

- **Conformal covariance.** There exist boundary and interior scaling exponents  $b, \tilde{b}$  such that if  $f : D \rightarrow f(D)$  is a conformal transformation,

$$f \circ \mu_D(z, w) = |f'(z)|^{b_z} |f'(w)|^{b_w} \mu_{f(D)}(f(z), f(w)),$$

where  $b_\zeta = b$  if  $\zeta \in \partial D$  and  $b_\zeta = \tilde{b}$  if  $\zeta \in D$ .

This implies conformal *invariance* of the probability measures,

$$f \circ \mu_D^\#(z, w) = \mu_{f(D)}^\#(f(z), f(w)).$$

If one is only considering the probability measures, then one does not need to make smoothness assumptions at the boundary. The domain Markov property below uses the probability measures for nonsmooth boundary points.

There are three other assumptions we will discuss. It turns out that they are redundant, so we do not need to make all of them assumptions, but this is not obvious.

- **Reversibility.** The measure  $\mu_D(w, z)$  can be obtained from  $\mu_D(z, w)$  by reversing the paths.
- **Domain Markov property.** In the probability measure  $\mu_D^\#(z, w)$ , given an initial segment of the curve  $\gamma_t = \gamma(0, t]$ , the conditional distribution of the remainder of the curve is  $\mu_{D \setminus \gamma_t}^\#(\gamma(t), w)$ .
- **Boundary perturbation.** Suppose  $D_1 \subset D$  and the domains agree in neighborhoods of  $z, w$ . Then  $\mu_{D_1}(z, w)$  is absolutely continuous with respect to  $\mu_D(z, w)$ . In fact, if  $\gamma$  is a curve connecting  $z$  and  $w$  in  $D_1$ , then the Radon-Nikodym derivative is given by

$$\exp \left\{ \frac{\mathbf{c}}{2} m_D(\gamma, D \setminus D_1) \right\},$$

where  $m_D(\gamma, D \setminus D_1)$  denotes the (Brownian) loop measure of loops in  $D$  that intersect both  $\gamma$  and  $D_1$ .

Schramm [25] studied the probability measures  $\mu_D^\#(z, w)$  where  $z \in \partial D$  and  $w \in D$  or  $w \in \partial D$ . He showed that for simply connected  $D$ , there is only a one-parameter family of measures satisfying conformal invariance and the domain Markov property. He used  $\kappa$  as the parameter and these are now called radial and chordal  $SLE_\kappa$  (in  $D$  from  $z$  to  $w$ ), respectively. It is known [26, 4] that for  $\kappa \leq 4$ , the measure is supported on simple curves of Hausdorff dimension  $d = 1 + \frac{\kappa}{8} \in (1, \frac{3}{2}]$ . The following has been proved for  $SLE_\kappa$ ,  $0 < \kappa \leq 4$  in simply connected domains.

- Let

$$b = \frac{6 - \kappa}{2\kappa}, \quad \tilde{b} = \frac{b(\kappa - 2)}{4} = \frac{2b + \mathbf{c}}{12}.$$

Let  $\Psi_{\mathbb{H}}(0, 1) = 1, \Psi_{\mathbb{D}}(1, 0) =$  and define  $\Psi_D(z, w)$  for other simply connected domains by the scaling rule

$$\Psi_D(z, w) = |f'(z)|^b |f'(w)|^{b_w} \Psi_{f(D)}(f(z), f(w)),$$

where  $b_w = b$  if  $w \in \partial D$  and  $b_w = \tilde{b}$  if  $w \in D$ . Then [14, 24, 19] if  $\mu_D(z, w) = \Psi_D(z, w) \mu_D^\#(z, w)$ , the family  $\{\mu_D(z, w)\}$  restricted to simply connected domains satisfies conformal covariance, domain Markov property, and the boundary perturbation rule.

- If  $w \in \partial D$ , then [30]  $\mu_D^\#(w, z)$  is the same as the reversal of  $\mu_D^\#(z, w)$ .

In the chordal case,  $\Psi_D(z, w) = H_{\partial D}(z, w)^b$  where  $H_{\partial D}(z, w)$  denotes a multiple of the boundary Poisson kernel. This follows from the scaling rule for the kernel,

$$H_{\partial D}(z, w) = |f'(z)| |f'(w)| H_{\partial f(D)}(f(z), f(w)).$$

If  $w \in D$ , the Poisson kernel satisfies

$$H_D(w, z) = |f'(z)| H_{f(D)}(f(w), f(z)),$$

and hence  $\Psi_D(w, z)$  is not given by a power of the Poisson kernel. If  $\kappa = 2$ , for which  $b = 1, \tilde{b} = 0$ , the partition function is given by the boundary Poisson kernel (chordal case) or Poisson kernel (radial case). One can also see this from the relationship with loop-erased random walk.

In his argument, Schramm uses the fact that if one slits a simply connected domain  $D$  at its boundary then the resulting domain  $D \setminus \gamma_t$  is also simply connected and hence by the Riemann mapping theorem is conformally equivalent to the original domain. If  $D$  is not simply connected, or  $D$  is “slit on the inside”, this is no longer true. For this reason, conformal invariance of the probability measures and the domain Markov property are not sufficient to determine the measures  $\mu_D^\#(z, w)$  for nonsimply connected domains. In [18] it was suggested to use the boundary perturbation rule to extend the definition. We continue this approach in this paper. There have been other approaches, see, e.g., [2, 3, 9, 8], but none have directly used the boundary perturbation rule.

We will show the following. (If  $z$  or  $w$  are boundary points, we implicitly assume sufficient smoothness at the boundary.)

- There is a unique way (up to some arbitrary multiplicative constants) to extend the measures  $\mu_D(z, w)$  so that it satisfies conformal covariance and the boundary perturbation rule.
- If  $\kappa \leq 8/3$  ( $\mathbf{c} \leq 0$ ), then  $\Psi_D(z, w) < \infty$ , and the probability measures satisfy the domain Markov property.
- If  $8/3 < \kappa \leq 4$ , and  $D$  is 1-connected,  $\Psi_D(z, w) < \infty$ .

The key observation is that the restriction property for the Brownian loop measure holds for multiply connected domains. We conjecture that  $\Psi_D(z, w) < \infty$  for all  $\kappa \leq 4$ , but have not shown this. However, we prove a weaker fact.

- If  $\kappa \leq 4$  and  $D_1$  is a simply connected subdomain and  $\mu_D(z, w; D_1)$  denotes the measure  $\mu_D(z, w)$  restricted to curves staying in  $D_1$ , then

$$\|\mu_D(z, w; D_1)\| < \infty.$$

- The probability measures  $\mu_D^\#(z, w; D_1)$  satisfy the domain Markov property.

- If  $\Psi_D(z, w) < \infty$  for all  $k$ -connected domains, then the measures  $\mu_D^\#(z, w)$ , restricted to  $k$ -connected domains, satisfy the domain Markov property.

The next property will follow from the definition and Zhan's result for simply connected domains [30].

- The measure  $\mu_D(w, z)$  is the reversal of  $\mu_D(z, w)$ .

Zhan [31] recently took a different approach to extending  $SLE_\kappa$  in the case of an annulus. Roughly speaking, he shows that there is a unique way of defining  $\mu_D^\#(z, w)$  for conformal annuli so that it satisfies the domain Markov property and reversibility. (Note that the combination of the two properties allows one to describe conditional distributions given both an initial segment and a terminal segment of the path.)

In this paper, we consider our process for 1-connected domains and show that it is the same as that defined by Zhan. In particular, reversibility of the process follows. We use the boundary perturbation rule to give an equation for the partition function and give a somewhat more direct proof of existence of the solution. Although this paper does not directly use the results in [31], it does use an idea from that paper. In particular, the annulus Loewner equation is used to find PDEs and the Feynman-Kac formula is used to analyze PDEs that arise.

We now summarize the contents of the paper. We describe in Section 2 a model introduced in [13] called the  $\lambda$ -SAW. It is a two-parameter family of lattice models for which it is conjectured that there is a one-parameter subfamily of critical models. One of the parameters in [13] was denoted  $\lambda$  but we have chosen to set  $\lambda = -\mathbf{c}/2$  here. It is a generalization of the loop-erased walk ( $\mathbf{c} = -2$ ) and self-avoiding walk ( $\mathbf{c} = 0$ ). This model was created after studying  $SLE$ . While we cannot prove that this has a limit at the moment (except for  $\mathbf{c} = -2$  and a somewhat different version for  $\mathbf{c} = 1$  for which we can use current results), it is useful for heuristic understanding of our definition of  $SLE$  in multiply connected domains.

Section 3 contains many results that are needed in the paper most of which have been proved elsewhere. This can be skimmed at first reading and referred back to as needed. Section 3.1 reviews facts about the Poisson kernel and sets some notation; this is followed by discussion of the annulus version. The annulus Poisson kernel is often written in terms of theta functions. We choose instead to write the functions in terms of infinite sums which arise naturally when raising the annulus to the covering space of an infinite strip. The next three subsections review the important tools in this area:  $SLE$  in  $\mathbb{H}$ , the Brownian bubble measure, and the Brownian loop measure. Section 3.6 reviews the methods to analyze  $SLE$  in simply connected domains in terms of the Brownian loop measure and extends this idea to shrinking domains. This will allow us to view radial  $SLE$  or annulus  $SLE$  in terms of chordal  $SLE$  in  $\mathbb{H}$  where the domain is shrinking by all the translates of the path. In the case of annulus  $SLE$  we get a process that we call locally chordal  $SLE_\kappa$ . We write this using an annulus parametrization and this leads to the annulus Loewner equation which we write as an equation in the covering infinite strip.

The definition of  $SLE$  is given in Section 4. In the boundary to boundary case, this is essentially the same definition as in [18]. We extend this to boundary/bulk and bulk/bulk cases. One nice thing about our definition is that reversibility is immediate, given reversibility for chordal  $SLE$  in simply connected domains. There are some subtleties in defining the bulk/bulk measure in subdomains of  $\mathbb{C}$  in terms of the measure on  $\mathbb{C}$ , see Proposition 4.8. The definitions make use of facts about

annulus  $SLE$  that are discussed in the next section. The extension of the definition to multiple paths with disjoint endpoints is immediate as in [13].

The next two sections discuss the results about annulus  $SLE_\kappa$ . Most of the results in this section were proved in [31], but there are some differences in our approach. We focus on the “crossing” case although the “chordal” case can be done similarly as we point out. In Section 5 we study annulus  $SLE_\kappa$  with a given winding number. By taking its preimage under the logarithm, we can consider it as a measure on curves connecting points of an infinite strip, and we in turn can compare this measure to chordal  $SLE_\kappa$  in the strip. This requires comparing the loop measures in the strip to the preimage of the loop measure in the annulus. (Although the loop measure is conformally invariant, the logarithm is a multi-valued function, so some care is needed.) At an intermediate step we consider the locally chordal  $SLE_\kappa$  discussed in Section 3. Although this latter process is not the same as annulus  $SLE_\kappa$ , it turns out that the partition function for annulus  $SLE_\kappa$  can be given in terms of a functional of this process. As in [31], we can then use the Feynman-Kac theorem to write a PDE for the partition function and this allows us to show that it gives the quantity we want.

Section 7 takes a different approach and derives the differential equation for the partition function in the annulus by comparing annulus  $SLE_\kappa$  to radial  $SLE_\kappa$ . Smoothness of the partition function follows from the work of the previous section, so only the Itô formula calculation is needed. The work here shows that the process we get is the same as the process in [31]. Our approach gives a little more than what is stated explicitly in [31]. The annulus partition function is of the form  $\Psi(r, x)$ , which denotes the total mass of  $SLE_\kappa$  from 1 to  $e^{-r+ix}$  in the annulus  $A_r = \{e^{-r} < |z| < 1\}$ . The probability measure is obtained by normalization. Multiplying the partition function by a function of  $r$  does not change the probability measure. Here we get not only the probability measure but the correct  $r$  dependence.

I would like to thank Dapeng Zhan for useful conversations.

## 2. THE LATTICE MODEL

Here we describe a lattice model for random walks called the  $\lambda$ -SAW [13]. For simplicity, we will start with the bulk/bulk version in a bounded domain  $D$ . For convenience, we will use the integer lattice  $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z}$ , but the scaling limit should be independent of the lattice.

A self-avoiding walk (SAW)  $\omega = [\omega_0, \dots, \omega_n]$  of length  $n$  is a finite nearest neighbor path in  $\mathbb{Z}^2$  such that  $\omega_j \neq \omega_k$  for  $j < k$ . Let  $|\omega| = n$  denote the length.

A rooted (random walk) loop  $\eta = [\eta_0, \dots, \eta_{2n}]$  of length  $2n > 0$  is a finite nearest neighbor path (not necessarily self-avoiding) with  $\eta_0 = \eta_{2n}$ . Again we write  $|\eta| = 2n$  for the length. An unrooted loop is an equivalence class of loops under the equivalence relation

$$[\eta_0, \dots, \eta_{2n}] \sim [\eta_j, \eta_{j+1}, \dots, \eta_{2n}, \eta_1, \dots, \eta_j]$$

for each  $j$ . The rooted random walk loop measure is the measure on rooted loops, which assigns measure  $4^{-|\eta|}/|\eta|$  to each loop  $\eta$  with  $|\eta| > 0$ . This induces a measure  $m^{RW}$  on unrooted loops called the *random walk loop measure* by giving each unrooted loop the sum of the weights of the different rooted loops that give the unrooted loop.

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♣ One may think of the unrooted loop measure as assigning measure  $4^{-n}$  to each unrooted loop  $\eta$  with  $|\eta| = n$ . However, this is not exactly correct. For example, if  $n = 4$  and  $\eta = [x, y, x, y, x]$ , then there are only two different rooted loops that generate the unrooted loop, and hence this unrooted loop has measure  $4^{-n}/2$ .

Suppose  $D$  is a bounded domain in  $\mathbb{C}$  and  $z, w$  are distinct points in  $D$ . Let  $\beta, \lambda$  be fixed constants which are the parameters of the model. For each  $n$ , let  $L_n = n^{-1} \mathbb{Z}^2 \cap D$  and let  $z_n, w_n$  be points in  $L_n$  closest to  $z, w$  (if there is a tie for “closest”, we can choose arbitrarily). Define the measure  $\nu_n = \nu_{n,D,z,w}$  on SAWs  $\omega$  in  $L_n$  with endpoints  $z_n, w_n$  which gives  $\omega$  measure

$$\exp \{ -\beta |\omega| + \lambda m^{RW}(\omega, D, n) \},$$

where  $m^{RW}(\omega, D, n)$  denotes the total  $m^{RW}$  measure of (unrooted) loops  $\eta$  in  $L_n$  that intersect  $\omega$ . Let  $Z_n(D) = Z_n(D; \beta, \lambda)$  denote the total mass of the measure. This is also called the *partition function*.

This model has two parameters but the conjecture is that there is a one-parameter family of critical models. Let us write  $\lambda = -c/2$  and write  $\beta = \beta_c$  for the corresponding value of  $\beta$ .

♣ The value of the critical  $\beta$  is a lattice dependent quantity. The value  $\lambda = -c/2$  is not lattice dependent as long as we define the random walk loop measure correctly. For a given lattice, the rooted loop measure is defined to give measure  $p(\eta)/|\eta|$  to every loop  $\eta$  where  $p(\eta)$  is the probability that simple random walk in the lattice starting at  $\eta_0$  produces the loop  $\eta$ . The value  $c$  is the “central charge” but we can think of it as a free parameter.

**Conjecture 2.1.** *For each  $c \leq 1$ , there corresponds a (lattice dependent)  $\beta$  and a (lattice independent) scaling exponent  $\tilde{b}$  such the measure  $\nu_n$  has the following properties.*

- *For each bounded  $D$  and distinct  $z, w$  in  $D$  there exists  $\Psi_D^*(z, w) \in (0, \infty)$  such that*

$$Z_n \sim n^{-2\tilde{b}} \Psi_D^*(z, w), \quad n \rightarrow \infty.$$

- *There exists a limit measure on simple curves*

$$\nu_D(z, w) = \lim_{n \rightarrow \infty} n^{2\tilde{b}} \nu_n.$$

- *The family of measures  $\{\nu_D(z, w)\}$  satisfies the conformal covariance relation: if  $f : D \rightarrow f(D)$  is a conformal transformation,*

$$f \circ \nu_D(z, w) = |f'(z)|^{\tilde{b}} |f'(w)|^{\tilde{b}} \nu_{f(D)}(f(z), f(w)).$$

There is also a boundary version of this conjecture. Suppose  $z$  is a boundary point of  $D$  and let us assume that  $\partial D$  is analytic near  $z$ . One can define the measure  $\nu_n$  as above, but there are lattice issues involved. We will not deal with them here and just state the following rough conjecture; see [11] for a more precise statement including lattice issues. We also assume smoothness near the appropriate boundary points.

**Conjecture 2.2.** *For each  $c \leq 1$ , there corresponds a (lattice dependent)  $\beta$  and (lattice independent) scaling exponents  $b, \tilde{b}$  such the measure  $\nu_n$  has the following properties.*

- For each bounded  $D$  and distinct  $z, w$  in  $\bar{D}$  there exists  $\Psi_D^*(z, w)$  such that

$$Z_n \sim n^{-(b_z+b_w)} \Psi_D^*(z, w), \quad n \rightarrow \infty.$$

- There exists a limit measure on simple curves

$$\nu_D(z, w) = \lim_{n \rightarrow \infty} n^{b_z+b_w} \nu_n.$$

- The family of measures  $\{\nu_D(z, w)\}$  satisfies the conformal covariance relation:

$$f \circ \nu_D(z, w) = |f'(z)|^{b_z} |f'(w)|^{b_w} \nu_{f(D)}(f(z), f(w)).$$

Here  $b_\zeta = b$  or  $\tilde{b}$ , respectively, if  $\zeta$  is a boundary point or an interior point.

The conjectures are open, but let us assume that the conjectures do hold. Let

$$\nu_D^\#(z, w) = \frac{\nu_D(z, w)}{\Psi_D^*(z, w)}$$

be the corresponding probability measures which are conformally *invariant*:

$$f \circ \nu_D^\#(z, w) = \nu_{f(D)}^\#(f(z), f(w)).$$

Schramm [25] showed that if  $D$  is simply connected and  $z \in \partial D$ , there is only a one-parameter family of possible limit measures for  $\nu_D^\#(z, w)$  which are now called *chordal* (if  $w \in \partial D$ ) or *radial* (if  $w \in D$ ) *Schramm-Loewner evolution with parameter  $\kappa$  ( $SLE_\kappa$ )*. Analysis of *SLE* [26, 14] shows that  $0 < \kappa \leq 4$  (if we want a measure on simple curves) and the other parameters are given by

$$(1) \quad b = \frac{6 - \kappa}{2\kappa}, \quad \tilde{b} = \frac{b(\kappa - 2)}{4}, \quad \mathbf{c} = 6\tilde{b} - b = b(3\kappa - 8).$$

Suppose  $z, w \in D$  and  $D_1 \subset D$ , and let  $\nu_n, \nu_n^1$  be the corresponding measures as above and  $L_n = D \cap n^{-1}\mathbb{Z}^2, L_n^1 = D_1 \cap n^{-1}\mathbb{Z}^2$ . Then if  $\omega$  is a SAW in  $L_n$  connecting  $z_n$  and  $w_n$ ,

$$\frac{\nu_n^1(\omega)}{\nu_n(\omega)} = \exp \left\{ \frac{\mathbf{c}}{2} [m^{RW}(\omega, D, n) - m^{RW}(\omega, D_1, n)] \right\}.$$

As  $n \rightarrow \infty$ , the quantity on the right has a limit [23] in terms of the Brownian loop measure

$$\lim_{n \rightarrow \infty} [m^{RW}(\omega, D, n) - m^{RW}(\omega, D_1, n)] = m_D(\omega, D \setminus D_1),$$

where the right-hand side denotes the Brownian loop measure [24] of loops in  $D$  that intersect both  $\omega$  and  $D \setminus D_1$ . Hence the limit measures should satisfy for  $\gamma \subset D_1$ ,

$$(2) \quad \frac{d\nu_{D_1}(z, w)}{d\nu_D(z, w)}(\gamma) = \exp \left\{ \frac{\mathbf{c}}{2} m_D(\gamma, D \setminus D_1) \right\}.$$

For simply connected  $D, D_1$  with  $z \in \partial D$ , this was established in [14, 24].

Schramm's construction of *SLE* makes generalizations to nonsimply connected domains difficult. The purpose of this paper is to show that one can use the relation (2) to define it. This requires some work. While we do not prove the conjectures stated in this section, it is helpful to remember that the definitions we give in this paper are those of the conjectured scaling limit of the  $\lambda$ -SAW with  $\lambda = -\mathbf{c}/2$ .

## 3. PRELIMINARIES

In this paper, we assume that  $\kappa \in (0, 4]$  and  $\mathbf{c}, b, \tilde{b}$  are as in (1). We also set

$$a = \frac{2}{\kappa}.$$

Constants throughout may depend implicitly on  $\kappa$ .

**3.1. Poisson kernel and related.** We establish notation and review facts about the Poisson kernel.

- $\mathbb{H}$  denotes the open upper half plane,  $\mathbb{D}$  the open unit disk, and if  $r > 0$ ,

$$A_r = \{z \in \mathbb{D} : e^{-r} < |z|\}, \quad S_r = \{z \in \mathbb{H} : \text{Im}(z) < r\},$$

$$\mathbb{D}_r = e^{-r}\mathbb{D}, \quad C_r = \partial\mathbb{D}_r.$$

Under this notation  $A_r = \mathbb{D} \setminus \overline{\mathbb{D}_r}$ ,  $\partial A_r = C_0 \cup C_r$ . Throughout this paper we fix

$$\psi(z) = e^{iz}$$

and note that  $\psi$  maps  $S_r$  (many-to-one) onto  $A_r$ . We write  $+\infty, -\infty$  for the two infinite points in  $\partial S_r$ .

- If  $D$  is a domain, then  $z$  is  $\partial D$ -analytic if  $z \in \partial D$  and there is a neighborhood  $N$  of  $z$  and a conformal transformation

$$\phi : N \rightarrow \phi(N)$$

with  $\phi(z) = 0$  and  $\phi(N \cap D) = \phi(N) \cap \mathbb{H}$ . We say that  $z$  is  $D$ -analytic if  $z \in D$  or  $z$  is  $\partial D$ -analytic.

- If  $\gamma$  is a curve, we write  $\gamma_t$  for  $\gamma[0, t]$ .
- If  $z, w \in \partial D$  and  $\gamma : [0, t_0] \rightarrow D$  is a curve with  $\gamma(0) = z, \gamma(t_0) = w$  we abuse notation by writing  $\gamma \subset D$  if  $\gamma(0, t_0) \subset D$ . If  $t < t_0$ , we write  $\gamma_t \subset D$  if  $\gamma(0, t] \subset D$ .
- If  $z \in D$  and  $w$  is  $\partial D$ -analytic, let  $H_D(z, w)$  denote the Poisson kernel (that is, the inward normal derivative of the Green's function at  $w$ ) normalized so that

$$H_{\mathbb{H}}(x + iy, 0) = \frac{y}{x^2 + y^2}.$$

It satisfies the scaling rule

$$H_D(z, w) = |f'(w)| H_{f(D)}(f(z), f(w)).$$

(When writing rules like this, it will be implicitly assumed that the quantities are well defined. For example, in this case  $z \in D$ ,  $w$  is  $\partial D$ -analytic, and  $f(w)$  is  $\partial f(D)$ -analytic.) Under our normalization, the probability that a complex Brownian motion starting at  $z$  exits  $D$  at  $V \subset \partial D$  is

$$\frac{1}{\pi} \int_V H_D(z, w) |dw|.$$

- If  $z, w$  are distinct  $\partial D$ -analytic points, we write  $H_{\partial D}(z, w)$  for the *boundary* or *excursion Poisson kernel* given by

$$H_{\partial D}(z, w) = \partial_{\mathbf{n}} H_D(z, w) = H_{\partial D}(w, z),$$

where  $\mathbf{n} = \mathbf{n}_z$  denotes the (inward) normal derivative at  $z$ . It satisfies the scaling rule:

$$(3) \quad H_{\partial D}(z, w) = |f'(z)| |f'(w)| H_{\partial f(D)}(f(z), f(w)).$$



- If  $D$  is simply connected, there is a complex form of the Poisson kernel  $\mathcal{H}_D(z, w)$  such that  $H_D(z, w) = \text{Im}\mathcal{H}_D(z, w)$ . This is defined up to a real translation, and we choose the translation so that

$$\mathcal{H}_{\mathbb{H}}(z, 0) = -\frac{1}{z}.$$

The function  $f(z) = \mathcal{H}_D(z, w)$  can be characterized as the unique conformal transformation  $f : D \rightarrow \mathbb{H}$  such that

$$f(w + \epsilon \mathbf{n}_w) = \frac{i}{\epsilon} + o(1), \quad \epsilon \downarrow 0+.$$

- The Poisson and boundary Poisson kernel for the strip  $S_r$  can be computed using conformal invariance,

$$(4) \quad H_{\partial S_r}(z, 0) = -\frac{\pi}{2r} \coth\left(\frac{\pi z}{2r}\right),$$

$$(5) \quad H_{\partial S_r}(0, x) = \frac{\pi^2}{4r^2} \left[ \sinh\left(\frac{\pi x}{2r}\right) \right]^{-2},$$

$$(6) \quad H_{\partial S_r}(0, x + ir) = \frac{\pi^2}{4r^2} \left[ \cosh\left(\frac{\pi x}{2r}\right) \right]^{-2}.$$

- If  $z, w$  are distinct boundary points of  $D$ ,  $D_1 \subset D$  with  $\text{dist}(z, D \setminus D_1) > 0$ ,  $\text{dist}(w, D \setminus D_1) > 0$ , let

$$Q_D(z, w; D_1)$$

denote the probability that a Brownian excursion in  $D$  from  $z$  to  $w$  stays in  $D_1$ . (A Brownian excursion in  $D$  is a Brownian motion starting at  $z$  and conditioned to go immediately into  $D$  and exit at  $w$ . It is not difficult to make this precise.) We note that  $Q_D(z, w; D_1)$  is invariant under conformal transformations of  $D$ , and if  $z, w$  are  $\partial D$ -analytic,

$$Q_D(z, w; D_1) = \frac{H_{\partial D_1}(z, w)}{H_{\partial D}(z, w)}.$$

If  $D \subset \mathbb{H}$  is simply connected with  $\mathbb{H} \setminus D$  bounded and  $\text{dist}(0, \mathbb{H} \setminus D) > 0$ , then [17, Proposition 5.15]

$$Q_{\mathbb{H}}(0, \infty; D) = \Phi'_D(0),$$

where  $\Phi_D : D \rightarrow \mathbb{H}$  is a conformal transformation with  $\Phi(z) \sim z$  as  $z \rightarrow \infty$ .

When studying *SLE* it is useful to consider subdomains of  $\mathbb{H}$  and the boundary point infinity. In order to make a number of formulas work in this case, it is useful to adapt the following “abuse of notation” about derivatives. This can be considered a kind of normalization at infinity.

- When we consider the conformal transformation  $g : \mathbb{H} \rightarrow \mathbb{H}$  given by  $g(z) = -1/z$ , then we write

$$(7) \quad g'(0) = g'(\infty) = -1.$$

- If  $D \subset \mathbb{H}$  and  $\mathbb{H} \setminus D$  is bounded, then we say that  $\infty$  is  $\partial D$ -analytic. If  $D_1, D_2$  are two such domains and  $f : D_1 \rightarrow D_2$  is a conformal transformation with  $f(\infty) = \infty$ , we define  $f'(\infty)$  by

$$f(z) \sim \frac{z}{f'(\infty)}, \quad z \rightarrow \infty.$$

Equivalently, if  $F(z) = -1/f(-1/z) = g \circ f \circ g(z)$ , then

$$f'(\infty) = F'(0).$$

- More generally, if  $F : D \rightarrow D'$  is a conformal transformation with  $F(z) = \infty$  or  $F(\infty) = z$ , we compute derivatives using the chain rule and (7).

The boundary Poisson kernel  $H_{\partial D}(z, w)$  can be defined if  $z$  or  $w$  equals infinity using the scaling rule (3). Under our normalization  $H_{\partial \mathbb{H}}(x, \infty) = 1$ .

---

♣ If  $D, D'$  are simply connected domains,  $z, w$  are distinct  $\partial D$ -analytic points, and  $z', w'$  are distinct  $\partial D'$ -analytic points, then there is a one parameter family of conformal transformations  $f : D \rightarrow D'$  with  $f(z) = z', f(w) = w'$ . The quantity  $f'(z)f'(w)$  is invariant of the choice of the transformation. Our definitions of derivatives at infinity are made so that this property holds as well when  $w = \infty$  or  $w' = \infty$ .

---

**3.2. The annulus.** The functions that arise from the Poisson kernel of the annulus will be important. By considering different winding numbers, using the scaling rule, and applying (6), we can see that

$$H_{\partial A_r}(1, e^{-r+ix}) = e^r \sum_{k=-\infty}^{\infty} H_{\partial S_r}(0, x + ir) = \frac{e^r}{2} \mathbf{J}(r, x),$$

where  $\mathbf{J}(r, x)$  is defined by

$$(8) \quad \mathbf{J}(r, x) = \frac{\pi^2}{2r^2} \sum_{k=-\infty}^{\infty} \left[ \cosh \left( \frac{\pi(x + 2k\pi)}{2r} \right) \right]^{-2}.$$

We will view  $\mathbf{J}(r, x)$  as a function on  $(0, \infty) \times \mathbb{R}$  satisfying  $\mathbf{J}(r, x) = \mathbf{J}(r, x + 2\pi)$ . Under our normalization of the Poisson kernel,

$$(9) \quad e^{-r} \int_0^{2\pi} H_{\partial A_r}(1, e^{-r+ix}) dx = \frac{\pi}{r},$$

which implies

$$\int_0^{2\pi} \mathbf{J}(r, x) dx = 2 \int_0^{\pi} \mathbf{J}(r, x) dx = \frac{2\pi}{r}.$$

Indeed,  $(r/2\pi)\mathbf{J}(r, x)$  has the interpretation as the density of the angle of the hitting point of an  $h$ -process in  $A_r$  started at 1 conditioned to leave  $A_r$  at  $C_r$  (in other words, the  $h$ -process associated to the harmonic function  $h(z) = -\log|z|$ ). Using this interpretation, we can see that there exists  $\rho > 0$  such that for all  $r$  sufficiently small

$$(10) \quad \frac{\rho\pi}{r} \leq \int_0^r \mathbf{J}(r, x) dx \leq \frac{(1-\rho)\pi}{r}.$$

---

♣ To see (9), recall that under our normalization of the Poisson kernel

$$e^{-r} \int_0^{2\pi} H_{A_r}(e^{-\epsilon}, e^{-r+ix}) dx$$

is  $\pi$  times the probability that a Brownian motion starting at  $e^{-\epsilon} = 1 - \epsilon + O(\epsilon^2)$  leaves  $A_r$  at  $C_r$ . A standard estimate for Brownian motion tells us that this probability equals  $\epsilon/r$ .

---

**Lemma 3.1.** *There exist  $c < \infty$  such that if  $r \geq 1, x \in \mathbb{R}$ ,*

$$\left| \mathbf{J}(r, x) - \frac{1}{r} \right| \leq c e^{-r}.$$

*Proof.* We will assume  $r \geq 2$  (the case  $1 \leq r \leq 2$  is easy). Let  $V$  be a subset of  $[0, 2\pi)$  which can also be viewed as a periodic subset of  $\mathbb{R}$ . We need to show that

$$\frac{1}{2\pi} \int_V \mathbf{J}(r, x) dx = \frac{l(V)}{r} [1 + O(re^{-r})],$$

where  $l$  denotes length. By definition,

$$\begin{aligned} \frac{1}{2\pi} \int_V \mathbf{J}(r, x) dx &= \frac{e^{-r}}{\pi} \int_V H_{\partial A_r}(1, e^{-r+ix}) dx \\ &= \frac{e^{-r}}{\pi} \int_V H_{\partial A_r}(e^{-r}, e^{ix}) dx. \end{aligned}$$

Let  $B_t$  denote a complex Brownian motion and  $T_s = \inf\{t : B_t \in C_s\}$ . Let

$$p(z; V) = \mathbb{P}^z\{B_{T_0} \in V\}, \quad q(z; V) = \mathbb{P}^z\{B_{T_0} \in V \mid T_0 < T_r\},$$

and let  $q_{\pm}(r; V)$  be the maximum and minimum of  $q(z, V)$  on  $C_{r-1}$ . Then,

$$q_-(r, V) \leq \frac{re^{-r}}{\pi} \int_V H_{\partial A_r}(e^{-r}, e^{ix}) dx \leq q_+(r, V).$$

Hence it suffices to show that if  $z \in C_{r-1}$ ,

$$q(z, V) = l(V) [1 + O(re^{-r})],$$

where the error term is uniform in  $z$ . If  $z \in C_{r-1}$ , then  $\mathbb{P}^z\{T_0 < T_r\} = 1/r$ , and hence

$$p(z, V) = r^{-1} q(z, V) + (1 - r^{-1}) \mathbb{P}^z\{B_{T_0} \in V \mid T_r < T_0\}.$$

Using the strong Markov property and the exact form of the Poisson kernel in the disk, we see that

$$p(z, V) = l(V) [1 + O(|z|)], \quad \mathbb{P}^z\{B_{T_0} \in V \mid T_r < T_0\} = l(V) [1 + O(|z|)],$$

and hence if  $z \in C_{r-1}$ ,

$$\begin{aligned} r^{-1} q(r, V) &= l(V) [1 + O(|z|)] - (1 - r^{-1}) l(V) [1 + O(|z|)] \\ &= l(V) [r^{-1} + O(e^{-r})] \\ &= r^{-1} l(V) [1 + O(re^{-r})]. \end{aligned}$$

□

Another important function will be

$$\mathbf{H}_I(r, x) = -\frac{x}{r} + \int_0^x \mathbf{J}(r, y) dy = \int_0^x \left[ \mathbf{J}(r, y) - \frac{1}{r} \right] dy,$$

which satisfies  $\mathbf{H}_I(r, x) = \mathbf{H}_I(r, x + 2\pi)$  and

$$\mathbf{H}'_I(r, x) = J(r, x) - \frac{1}{r}.$$

Here we are using the notation from [31], and the prime denotes an  $x$ -derivative.

**Lemma 3.2.** *Let  $K(r, x) = r\mathbf{H}_I(r, x)$ . Then for all  $r$ ,  $K$  is an odd function of period  $2\pi$  satisfying  $K(r, \pi - x) = K(r, \pi + x)$ ,  $K(0) = K(\pi) = 0$ , and*

$$K(r, x) \leq \pi - x, \quad 0 \leq x \leq \pi.$$

*Moreover, there exists  $\epsilon > 0$  such that for all  $r$  sufficiently small and all  $x$ ,*

$$K(r, x) \leq \pi - \epsilon r.$$

*Proof.* This is straightforward. The last estimate uses (10).  $\square$

♣ Although we will not need it for our main theorem, in a comment in Section 7.1 we will use the fact that the function  $\Phi(r, x) = r\mathbf{J}(r, x) = r\mathbf{H}'_I(r, x) + 1$  satisfies the differential equation

$$(11) \quad \dot{\Phi} = \Phi'' + \mathbf{H}_I \Phi + \mathbf{H}'_I \Phi.$$

Here, as later in the paper, we use dots for  $r$ -derivatives and primes for  $x$ -derivatives. To see this, we will need the following fact from [31]:

$$\dot{\mathbf{H}}_I = \mathbf{H}''_I + \mathbf{H}'_I \mathbf{H}_I.$$

Hence  $G = \mathbf{H}'_I$  satisfies

$$\dot{G} = G'' + \mathbf{H}_I G' + \mathbf{H}'_I G,$$

and

$$\begin{aligned} \dot{\Phi} &= G + rG'' + r\mathbf{H}_I G' + r\mathbf{H}'_I G \\ &= G + \Phi'' + \mathbf{H}_I \Phi' + \mathbf{H}'_I (\Phi - 1) \\ &= \Phi'' + \mathbf{H}_I \Phi' + \mathbf{H}'_I \Phi \end{aligned}$$

**Lemma 3.3.** *There exists  $c > 0$  such that the following holds. Suppose  $r \geq 1$  and  $f : D \rightarrow A_r$  is a conformal transformation with  $f(C_0) = C_0$  where  $D = \mathbb{D} \setminus K$  and  $K$  is a compact subset containing the origin. Then for  $|z| = 1$ ,*

$$||f'(z)| - 1| \leq ce^{-r}, \quad |f''(z)| \leq ce^{-r}.$$

*Proof.* Let  $\phi_D$  be the harmonic function on  $D$  with boundary values 0 on  $C_0$  and 1 on  $K$  and let  $\phi_r = \phi_{A_r}$ . By conformal invariance,

$$\phi_D(z) = \phi_r(f(z)) = \frac{-\log |f(z)|}{r}.$$

Since  $f$  maps  $C_0$  to  $C_0$ , this implies

$$r \partial_{\mathbf{n}} \phi_D(z) = |f'(z)|,$$

where  $\mathbf{n}$  denotes the inward unit normal. Also, conformal invariance of excursion measure gives

$$\int_{C_0} \partial_{\mathbf{n}} \phi_D(z) |dz| = \int_{C_0} \partial_{\mathbf{n}} \phi_r(z) |dz| = \frac{2\pi}{r}.$$

Hence to prove the first estimate, it suffices to show for  $z, w \in C_0$ ,

$$(12) \quad \partial_{\mathbf{n}} \phi_D(z) = \partial_{\mathbf{n}} \phi_D(w) [1 + O(e^{-r})].$$

Using Koebe estimates, we can find a universal  $s$  such that for  $r$  sufficiently large,  $K \subset \mathbb{D}_{r-s}$ . Suppose we start Brownian motions at  $e^{-\epsilon} z$  and  $e^{-\epsilon} w$ , respectively. The probability that they reach  $C_{r-s}$  without hitting  $C_0$  is  $\epsilon/(r-s)$ . On  $C_{r-s}$ ,

$\phi_D) = 1 - O(r^{-1})$ . Using Lemma 3.1, we can see that the conditional distributions on  $C_{r-s}$  given that the Brownian motions reach  $C_{r-s}$  are the same for  $z, w$  up to an error of order  $O(re^{-r})$ . Hence, if  $q(z) = q(z, r, \epsilon)$  denotes the probability that the Brownian motion starting at  $z$  reaches  $C_{r-s}$  before  $C_0$  but does not hit  $K$  before  $C_0$ , then

$$|q(z) - q(w)| \leq c \frac{\epsilon}{r-s} r^{-1} O(re^{-r}) \leq \frac{\epsilon}{r} O(e^{-r}),$$

from which we conclude (12). Indeed, we conclude the stronger fact,

$$q(z) = -\frac{\log |z|}{r} [1 + O(e^{-r})], \quad e^{-1} < |z| < 1.$$

This implies

$$|f(z)| = |z| [1 + O(e^{-r})], \quad e^{-1} < |z| < 1.$$

For the second estimate, fix  $z$  and assume without loss of generality that  $z = 1$  and  $f(1) = 1$ . By Schwarz reflection, we can extend  $f$  to a neighborhood of radius  $1/2$  about 1. Let  $L(z) = \log z, g(z) = \log f$ . where  $L(1) = g(1) = 0$ . We have  $|\operatorname{Reg}(z) - \operatorname{Re} L(z)| \leq e^{-r}$  and  $g(1) = L(1)$ . From this we can use standard arguments to conclude that  $|g(z) - L(z)| = O(e^{-r})$ . Using the Cauchy integral formula, we get  $|g'(z) - L'(z)|, |g''(z) - L''(z)| \leq c O(e^{-r})$ .  $\square$

A computation that we will do a little later will give us a particular annulus function  $\mathbf{A}(r, x)$  which we now define. Suppose that  $D = S_r, z = 0, w = x + ri$  and let  $\gamma_t$  be a curve starting at the origin parametrized so that  $\operatorname{hcap}[\gamma_t] = t$ . Let  $D_t$  be the domain obtained by splitting  $\mathbb{H}$  by the nontrivial  $2\pi k$  translates of  $\gamma_t$ ,

$$D_t = S_r \setminus \bigcup_{k \in \mathbb{Z} \setminus \{0\}} [\gamma_t + 2\pi k],$$

and let

$$Q_t = Q_D(0, w; D_t).$$

Then (see the end of Section 3.8), one can check that as  $t \rightarrow 0$ ,

$$(13) \quad Q_t = 1 - \mathbf{A}(r, x) t + o(t),$$

where

$$\mathbf{A}(r, x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{H_{\partial S_r}(0, 2\pi k) H_{\partial S_r}(2\pi k, x + ir)}{H_{\partial S_r}(0, x + ir)}.$$

Using (5) and (6), we get

$$(14) \quad \mathbf{A}(r, x) = \frac{\pi^2}{4r^2} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\cosh^2(\pi x/2r)}{\sinh^2(\pi^2 k/r) \cosh^2(\pi(x - 2\pi k)/2r)},$$

**Proposition 3.4.** *For fixed  $r$ ,  $\mathbf{A}(r, \cdot)$  is a positive, even function, that is increasing in  $|x|$ . There exists  $c > 0$  such that If  $0 < r \leq 1$  and  $0 \leq x \leq \pi$ ,*

$$(15) \quad \mathbf{A}(r, x) \leq \frac{c}{r^2} \exp \left\{ -\frac{2\pi}{r} (\pi - x) \right\}.$$

*Proof.* The definition implies  $\mathbf{A}(r, x) = \mathbf{A}(r, -x)$ . For  $r \leq 1$ ,  $0 \leq x < \pi$ ,

$$\cosh^2(\pi x/2r) \asymp e^{\pi x/r},$$

$$\sinh^2(\pi^2 k/r) \cosh^2(\pi(x - 2\pi k)/2r) \asymp e^{2|k|\pi^2/r} e^{\pi|2\pi k - x|/r} \geq e^{2|k|\pi^2/r} e^{\pi(2\pi - x)/r}.$$

By summing over  $k$ , we get (15). The monotonicity in  $|x|$  will follow if we show that that for each integer  $k$ ,

$$\frac{\cosh^2(\pi x/2r)}{\cosh^2(\pi(x - 2\pi k)/2r)} + \frac{\cosh^2(\pi x/2r)}{\cosh^2(\pi(x + 2\pi k)/2r)}$$

is an increasing function of  $|x|$ .

Indeed, we will now show that if  $y \in \mathbb{R}$  and

$$f(x) = \frac{\cosh^2 x}{\cosh^2(x - y)} + \frac{\cosh^2 x}{\cosh^2(x + y)},$$

then  $f$  is increasing for  $x \geq 0$ . Since

$$f(x) = \frac{\cosh(2x) + 1}{\cosh(2x - 2y) + 1} + \frac{\cosh(2x) + 1}{\cosh(2x + 2y) + 1},$$

it suffices to show for every  $y \in \mathbb{R}$ , that

$$F(x) = \frac{\cosh x + 1}{\cosh(x - y) + 1} + \frac{\cosh x + 1}{\cosh(x + y) + 1},$$

is increasing for  $x \geq 0$ . Using the sum rule, we get

$$\cosh(x - y) + 1 + \cosh(x + y) + 1 = 2 \cosh x \cosh y + 2,$$

Letting  $r = \cosh y \geq 1$ , we get

$$\begin{aligned} [\cosh(x - y) + 1] [\cosh(x + y) + 1] &= (\cosh x \cosh y + 1)^2 - \sinh^2 x \sinh^2 y \\ &= (r \cosh x + 1)^2 - (r^2 - 1)(\cosh^2 x - 1) \\ &= \cosh^2 x + 2r \cosh x + r^2 \\ &= (\cosh x + r)^2. \end{aligned}$$

Therefore,

$$F(x) = \frac{2r (\cosh x + r^{-1}) (\cosh x + 1)}{(\cosh x + r)^2} = 2r e^{G(\cosh x)},$$

where

$$G(t) = \log(t + \frac{1}{r}) + \log(t + 1) - 2 \log(t + r).$$

Since  $r \geq 1$ ,  $G'(t) > 0$  for  $t > 0$  and hence  $G$  and  $F$  are increasing.  $\square$

**3.3.  $SLE_\kappa$  in  $\mathbb{H}$ .** If  $\kappa = 2/a \in (0, 4]$ , then *chordal  $SLE_\kappa$  (in  $\mathbb{H}$  from 0 to  $\infty$ )* is the solution to the *chordal Loewner equation*

$$(16) \quad \partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z,$$

where  $U_t = -B_t$  is a standard Brownian motion. With probability one [26], this generates a random path  $\gamma : (0, \infty) \rightarrow \mathbb{H}$  such that the domain of  $g_t$  is  $\mathbb{H} \setminus \gamma_t$ . The curve is parametrized so that  $\text{hcap}[\gamma_t] = at$  (see [17, Chapter 3] for definitions); in other words,

$$g_t(z) = z + \frac{at}{z} + O(|z|^{-2}), \quad z \rightarrow \infty.$$

For every  $r > 0$ , let

$$\tau_r = \inf\{t > 0 : \gamma(t) \notin S_r\} = \inf\{t > 0 : \text{Im}\gamma(t) = r\}.$$

---

♣  $SLE_\kappa$  for  $\kappa > 4$  is also very interesting, but the paths are not simple. We restrict in this paper to  $\kappa \leq 4$ .

---

Chordal  $SLE_\kappa$  produces a probability measure on curves, modulo (increasing) reparametrization, from 0 to  $\infty$ . By conformal transformation, we get a probability measure on curves connecting distinct boundary points  $z, w$  of simply connected domains  $D$ . We will denote this measure by  $\mu_D^\#(z, w)$ .

---

♣ To get a measure on parametrized curves, one should use the natural parametrization as described in [22]. This parametrization satisfies a conformal covariance rule under conformal transformations. We would extend our definitions in this paper to parametrized curves, but it would not add anything to our arguments here. For this reason we will consider curves modulo reparametrization as in [25].

---

*Radial  $SLE_\kappa$  from 1 to 0 in  $\mathbb{D}$*  is defined by the transformations on the disk

$$\tilde{g}_t(e^{iz}) = e^{h_t(e^{iz})}$$

where  $h_t$  satisfies

$$(17) \quad \partial_t h_t(z) = \frac{a}{2} \cot_2(h_t(z) - U_t),$$

where, as in [31], we write  $\cot_2(z) = \cot(z/2)$ , and  $U_t$  is a standard Brownian motion. By conformal invariance, this gives a probability measure on curves  $\mu^\#(z, w)$  connecting one boundary point  $z$  and one interior point  $w$ .

**3.4. Brownian bubble measure.** Our main interest is the Brownian loop measure. However, computations of the measure lead to considering excursions and the boundary bubble measure.

Suppose  $D$  is a domain with smooth (not necessarily connected) boundary. For each  $z \in \partial D$ ,  $V, V_1 \subset \partial D$ , we define (*Brownian excursion measures*) by

$$\begin{aligned} \mathcal{E}_D(z, V) &= \int_V H_{\partial D}(z, w) |dw|, \\ \mathcal{E}_D(V_1, V) &= \int_{V_1} \mathcal{E}_D(z, V) |dz| = \int_V \int_{V_1} H_{\partial D}(z, w) |dz| |dw|. \end{aligned}$$

They satisfy the scaling rules

$$\begin{aligned} \mathcal{E}_D(z, V) &= |f'(z)| \mathcal{E}_{f(D)}(f(z), f(V)), \\ \mathcal{E}_D(V_1, V) &= \mathcal{E}_{f(D)}(V_1, V). \end{aligned}$$

In particular  $\mathcal{E}_D(V_1, V)$  is a conformal invariant and hence is well defined even if the boundaries are not smooth. The quantity  $\mathcal{E}_D(z, V)$  needs local smoothness at  $z$  to be defined.

Boundary bubbles in  $D$  are loops rooted at  $z \in \partial D$  and otherwise staying in  $D$ . We review the definitions (see [17, Section 5.5]). The bubble measure is a  $\sigma$ -finite measure on bubbles. In  $\mathbb{H}$  we can define the measure, by specifying for each simply connected domain  $D \subset \mathbb{H}$  with  $\text{dist}(0, \mathbb{H} \setminus D) > 0$ , the measure of the set of bubbles at 0 that do not lie in  $D$ .

**Definition** If  $D \subset \mathbb{H}$  is a subdomain,  $x \in \mathbb{R}$ , and  $\text{dist}(x, \mathbb{H} \setminus D) > 0$ , then

$$\Gamma(x; D) = \Gamma_{\mathbb{H}}(x; D) = \partial_y [H_{\mathbb{H}}(z, x) - H_D(z, x)]|_{z=x}.$$

The quantity  $\Gamma(x; D)$  is the bubble measure (in  $\mathbb{H}$  rooted at  $x$ ) of bubbles that intersect  $\mathbb{H} \setminus D$ . Alternatively, we can write

$$(18) \quad \Gamma(x; D) = \lim_{\epsilon \downarrow 0} \epsilon^{-1} \mathbb{E}^{x+i\epsilon} [H(B_\tau, x)],$$

where  $B$  is a complex Brownian motion and  $\tau = \tau_D = \inf\{t : B_t \notin D\}$ . Note that

$$\Gamma(x; D_1) - \Gamma(x; D_2) = \partial_y [H_{D_2}(z, x) - H_{D_1}(z, x)]|_{z=x}.$$

We can similarly define  $\Gamma_D(z; D')$  if  $z$  is  $\partial D$ -analytic,  $D' \subset D$ , and  $\text{dist}(z, D \setminus D') > 0$ . It is defined as in (18), which we can also write as

$$\Gamma_D(z; D') = \int_{D \cap \partial D'} H_D(w, z) d\mathcal{E}_D(z, w).$$

It satisfies the following scaling rule: if  $f : D \rightarrow f(D)$  is a conformal transformation, then

$$\Gamma_D(z; D') = |f'(z)|^2 \Gamma_{f(D)}(f(z), f(D')).$$

If  $D \subset \mathbb{H}$  is simply connected, this quantity can be computed [17, Proposition 5.22]: if  $f : D \rightarrow \mathbb{H}$  is a conformal transformation with  $f(x) = x$ , then

$$(19) \quad \Gamma(x; D) = -\frac{1}{6} S f(x),$$

where  $S$  denotes the Schwarzian derivative. Particular cases of importance to us are considered in the following proposition.

**Proposition 3.5.**

$$\Gamma_{\mathbb{H}}(0; S_r) = \frac{\pi^2}{12r^2},$$

If  $\Gamma(r) = \Gamma_{\mathbb{D}}(1; A_r)$ , then as  $r \rightarrow \infty$ ,

$$(20) \quad \Gamma(r) = \frac{1 + O(e^{-r})}{2r}.$$

Moreover,

$$\Gamma(r) = \frac{\pi^2}{12r^2} + \delta(r),$$

where

$$(21) \quad \delta(r) = \sum_{k \in \mathbb{Z} \setminus \{0\}} [H_{\partial \mathbb{H}}(0, 2\pi k) - H_{\partial S_r}(0, 2\pi k)] = \frac{1}{12} - \frac{\pi^2}{2r^2} \sum_{k=1}^{\infty} \left[ \sinh\left(\frac{k\pi^2}{r}\right) \right]^{-2}.$$

In particular,

$$\delta(r) = \frac{1 + O(e^{-r})}{2r} - \frac{\pi^2}{12r^2}.$$

*Proof.* Since  $S_r$  is simply connected, we can use (19) with  $f(z) = e^{\pi z/r} - 1$  to get

$$\Gamma_{\mathbb{H}}(0; S_r) = -\frac{S f(0)}{6} = \frac{\pi^2}{12r^2}.$$

The second equality follows from (18). Indeed, as noted previously

$$\lim_{\epsilon \downarrow 0} \epsilon^{-1} \mathbb{P}^{1-\epsilon} \{B_\tau \in C_r\} = 1/r,$$



and the exact form of Poisson kernel in  $\mathbb{D}$  shows that

$$H_{\mathbb{D}}(z, 1) = \frac{1}{2} + O(e^{-r}), \quad z \in C_r.$$

To each Brownian bubble in  $\mathbb{D}$  rooted at 1 that intersects  $C_r$ , there is a corresponding path in  $\mathbb{H}$  that starts at 0, ends at  $2\pi k$  for some integer  $k$ , and does not stay in  $S_r$ . Only those paths that end at 0 are Brownian bubbles in  $\mathbb{H}$  rooted at 0. Therefore, to compute  $\Gamma_{\mathbb{H}}(1; S_r)$  we can subtract the measure of the other bubbles. To get the measures of the bubbles to be subtracted we consider the measure of excursions in  $\mathbb{H}$  minus the measure of excursions in  $S_r$ . We therefore get

$$\begin{aligned} \Gamma_{\mathbb{H}}(0; S_r) &= \Gamma_{\mathbb{D}}(1; A_r) - \sum_{k \in \mathbb{Z} \setminus \{0\}} [H_{\partial\mathbb{H}}(0, 2\pi k) - H_{\partial S_r}(0, 2\pi k)] \\ &= \Gamma_{\mathbb{D}}(1; A_r) - \delta(r). \end{aligned}$$

Using (5) and  $H_{\partial\mathbb{H}}(0, x) = x^{-2}$ , we see that

$$\begin{aligned} \delta(r) &= 2 \sum_{k=1}^{\infty} \left( \frac{1}{(2\pi k)^2} - \frac{\pi^2}{4r^2} \left[ \sinh\left(\frac{k\pi^2}{r}\right) \right]^{-2} \right) \\ &= \frac{1}{12} - \frac{\pi^2}{2r^2} \sum_{k=1}^{\infty} \left[ \sinh\left(\frac{k\pi^2}{r}\right) \right]^{-2}. \end{aligned}$$

□

**3.5. Brownian loop measure.** In order to describe  $SLE_{\kappa}$  in other domains, we introduce the Brownian loop measure as first introduced in [24].

**Definition** The *rooted Brownian loop measure* on  $\mathbb{C}$  is the measure on loops given by

$$(22) \quad \frac{1}{2\pi t^2} dt \times \text{area} \times \nu^{BB},$$

where  $\nu^{BB}$  denotes the probability measure induced by a Brownian bridge of time duration one at the origin.

To be more precise, a rooted loop is a continuous function  $\eta : [0, t_{\eta}] \rightarrow \mathbb{C}$  with  $\eta(0) = \eta(t_{\eta})$ . Such a loop can be described by a triple  $(t, z, \bar{\eta})$  where  $t > 0$  is the time duration,  $z = \eta(0)$  is the root, and  $\bar{\eta} : [0, 1] \rightarrow \mathbb{C}$  is a loop of time duration one starting at the origin. The rooted loop measure is obtained by choosing  $(t, z, \bar{\eta})$  according to the measure (22). If  $D \subset \mathbb{C}$ , the the rooted loop measure in  $D$  is the rooted loop measure in  $\mathbb{C}$  restricted to loops that lie in  $D$ .

**Definition** The rooted loop measure on a domain  $D$  induces a measure on *unrooted loops* which we denote by  $m_D$ . We consider this as a measure on unrooted loops modulo reparametrization. (However, the proof of conformal invariance requires considering the parametrized loops.)

For the purposes of this paper, we do not need to worry about the time parametrization of the loops. The fundamental fact that explains the importance of the loop measure is the following. We do this to emphasize that we do not need to assume that  $D$  is simply connected.

**Proposition 3.6** (Conformal invariance). *If  $f : D \rightarrow f(D)$  is a conformal transformation, then*

$$f \circ m_D = m_{f(D)}.$$

---

♣ We have stated the proposition for loops, modulo reparametrization. One can get a similar result for parametrized loops but then one must change the parametrization as in the conformal invariance of Brownian motion.

---

*Sketch of proof.* Let  $\rho(z, z; t)$  be the measure on paths associated to Brownian loops at  $z$  of time duration  $t$ . It is a measure of total mass  $p_t(z, z) = (2\pi t)^{-1}$  that can be defined using standard Brownian bridge techniques. Let

$$\rho(z, z) = \int_0^\infty \rho(z, z; t) dt,$$

which is an infinite measure. For any  $D$ , we define  $\rho_D(z, z; t), \rho_D(z, z)$  by restriction. If  $f : D \rightarrow f(D)$  is a conformal transformation, and  $\eta$  is a loop in  $D$ , we write  $f \circ \eta$  for the corresponding loop in  $f(D)$  obtained using Brownian scaling on the parametrization. In other words, if  $\eta$  has time duration  $t_\eta$ , then  $f \circ \eta$  has time duration

$$\int_0^{t_\eta} |f'(\eta(s))|^2 ds.$$

The measure  $\rho_D(z, z)$  induces a measure  $f \circ \rho_D(z, z)$  by considering  $f \circ \eta$ . Using the conformal invariance of Brownian motion, one can check that

$$(23) \quad f \circ \rho_D(z, z) = \rho_{f(D)}(f(z), f(z)).$$

Suppose  $h$  is a continuous, nonnegative function on  $D$ . Then  $h$  induces a measure on (rooted) loops by

$$\rho_{D,h} = \int_D \rho_D(z, z) h(z) dA(z),$$

where  $A$  denotes area. We can also consider this as a measure on unrooted loops by forgetting the root. We write  $\rho_D$  for  $\rho_{D,h}$  with  $h \equiv 1$ . Another way to define the Brownian loop measure  $\mu_D$  on unrooted loops is

$$\frac{d\mu_D}{d\rho_D}(\eta) = \frac{1}{t_\eta},$$

where  $t_\gamma$  denotes the time duration of  $\gamma$ . More generally,

$$\frac{d\mu_D}{d\rho_{D,h}}(\eta) = \left[ \int_0^{t_\eta} h(\eta(s)) ds \right]^{-1}.$$

Suppose  $h(z) = |f'(z)|^2$ . Then (23) implies that

$$\begin{aligned} f \circ \rho_{D,h} &= \int_D \rho_{f(D)}(f(z), f(z)) |f'(z)|^2 dA(z) \\ &= \int_{f(D)} \rho_{f(D)}(w, w) dA(w) = \rho_{f(D)}. \end{aligned}$$

Also,

$$\int_0^{t_\eta} h(\eta(s)) ds = \int_0^{t_\eta} |f'(\eta(s))|^2 ds = \frac{1}{t_{f \circ \eta}}.$$

□

By construction,  $m_D$  also satisfies the restriction property.

- If  $V_1, V_2$  are subsets, we write either  $m(V_1, V_2; D)$  or  $m_D(V_1, V_2)$  for the  $m_D$  measure of the set of loops in  $D$  that intersect both  $V_1$  and  $V_2$ .
- Suppose  $D \subset \mathbb{H}$  is a domain (not necessarily simply connected) with  $\text{dist}(0, \mathbb{H} \setminus D) > 0$ . Suppose  $\gamma$  satisfies (16) and  $t < T := \inf\{t : \text{dist}(\gamma(t), \mathbb{H} \setminus D) = 0\}$ . Then

$$(24) \quad m(\gamma_t, \mathbb{H} \setminus D; \mathbb{H}) = a \int_0^t \Gamma(U_s; g_s(D)) ds.$$

If  $D$  is simply connected, we can use (19) to write

$$(25) \quad m(\gamma_t, \mathbb{H} \setminus D; \mathbb{H}) = -\frac{a}{6} \int_0^t S f_s(U_s) ds,$$

where  $f_s$  is a conformal transformation of  $g_s(D)$  onto  $\mathbb{H}$  with  $f(U_s) \in \mathbb{R}$ .

♣ The only functionals of the Brownian loop measure that we will need are of the type on the left-hand side of (24). We might consider using the right-hand side of (24) as the *definition* of  $m(\gamma_t, \mathbb{H} \setminus D; \mathbb{H})$ . However, it is not so easy to see from this formulation to see that if  $\gamma$  is a curve in  $\mathbb{H}$  connecting boundary points  $0, x$ , then

$$m(\gamma_t, \mathbb{H} \setminus D; \mathbb{H}) = m(\gamma_t^R, \mathbb{H} \setminus D; \mathbb{H}),$$

where  $\gamma_t^R$  denotes the reversal of the path. This is immediate from the loop measure description of the quantity.

♣ The formula (24) comes from a Brownian bubble analysis of the Brownian loop measure. Suppose  $\gamma$  is a simple curve from  $0$  to  $\infty$  in  $\mathbb{H}$ . If  $l$  is a loop in  $\mathbb{H}$  that intersects  $\gamma$ , we can consider the first time (using the time scale of  $\gamma$ ) that the loop intersects  $\gamma$ . If  $l$  intersects  $\gamma$  first at time  $t$ , then  $l$  is a “boundary bubble” in  $\mathbb{H} \setminus \gamma_t$  rooted at  $\gamma(t)$ . We therefore can write the Brownian loop measure, restricted to loops intersecting  $\gamma$ , as an integral of the Brownian bubble measure in decreasing family of domains  $\mathbb{H} \setminus \gamma_t$ . We can think of  $\gamma$  as an “exploration process” for the Brownian loop measure. This idea is used in the construction of conformal loop ensembles by Sheffield and Werner [28]. This exploration idea is important in our analysis of  $SLE_\kappa$  in an annulus.

Although the Brownian loop measure is a measure on unrooted loops, it is often convenient to choose roots of the loops. For example, if  $\eta$  is an unrooted loop, we can choose the root to be the closest point to the origin, say  $e^{-r+i\theta}$ . (Except for a set of measure zero, this point will be unique). The rooted loop is then a Brownian bubble in the domain  $O_r := \mathbb{C} \setminus \overline{\mathbb{D}_r}$ . This is the basis for the following computation.

**Proposition 3.7.** *Suppose  $D \subset \mathbb{D}$  is a simply connected domain with  $\text{dist}(0, \partial D) > e^{-r}$ . Then,*

$$m(\overline{\mathbb{D}_r}, \mathbb{D} \setminus D; \mathbb{D}) = \frac{1}{\pi} \int_r^\infty \int_0^{2\pi} \Gamma_{O_s}(e^{-s+i\theta}; D) ds d\theta,$$

where  $O_s = \mathbb{D} \setminus \overline{\mathbb{D}_s}$ .

**Lemma 3.8.** *There exists  $c < \infty$  such that the following is true. Suppose  $D \subset \mathbb{D}$  is a simply connected domain containing the origin and  $g : D \rightarrow \mathbb{D}$  is the conformal transformation with  $g(0) = 0, g'(0) > 0$ . Suppose that  $r \geq \log g'(0) + 2$ . Let  $\phi : g(A_r \cap D) \rightarrow A_s$  be a conformal transformation sending  $C_0$  to  $C_0$  and let  $h = \phi \circ g$  which maps  $A_r \cap D$  onto  $A_s$ . Then if  $u = r - \log g'(0)$ ,  $z \in C_r, w \in C_0$ ,*

$$|s - u| \leq c e^{-u}, \quad |\phi'(w) - 1| \leq c e^{-u}, \quad |h'(z) - g'(0)| \leq c g'(0) e^{-u},$$

$$|m_{\mathbb{D}}(\overline{\mathbb{D}}_r, \mathbb{D} \setminus D) - \log(r/u)| \leq c e^{-u}.$$

*Proof.* The Koebe-1/4 theorem applied to  $g^{-1}$  shows that  $\text{dist}(0, \partial D) \geq [4g'(0)]^{-1}$ . Applying the distortion theorem to  $g$  restricted to  $\mathbb{D}_{u+\frac{3}{2}}$ , we see that there exists  $c < \infty$  such that if  $|w| \leq e^{-r}$ ,

$$|g(w) - g'(0)w| \leq c e^{-2u},$$

$$|g'(w) - g'(0)| \leq c e^{-u}.$$

In particular, if  $|w| = e^{-r}$ , then

$$(26) \quad |g(w)| = e^{-u} [1 + O(e^{-u})].$$

Using this and monotonicity, we see that

$$(27) \quad s = u + O(e^{-u}).$$

Let  $U$  denote the conformal annulus  $g(A_r \cap D)$  so that  $\phi$  maps  $U$  onto the annulus  $A_s$ . By conformal invariance we see that  $\log |g(z)|/s$  equals the probability that a Brownian motion starting at  $z$  exits  $U$  at  $g(C_r)$ . However, we know that the inner boundary of  $U$  lies within distance  $O(e^{-2u})$  of  $C_u$ . If the Brownian motion gets that close to  $C_u$ , the probability that it does not exit at  $C_u$  is  $O(e^{-u}/u)$ . Therefore,

$$\frac{\log |g(z)|}{s} = \frac{\log |z|}{u} [1 + O(e^{-u}/u)].$$

Hence, from (27), we get

$$\log |g(z)| = \log |z| [1 + O(e^{-u})],$$

which implies  $|g'(e^{i\theta})| = 1 + O(e^{-u})$ . The argument to show that  $|h'(e^{-r+i\theta})| = g'(0)[1 + O(e^{-u})]$  is similar.

By conformal invariance and symmetry,

$$\mathcal{E}_{A_r \cap D}(C_r, \partial D) = \mathcal{E}_{A_s}(C_s, C_0) = \mathcal{E}_{A_s}(C_0, C_s) = 2\pi s^{-1}.$$

Similarly, if

$$\hat{v}(z) = \mathbb{P}^z\{B_{\hat{\sigma}} \in C_0\} = 1 - \frac{\log |z|}{r},$$

where  $\hat{\sigma} = \inf\{t : B_t \notin A_r\}$ , then

$$2\pi r^{-1} = \mathcal{E}_{A_r}(C_r, C_0) = \int_{C_r} \partial_n \hat{v}(z) |dz|.$$

By the strong Markov property, we can write

$$\mathcal{E}_{A_r}(C_r, C_0) = \int_{\mathbb{D} \cap \partial D} \left[1 - \frac{\log |z|}{r}\right] d\mathcal{E}_{A_r}(C_r, dz).$$

The term  $1 - \frac{\log|z|}{r}$  is the probability that a Brownian motion starting at  $z$  exits  $A_r$  at  $C_0$ . Therefore, using (27),

$$\int_{\mathbb{D} \cap \partial D} \frac{\log|z|}{r} d\mathcal{E}_{A_r}(C_0, dz) = 2\pi[s^{-1} - r^{-1}] = 2\pi[u^{-1} - r^{-1} + O(e^{-u}/u^2)].$$

Lemma 3.1 implies that if  $V \subset \partial D$  and  $z, w \in C_r$ ,

$$\mathcal{E}_{A_r}(z, V) = \mathcal{E}_{A_r}(w, V) [1 + O(e^{-u})],$$

and hence

$$\mathcal{E}_{A_r}(z, V) = \frac{1}{2\pi} \mathcal{E}_{A_r}(C_0, V) [1 + O(e^{-u})].$$

Lemma 3.1 can also be used to see that if  $w \in \mathbb{D} \cap \partial D, z \in C_r$ ,

$$H_{A_r}(w, C_r) = \frac{1}{2} \frac{\log|w|}{r} [1 + O(e^{-u})].$$

Therefore, using (27),

$$\Gamma_{A_r}(z, A_r \cap D) = \int_{\mathbb{D} \cap \partial D} H_{A_r}(w, z) d\mathcal{E}_{A_r \cap D}(z, w) = \frac{u^{-1} - r^{-1}}{2} [1 + O(e^{-u})].$$

From Proposition 3.7 we know that the quantity we are interested in can be written as

$$\frac{1}{\pi} \int_0^{2\pi} \int_r^\infty \Gamma_{A_t}(e^{-t+i\theta}; A_t \cap D) dt d\theta = \int_0^\infty \left[ \frac{1}{u+t} - \frac{1}{r+t} \right] [1 + O(e^{-u-t})] dt.$$

By computing the integral we see that this quantity equals

$$\log(r/u) + O(e^{-u}).$$

□

We will need to consider the Brownian loop measure in an annulus. If we fix the origin as a marked point, we can divide loops into two sets: those with nonzero winding number around zero and those with zero winding number. If  $A$  is a conformal annulus such that 0 and  $\infty$  lie in different components of  $A^c$ , then the measure of the set of loops in  $A$  with nonzero winding number is finite. It is a conformal invariant which we calculate in the next proposition.

**Proposition 3.9.** *Let  $m^*(r)$  denote the Brownian loop measure of loops in  $A_r$  that have nonzero winding number. Then*

$$m^*(r) = \frac{r}{6} - 2 \int_0^r \delta(s) ds,$$

where  $\delta(s)$  is defined as in (21). In particular, there exists  $C > 0$  such that as  $r \rightarrow \infty$ ,

$$(28) \quad e^{m^*(r)} = C r^{-1} e^{r/6} [1 + O(r^{-1})].$$

*Proof.* By focusing on the point of the loop of largest radius (see the appendix of [18]), we can give the expression

$$m^*(r) = 2\pi \int_0^r \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{\pi} H_{S_s}(0, 2\pi k) = \frac{1}{6} - 2\delta(r).$$

Proposition 3.5 implies that there exists  $c$  such that

$$m^*(r) = \frac{r}{6} - \log r + c + O(r^{-1}), \quad r \rightarrow \infty,$$

from which (28) follows with  $C = e^c$ . □

**Corollary 3.10.**

- Suppose  $D \subset \mathbb{D}$  is a simply connected domain containing the origin and suppose that  $\text{dist}(0, \partial D) > e^{-r}$ . Let  $0 \leq s < r$  be defined by saying that the annulus  $D \setminus \overline{\mathbb{D}}_r$  is conformally equivalent to  $A_s$ . Then the Brownian loop measure of loops in  $A_r$  of nonzero winding number that intersect  $\mathbb{D} \setminus D$  is  $m^*(r) - m^*(s)$ .
- Under the same assumptions, the Brownian loop measure of loops in  $\mathbb{D}$  of nonzero winding number that intersect  $\mathbb{D} \setminus D$  is  $\log g'(0)/6$  where  $g : D \rightarrow \mathbb{D}$  is the conformal transformation with  $g(0) = 0, g'(0) > 0$ .

*Proof.* The first assertion follows immediately and the second is obtained by considering comparing  $\mathbb{D} \setminus A_r$  and  $D \setminus A_r$  as  $r \rightarrow \infty$ . □

We will use the following estimate in the discussion in the next section but it will not figure in our main results. See [15] for a proof.

**Proposition 3.11.** *Let  $k(r)$  denote the  $m_{\mathbb{D}_{-r}}$  measure of loops that intersect both  $A_{-r} \setminus A_{-r+1}$  and  $\mathbb{D}$ . Let  $k'(r)$  be the measure of such loops that do not separate the origin from  $C_0$ . Then as  $r \rightarrow \infty$ ,*

$$k(r) = r^{-1} + O(r^{-2}), \quad k'(r) = O(r^{-2}).$$

*In particular, if  $V_1, V_2$  are disjoint compact sets, then there exists  $\Lambda(V_1, V_2)$  such that as  $r \rightarrow \infty$ ,*

$$m_{\mathbb{D}_{-r}}(V_1, V_2) = \log r - \Lambda(V_1, V_2) + o(1).$$

**3.6. Chordal  $SLE_\kappa$  in simply connected domains.** We will review two equivalent ways to construct  $SLE_\kappa$  in simply connected domains for  $\kappa = 2/a \leq 4$ . See [14, 24, 17, 19] for more details. Suppose  $D$  is a simply connected subdomain of  $\mathbb{H}$  with  $\text{dist}(0, \mathbb{H} \setminus D) > 0$ . Let  $w$  be a nonzero  $\partial D$ -analytic point; we allow  $w = \infty$  as a possibility. Let  $\Phi : D \rightarrow \mathbb{H}$  be the unique conformal transformation with  $\Phi(0) = 0, \Phi(w) = \infty, |\Phi'(w)| = 1$ . Here we are using the conventions about derivatives as discussed in Section 3.1. The most important example for this paper is  $D = S_r$  and  $w = x + ir$  for some  $x \in \mathbb{R}$ .

Let  $g_t$  be the solution of the Loewner equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z,$$

where  $U_t = -B_t$  is a standard Brownian motion defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Then the corresponding curve  $\gamma$  is  $SLE_\kappa$  in  $\mathbb{H}$  from 0 to  $\infty$  which with  $\mathbb{P}$ -probability one is a simple curve with  $\gamma(0, \infty) \subset \mathbb{H}$ .

Let

$$T = T_D = \inf\{t > 0 : \gamma(t) \notin D\}.$$

For  $t < T$ , let

$$w_t = g_t(w), \quad \gamma_t^* = \Phi \circ \gamma_t,$$

and let  $\hat{g}_t$  be the unique conformal transformation of  $\mathbb{H} \setminus \gamma_t^*$  onto  $\mathbb{H}$  with  $\hat{g}_t(z) = z + o(1)$  as  $z \rightarrow \infty$ . Let

$$\Phi_t = \hat{g}_t \circ \Phi \circ g_t^{-1}.$$

Then  $\hat{g}_t$  satisfies the Loewner equation

$$\partial_t \hat{g}_t(z) = \frac{a \Phi'_t(U_t)^2}{\hat{g}_t(z) - \hat{U}_t}, \quad \hat{g}_0(z) = z,$$

where  $\hat{U}_t = \hat{g}_t(\gamma^*(t)) = \Phi_t(U_t)$ . Then  $\Phi_t$  is the unique conformal transformation of  $g_t(D \setminus \gamma_t)$  onto  $\mathbb{H}$  with  $\Phi_t(U_t) = \hat{U}_t$ ,  $\Phi_t(w_t) = \infty$ ,  $|\Phi'_t(w_t)| = |g'_t(w)|^{-1}$ . Moreover, using only the Loewner equation, one can show that

$$(29) \quad \dot{\Phi}_t(U_t) = -\frac{3b}{2} \Phi_t''(U_t), \quad \dot{\Phi}'_t(U_t) = \frac{a \Phi_t''(U_t)^2}{4 \Phi_t'(U_t)} - \frac{2a \Phi_t'''(U_t)}{3}$$

where  $\dot{\Phi}_t(U_t)$ ,  $\dot{\Phi}'_t(U_t)$  denote  $\partial_t \Phi_t(x)$ ,  $\partial_t \Phi'_t(x)$  evaluated at  $x = U_t$ .

Let

$$(30) \quad H_t = H_{\partial g_t(D \setminus \gamma_t)}(x, w_t), \quad K_t = |g'_t(w)|^b H_t(U_t)^b = \Phi'_t(U_t)^b.$$

The second equality for  $K_t$  follows from the scaling rule for the Poisson kernel. A straightforward Itô's formula calculation using (29) shows that

$$dK_t = K_t \left[ \frac{ac}{12} S \Phi_t(U_t) dt + \frac{b H'_t(U_t)}{H_t(U_t)} dU_t \right],$$

where  $S$  denotes the Schwarzian derivative. Let

$$\begin{aligned} M_t &= \exp \left\{ -\frac{ac}{12} \int_0^t S \Phi_s(U_s) ds \right\} K_t \\ &= \exp \left\{ \frac{c}{2} m_{\mathbb{H}}(\gamma_t, \mathbb{H} \setminus D) \right\} |g'_t(w)|^b H_t(U_t)^b. \end{aligned}$$

(To check the second equality, recall that we have parametrized so that  $\text{hcap}(\gamma_t) = at$ .) Then  $M_t$  is a local martingale satisfying

$$dM_t = \frac{b H'_t(U_t)}{H_t(U_t)} M_t dU_t = \frac{b \Phi'_t(U_t)}{\Phi'_t(U_t)} M_t dU_t.$$

We can use Girsanov theorem to define a new probability measure  $\mathbb{P}^*$  obtained by weighting by the local martingale  $M_t$ . (The Girsanov theorem is stated for nonnegative martingales; since we only have a local martingale, we need to use stopping times. However, as long as  $t < T$ , there is no problem.) The Girsanov theorem states that

$$(31) \quad dU_t = \frac{b H'_t(U_t)}{H_t(U_t)} dt + dW_t, \quad t < T,$$

where  $W_t$  is a standard Brownian motion with respect to  $\mathbb{P}^*$ .

Another application of Itô's formula using (29) shows that if  $U_t$  satisfies (31), then  $\hat{U}_t = \Phi_t(U_t)$  satisfies

$$d\hat{U}_t = \Phi'_t(U_t) dW_t.$$

The upshot is that, *with respect to the measure  $\mathbb{P}^*$* ,  $\eta_t$  has the distribution of (a time change of)  $SLE_\kappa$  from 0 to  $\infty$  in  $\mathbb{H}$ . Since  $\gamma_t = \Phi^{-1} \circ \eta_t$ , this implies that with respect to  $\mathbb{P}^*$ ,  $\gamma_t$  has the distribution of  $SLE_\kappa$  from 0 to  $w$  in  $D$ . The Girsanov transformation (31) is sufficient for understanding the probability measure  $\mu_D^\#(0, w)$ . Note that it is determined by the logarithmic derivative of  $H_t$ ; the “compensator” terms do not need to be computed.

The example of importance in this paper is  $D = S_r$  and  $w = x + ir$ . It will suffice for us to consider the probability measure  $\mu_{S_r}^\#(0, w)$ . The drift term in (31)

is somewhat complicated to write down; however, at time  $t = 0$ , we can use (6) to see that it equals  $b\mathbf{L}(r, x)$  where

$$(32) \quad \mathbf{L}(r, x) = \frac{H'_{\partial S_r}(0, x + ir)}{H_{\partial S_r}(0, x + ir)} = \frac{\pi}{r} \tanh\left(\frac{\pi x}{2r}\right),$$

where the prime denotes derivative in the *first* component. This measure is the same (modulo time change) as the conformal image of  $SLE_\kappa$  from 0 to  $\infty$  in  $\mathbb{H}$ ; in particular, with probability one, the path leaves  $S_r$  at  $w$ .

In analyzing annulus  $SLE_\kappa$  we will be studying measures that will turn out to be absolutely continuous with respect to  $\mu_{S_r}^\#(0, x + ir)$ . To review the issues that we need to address, let us recall the case of  $SLE_\kappa$  from 0 to  $\infty$  in a simply connected domain  $D$  with  $\mathbb{H} \setminus D$  bounded and  $\text{dist}(0, \mathbb{H} \setminus D) > 0$ . In this case, when we weight by the appropriate local martingale  $M_t$ , then with  $\mathbb{P}^*$ -probability one,  $T = \infty$  and  $\gamma(t) \rightarrow \infty$ . If  $T = \infty$  and  $\gamma(t) \rightarrow \infty$ , then a deterministic estimate gives

$$M_\infty = \exp\left\{\frac{\mathbf{c}}{2} m_{\mathbb{H}}(\gamma, \mathbb{H} \setminus \gamma)\right\} 1\{\gamma \subset D\},$$

and since this happens with  $\mathbb{P}^*$ -probability one,

$$(33) \quad \mathbb{E}[M_\infty] = M_0 = \Phi'(0)^b.$$

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♣ The argument we will use for the annulus is similar to the proof for simply connected domains, so it is worth reviewing the main steps. Suppose  $D$  is a simply connected domain with  $\mathbb{H} \setminus D$  bounded and  $w = \infty$ . Here we were able to guess the exact form for the partition function for  $\mu_D(0, \infty)$ ,  $\Phi'_D(0)^b$ . Direct Itô's formula calculation shows that  $M_t$  as above gives a local martingale. However, to justify (33), we need that fact that the curve *weighted by the local martingale* goes to infinity without leaving the domain. This gives the necessary "uniform integrability".

In the annulus case, we will consider two measures on curves from 0 to  $w = x + ir$  in  $S_r$ . We will use the Feynman-Kac theorem applied to a slightly different process to give a candidate for the partition function. Although we will not have an explicit form of it, we will know that it satisfies a certain PDE and hence gives us a local martingale. Having a local martingale is not sufficient; we will also need to show that the process weighted by the local martingale leaves the domain at  $w$ . This will give the analogue of (33). The argument for the annulus, as well as the argument here, will require  $\kappa \leq 4$ .

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**3.7. Shrinking domains.** We will need a generalization of this where the domain  $D$  is replaced with a decreasing family of domains  $\{D_t : t > 0\}$ . Although what we describe can be done more generally, we will restrict to the case that we need in this paper. This will lead to a process that we call *locally chordal  $SLE_\kappa$  in an annulus*.

Let  $D = S_r$  and  $w \in \partial S_r \setminus \{0\}$ . (The case  $S_\infty = \mathbb{H}$ ,  $w = \infty$  corresponds to radial  $SLE$  and is discussed in the next subsection.) Let

$$(34) \quad \tilde{\gamma}_t = \bigcup_{k \in \mathbb{Z} \setminus \{0\}} (\gamma_t + 2\pi k), \quad D_t = D \setminus \tilde{\gamma}_t.$$

and

$$\hat{D}_t = D \setminus (\tilde{\gamma}_t \cup \gamma_t) = \psi^{-1}[\mathbb{D} \setminus \eta_t],$$



where  $\eta_t = \psi \circ \gamma_t$ . In other words, when we slit the domain  $D = S_r$  by  $\gamma_t$  we also add slits at the  $2\pi k$  translates of  $\gamma_t$ .

Let  $T$  denote the first  $t > 0$  such that either  $\gamma(t) \in \partial S_r$  or  $\eta_t$  disconnects the origin from the unit circle,

$$T = \inf\{t > 0 : \gamma_t \not\subset D_t\}.$$

Let  $\tilde{D}_t = g_t(\hat{D}_t)$ , and, as before,  $U_t = g_t(\gamma(t))$ . We want to study the process that evolves at time  $t$  like chordal  $SLE_\kappa$  from  $\gamma(t)$  to  $w$  in the domain  $D_t$ . Equivalently, the process after conformal transformation by  $g_t$  evolves like chordal  $SLE_\kappa$  from  $U_t$  to  $w_t = g_t(w)$  in  $\tilde{D}_t$ . The latter process can be defined in two equivalent ways. Let  $H_t(x) = H_{\partial g_t(D \setminus \gamma_t)}(x, w_t)$  as in the previous section and let

$$\tilde{H}_t(x) = H_{\partial \tilde{D}_t}(x, w_t), \quad Q_t(x) = \frac{\tilde{H}_t(x)}{H_t(x)}.$$

The process can be considered as either of the following.

- $SLE_\kappa$  in  $\mathbb{H}$  from 0 to  $\infty$  weighted by  $\tilde{H}_t(U_t)^b$ .
- $SLE_\kappa$  in  $S_r$  from 0 to  $w$  weighted by  $Q_t(U_t)^b$ .

♣ If  $J_t$  is a positive process, then “weighting by  $J_t$ ” is in the sense of the Girsanov theorem. If  $J_t$  satisfies

$$dJ_t = J_t [R_t dt + A_t dU_t].$$

then

$$N_t := \exp \left\{ - \int_0^t R_s ds \right\} J_t,$$

is a local martingale satisfying

$$dN_t = A_t N_t dU_t.$$

When we use the Girsanov theorem (using stopping times so that the local martingale is a martingale), then

$$dU_t = A_t dt + dW_t,$$

where  $W_t$  is a Brownian motion in the new measure.

Let

$$(35) \quad \Delta_t = \frac{\dot{Q}_t(U_t)}{Q_t(U_t)},$$

where  $\dot{Q}_t(U_t)$  denotes  $\partial_t Q_t(x)$  evaluated at  $x = U_t$ . Our assumptions allow us to conclude that  $\Delta_t$  is well defined and continuous for  $t < T$ .

As in (30), we define

$$K_t = |g'_t(w)|^b \tilde{H}_t(U_t)^b = |g'_t(w)|^b H_t(U_t)^b Q_t(U_t)^b.$$

Using the previous calculation and the chain rule, we see that  $K_t$  satisfies

$$dK_t = K_t \left[ \left( -b\Delta_t + \frac{a\mathbf{c}}{12} S\Phi_t(U_t) \right) dt + \frac{b\tilde{H}'_t(U_t)}{\tilde{H}_t(U_t)} dU_t \right].$$

If

$$C_t = \exp \left\{ \int_0^t \Delta_s ds \right\},$$

$$M_t = C_t^b \exp \left\{ -\frac{a\mathbf{c}}{12} \int_0^t S\Phi_s(U_s) ds \right\} K_t,$$

then  $M_t$  is a local martingale satisfying

$$dM_t = \frac{b \tilde{H}'_t(U_t)}{\tilde{H}_t(U_t)} M_t dU_t.$$

The term

$$-\frac{a}{6} \int_0^t S\Phi_s(U_s) ds$$

can be interpreted in terms of Brownian loops, but we need to be careful. At time  $s$ ,  $-S\Phi_s(U_s)/6$  represents the measure of Brownian bubbles in  $\mathbb{H}$  rooted at  $U_s$  that intersect  $g_s(D_s)$ . For every Brownian loop  $l$ , let  $s(l)$  be the smallest  $s$  such that  $s(l) \cap \gamma_s \neq \emptyset$ . Then

$$-\frac{a}{6} \int_0^t S\Phi_s(U_s) ds = \tilde{m}_t,$$

where  $\tilde{m}_t = \log \tilde{\Lambda}_t$  is the Brownian loop measure of  $l$  in  $D$  with  $s(l) \leq t$  and  $l \cap D \setminus D_{s(l)} \neq \emptyset$ . Then the local martingale is

$$M_t = C_t^b \tilde{\Lambda}_t^{c/2} H_t(U_t)^b Q_t^b = C_t^b \tilde{\Lambda}_t^{c/2} \tilde{H}_t(U_t)^b.$$

Note that the only term in  $M_t$  that has nontrivial quadratic variation is  $\tilde{H}_t(U_t)^b$ . Therefore, when we weight by the local martingale, the process looks locally like  $SLE_\kappa$  from  $\gamma(t)$  to  $w$  in  $D_t$ . We call it *locally chordal*  $SLE_\kappa$  (we have defined it only for  $\kappa \leq 4$ .) This gives a probability measure on paths starting at 0 in  $S_r$ . We will use  $\kappa \leq 4$  to show that with probability one the paths leave  $S_r$  at  $w$ . We can also view the paths as living in the annulus  $A_r$  and going from 1 to  $e^{-r+ix}$  with a known total winding number. In Section 3.9 we will use an annulus reparametrization of the curve.

**3.8. Radial  $SLE_\kappa$  raised to  $\mathbb{H}$ .** Suppose  $D$  is a simply connected domain,  $z \in \partial D$ ,  $w \in D$ , and  $\partial D$  is locally analytic at  $z$ . Radial  $SLE_\kappa$  in  $D$  from  $z$  to  $w$  is a measure on paths

$$\mu_D(z, w) = \Psi_D(z, w) \mu_D^\#(z, w),$$

that satisfies the conformal covariance rule

$$f \circ \mu_D(z, w) = |f'(z)|^b |f'(w)|^{\tilde{b}} \mu_{f(D)}^\#(f(z), f(w)).$$

The conformal covariance rule determine the total mass up to a multiplicative constant and for convenience we choose the constant so that  $\Psi_{\mathbb{D}}(1, 0) = 1$ .

To obtain the probability measure  $\mu_D^\#(0, w)$  where  $w \in \mathbb{H}$ , we weight chordal  $SLE_\kappa$  by a particular local martingale. Let  $g_t$  be the conformal maps for chordal  $SLE_\kappa$  from 0 to  $\infty$ , and let  $w \in \mathbb{H}$ . Let  $Z_t = g_t(w) - U_t$  and

$$M_t = |g'_t(w)|^{\tilde{b}} H_{\mathbb{H}}(Z_t, U_t)^b,$$

where  $b, \tilde{b}$  are the boundary and interior scaling exponents, respectively, as in (1). Then  $M_t$  is a local martingale and the measure on the paths obtained by weighting by this local martingale is that of radial  $SLE_\kappa$ . In the weighted measure, the path stops at finite (half plane capacity) time  $T_w$  at which  $\gamma(T_w) = w$ . This determines the probability measure  $\mu_{\mathbb{H}}^\#(0, w)$  and conformal invariance determines the measure for all simply connected  $D$ . Although this is not the same definition as originally given by Schramm [25], the Girsanov theorem shows that it is equivalent.

One can also understand the relationship between radial and chordal  $SLE_\kappa$  using the Brownian loop measure. Suppose that  $\gamma_t$  is a simple curve in  $\mathbb{H}$  starting at the

origin and let  $\eta_t = \psi \circ \gamma_t$ . We will assume that  $t$  is small so that  $\eta_t$  is also simple. Let

$$\tilde{h}_t : \mathbb{D} \setminus \eta_t \rightarrow \mathbb{D}$$

be the conformal transformation with  $\tilde{h}'_t(0) > 0$  and suppose the curve has been parametrized so that  $h'_t(0) = e^t$ . Let  $g_t : \mathbb{H} \setminus \gamma_t$  be the usual conformal transformation with driving function  $U_t$ ; one can show that

$$\partial_t \text{hcap}[\gamma_t] \big|_{t=0} = 2,$$

which is why this is a standard choice of parametrization for chordal *SLE*. Let  $\tilde{\gamma}_t, \hat{\gamma}_t$  be as in the previous subsection and let  $h_t$  be a conformal transformation  $h_t : \mathbb{H} \setminus \hat{\gamma}_t \rightarrow \mathbb{H}$  such that  $\psi(h_t(z)) = \tilde{h}_t(\psi(z))$ . This transformation is determined uniquely by requiring that

$$h_t(iy) = i[y - t] + o(1), \quad y \rightarrow \infty.$$

We define  $\phi_t$  by

$$h_t = \phi_t \circ g_t.$$

Let  $\mu_1, \mu_2$  denote  $\mu_{\mathbb{H}}(0, \infty)$  and  $\mu_{\mathbb{D}}(1, 0)$ . The latter measure can be viewed as a measure on curves  $\gamma_t$  by pulling back by  $\psi$ . (Note that  $|\psi'(0)| = 1$  so the derivative factor in the scaling rule equals one.) We view these measures on the initial segment  $\gamma_t$ . The measure  $\mu_2$  is supported on curves such that  $\gamma_t \cap \tilde{\gamma}_t \neq \emptyset$ . Note that  $\mu_2 \ll \mu_1$ , and let  $Y_t(\gamma_t)$  denote the Radon-Nikodym derivative so that  $d\mu_2 = Y d\mu_1$ . Let  $\Psi^*$  denote the partition function for the raised radial *SLE*; in particular,  $\Psi_{\mathbb{H}}^*(0, \infty) = 1$ .

Although the loop measure is conformally invariant, we must be careful here because  $\psi : \mathbb{H} \rightarrow \mathbb{D}$  is not one-to-one. Indeed, each loop  $l'$  in  $\mathbb{D}$  has an infinite number of preimages in  $\mathbb{H}$ . If  $l'$  is a loop in  $\mathbb{D}$  that intersects  $\eta_t$ , we can specify a unique preimage by considering the smallest  $s$  such that  $\eta_s \in l'$  and then rooting  $l'$  at  $\eta_s$ . We associate to  $l'$  the corresponding loop  $l$  in  $\mathbb{H}$  rooted at  $\gamma_s$ .

Also, the loops of nonzero winding number in  $\mathbb{D}$  have preimages that are not loops in  $\mathbb{H}$ . Since the paths have been parametrized so that  $\tilde{h}'(0) = e^t$ , Corollary 3.10 implies that the measure of such loops is deterministic and equal to  $t/6$ . Using this idea, we get the formal expression

$$Y(\gamma_t) = C_t \exp \left\{ \frac{c}{2} [\hat{m}(\gamma_t) - (t/6)] \right\} \frac{\Psi_{\mathbb{H} \setminus \hat{\gamma}_t}^*(\gamma(t), 0)}{\Psi_{\mathbb{H} \setminus \gamma_t}(\gamma(t), 0)}.$$

Here  $C_t$  is a normalization to make this a probability measure and  $\hat{m}(\gamma_t)$  denotes the measure of loops  $l$  in  $\mathbb{H}$  that intersect  $\gamma_t$  with the following property.

- Let  $s$  be the smallest time with  $\gamma_t \in l$ . Then

$$l \cap \tilde{\gamma}_s \neq \emptyset.$$

In other words, the loop hits a translate of  $\gamma_t$  before it hits  $\gamma_t$  where time is measured on the curve  $\gamma_t$ .

The ratio of partition functions is only formal but we can make sense of it by writing

$$\frac{\Psi_{\mathbb{H} \setminus \hat{\gamma}_t}^*(\gamma(t), \infty)}{\Psi_{\mathbb{H} \setminus \gamma_t}(\gamma(t), \infty)} = \frac{\Psi_{\mathbb{H} \setminus \hat{\gamma}_t}^*(\gamma(t), \infty)}{\Psi_{\mathbb{H} \setminus \hat{\gamma}_t}(\gamma(t), \infty)} \frac{\Psi_{\mathbb{H} \setminus \hat{\gamma}_t}(\gamma(t), \infty)}{\Psi_{\mathbb{H} \setminus \gamma_t}(\gamma(t), \infty)}.$$

The first term on the right equals one since, formally,

$$\frac{\Psi_{\mathbb{H} \setminus \hat{\gamma}_t}^*(\gamma(t), \infty)}{\Psi_{\mathbb{H} \setminus \hat{\gamma}_t}(\gamma(t), \infty)} = \frac{|h'_t(\gamma(t))|^b \Psi_{\mathbb{H}}^*(h_t(\gamma(t)), \infty)}{|h'_t(\gamma(t))|^b \Psi_{\mathbb{H}}(h_t(\gamma(t)), \infty)} = 1.$$

For the second term, we use the formal computation

$$\frac{\Psi_{\mathbb{H} \setminus \hat{\gamma}_t}(\gamma(t), \infty)}{\Psi_{\mathbb{H} \setminus \gamma_t}(\gamma(t), \infty)} = \frac{|g'_t(\gamma(t))|^b \Psi_{g_t(\mathbb{H} \setminus \hat{\gamma}_t)}(g_t(\gamma(t)), \infty)}{|g'_t(\gamma(t))|^b \Psi_{\mathbb{H}}(U_t, \infty)} = \Psi_{g_t(\mathbb{H} \setminus \hat{\gamma}_t)}(U_t, \infty),$$

and conformal covariance,

$$\Psi_{g_t(\mathbb{H} \setminus \hat{\gamma}_t)}(U_t, \infty) = \phi'_t(U_t)^b.$$

Therefore,

$$Y_t(\gamma_t) = C_t e^{-ct/12} \exp \left\{ \frac{\mathbf{c}}{2} \hat{m}(\gamma_t) \right\} \phi'_t(U_t)^b.$$

This is a local martingale (and a martingale for  $\kappa \leq 4$ ) for chordal  $SLE_\kappa$  and when we weight by the martingale we get locally chordal  $SLE_\kappa$  from  $\gamma(t)$  to  $\infty$  in  $\mathbb{H} \setminus \hat{\gamma}_t$ . Although we are considering chordal  $SLE_\kappa$ , we are using the radial parametrization. This is the same as radial  $SLE_\kappa$  viewed on the covering space  $\mathbb{H}$ . It remains to find the normalization factor  $C_t$ . Since the weighted measure locally looks like chordal  $SLE_\kappa$  in the infinitely slit domain and hence after mapping by  $h_t$  looks like chordal  $SLE_\kappa$ , we get that  $C_t = e^{\tilde{b}t}$  for some  $\tilde{b}$ . To find the exponent we need only differentiate at 0. The measure of loops that hit both  $\gamma_t$  and a translate of  $\gamma_t$  is of order  $t^2$  and hence

$$\partial_t \hat{m}(\gamma_t) \big|_{t=0} = 0.$$

We claim that

$$(36) \quad \partial_t \phi'_t(U_t) \big|_{t=0} = -\frac{1}{6},$$

and hence

$$\tilde{b} = \frac{\mathbf{c}}{12} + \frac{b}{6}.$$

Let us sketch the proof of (36). We write “small error” for errors that are  $o(t)$  as  $t \downarrow 0$ . The quantity  $\phi'_t(U_t)$  is the probability that a Brownian excursion in  $\mathbb{H} \setminus \hat{\gamma}_t$  from  $\gamma(t)$  to  $\infty$  does not hit  $\tilde{\gamma}_t$ . Up to small error, it is the probability that an excursion in  $\mathbb{H}$  from 0 to  $\infty$  does not hit  $\tilde{\gamma}_t$ . The set  $\tilde{\gamma}_t$  is a union of curves of half-plane capacity  $2t$  rooted at the points  $2\pi k$ ,  $k \in \mathbb{Z} \setminus \{0\}$ . The probability that an excursion hits the translate  $\gamma_t + 2\pi k$  is exactly

$$\partial_y q(iy)$$

where  $q(z) = \mathbb{E}^z[\text{Im}[B_\tau]]$ ,  $B$  is a standard Brownian motion and  $\tau$  is the first time that it leaves  $\mathbb{H} \setminus [\gamma_t + 2\pi k]$ . As  $t \downarrow 0$ , up to small error this equals

$$\frac{1}{(2\pi k)^2} \text{hcap}[\gamma_t] = \frac{t}{2\pi^2}.$$

The probability of hitting more than one translate is  $O(t^2)$ , and hence, up to small error, the probability that the excursion hits  $\tilde{\gamma}_t$  is

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{t}{2\pi^2} = \frac{t}{6}.$$

♣ In the last computation we use the fact that for a small curve rooted at  $x \in \mathbb{R}$ , the expected value of  $\text{Im}(B_\tau)$  is given by the half-plane capacity times a multiplicative constant of the Poisson kernel. In order to keep track of constants (perhaps made especially confusing by our definition of  $H$ ), it is useful to remember that for large  $y$  if  $D = \mathbb{H} \setminus \overline{\mathbb{D}}$ ,

$$\mathbb{E}^{iy}[\text{Im}(B_\tau)] \sim \frac{1}{y} = H_{\mathbb{H}}(y, 0).$$

Hence, we get the general relation,

$$\mathbb{E}^z[\text{Im}(B_\tau)] \sim H_{\mathbb{H}}(z, x) \text{hcap}[\gamma_t].$$

The estimate (13) is done similarly. In this case, the probability that an excursion from 0 to  $x + ir$  in  $S_r$  hits the translate  $\gamma_t + 2\pi k$  is exactly,  $\partial_y q(y)$  where

$$q(z) = \frac{\mathbb{E}^z[H_{S_r}(B_\tau, x + ir)]}{H_{\partial S_r}(0, x + ir)}.$$

Here  $\tau$  is the first time that the Brownian motion leaves  $S_r \setminus [\gamma_t + 2\pi k]$ . Up to small error, if  $B_\tau \notin \partial S_r$ ,

$$H_{S_r}(B_\tau, x + ir) = \text{Im}[B_\tau] H_{S_r}(2\pi k, x + ir).$$

Also, as  $y \downarrow 0$ ,

$$\partial_y \mathbb{E}^{iy}[\text{Im}(B_\tau)]|_{y=0} = \text{hcap}[\gamma_t] H_{S_r}(0, 2\pi k) [1 + o(1)].$$

**3.9. Annulus Loewner equation.** We will need to consider the annulus Loewner equation which is similar to the chordal equation (16). We will need to define the annulus equation in the covering space  $S_r$ . We start with some definitions. Assume  $U : [0, \infty) \rightarrow \mathbb{R}$  is continuous with  $U_0 = 0$  and such that the chordal equation (16) produces a simple curve. Recall that  $\psi(z) = e^{iz}$ ,  $\tau_r = \inf\{t : \text{Im}\gamma(t) = r\}$ , and let  $\eta_t = \psi \circ \gamma_t$ . Let

$$\begin{aligned} \tilde{\gamma}_t &= \bigcup_{k \in \mathbb{Z} \setminus \{0\}} (\gamma_t + 2\pi k), \quad \hat{\gamma}_t = \gamma_t \cup \tilde{\gamma}_t, \\ T &= \inf\{t : \gamma_t \cap \tilde{\gamma}_t \neq \emptyset\}. \end{aligned}$$

Equivalently,  $T$  is the first time that the curve  $\eta_t$  is not simple. Note that  $T \neq \tau_r$  for each  $r$ ; indeed, by the definition of  $T$ , there must be an  $s < T$  with  $\text{Im}\gamma(s) = \text{Im}\gamma(T)$ . Let

$$S_{r,t} = S_r \setminus \gamma_t, \quad \hat{S}_{r,t} = S_r \setminus \hat{\gamma}_t.$$

If  $t < T \wedge \tau_r$ , there is a unique  $r(t) = r(t, \gamma_t) \in (0, r]$  such that there is a conformal transformation

$$\bar{h}_t : A_r \setminus \eta_t \rightarrow A_{r(t)},$$

with  $\bar{h}_t(C_0) = C_0$ . The transformation  $\bar{h}_t$  is unique up to a rotation. This transformation can be raised to the covering space  $S_r$  to give a conformal transformation

$$h_t : \hat{S}_{r,t} \rightarrow S_{r(t)}$$

with  $h_t(\pm\infty) = \pm\infty$ . This transformation is unique up to a real translation, and we specify it uniquely by requiring

$$h_t(U_t) = U_t.$$

We define  $\phi_t$  by

$$h_t = \phi_t \circ g_t.$$

Note that  $\phi_t$  is the unique conformal transformation of  $g_t(S_{r,t})$  onto  $S_{r(t)}$  with  $\phi_t(\pm\infty) = \pm\infty$  and  $\phi_t(U_t) = U_t$ . Although  $r(t)$  depends on the curve  $\gamma$ , the next lemma shows that its derivative at 0 is independent of  $\gamma$  assuming  $\gamma$  has the capacity parametrization.

**Lemma 3.12.** *If  $\gamma$  is a curve with  $\text{hcap}[\gamma_t] = at$ , then  $\dot{r}(0) = -a/2 = -1/\kappa$ .*

*Proof.* We will consider excursion measure defined by

$$\mathcal{E}_D(V_1, V_2) = \frac{1}{2\pi^2} \int_{V_1} \int_{V_2} H_D(z, w) |dz| |dw|.$$

This definition assumes  $V_1, V_2$  are nice boundaries, but this is a conformal invariant (see [17, Chapter 5]) and hence is defined for rough boundaries as well. In this normalization,  $\mathcal{E}_r := \mathcal{E}_{A_r}(C_0, C_r) = 1/r$ . Consider  $D_t = A_r \setminus \eta_t$  where  $\eta = \psi \circ \gamma$ . We only need to consider small  $t$  for which  $\eta$  is a simple curve in  $A_r$ . Let  $\mathcal{E}(t) = \mathcal{E}_{D_t}(C_r, C_0 \cup \eta_t)$ . By definition of  $r(t)$  and conformal invariance of excursion measure,  $\mathcal{E}(t) = 1/r(t)$ . Therefore, by the chain rule

$$(37) \quad \dot{\mathcal{E}}(0) = \frac{\dot{r}(0)}{r^2}.$$

Suppose  $r > 1$  and  $t$  is sufficiently small so that  $\mathbb{D}_1 \subset D_t$ . Then using the strong Markov property,

$$\mathcal{E}_{A_r}(C_r, C_1) - \mathcal{E}_D(C_r, D_t) = \mathcal{E}_{\mathbb{D}_1 \cap A_r}(C_r, C_1) \mathbb{E}[q(B_{\tau_t})] = \frac{1}{r-1} \mathbb{E}\left[-\frac{\log |B_{\tau_t}|}{r}\right].$$

Here  $B$  is a Brownian motion started uniformly on  $C_s$ ,  $\tau_t$  is the first time that it leaves  $D_t$  and  $q(z)$  denotes the probability that a Brownian motion starting at  $z$  hits  $C_r$  before  $C_0$ ,

$$q(z) = \frac{-\log |z|}{r}.$$

Therefore,

$$\dot{\mathcal{E}}(0) = \frac{1}{r^2} \frac{r}{r-1} \partial_t \mathbb{E}[\log |B_{\rho_t \wedge \sigma_r}|] |_{t=0}.$$

where  $\rho_t$  is the first time to leave  $D_t$  and  $\sigma_r$  is the first time to hit  $C_r$ . We claim that

$$(38) \quad \partial_t \mathbb{E}[\log |B_{\rho_t \wedge \sigma_r}|] |_{t=0} = \frac{r-1}{r} \partial_t \mathbb{E}[\log |B_{\rho_t}|] |_{t=0}.$$

To see this, we first note that the probability starting at  $C_1$  of hitting  $C_r$  before  $C_0$  is  $1/r$ . Also, given  $\rho_t < \sigma_r$ , the probability of hitting  $C_r$  before  $C_0$  is  $O(d_t/r)$  where  $d_t = \text{diam}(\gamma_t) = o(1)$ . Also, since we start with the uniform distribution on  $C_1$ , the distribution of  $\sigma_r$  given that  $\sigma_r < \sigma_0$  is also uniform. Therefore,

$$\mathbb{E}[\log |B_{\rho_t}| ; \sigma_r < \rho_t] = \frac{1}{r} \mathbb{E}[\log |B_{\rho_t}|] [1 + O(d_t)].$$

and hence

$$\partial_t \mathbb{E}[\log |B_{\rho_t}| ; \sigma_r < \rho_t] |_{t=0} = \frac{1}{r} \partial_t \mathbb{E}[\log |B_{\rho_t}|] |_{t=0}.$$

from which (38) follows. Note that the right-hand side of (38) is the same if we start the Brownian motion at the origin.

By comparison with (37), we see that  $\dot{r}(0)$  is independent of  $r(0)$ , and we can compute  $\dot{r}(0)$  by letting  $r \downarrow 0$ . In this case, we get the comparison of the chordal Loewner equation to the radial Loewner equation.  $\square$

We define

$$\sigma_s = \inf\{t : r(t) = s\}.$$

Let  $\gamma^*$  be  $\gamma$  with the “annulus parametrization”

$$\gamma^*(s) = \gamma(\sigma_s), \quad 0 \leq s \leq r,$$

and let

$$U_s^* = U_{r(s)}, \quad h_s^* = h_{\sigma_s}.$$

The direction of “time” is reversed so one must be careful with minus signs.

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♣ In the annulus parametrization, the radius takes the place of time. However, the direction of “time” is reversed, so one must take some care with minus signs.

---

We will just state the annulus Loewner equation (see, e.g., [1, 12]). It can also be described in terms of excursion reflection Brownian motion (this helps motivate the formulas), see [5, 16]. We review the facts here. Let  $\mathcal{H}_{S_r}(z, x) = \mathcal{H}_{S_r}(z - x)$  denote the complexification of the Poisson kernel in  $S_r$  which recall by (4) is given by

$$\mathcal{H}_{S_r}(z) = -\frac{\pi}{2r} \coth\left(\frac{\pi z}{2r}\right),$$

and satisfies

$$\begin{aligned} \operatorname{Im}\mathcal{H}(z) &= H_{S_r}(z, 0), \\ \mathcal{H}_{S_r}(z) &= -\frac{1}{z} + O(|z|), \quad z \rightarrow 0, \end{aligned}$$

and if  $x \in \mathbb{R}$ ,

$$\operatorname{Re}\mathcal{H}_{S_r}(x) = -\frac{\pi}{2r} \coth\left(\frac{\pi x}{2r}\right), \quad \operatorname{Re}\mathcal{H}_{S_r}(x + ir) = -\frac{\pi}{2r} \tanh\left(\frac{\pi x}{2r}\right).$$

There exists a unique holomorphic function with period  $2\pi$

$$\mathcal{H}_r : S_r \rightarrow \mathbb{H}_r,$$

such that

$$\mathcal{H}_r(z) = -\frac{1}{z} + o(1), \quad z \rightarrow 0,$$

and such that the induced map

$$\bar{\mathcal{H}}_r(e^{iz}) = \mathcal{H}_r(z)$$

is a conformal transformation of  $A_r$  onto a domain of the form  $\mathbb{H} \setminus L$  for some horizontal line segment  $L$ . One can find this using excursion reflected Brownian motion (ERBM) as we now sketch. The imaginary part  $H_r = \operatorname{Im}\mathcal{H}_r$  will be the Poisson kernel for ERBM in the annulus. We can write

$$(39) \quad H_r(z) = \frac{\operatorname{Im}(z)}{2r} + H_{A_r}(e^{iz}, 1) = \frac{\operatorname{Im}(z)}{2r} - \frac{\pi}{2r} \sum_{k \in \mathbb{Z}} \operatorname{Im} \coth\left(\frac{\pi z}{2r}\right).$$

In this formula, the infinite sum represents the contribution to the ERBM Poisson kernel by paths that do not hit the “hole”  $\mathbb{D} \setminus A_r$ . The first term gives the contribution of paths that hit the hole first. The probability of hitting the hole before hitting  $C_0$  is  $\operatorname{Im}(z)/r$ . Given that it hits the hole, the distribution of the first visit to  $C_0$  is uniform on the circle and hence the value of the kernel is  $1/2$  (recall that in our normalization,  $H_{\mathbb{D}}(0, 1) = 1/2$ .)

One can check that the sum in (39) is absolutely convergent. However, the real parts are not absolutely convergent so we must take a little care in the definition of  $\mathcal{H}_r$ . We write

$$\begin{aligned}\mathcal{H}_r(z) &= \frac{z}{2r} - \frac{\pi}{2r} \coth\left(\frac{z\pi}{2r}\right) \\ &\quad - \frac{\pi}{2r} \sum_{k=1}^{\infty} \left[ \coth\left(\frac{(z+2k\pi)\pi}{2r}\right) + \coth\left(\frac{(z-2k\pi)\pi}{2r}\right) \right] \\ &= \frac{z}{2r} - \frac{\pi}{2r} \sum_k^{PP} \coth\left(\frac{(z+2k\pi)\pi}{2r}\right),\end{aligned}$$

where we write

$$\sum_k^{PP} f(k) = \lim_{N \rightarrow \infty} \sum_{k=-N}^N f(k).$$

**Lemma 3.13.** *As  $z \rightarrow 0$ ,*

$$(40) \quad \mathcal{H}_r(z) = -\frac{1}{z} + z \left( \frac{1}{2r} - \Gamma(r) + \frac{1}{12} \right) + O(|z|^3),$$

where  $\Gamma(r)$  is as defined in (20).

*Proof.* We use the first expression for the definition of  $\mathcal{H}_r$ . Note that as  $z \rightarrow 0$ ,

$$\coth z = \frac{1}{z} + \frac{z}{3} + O(|z|^3),$$

and hence

$$\frac{\pi}{2r} \coth\left(\frac{z\pi}{2r}\right) = \frac{\pi}{2r} \left[ \frac{2r}{z\pi} + \frac{z\pi}{6r} + O(|z|^3) \right] = \frac{1}{z} + \frac{\pi^2 z}{12r^2} + O(|z|^3)$$

Also the derivative at  $z = 0$  of

$$-\frac{\pi}{2r} \sum_{k=1}^{\infty} \left[ \coth\left(\frac{(z+2k\pi)\pi}{2r}\right) + \coth\left(\frac{(z-2k\pi)\pi}{2r}\right) \right]$$

is  $\frac{1}{12} - \delta(r)$ . □

Note that

$$\mathcal{H}_r(z + ir) = \frac{z + ir}{2r} - \frac{\pi}{2r} \sum_k^{PP} \tanh\left(\frac{(z+2k\pi)\pi}{2r}\right) = -\frac{\mathbf{H}_I(r, x)}{2} + \frac{i}{2},$$

where  $\mathbf{H}_I$  is as defined in Section 3.2.

The chordal equation (16) can be written as

$$\partial_t g_t(z) = -a \mathcal{H}_{\mathbb{H}}(g_t(z) - U_t).$$

The annulus Loewner equation is similar,

$$\partial_t h_t(z) = 2 \dot{r}(t) \mathcal{H}_{r(t)}(h_t(z) - U_t),$$

or equivalently,

$$(41) \quad \partial_r h_r^*(z) = 2 \mathcal{H}_r(h_r^*(z) - U_r^*).$$



An important observation is that if  $r(0) = r$ , then for small  $t$ , the functions  $g_t$ ,  $h_t$ , and  $h_{r-\frac{at}{2}}^*$  are very close near the origin. For future reference, we also note that

$$(42) \quad \partial_s \log(h_s^*)'(x + ir) \big|_{s=r} = 2\mathcal{H}'_r(x + ir) = -\mathbf{H}'_I(r, x).$$

---

♣ There may appear to be some arbitrariness in the choice of the real translation for the complex kernel  $\mathcal{H}_{\mathbb{H}}(g_t(z) - U_t)$ . It turns out that this choice is not so important. We will write

$$d[h_r^*(z) - U_r^*] = 2\mathcal{H}_r(h_r^*(z) - U_r^*) - dU_r^*.$$

If we had chosen a different real translation of  $\mathcal{H}_r$ , it would cancel here when we took the difference.

---



---

♣ We have written the annulus equation in the covering space  $S_r$ . We would also consider the function given by

$$f_s(e^{iw}) = e^{ih_s(w)}, \quad 0 \leq s \leq r.$$

There is a curve  $\eta : (0, r) \rightarrow A_r$  with  $\eta(0+) = 1$  such that  $f_s$  is a conformal transformation of  $A_r \setminus \gamma_s$  onto  $A_{r-s}$ . Such a transformation is defined up to a rotation, but specifying continuity and  $f_s(\eta(r-s)) = U_s^*$  determines the rotation.

---

We will need to compare the chordal and annulus equations at time  $t = 0$ . Recall that  $\phi_t$  is defined by

$$h_t(z) = \phi_t(g_t(z)),$$

and that  $\phi_t(U_t) = U_t = g_t(\gamma(t))$ . Although  $g_t$  is not smooth at  $\gamma(t)$ , it is not difficult to show that  $\phi_t$  is analytic in a neighborhood of  $U_t$  and we can give the derivatives. We summarize the facts we need in this lemma whose simple prove we omit.

**Lemma 3.14.** *Suppose  $K_{j,t}(z)$ ,  $j = 1, 2$ ,  $t \in [0, \epsilon]$  are analytic functions in a punctured neighborhood of the origin and are continuous in  $t$ . Suppose  $U_t$  is a continuous function with  $U_0 = 0$  and  $g_t, h_t$  satisfy*

$$\partial_t g_t(z) = K_{1,t}(g_t(z) - U_t), \quad \partial_t h_t(z) = K_{2,t}(h_t(z) - U_t),$$

*with  $g_0(z) = h_0(z)$ . Suppose that for all  $t$ ,  $K_{1,t} - K_{2,t}$  is analytic in the (unpunctured) neighborhood. If  $\phi_t$  is defined by  $h_t(z) = \phi_t(g_t(z))$ , then*

$$(43) \quad \dot{\phi}_0(z) = [K_{2,0} - K_{1,0}](z), \quad \dot{\phi}'_0(z) = [K_{2,0} - K_{1,0}]'(z).$$

We now return to the locally chordal  $SLE_\kappa$  from 0 to  $z_0 = x + ir$  in  $S_r$ . Given the path  $\gamma_t$ , the process is moving infinitesimally like  $SLE_\kappa$  in  $\hat{S}_{r,t}$  from  $\gamma(t)$  to  $z_0$ . By conformal invariance we can also view it in  $g_t(\hat{S}_{r,t})$  from  $U_t$  to  $g_t(z_0)$  or in  $h_t(\hat{S}_{r,t}) = S_{r(t)}$  from  $U_t$  to  $h_t(z_0)$ . Using the last perspective and (31) and (32), we see that

$$dU_t = b\mathbf{L}(r(t), R_t) dt - dW_t,$$

where  $R_t = \text{Re}[h_t(z_0)] - U_t$  and  $W_t$  is a standard Brownian motion. We choose a time parametrization so that the radius evolves linearly. If  $U_t^* = U_{\sigma(t)}$  as above,

$$dU_t^* = b\kappa\mathbf{L}(r - t, R_t^*) dt - \sqrt{\kappa} dB_t.$$

Using (42), we see that if  $f_t = h_{r-t}^*$ ,

$$\partial_t[\text{Ref}_t(z_0)] = \mathbf{H}_I(r - t, R_t^*),$$

and hence

$$(44) \quad dR_t^* = [\mathbf{H}_I(r-t, R_t^*) - b\kappa \mathbf{L}(r-t, R_t^*)] dt + \sqrt{\kappa} dB_t.$$

We have written locally chordal  $SLE_\kappa$  in the annulus as a one-dimensional SDE stopped at a finite time  $r$ . The next lemma shows that the process leaves  $S_r$  at  $z_0$ . The equivalent statement is the following.

**Lemma 3.15.** *If  $X_t$  satisfies*

$$dX_t = [\mathbf{H}_I(r-t, X_t) - b\kappa \mathbf{L}(r-t, X_t)] dt + \sqrt{\kappa} dB_t, \quad 0 \leq t < r,$$

*then with probability one  $X_{r-} = 0$ .*

---

♣ This lemma should not be surprising. If we considered chordal  $SLE_\kappa$  from 0 to  $x + ir$  in  $S_r$  we know that (for  $\kappa \leq 4$ ) the path leaves the domain at  $x + ir$ . This lemma states that the same thing for locally chordal  $SLE_\kappa$ . Since for  $r$  near zero, locally chordal and chordal  $SLE_\kappa$  are almost the same, the lemma has to be true. One should expect  $\kappa \leq 4$  to come into the proof, and this is the case.

---

*Proof.* We discuss the most delicate case,  $\kappa = 4$  for which  $b\kappa = 1$ ; if  $\kappa < 4$ , then  $b\kappa > 1$  and the argument is easier. Our equation is

$$dX_t = [\mathbf{H}_I(r-t, X_t) - \mathbf{L}(r-t, X_t)] dt + 2 dB_t.$$

If  $Y_s = X_{r-e^{-s}}$ , then  $Y_s$  satisfies

$$dY_s = m(s, Y_s) ds + 2e^{-s/2} dW_s,$$

where

$$m(s, y) = e^{-s} [\mathbf{H}_I(e^{-s}, y) - \mathbf{L}(e^{-s}, y)],$$

and  $W_s$  is a standard Brownian motion. It suffices to show that for every  $\epsilon > 0$ , with probability one,  $|Y_s| \leq \epsilon$  for all  $s$  sufficiently large. By symmetry it suffices to show that  $\limsup Y_s \leq 0$ . Let

$$Z_s = \int_0^s 2e^{-r/2} dW_r,$$

and note that with probability one  $Z_\infty$  exists and is finite.

Using Lemma 3.2, we can see that there exists  $s_\epsilon$  such that  $m(s, y) \leq 0$  for  $s \geq s_\epsilon, y \geq \epsilon/2$ . Therefore, if  $Y_s \geq \epsilon$  and  $s \geq s_\epsilon$ ,

$$Y_r \leq \epsilon + \max_{t \geq s_\epsilon} |Z_t - Z_{s_\epsilon}|.$$

Therefore, it suffices to show that with probability one  $\liminf Y_n \leq 0$ . In other words, for every  $\epsilon > 0, s < \infty, y > 0$ , the probability that the process reaches  $\epsilon$  given  $Y_s = y$  equals one.

Although the drift  $m(s, y)$  is negative, the absolute value is very small at  $y$  slightly larger than an integer multiple of  $2\pi$ . However, we also know from Lemma 3.2 that for all  $y, m(s, y) \leq -ce^{-s}$ . Given this, we can see that if we start near  $2\pi k$ , there is at least a positive probability that there will exist  $s$  with  $Y_s \leq 2\pi k - c_1 e^{-s}$ . Given this, there is a positive probability that the process will never return to  $\{y \geq 2\pi k - (c_1/2)e^{-s}\}$  and since the drift is negative, this will imply that it will get near  $2\pi(k-1)$ . This happens with positive probability, but if it fails and we

are near  $2\pi k$  at a larger time  $s'$  we can find  $s'' > s'$  for which  $Y_{s''} \leq 2\pi k - c_1 e^{-s''}$ . Eventually we will succeed and get to  $2\pi(k-1)$ . We can iterate this argument.  $\square$

#### 4. DEFINITION OF $\mu_D(z, w)$

**4.1. Definition of boundary  $SLE_\kappa$  for  $\kappa \leq 4$ .** We fix  $\kappa \in (0, 4]$ . In this section, we will define boundary  $SLE_\kappa$  as proposed in [18]. It is a (positive) measure  $\mu_D(z, w)$  on simple curves  $\gamma$  in a domain  $D$  connecting distinct  $\partial D$ -analytic boundary points  $z$  and  $w$ . If  $D$  is simply connected, then the definition is the same as that of chordal  $SLE_\kappa$ . We write

$$\Psi_D(z, w) = \|\mu_D(z, w)\|$$

for the total mass of the measure. We *conjecture* that  $\Psi_D(z, w) < \infty$  for all  $D, z, w$ . In the case of simply connected domains, we know this is true, and in this paper we will show it for 1-connected domains for  $\kappa \leq 4$ . From the construction it will follow that  $\Psi_D(z, w) < \infty$  for all domains if  $\kappa \leq 8/3$  ( $\mathbf{c} \leq 0$ ).

Suppose  $D_1 \subset D$  is a subdomain of  $D$  that agrees with  $D$  in neighborhoods of  $z$  and  $w$ . We let  $\mu_D(z, w; D_1)$  be  $\mu_D$  restricted to curves  $\gamma \subset D_1$ . Let

$$\Psi_D(z, w; D_1) = \|\mu_D(z, w; D_1)\|.$$

We will show that  $\mu_D(z, w; D_1) < \infty$  for all such simply connected  $D_1$  for  $\kappa \leq 4$ . The measure  $\mu_D^\#(z, w; D_1)$  is defined to be the probability measure obtained by normalization

$$\mu_D^\#(z, w; D_1) = \frac{\mu_D(z, w; D_1)}{\Psi_D(z, w; D_1)}.$$

If  $\Psi_D(z, w) < \infty$ , we write  $\mu^\#(z, w)$  for the probability measure.

---

♣What we call boundary  $SLE$  should really be called boundary/boundary  $SLE$ , but this terminology is a bit cumbersome. In later subsections, we also discuss boundary/bulk, bulk/boundary, and bulk/bulk cases.

---

In this definition and later on we use the convention as described below equation (3) that if formulas are written with derivatives, then sufficient smoothness is assumed.

**Definition** If  $\kappa \leq 4$  and  $b, \mathbf{c}$  are as in (1), *boundary  $SLE_\kappa$*  is the unique family of measures (modulo reparametrization)  $\{\mu_D(z, w)\}$ , where  $D \subset \mathbb{C}$  and  $z, w$  are distinct  $\partial D$ -analytic points, satisfying the following.

- For each  $D, z, w$ ,  $\mu_D(z, w)$  is a positive measure on curves  $\gamma : [0, t_\gamma] \rightarrow D$  with  $\gamma(0) = z, \gamma(t_\gamma) = w, \gamma \subset D$ . The total mass is denoted by

$$\Psi_D(z, w) = \|\mu_D(z, w)\|.$$

The normalization is chosen so that  $\Psi_{\mathbb{H}}(0, 1) = 1$ .

- **Conformal covariance** If  $f : D \rightarrow f(D)$  is a conformal transformation, then

$$(45) \quad f \circ \mu_D(z, w) = |f'(z)|^b |f'(w)|^b \mu_{f(D)}(z, w).$$

- It follows from (45) that the probability measures are conformally invariant,

$$f \circ \mu_D^\#(z, w; D_1) = \mu_{f(D)}^\#(f(z), f(w); f(D_1)),$$

and if  $\Psi_D(z, w) < \infty$ ,

$$(46) \quad f \circ \mu_D^\#(z, w) = \mu_{f(D)}^\#(z, w).$$

In particular,  $\mu_D^\#(z, w; D_1)$  (resp.,  $\mu_D^\#(z, w)$ ) can be defined for nonanalytic boundary points provided that there is a conformal transformation  $f : D \rightarrow f(D)$  such that  $f(z), f(w)$  are  $\partial f(D)$ -analytic (resp., with  $\Psi_{f(D)}(f(z), f(w)) < \infty$ ).

- **Domain Markov property.** If  $\Psi_D(z, w) < \infty$ , then for the probability measure  $\mu_D^\#(z, w)$ , the conditional probability measure of the remainder of a curve  $\gamma$  given an initial segment  $\gamma_t$ , is that of  $\mu_{D \setminus \gamma_t}^\#(\gamma(t), w)$ . If  $D_1 \subset D$  is simply connected, for the probability measure  $\mu_D^\#(z, w; D_1)$ , the conditional probability measure of the remainder of a curve  $\gamma$  given an initial segment  $\gamma_t$ , is that of  $\mu_{D \setminus \gamma_t}^\#(\gamma(t), w; D_1 \setminus \gamma_t)$ .
- **Boundary perturbation.** Suppose  $D' \subset D$  are domains that agree in neighborhoods of  $\partial D'$ -analytic boundary points  $z, w$ . Then  $\mu_{D'}(z, w)$  is absolutely continuous with respect to  $\mu_D(z, w)$  with Radon-Nikodym derivative  $Y = Y_{D, D', z, w}$  given by

$$(47) \quad Y(\gamma) = \frac{d\mu_{D'}(z, w)}{d\mu_D(z, w)}(\gamma) = 1\{\gamma \subset D'\} \exp \left\{ \frac{\mathbf{c}}{2} m_D(\gamma, D \setminus D') \right\}.$$

We will now construct the measure and in the process show uniqueness. For simply connected domains, we set  $\Psi_D(z, w) = H_{\partial D}(z, w)^b$  and  $\mu_D^\#(z, w)$  to be the conformal image of  $\Psi_{\mathbb{H}}(0, \infty)$  under a conformal transformation. The discussion in Section 3.6 shows that this is the unique family of measures that satisfy the conditions above for simply connected  $D$ .

**Definition** Suppose  $D$  is a domain and  $z, w$  are distinct  $\partial D$ -analytic boundary points. Let  $D_1$  be a simply connected subdomain of  $D$  that agrees with  $D$  in neighborhoods of  $z, w$ . Then  $\hat{\mu}_D(z, w; D_1)$  is the measure absolutely continuous with respect to  $\mu_{D_1}(z, w)$  with Radon-Nikodym derivative

$$(48) \quad \frac{d\hat{\mu}_D(z, w; D_1)}{d\mu_{D_1}(z, w)}(\gamma) = 1\{\gamma \subset D_1\} \exp \left\{ -\frac{\mathbf{c}}{2} m_D(\gamma, D \setminus D_1) \right\}.$$

---

♣A minus sign appears on the right-hand side above. This is because we are writing the derivative of the measure on the larger domain with respect to that on the smaller domain.

---

The next proposition establishes a necessary consistency condition for the measures  $\hat{\mu}_D(z, w; D_j)$  in order to define  $\mu_D(z, w)$ .

**Proposition 4.1.** *Suppose  $D$  is a domain and  $z, w$  are distinct  $\partial D$ -analytic boundary points. Let  $D_1, D_2$  be simply connected subdomains of  $D$  that agree with  $D$  in neighborhoods of  $z, w$ . For  $j = 1, 2$ , let  $\nu_j$  be  $\hat{\mu}_D(z, w; D_j)$  restricted to curves  $\gamma$  with  $\gamma \subset D_1 \cap D_2$ . Then  $\nu_1 = \nu_2$ .*

*Proof.* Suppose  $\gamma \subset D_1 \cap D_2$ . Then there exists simply connected  $\hat{D} \subset D_1 \cap D_2$  that agrees locally with  $D$  near  $z, w$  such that  $\gamma \subset \hat{D}$ . Hence it suffices to show that for every simply connected domain  $\hat{D}$ ,  $\nu_1$  and  $\nu_2$ , restricted to curves in  $\hat{D}$ , agree. Suppose  $\gamma \subset \hat{D}$ . Since  $D_j, \hat{D}$  are simply connected,

$$\frac{d\mu_{D_j}(z, w)}{d\mu_{\hat{D}}(z, w)}(\gamma) = \exp \left\{ -\frac{\mathbf{c}}{2} m_{D_j}(\gamma, D_j \setminus \hat{D}) \right\}.$$

Combining this with (48), we get

$$\frac{d\hat{\mu}_D(z, w; D_j)}{d\mu_{\hat{D}}(z, w)}(\gamma) = \exp \left\{ -\frac{\mathbf{c}}{2} m_D(\gamma, D \setminus \hat{D}) \right\}.$$

Here we use the fact that the loops in  $D$  that intersect  $\gamma$  and  $D \setminus \hat{D}$  can be partitioned into two sets: those that intersect  $D \setminus D_1$  and those that are contained in  $D_1$ .  $\square$

Given Proposition 4.1 we can make the following definition.

**Definition** Suppose  $D$  is a domain and  $z, w$  are distinct  $\partial D$ -analytic boundary points. Then  $\mu_D(z, w)$  is the measure on simple paths (modulo parametrization) such that for each simply connected  $D_1 \subset D$ ,  $\mu_D(z, w)$  restricted to curves  $\gamma \subset D_1$  is  $\hat{\mu}_D(z, w; D_1)$ .

In other words,  $\mu_D(z, w; D_1) = \hat{\mu}_D(z, w; D_1)$  for simply connected  $D_1$ . It follows immediately from the definition that the family of measures  $\{\mu_D(z, w)\}$  satisfies (47). Suppose  $D$  is a domain and  $z, w$  are distinct  $\partial D$ -analytic points and  $D_1$  is a simply connected domain as above. Suppose  $f : D \rightarrow f(D)$  is a conformal transformation. Then  $f : D_1 \rightarrow f(D_1)$  is also a conformal transformation, and hence

$$f \circ \mu_{D_1}(z, w) = |f'(z)|^b |f'(w)|^b \mu_{f(D_1)}(f(z), f(w)).$$

Conformal invariance of the loop measure then implies that

$$f \circ \mu_D(z, w; D_1) = |f'(z)|^b |f'(w)|^b \mu_{f(D)}(z, w; f(D_1)).$$

Since this is true for every simply connected  $D_1$ , the family  $\{\mu_D(z, w)\}$  satisfies (45).

In this paper, we will show the following. (While we prove it in this paper, we could also derive this from [31].)

**Proposition 4.2.** *If  $D$  is a conformal annulus, then  $\Psi_D(z, w) < \infty$  and the family  $\{\mu_D(z, w)\}$  restricted to conformal annuli satisfies the domain Markov property.*

When considering the measure  $\mu_D(z, w)$  for multiply connected domains, there are two cases.

- The *chordal* case:  $z, w$  in the same component of  $\partial D$ . Then there exists simply connected  $\hat{D}$  such that  $D \subset \hat{D}$ .
- The *crossing* case:  $z, w$  in different components of  $\partial D$ . Then there exists 1-connected  $\hat{D}$  such that  $D \subset \hat{D}$ .

**Proposition 4.3.** *Suppose  $D$  is a domain and  $z, w$  are distinct  $\partial D$ -analytic points.*

- If  $\kappa \leq 8/3$ , then  $\Psi_D(z, w) < \infty$ .
- If  $8/3 < \kappa \leq 4$ , then for every simply connected  $D_1 \subset D$  that agrees with  $D$  near  $z, w$ ,  $\Psi_D(z, w; D_1) < \infty$ .

*Proof.* If  $\kappa \leq 8/3$ , we can consider  $D$  as a subdomain of a simply connected or 1-connected domain  $\hat{D}$  and since  $\mathbf{c} \leq 0$ , (47) implies that  $\Psi_D(z, w) \leq \Psi_{\hat{D}}(z, w) < \infty$ . If  $8/3 < \kappa \leq 4$ , then  $\mathbf{c} > 0$ , and (48) implies that  $\Psi_D(z, w; D_1) \leq \Psi_{D_1}(z, w) < \infty$ .  $\square$

**Proposition 4.4.** *The family  $\{\mu_D(z, w)\}$  satisfies the domain Markov property.*

*Proof.* Without loss of generality we may assume that  $D$  is a subdomain of  $\mathbb{H}$  whose boundary includes  $\mathbb{R}$  and  $z = 0$ . Let  $D_1$  be a simply connected domain as above for which we know  $\Psi_D(z, w; D_1) < \infty$  and let  $\gamma_t$  be an initial segment. To be more precise, let  $t$  be a finite stopping time for chordal  $SLE_\kappa$  in  $D_1$ . Let  $\mathcal{F}_t$  be the corresponding  $\sigma$ -algebra generated by  $\gamma_t$ . For  $\gamma \subset D_1$ , let

$$Y(\gamma) = \frac{\mu_D(z, w; D_1)}{\mu_{D_1}(z, w)}(\gamma) = \exp \left\{ \frac{\mathbf{c}}{2} m_D(\gamma, D \setminus D_1) \right\}.$$

Let  $\mathbb{P}, \mathbb{E}$  denote probability and expectation with respect to the probability measure  $\mu_{D_1}^\#(z, w)$ . Then,

$$\Psi_D(z, w; D_1) = \Psi_{D_1}(z, w) \mathbb{E}[Y].$$

By the domain Markov property for  $SLE_\kappa$  in simply connected domains,

$$\mathbb{E}[Y \mid \mathcal{F}_t] = \exp \left\{ \frac{\mathbf{c}}{2} m_D(\gamma_t, D \setminus D_1) \right\} \mathbb{E}_t^*[Y],$$

where  $\mathbb{E}_t^*$  denotes expectation with respect to  $\mu_{D_1 \setminus \gamma_t}^\#(\gamma(t), w)$ .

We will do the chordal case comparing to simple connected domains. The crossing case is similar using conformal annuli. Suppose  $z, w$  are in the same component of  $\partial D$ . Without loss of generality, we may assume that  $D$  is a subdomain of  $\mathbb{H}$  and  $z, w \in \mathbb{R}$ . We know that

$$\frac{d\mu_D(z, w)}{d\mu_{\mathbb{H}}(z, w)}(\gamma) = 1_{\{\gamma \in D\}} \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{H}}(\gamma, \mathbb{H} \setminus D) \right\}.$$

Let  $\mathbb{P}, \mathbb{E}$  denote probabilities and expectations with respect to the measure  $\mu_{\mathbb{H}}^\#(z, w)$ . Let

$$Y_t = 1_{\{\gamma_t \subset D\}} \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{H}}(\gamma_t, \mathbb{H} \setminus D) \right\}, \quad Y = Y_\infty.$$

Suppose we are given an initial segment  $\gamma_t$  and let  $H_t = \mathbb{H} \setminus \gamma_t$ . Here  $t$  can be a stopping time and we assume that  $t < T = \inf\{s > 0 : \gamma(s) \in \mathbb{R}\} = \inf\{s > 0 : \gamma(s) = w\}$ . (The equality is true with  $\mathbb{P}$  probability one.) Let  $g_t$  denote the corresponding map and let  $\mathcal{F} = \mathcal{F}_t$  denote the  $\sigma$ -algebra generated by  $t$ . By the domain Markov property of  $SLE_\kappa$  in simply connected domains,

$$\mathbb{E}[Y \mid \mathcal{F}] = Y_t \mathbb{E}_t^* \left[ \exp \left\{ \frac{\mathbf{c}}{2} m_{H_t}(\gamma, H_t \setminus D) \right\} \right],$$

where  $\mathbb{E}_t^*$  denotes expectations with respect to  $\mu_{H_t}^\#(\gamma(t), w)$ . More generally if  $E$  is an event depending on the path  $\gamma \setminus \gamma_t$ ,

$$\mathbb{E}[Y 1_E \mid \mathcal{F}] = Y_t \mathbb{E}_t^* \left[ 1_E \exp \left\{ \frac{\mathbf{c}}{2} m_{H_t}(\gamma, H_t \setminus D) \right\} \right],$$

If  $\Psi_D(z, w) < \infty$ , the proof for  $\mu_D^\#(z, w)$  is similar and we omit it.  $\square$

**Proposition 4.5.** *If  $z, w$  are  $\partial D$ -analytic, then  $\mu_D(w, z)$  is the same as the reversal of  $\mu_D(z, w)$ .*

*Proof.* In the case of simply connected domains, this was proved by Zhan [30]. Given this, the general case follows.  $\square$

We end this section with a number of remarks.

- In our definition we have started with the parameter  $\kappa$  and defined the quantities  $b, \mathbf{c}$  in terms of  $\kappa$ . We could have made  $b, \mathbf{c}$  free parameters, but then we would find out that there was only a one-dimensional family of pairs  $(b, \mathbf{c})$  for which we could define such measures. To establish this fact, we would use Schramm's argument and  $\kappa$  (as a function of  $b$  or  $\mathbf{c}$ ) would be introduced.
- Implicit in the domain Markov property is the assumption that the initial segment may be chosen using a stopping time. This makes it a condition on curves modulo reparametrization. Perhaps this should be called the *strong* domain Markov property.
- It is also useful to have the measures  $\mu_D(x, \infty)$  where  $D \subset \mathbb{H}$  with  $\mathbb{H} \setminus D$  bounded and  $\text{dist}(x, \mathbb{H} \setminus D) > 0$ . To get this we find a conformal transformation

$$f : D' \rightarrow D$$

with  $f(z) = 0, f(w) = \infty$  and use the conventions about derivatives as in Section 3.1. Under this convention, we see that  $\Psi_{\mathbb{H}}(0, \infty) = 1$ . If  $D \subset \mathbb{H}$  is simply connected with  $\mathbb{H} \setminus D$  bounded and  $\text{dist}(0, \mathbb{H} \setminus D) > 0$ , then  $\Psi_D(0, \infty) = \Phi'_D(0)^b$  where  $\Phi_D : D \rightarrow \mathbb{H}$  is a conformal transformation with  $\Phi_D(\infty) = \infty, \Phi'_D(\infty) = 1$ .

**4.2. Definition of boundary/bulk and bulk/bulk  $SLE_\kappa$  for  $\kappa \leq 4$ .** The boundary  $SLE_\kappa$  is a measure on curves connecting two boundary points in a domain  $D$ . We extend this definition to allow one boundary point and one interior point (the radial or reverse radial case) or two interior points (the bulk case). In all the cases we will write  $\mu_D(z, w)$  for the measure,  $\Psi_D(z, w)$  for the total mass, and if  $\Psi_D(z, w) < \infty$   $\mu_D^\#(z, w)$  for the corresponding probability measure. The definition will be the same as the first definition in Section 4.1 except that (45) is replaced with the following more general formula. Note that this definition subsumes the previous one.

- **Conformal covariance** If  $f : D \rightarrow f(D)$  is a conformal transformation,  $z, w$  are  $D$ -analytic, and  $f(z), f(w)$  are  $f(D)$ -analytic, then

$$(49) \quad f \circ \mu_D(z, w) = |f'(z)|^{b_z} |f'(w)|^{b_w} \mu_{f(D)}(z, w),$$

where  $b_\zeta = b$  if  $\zeta$  is a boundary point and  $b_\zeta = \tilde{b}$  if  $\zeta$  is an interior point.

---

♣ We are writing  $\mu_D(z, w)$  for all the cases in order not to add more notation. It is important to remember that the definitions of these measures are different (although related, of course) depending on whether  $z, w$  are boundary or interior points.

---

If  $D$  is simply connected,  $z$  is  $\partial D$ -analytic and  $w \in D$ , then we define  $\mu_D(z, w)$  by

$$\mu_D(z, w) = \Psi_D(z, w) \mu_D^\#(z, w),$$

where  $\mu_D^\#(z, w)$  is radial  $SLE_\kappa$  as in Section 3.3. The partition function  $\Psi_D(z, w)$  is determined up to a multiplicative constant by the rule (49), and we choose

the constant so that  $\Psi_{\mathbb{D}}(1, 0) = 1$ . Using the relationship in Section 3.8, one can check that this satisfies the necessary conditions. In particular, the boundary perturbation rule (47) holds for simply connected domains.

It was essentially shown in [31], and we will reprove it here, that radial  $SLE_{\kappa}$  can be given as a limit of boundadry/boundary  $SLE_{\kappa}$  in the annulus. The following theorem makes a more precise estimate.

**Theorem 4.6.** *There exists  $c < \infty, q > 0$  such that the following holds. Let  $t > 0$  and let  $\gamma_t$  denote an initial segment of a path in  $\mathbb{D}$  starting at 1 such that if  $g : \mathbb{D} \setminus \gamma_t \rightarrow \mathbb{D}$  is a conformal transformation with  $g(0) = 0, g'(0) > 0$ , then  $g'(0) = e^t$ . Suppose that  $r \geq t + 2, 0 \leq \theta < 2\pi$ , and let  $\mu_1 = \mu_{\mathbb{D}}(1, 0), \mu_2 = \mu_{A_r}^{\#}(1, e^{-r+i\theta})$ , both considered as probability measures on initial segments  $\gamma_t$ . Let  $Y = d\mu_2/d\mu_1$ . Then*

$$(50) \quad |Y(\gamma_t) - 1| \leq c e^{(t-r)q}.$$

Moreover, there exists  $c_0 \in (0, \infty)$  such that

$$(51) \quad \Psi(1, e^{-r+ix}) = c_0 e^{(b-\tilde{b})r} r^{c/2} [1 + O(e^{-qr})].$$

We will write

$$\mu_{A_r}(1, e^{-r+ix}) = c_0 e^{(b-\tilde{b})r} r^{c/2} \mu_{\mathbb{D}}(1, 0) [1 + O_t(e^{-qr})],$$

as shorthand for (50) and (51).

♣ We can see the interior scaling exponent as coming from a computation from the annulus partition function. Suppose  $D$  is a bounded domain,  $0 \in D$  and  $w \in \partial D$  is  $D$ -analytic. Suppose that  $\epsilon$  is small and  $|z| = \epsilon$ . Let  $D_{\epsilon}$  denote the conformal annulus obtained by removing the closed disk of radius  $\epsilon$ . Then by analysis of the annulus partition function which is a boundary/boundary quantity, we see as  $\epsilon \rightarrow 0$ ,

$$\Psi_{D_{\epsilon}}(1, z) \sim c \epsilon^{\tilde{b}-b} [\log(1/\epsilon)]^{c/2},$$

and hence we can define  $\Psi_D(1, 0)$  (up to an arbitrary multiplicative constant) by

$$\Psi_D(1, 0) \sim \epsilon^{b-\tilde{b}} [\log(1/\epsilon)]^{-c/2} \Psi_{D_{\epsilon}}(1, \epsilon).$$

If  $f : D \rightarrow f(D)$  is a conformal transformation with  $f(0) = 0$ , then  $f(D_{\epsilon})$  is approximately the disk of radius  $f'(0)\epsilon$ , and

$$\begin{aligned} \Psi_{D_{\epsilon}}(1, z) &\sim |f'(1)|^b |f'(z)|^b \Psi_{D_{\epsilon f'(0)}}(f(1), f(z)) \\ &\sim |f'(1)|^b |f'(0)|^b \Psi_{D_{\epsilon f'(0)}}(f(1), f(z)) \end{aligned}$$

Therefore, if  $u = |f'(0)|$ ,

$$\begin{aligned} \Psi_D(1, 0) &\sim \epsilon^{b-\tilde{b}} [\log(1/\epsilon)]^{-c/2} \Psi_{D_{\epsilon}}(1, z) \\ &\sim |f'(1)|^b u^{\tilde{b}} (u\epsilon)^{b-\tilde{b}} [\log(1/\epsilon)]^{-c/2} \Psi_{f(D)_{u\epsilon}}(f(1), f(uz)) \end{aligned}$$

Note that the logarithmic term which includes the central charge does not contribute to the scaling exponent.

We now define boundary/bulk and bulk/boundary  $SLE$ . The consistency of this definition follows from the fact that (47) holds for simply connected domains.

**Definition** If  $z \in D$  and  $w$  is a  $\partial D$ -analytic boundary point, then  $\mu_D(w, z)$  and  $\mu_D(z, w)$  are defined as follows.



- If  $D$  is simply connected,

$$\mu_D(w, z) = |f'(w)|^{-b} |f'(z)|^{-\bar{b}} f \circ \mu_{\mathbb{D}}(1, 0),$$

where  $f : \mathbb{D} \rightarrow D$  is the conformal transformation with  $f(1) = w, f(0) = z$ .

- If  $D \subset D_1$  where  $D_1$  is simply connected and agrees with  $D$  near  $z$  and  $w$ , then

$$\frac{d\mu_D(w, z)}{d\mu_{D_1}(w, z)}(\gamma) = 1\{\gamma \subset D\} \exp \left\{ \frac{c}{2} m_D(\gamma, D_1 \setminus D) \right\}.$$

- $\mu_D(z, w)$  is defined to be the measure obtained from  $\mu_D(w, z)$  by reversing the paths.

We can define bulk/bulk  $SLE_\kappa$  similarly. There is technical issue if  $D$  is all of  $\mathbb{C}$ . Let us define  $D$  to be *regular* if with probability one a Brownian motion exits the domain  $D$ .

**Definition** If  $z, w$  are distinct points of a regular domain  $D$ , then  $\mu_D(z, w)$  is defined by

$$\mu_D(z, w) = c_0^{-1} \lim_{r \rightarrow \infty} e^{2(\bar{b}-b)r} r^{c/2} \mu_{D_r}(z + e^{-r}, w + e^{-r}),$$

where

$$D_r = \{\zeta \in D : |\zeta - z| > e^{-r}, |\zeta - w| > e^{-r}\}.$$

We could also have defined

$$\mu_D(z, w) = c_0^{-1} \lim_{r \rightarrow \infty} e^{2(\bar{b}-b)r} r^{c/2} \mu_{D_r}(z + e^{-r+i\theta}, w + e^{-r+i\theta'}),$$

for any  $\theta, \theta'$ . Alternatively, we could define

$$\mu_D(z, w) = c' \lim_{r \rightarrow \infty} e^{(\bar{b}-b)r} \mu_{D_{r,z}}(z + e^{-r+i\theta}, w + e^{-r+i\theta'}),$$

where

$$D_{r,z} = \{\zeta \in D : |\zeta - z| > e^{-r}\}.$$

Our choice of definition has the advantage that it follows immediately that  $\mu_D(w, z)$  is the reversal of  $\mu_D(z, w)$ . If we want to let  $D = \mathbb{C}$ , we have to renormalize.

**Proposition 4.7.** *If  $z, w \in \mathbb{D}$ , then There exists  $\Psi(z, w) \in (0, \infty)$  such that*

$$\Psi_{\mathbb{D}_{-r}}(z, w) = \Psi(z, w) r^{-c/2} [1 + O(r^{-1})].$$

*Proof.* This essentially follows from Proposition 3.11. □

Using this as a guide, we define

$$\mu(z, w) = c' \lim_{r \rightarrow \infty} r^{c/2} \mu_{A_{-r}}(z, w).$$

This satisfies the conformal covariance rule

$$f \circ \mu(z, w) = |f'(z)|^{\bar{b}} |f'(w)|^{\bar{b}} \mu(f(z), f(w)),$$

where  $f$  is a linear fractional transformation (conformal transformation of the Riemann sphere). Conformal covariance implies that there exists  $c'' \in (0, \infty)$  such that for all  $z, w$ ,

$$\Psi(z, w) = c'' |z - w|^{-2\bar{b}}.$$

The probability measure  $\mu^\#(z, w)$ , which is invariant under linear fractional transformations, is called *whole plane  $SLE_\kappa$* . While we have defined  $\mu(z, w)$  as a limit, we could also imagine being able to define it directly. In this case, we get  $\mu_D(z, w)$  by a (normalized) boundary perturbation rule.

**Proposition 4.8.** *If  $D$  is a domain and  $z, w \in D$  are distinct, then*

$$\frac{d\mu_D(z, w)}{d\mu(z, w)}(\gamma) = 1\{\gamma \subset D\} \exp\left\{-\frac{\mathbf{c}}{2} \Lambda(\gamma, \partial D)\right\}$$

where  $\Lambda(\gamma, \partial D)$  is as defined in Proposition 3.11.

*Proof.* For  $r$  sufficiently large so that  $\gamma \subset \mathbb{D}_{-r}$ ,

$$\frac{d\mu_D(z, w)}{d\mu_{A_{-r}}(z, w)}(\gamma) = \exp\left\{\frac{\mathbf{c}}{2} m_{A_{-r}}(\gamma, A_{-r} \setminus D)\right\}.$$

Proposition 3.11 implies that as  $r \rightarrow \infty$ ,

$$m_{A_{-r}}(\gamma, A_{-r} \setminus D) = \log r - \Lambda(\gamma, \partial D) + o(1).$$

Therefore,

$$\frac{d\mu_D(z, w)}{r^{\mathbf{c}/2} d\mu_{A_{-r}}(z, w)}(\gamma) \sim \exp\left\{-\frac{\mathbf{c}}{2} \Lambda(\gamma, \partial D)\right\}, \quad r \rightarrow \infty.$$

□

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♣ While it might seem natural to define  $\mu(z, w)$  using whole plane *SLE* and then the proposition to define  $\mu_D(z, w)$ , there is a disadvantage in this approach. The reason is that it is not so easy to prove that  $\mu_D(z, w)$  satisfies the conformal covariance relation for conformal transformations of  $D$  since the quantity  $\Lambda(\gamma, \partial D)$  is not conformally invariant under transformations of  $D$ .

---

**Example** If  $\kappa = 2$ , then  $\Psi_D(z, w)$  is proportional to the usual Green's function for Brownian motion with Dirichlet boundary conditions. For this, it is well known that

$$\Psi_{A_{-r}}(0, 1) \sim r,$$

which agrees with the formula since  $\mathbf{c} = -2$ . Also,  $\tilde{b} = 0$  which implies that  $\Psi_D(z, w)$  is a conformal *invariant*. This is well known for the Green's function.

**4.3. Multiple paths.** Extending the definition of *SLE* $_{\kappa}$  to multiple is straightforward as in [13]. Suppose  $\mathbf{z} = (z^1, \dots, z^k)$ ,  $\mathbf{w} = (w^1, \dots, w^k)$  are distinct analytic points in a domain  $D$ . The points can be bulk or boundary points. The measure  $\mu_D(\mathbf{z}, \mathbf{w})$  is defined by giving its Radon-Nikodym derivative  $Y$  with respect to the product measure

$$\mu_D(z^1, w^1) \times \dots \times \mu_D(z^k, w^k).$$

Let  $\bar{\gamma} = (\gamma^1, \dots, \gamma^k)$  be a  $k$ -tuple of paths (modulo reparametrization) in  $D$  where  $\gamma^j$  goes from  $z^j$  to  $w^j$ . Then

$$(52) \quad Y = 1\{\gamma^j \cap \gamma^l = \emptyset, j \neq l\} \exp\left\{\frac{\mathbf{c}}{2} \sum_{j=2}^k m_D(\gamma^j, \gamma^1 \cup \dots \cup \gamma^{j-1})\right\}.$$

---

♣ One can consider the measure on multiple paths in the context of the  $\lambda$ -SAW. On the discrete level, the measure on a  $k$ -tuple of paths  $\bar{\omega} = (\omega^1, \dots, \omega^k)$  is

$$\exp\left\{-\beta(|\omega^1| + \dots + |\omega^k|) + \lambda m^{RW}(\omega^1 \cup \dots \cup \omega^k, D, n)\right\}.$$

The exponential factor on the right hand side of (52) compensates for overcounting of loops that intersect  $\bar{\gamma}$ .

5. CROSSING  $SLE_\kappa$  IN AN ANNULUS

In this section we study the measure  $\mu_{A_r}(1, e^{-r+i\theta})$  which is a measure on simple paths (modulo reparametrization)  $\eta$  from 1 to  $e^{-r+i\theta}$  in  $A_r$ . Let us recall the definition. Suppose  $D'$  is a simply connected subdomain of  $A_r$  that agrees with  $A_r$  in neighborhoods of 1 and  $w = e^{-r+i\theta}$ . Then if  $\eta$  is a curve in  $D'$  connecting 1 and  $w$ ,

$$\frac{d\mu_{A_r}(1, w)}{d\mu_{D'}(1, w)}(\eta) = \exp \left\{ -\frac{\mathbf{c}}{2} m_{A_r}(\eta, A_r \setminus D') \right\}.$$

We can write

$$(53) \quad m_{A_r}(\eta, A_r \setminus D') = \hat{m}_{A_r}(\eta, A_r \setminus D') + m^*(r),$$

where  $m^*(r)$  denotes the measure of the set of loops in  $A_r$  of nonzero winding number and  $\hat{m}_{A_r}(\eta, A_r \setminus D')$  is the measure of the set of loops of zero winding number that intersect both  $\eta$  and  $A_r \setminus D'$ . Here we use the fact that every loop of nonzero winding number intersects both  $\eta$  and  $A_r \setminus D'$ . (This construction assumes that there is a unique point on the Brownian loop that goes through the point  $\eta(t)$ . For each curve  $\eta$  this is true up to a set of loops of measure zero. See the discussion after Theorem 12 in [24].)

Let  $\gamma$  be the continuous preimage under  $\psi$  of  $\eta$  with  $\gamma(0) = 0$ , and let  $D$  be the simply connected domain containing  $\gamma$  such that  $\psi(D) = D'$ . Each loop  $\ell'$  in  $A_r$  has an infinite number of preimages under  $\psi$ . For each loop  $\ell'$  in  $A_r$  that intersects  $\eta$ , we choose a unique such preimage as follows. Consider the first time  $t$  such that  $\eta(t) \in \ell'$ . We make  $\ell'$  a rooted loop by choosing the root to be  $\eta(t)$ . Then we choose  $\ell$  to be the (rooted) preimage of  $\ell'$  that is rooted at  $\gamma(t)$ . The definition of  $\ell$  implies that if it is rooted at  $\gamma(t)$ , then

$$(54) \quad \ell \cap \tilde{\gamma}_t = \emptyset,$$

where, as before,

$$\tilde{\gamma}_t = \bigcup_{k \in \mathbb{Z} \setminus \{0\}} (\gamma_t + 2\pi k).$$

We will call a loop  $\ell$   $\gamma$ -good if it intersects  $\gamma$  and satisfies (54). Then  $\ell \leftrightarrow \ell'$  gives a bijection between  $\gamma$ -good loops in  $S_r$  and loops in  $A_r$  of zero winding number that intersect  $\eta$ .

If  $r > 0$ ,  $x \in \mathbb{R}$ , we define the measure  $\nu_{S_r}(0, x + ir)$  by the relation

$$\frac{d\nu_{S_r}(0, x + ir)}{d\mu_D(0, x + ir)}(\gamma) = \exp \left\{ -\frac{\mathbf{c}}{2} m_{S_r}(\gamma, S_r \setminus D; *) \right\}, \quad \gamma \subset D$$

where  $m_{S_r}(\gamma, S_r \setminus D; *)$  denotes the Brownian loop measure of  $\gamma$ -good loops in  $S_r$  that intersect both  $\gamma$  and  $S_r \setminus D$ . Recall that

$$\frac{d\mu_{S_r}(0, x + ir)}{d\mu_D(0, x + ir)}(\gamma) = \exp \left\{ -\frac{\mathbf{c}}{2} m_{S_r}(\gamma, S_r \setminus D) \right\},$$

This leads to an alternative, equivalent definition of  $\nu_{S_r}(0, x + ir)$ . Note that  $\psi \circ \gamma$  is a simple curve if and only if  $\gamma \cap \tilde{\gamma} = \emptyset$ .

**Definition** The measure  $\nu_{S_r}(0, x + ir)$  is the measure absolutely continuous with respect to  $\mu_{S_r}(0, x + ir)$  with Radon-Nikodym derivative

$$(55) \quad \frac{d\nu_{S_r}(0, x + ir)}{d\mu_{S_r}(0, x + ir)}(\gamma) = 1\{\gamma \cap \tilde{\gamma} = \emptyset\} \exp \left\{ \frac{\mathbf{c}}{2} m_{S_r}(\gamma) \right\},$$

where  $m_{S_r}(\gamma)$  is the measure of loops in  $S_r$  that intersect  $\gamma$  but are not  $\gamma$ -good. We call this *annulus  $SLE_\kappa$  in  $S_r$  from 0 to  $x + ir$* .

We can relate annulus  $SLE_\kappa$  in  $S_r$  to  $SLE_\kappa$  in  $A_r$  by conformal covariance. We define  $\nu_{A_r}(1, x)$  by

$$(56) \quad \begin{aligned} \nu_{A_r}(1, x) &= |\psi'(0)|^{-b} |\psi'(x + ir)|^{-b} e^{-\mathbf{c} m^*(r)/2} \psi \circ \nu_{S_r}(0, x + ir) \\ &= e^{br} e^{-\mathbf{c} m^*(r)/2} \psi \circ \nu_{S_r}(0, x + ir), \end{aligned}$$

We think of this as annulus  $SLE_\kappa$  from 1 to  $e^{-r+ix}$  restricted to curves of a particular winding number. The term  $e^{-\mathbf{c} m^*(r)/2}$  is discussed in Proposition 3.9. Annulus  $SLE_\kappa$  is obtained by summing over all winding numbers

$$(57) \quad \mu_{A_r}(1, e^{-r+i\theta}) = \sum_{k \in \mathbb{Z}} \nu_{A_r}(1, \theta + 2\pi k).$$

**5.1. Main result.** We will show that the partition function for annulus  $SLE$  on  $S_r$  can be given in terms of a functional of locally chordal  $SLE_\kappa$ . Recall the functions  $\mathbf{H}_I$  from Section 3.2,  $\mathbf{A}$  from (14), and  $\mathbf{L}$  from (32).

**Theorem 5.1.** *If  $\tilde{\Psi}(r, x) = \|\nu_{S_r}(0, x + ri)\|$ , then*

$$\tilde{\Psi}(r, x) = V(r, x) \Psi_{S_r}(0, x + ri).$$

Here

$$(58) \quad V(r, x) = \mathbb{E}^x \left[ \exp \left\{ -2b \int_0^r \mathbf{A}(r-s, X_s) ds \right\} \right],$$

where  $X_t, 0 \leq t \leq r$  satisfies

$$(59) \quad dX_t = [\mathbf{H}_I(r-t, X_t) - b\kappa \mathbf{L}(r-t, X_t)] dt + \sqrt{\kappa} dB_t,$$

and  $B_t$  is a standard Brownian motion. In particular,  $\tilde{\Psi}(r, x)$  is  $C^1$  in  $r$ ,  $C^2$  in  $x$  and  $\tilde{\Psi}(r, x) \leq \Psi_{S_r}(0, x + ri)$ .

We used the functional in (58) as our definition, but as we show now, it is the solution of a PDE. Let us define  $V(0, x) \equiv 1$ .

**Proposition 5.2.** *The function  $V(r, x)$  satisfies  $0 \leq V(r, x) \leq 1$ , is continuous on  $[0, \infty) \times (-\pi, \pi)$  and for  $r > 0$  satisfies the equation*

$$(60) \quad \dot{V} = -2b \mathbf{A} V + [\mathbf{H}_I - b\kappa \mathbf{L}] V' + \frac{\kappa}{2} V'',$$

where dot refers to  $r$ -derivatives and primes refer to  $x$ -derivatives.

Moreover, for fixed  $r$ ,  $x \mapsto V(r, x)$  is an odd function that is decreasing in  $|x|$ .

*Proof.* For  $r > 0$ , the function  $\mathbf{H}_I, \mathbf{L}$  are smooth and  $\mathbf{A} \geq 0$ . Hence (60) follows from the Feynman-Kac formula, see, e.g. [7, Section 6.5] or [10, Section 5.7.b]. Combining (15) with Lemma 3.15, we see that  $V(0+, x) = 1$  for  $|x| < 2\pi$ . For the last assertion, we use Proposition 3.4 which states that  $\mathbf{A}(r, x)$  is an increasing function of  $|x|$ . It is not difficult to see that if  $0 < x_1 < x_2 < \infty$ , then we can couple

process  $X_t^1, X_t^2$  on the same probability space, each satisfying (74) with  $X_0^j = x_j$  and such that  $|X_t^1| \leq |X_t^2|$  for all  $t$ . In this coupling, we have

$$\int_0^r \mathbf{A}(r-s, X_s^1) ds \leq \int_0^r \mathbf{A}(r-s, X_s^2) ds.$$

□

**5.2. Radon-Nikodym derivative.** Similarly to the approach for simply connected domains as in Section 3.6, we will find an appropriate nonnegative local martingale and use the Girsanov theorem to analyze the process weighted by the local martingale. Suppose  $(\Omega, \mathcal{F}, \hat{\mathbb{P}})$  is a probability space under which  $U_t = -B_t$  is a standard Brownian motion. Let  $g_t$  be the solution to the Loewner equation (16) producing the random curve  $\gamma$ . Let  $\gamma_t, \tilde{\gamma}_t, \hat{\gamma}_t$  be as above and fix  $r, z_0 = x + ir$ . The following proposition is the particular case of Section 3.6 for  $D = D_r, w = x + ir$ .

**Proposition 5.3.** *If*

$$J_t = |g_t'(z_0)|^b H_{\partial g_t(S_r \setminus \gamma_t)}(U_t, g_t(z_0))^b \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{H}}(\gamma_t, \mathbb{H} \setminus S_r) \right\},$$

*then  $J_t$  is a local martingale for  $t < \tau_r$ . Moreover, if one uses Girsanov, then under the weighted measure  $\gamma$  has the distribution of  $SLE_\kappa$  from 0 to  $x + ir$ .*

Let  $\mathbb{P}, \mathbb{E}$  denote expectations in the weighted measure under which  $\gamma$  has the distribution of  $\mu_{S_r}^\#(0, x + ir)$ .

If  $\ell$  is an (unrooted) loop in  $S_r$ , let

$$\begin{aligned} \tilde{s}(\ell) &= \min\{t : \ell \cap \tilde{\gamma}_t \neq \emptyset\}, \\ s(\ell) &= \min\{t : \ell \cap \gamma_t \neq \emptyset\}. \end{aligned}$$

It is not hard to show, using the fact that two-dimensional Brownian motion does not hit points, that the loop measure of the set of loops with  $s(\ell) = \tilde{s}(\ell) < \infty$  is zero. Let

$$\Lambda_t = \Lambda_t(\gamma_t, r) = 1\{T > t\} \exp\{m_t\},$$

where  $m_t = m_{t,r}(\gamma_t)$  denotes the measure of the set of loops in  $S_r$  that satisfy

$$\tilde{s}(\ell) < s(\ell) \leq t.$$

Theorem 5.1 can be rephrased as follows.

**Theorem 5.4.** *If  $\gamma$  has distribution  $\mu_{S_r}^\#(0, x + ir)$ , then*

$$(61) \quad \mathbb{E} \left[ \Lambda_{\tau_r}^{\mathbf{c}/2} \right] = V(r, x).$$

We will prove (61) in a series of propositions. Recall the definition of  $\mathbf{A}$  from (14). Let

$$\begin{aligned} R_t &= \operatorname{Re}[h_t(z_0)] - U_t, \quad V_t = V(r(t), R_t), \\ Q_t &= Q_{S_r \setminus \gamma_t}(\gamma(t), z_0; S_r \setminus \hat{\gamma}_t), \quad K_t = \exp \left\{ 2 \int_0^t \dot{r}(s) \mathbf{A}(r(s), R_s) ds \right\}. \end{aligned}$$

$$(62) \quad \begin{aligned} N_t &= \Lambda_t^{\mathbf{c}/2} Q_t^b K_t^{ab}, \quad O_t = K_t^{-ab} V_t, \\ M_t &= N_t O_t = \Lambda_t^{\mathbf{c}/2} Q_t^b V_t. \end{aligned}$$

By conformal invariance,

$$H_{\partial g_t(S_r \setminus \hat{\gamma}_t)}(U_t, g_t(z_0)) = Q_t H_{\partial g_t(S_r \setminus \gamma_t)}(U_t, g_t(z_0)).$$

Therefore,

$$|\phi'_t(U_t)| |\phi'_t(g_t(z_0))| H_{\partial h_t(S_r \setminus \hat{\gamma}_t)}(U_t^*, h_t(z_0)) = Q_t H_{\partial g_t(S_r \setminus \gamma_t)}(U_t, g_t(z_0)),$$

and hence

$$|\phi'_t(U_t)| |h'_t(z_0)| H_{\partial h_t(S_r \setminus \hat{\gamma}_t)}(U_t^*, h_t(z_0)) = |g'_t(z_0)| Q_t H_{\partial g_t(S_r \setminus \gamma_t)}(U_t, g_t(z_0)).$$

Therefore, we can write

$$J_t N_t = C_t(z_0) H_{\partial h_t(S_r \setminus \hat{\gamma}_t)}(U_t^*, h_t(z_0))^b,$$

where

$$C_t(z_0) = \phi'_t(U_t)^{-b} |h'_t(z_0)|^b \exp \left\{ -\frac{\mathbf{c}}{2} m_{\mathbb{H}}(\gamma_t, \mathbb{H} \setminus S_r) \right\} \Lambda_t^{\mathbf{c}/2} K_t^{ab},$$

Important observations are that  $C_t(z_0)$  is  $C^1$  in  $t$  and  $C_t(z_0) = C_t(z_0 + 2\pi)$ .

**Lemma 5.5.** *Suppose  $\gamma$  is a parametrized with  $\text{hcap}[\gamma(0, t]] = at$ . Let  $\tilde{\gamma}_t$  be as above and  $Q_t = Q_{S_r \setminus \gamma_t}(\gamma(t), x + ir; S_r \setminus \hat{\gamma}_t)$ . Then*

$$\partial_t Q_t|_{t=0} = -a\mathbf{A}(r, x).$$

*Proof.* See (13). □

**Proposition 5.6.**

- $N_t$  is a local martingale with respect to  $\mathbb{P}$  for  $t < T \wedge \tau_r$ . In particular,  $J_t N_t$  is a  $\hat{\mathbb{P}}$ -local martingale.
- With respect to  $\mathbb{P}^*$ , the curve  $\gamma$  at time  $t$  grows like  $SLE_\kappa$  from  $\gamma(t)$  to  $z_0$  in  $\tilde{S}_{t,r}$ .

*Proof.* This is a particular case of Section 3.7. □

Let  $\mathbb{P}^*, \mathbb{E}^*$  denote the probabilities and expectations obtained from  $\mathbb{P}$  by weighting by the local martingale  $N_t$ . This is the same as the measure obtained from  $\tilde{\mathbb{P}}$  by weighting by  $J_t N_t$ . We have seen that this is locally chordal  $SLE_\kappa$  and we can consider the path in the annulus parametrization.

**Proposition 5.7.** *Suppose  $V$  is as defined in (58). Then*

$$M_t^* = \exp \left\{ -2b \int_0^t \mathbf{A}(r-s, R_s^*) ds \right\} V(r-t, R_t^*),$$

*is a local martingale satisfying*

$$(63) \quad dM_t^* = \sqrt{\kappa} \frac{V'(r-t, R_t^*)}{V(r-t, R_t^*)} M_t^* dB_t.$$

*Moreover, if we weight by the local martingale using Girsanov theorem then with probability one in the weighted measure,  $R_{r-}^* = 0$ .*

*Proof.* The relation (63) follows immediately from Itô's formula. For the second claim, we note that in the unweighted measure we have  $R_{r-}^* = 0$ . Since  $V$  is decreasing in  $|x|$ , the additional drift given by the weighting points toward the origin. □

**Proposition 5.8.** *Suppose  $\gamma$  is a simple curve in  $S_r$  from 0 to  $z_0$  with  $T > \tau_r$ . Then,*

$$M_{T-} = \Lambda_{T-}^{\mathbf{c}/2} \in (0, \infty).$$

*Proof.* Easy estimates show that under the assumptions,  $Q_{\tau-} = 1$ ,  $r(\tau-) = 0$ ,  $R_{\tau-} = 0$ . Proposition 5.2, then gives  $V_{\tau-} = 1$ . The assumptions also imply that  $\text{dist}(\gamma_\tau, \tilde{\gamma}_\tau) > 0$ , which implies  $0 < \Lambda_{\tau-} < \infty$ .  $\square$

**Proposition 5.9.**  *$O_t, t < \tau_r \wedge T$  is a local martingale with respect to  $\mathbb{P}^*$ . In particular,  $M_t = N_t O_t, t < \tau_r \wedge T$  is a local martingale with respect to  $\mathbb{P}$ , and  $J_t M_t$  is a local martingale with respect to  $\hat{\mathbb{P}}$ .*

*Proof.* This is a restatement of the previous proposition in terms of the original parametrization.  $\square$

Let  $\mathbb{P}'$  denote the probability measure obtained from weighting by the local martingale  $M_t$ .

**Proposition 5.10.** *With  $\mathbb{P}'$  probability one,  $\tau_r < T$  and*

$$(64) \quad M_0 = V(r, x), \quad M_{\tau_r} = \Lambda_{\tau_r}^{c/2},$$

*In particular,*

$$V(r, x) = M_0 = \mathbb{E}[M_{\tau_r}] = \mathbb{E}\left[\Lambda_{\tau_r}^{c/2}\right].$$

*Proof.* The drift given by weighting by this martingale has a stronger drift to the origin than for locally chordal  $SLE_\kappa$  and we know that the latter one is good.  $\square$

---

♣ There is a general principle that is being used here that is worth stressing. Suppose  $M_t$  is a positive local martingale for  $t < \tau$ . The martingale convergence theorem implies that with probability one the limit  $M_\tau = \lim_{t \rightarrow \tau-} M_t$  exists. However, one cannot conclude  $\mathbb{E}[M_0] = \mathbb{E}[M_\tau]$  without more assumptions. One way to establish this equality is to consider the paths weighted by the local martingale. If  $M_\tau$  exists and is finite with probability one *in the new measure*, then we have uniform integrability and  $\mathbb{E}[M_0] = \mathbb{E}[M_\tau]$ . In our case we establish that in the new measure we have  $R_{\tau-}^* = 0$ . If the latter fact holds, then we use an easy deterministic estimate about curves to see that  $M_\tau < \infty$ .

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♣ At this point of the paper, the argument went very quickly, so it is a good idea to explain what has happened. The goal was to estimate the expectation (with respect to chordal  $SLE_\kappa$  in  $S_\tau$  from 0 to  $z_0$ ) of a random variable which is the exponential of the measure of a certain set of bad loops. For a curve  $\gamma$  and a loop  $l$ , we say that  $l$  is bad if  $l$  intersects  $\gamma$ , say at first time  $s'$ , but also intersects  $\tilde{\gamma}$  at first time  $s < s'$ . Suppose we have seen  $\gamma_t$ . Then we can split the bad loops into three sets: those with  $s < s' \leq t$ ; those with  $s < t < s'$ ; and those with  $t < s < s'$ . When we weight only by the first two sets of loops, we get the local martingale  $N_t$ , and the probability measure is locally chordal  $SLE_\kappa$ . Lemma 3.15 shows that this is supported simple curves with  $\gamma \cap \tilde{\gamma} = \emptyset$ . We then weight again to include the third set of loops and this leads to the function  $V$ . Since we can show directly that  $V$  is decreasing in  $|x|$  (and here we were lucky with the monotonicity proved in Proposition 3.4), we can see that the extra drift given by weighting by these loops points towards the origin and hence this measure is also supported simple curves with  $\gamma \cap \tilde{\gamma} = \emptyset$ . This allows us to justify the equation  $\mathbb{E}[M_0] = \mathbb{E}[M_{\tau_r}]$ .

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6. ANNULUS  $SLE_\kappa$  FROM 0 TO  $x$  IN  $S_r$ 

The same ideas can be used to analyze  $\nu_{S_r}(0, x)$  where  $0 < |x| < 2\pi$ . For ease, we will assume  $x > 0$ , but the  $x < 0$  case is done the same way. We will only sketch the ideas, since this case is also considered in [31]. Topological constraints restrict the values of  $x$ ; if  $|x| \geq 2\pi$  and  $\gamma$  connects 0 and  $x$ , then  $\eta = \psi \circ \gamma$  is not simple. As before, we define the measure by giving the Radon-Nikodym derivative as in (55)

$$\frac{d\nu_{S_r}(0, x)}{d\mu_{S_r}(0, x)}(\gamma) = Y(\gamma) = 1_{\{\gamma \cap \tilde{\gamma} = \emptyset\}} \exp \left\{ \frac{c}{2} m(\gamma) \right\}.$$

The relevant functions are the following.

$$(65) \quad \tilde{\mathbf{A}}(r, x) = \frac{\pi^2}{4r^2} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\sinh^2(\pi x/2r)}{\sinh^2(\pi^2 k/r) \sinh^2(\pi(x - 2\pi k)/2r)},$$

$$(66) \quad \tilde{\mathbf{H}}_I(r, x) = \frac{\pi}{2r} \coth\left(\frac{\pi x}{2r}\right) + \frac{\pi}{2r} \sum_{k=1}^{\infty} \left[ \coth\left(\frac{\pi(x + 2\pi)}{2r}\right) + \coth\left(\frac{\pi(x - 2\pi)}{2r}\right) \right].$$

$$(67) \quad \tilde{\mathbf{L}}(r, x) = -\frac{\partial_x H_{S_r}(0, x)}{b H_{S_r}(0, x)} = \frac{\pi}{r} \coth\left(\frac{\pi x}{2r}\right).$$

**Lemma 6.1.** *If  $y \in \mathbb{R}$  and*

$$f(x) = \frac{\sinh^2 x}{\sinh^2(x - y)} + \frac{\sinh^2 x}{\sinh^2(x + y)},$$

*then  $f$  is increasing for  $0 \leq x < y$ .*

*Proof.* Since

$$f(x) = \frac{\cosh(2x) - 1}{\cosh(2x - 2y) - 1} + \frac{\cosh(2x) - 1}{\cosh(2x + 2y) - 1},$$

it suffices to show for every  $y \in \mathbb{R}$ , that

$$F(x) = \frac{\cosh x - 1}{\cosh(x - y) - 1} + \frac{\cosh x - 1}{\cosh(x + y) - 1},$$

is increasing for  $0 \leq x < y$ . Using the sum rule, we get

$$\cosh(x - y) - 1 + \cosh(x + y) - 1 = 2 \cosh x \cosh y - 2,$$

Letting  $r = \cosh y \geq 1$ , we get

$$\begin{aligned} [\cosh(x - y) - 1][\cosh(x + y) - 1] &= (\cosh x \cosh y - 1)^2 - \sinh^2 x \sinh^2 y \\ &= (r \cosh x - 1)^2 - (r^2 - 1)(\cosh^2 x - 1) \\ &= \cosh^2 x - 2r \cosh x + r^2 \\ &= (\cosh x - r)^2. \end{aligned}$$

Therefore,

$$F(x) = \frac{2r (\cosh x - r^{-1}) (\cosh x - 1)}{(\cosh x - r)^2} = 2r e^{G(\cosh x)},$$

where

$$G(t) = \log\left(t - \frac{1}{r}\right) + \log(t + 1) - 2 \log(t - r).$$



Since  $r \geq 1$ ,  $G'(t) > 0$  for  $0 < t < r$  and hence  $G$  and  $F$  are increasing.  $\square$

**Definition** The function  $\tilde{V}(r, x)$ ,  $0 \leq r < \infty$ ,  $0 < x < 2\pi$  is defined by

$$(68) \quad \tilde{V}(r, x) = \mathbb{E}^x \left[ \exp \left\{ -2b \int_0^\sigma \tilde{\mathbf{A}}(r-s, X_s) ds \right\} \right],$$

where  $X_t$ ,  $0 \leq t < \sigma$  satisfies

$$(69) \quad dX_t = \left[ 2\tilde{\mathbf{H}}_I(r-t, X_t) - b\kappa \tilde{\mathbf{L}}(r-t, X_t) \right] dt + \sqrt{\kappa} dB_t,$$

with  $B_t$  is a standard Brownian motion and

$$\sigma = \inf\{t : X_t = 0\}.$$

We define  $V(r, 0) = 1$ .

An important observation is that if  $X_t$  satisfies (69) with  $X_0 \in [0, 2\pi)$ , then with probability one  $\sigma < r$  and  $X_t \in [0, 2\pi)$  for  $0 \leq t \leq \sigma$ . Hence this is well defined. The function  $\tilde{V}$ ,  $0 < r < \infty$ ,  $0 < x < 2\pi$

$$(70) \quad \dot{\tilde{V}}(r, x) = -2b \tilde{\mathbf{A}}(r, x) \tilde{V}(r, x) + \left[ 2\tilde{\mathbf{H}}_I(r, x) - b\kappa \tilde{\mathbf{L}}(r, x) \right] \tilde{V}'(r, x) + \frac{\kappa}{2} \tilde{V}''(r, x),$$

where dot refers to  $r$ -derivatives and primes refer to  $x$ -derivatives.

The definition of  $\mu_{A_r}(1, e^{i\theta})$  for  $0 < \theta < 2\pi$  takes a little more thought. We write

$$\mu_{A_r}(1, e^{i\theta}) = \mu_{A_r}(1, e^{i\theta}; R) + \mu_{A_r}(1, e^{i\theta}; L)$$

where  $\mu_{A_r}(1, e^{i\theta}; R)$  denotes  $\mu_{A_r}(1, e^{i\theta})$  restricted to curves  $\eta$  such that the origin lies in the component of  $\mathbb{D} \setminus \eta$  whose boundary includes  $(e^{i\theta}, 1)$ . Then similarly to (56) we write

$$\frac{d\mu_{A_r}(1, e^{ix})}{d[\psi \circ \nu_{S_r}(0, x)]}(\eta) = e^{br} \exp \left\{ -\frac{\mathbf{c}}{2} m^*(r, \eta) \right\},$$

where  $m^*(r, \eta)$  denotes the measure of the set of loops in  $A_r$  of nonzero winding number that intersect  $\eta$ . Unlike the crossing case, the quantity on the right hand side depends on  $\eta$ . It is not hard to give an expression for this. Let  $\tilde{A}$  denote the component of  $A_r \setminus \eta$  that contains  $C_r$  on its boundary. let  $r_\gamma = r_{\gamma, r}$  be such that  $\tilde{A}$  is conformally equivalent to  $A_{r_\gamma}$ . Then  $m^*(r_\gamma)$  denotes the measure of loops of nonzero winding number in  $\tilde{A}$  and hence

$$m^*(r, \eta) = m^*(r) - m^*(r_\gamma).$$

We could have also defined  $\mu_{A_r}(1, e^{i\theta})$  by

$$\frac{d\mu_{A_r}(1, e^{i\theta})}{d\mu_{\mathbb{D}}(1, e^{i\theta})}(\gamma) = 1\{\gamma \subset A_r\} \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\gamma, \mathbb{D} \setminus A_r) \right\}.$$

Since these both satisfy (48), they must give the same measure.

There is a subtlety that is worth mentioning. Let  $J$  denotes the closed disk about 0 of radius  $e^{-r}$  so that  $A_r = \mathbb{D} \setminus J$  and  $f : A_r \rightarrow D \subset \mathbb{D}$  is a conformal transformation that sends  $\partial\mathbb{D}$  to  $\partial D$ . Informally we can write  $f(J) = K$  where  $D = \mathbb{D} \setminus K$ , but the conformal map  $f$  is not defined on  $J$ . If  $z, w \in \partial\mathbb{D}$ , then

$$f \circ \mu_{A_r}(z, w) = |f'(z)|^b |f'(w)|^b \mu_D(f(z), f(w)).$$

This gives one way to construct  $\mu_D(f(z), f(w))$ . But we also define it by the Radon-Nikodym derivative. Suppose  $\gamma \subset A_r$ , then  $f \circ \gamma \subset D$  and

$$\frac{d\mu_{A_r}(z, w)}{d\mu_{\mathbb{D}}(z, w)}(\gamma) = \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\gamma, J) \right\},$$

$$\frac{d\mu_D(f(z), f(w))}{d\mu_{\mathbb{D}}(f(w), f(w))}(f \circ \gamma) = \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(f \circ \gamma, K) \right\}.$$

However, since  $f$  is not a conformal transformation of the disk, we have no reason to believe that  $m_{\mathbb{D}}(\gamma, J) = m_{\mathbb{D}}(f \circ \gamma, K)$ .

## 7. ANNULUS $SLE_{\kappa}$ IN $A_r$

In the last section we considered the measure  $\nu_{S_r}(0, x + ir)$  which we called annulus  $SLE_{\kappa}$  in the strip  $S_r$ . This was analyzed by comparing the measure to chordal  $SLE_{\kappa}$  in  $S_r$ . Recall from (56) that the measure on paths given by annulus  $SLE_{\kappa}$  restricted to a particular winding number is

$$\nu_{A_r}(1, x) = e^{br} e^{-\mathbf{c}m^*(r)/2} \psi \circ \nu_{S_r}(0, x + ir).$$

The term  $e^{br} = |\psi'(x + ir)|^b$  comes from conformal covariance and  $m^*(r)$  is the Brownian loop measure of loops in  $A_r$  of nonzero winding number. Annulus  $SLE_{\kappa}$  in  $A_r$  from 1 to  $e^{-r+i\theta}$  is obtained from summing over all winding numbers

$$\mu_{A_r}(1, e^{-r+i\theta}) = \sum_{k \in \mathbb{Z}} \nu_{A_r}(1, \theta + 2\pi k).$$

In this section we will compare  $\nu_{A_r}(1, x)$  and  $\mu_{A_r}(1, e^{-r+i\theta})$  to radial  $SLE_{\kappa}$  in order to derive PDEs for the annulus partition functions. We will rederive an equation from [31].

**7.1. The differential equation.** Let  $\tilde{\Psi}(r, x) = |\nu_{S_r}(0, x + ir)|$  be as in the previous section, and let  $\hat{F}(r, x)$  and  $F(r, x)$  denote the partition functions associated to annulus  $SLE_{\kappa}$  and annulus  $SLE_{\kappa}$  restricted to a particular winding number, respectively. In other words,

$$F(r, x) = |\nu_{A_r}(0, x)| = \beta(r) \tilde{\Psi}(r, x),$$

where

$$\beta(r) = \exp \left\{ br - \frac{\mathbf{c}m^*(r)}{2} \right\} = e^{br} e^{-\mathbf{c}r/12} \exp \left\{ \mathbf{c} \int_0^r \delta(s) ds \right\},$$

and

$$\hat{F}(r, x) = \Psi_{A_r}(1, e^{-r+ix}) = \sum_{k=-\infty}^{\infty} F(r, x + 2\pi k).$$

Since

$$(71) \quad F(r, x) = \beta(r) \tilde{\Psi}(r, x) \leq \beta(r) \Psi_{S_r}(0, e^{x+ir}) \asymp \beta(r) r^{-2b} \left[ \cosh \left( \frac{\pi x}{2r} \right) \right]^{-2b},$$

we see that  $\hat{F}(r, x) < \infty$ . Recall the functions  $\mathbf{J}$  and  $\mathbf{H}_I$  from Section 3.2. As before, we will use dot for  $r$ -derivatives and primes for  $x$ -derivatives.

**Proposition 7.1.**  *$F$  satisfies the differential equation*

$$(72) \quad \dot{F} = \frac{\kappa}{2} F'' + \mathbf{H}_I F' + \left[ b \mathbf{H}'_I + b + \tilde{b} (6\Gamma(r) - 1) - \frac{b}{r} \right] F.$$

Moreover,  $\hat{F}$  satisfies the same equation.

---

♣As in [31], we check that this is consistent with what we know about  $\kappa = 2$  for which  $b = 1, \tilde{b} = 0$ . For  $\kappa = 2$ , from arguments based on the loop-erased walk we know that the  $SLE_2$  partition function for any domain  $D$  should be given by a multiple of the excursion Poisson kernel,  $H_{\partial D}(z, w)$ . Hence a solution to (72) should be

$$\begin{aligned} \hat{F}(r, x) &= H_{\partial A_r}(1, e^{-r+ix}) \\ &= e^r \sum_{k \in \mathbb{Z}} H_{\partial S_r}(0, x + 2\pi k + i) \\ &= \frac{1}{2} e^r \mathbf{J}(r, x). \end{aligned}$$

If this is so, then Proposition 7.1 implies that if  $\Phi(r, x) = 2re^{-r}\hat{F}(r, x) = r\mathbf{J}(r, x)$ , then  $\dot{\Phi} = \Phi'' + \mathbf{H}_I \Phi' + \mathbf{H}'_I \Phi$ . But we noted this relation in (11).

---

We set

$$\begin{aligned} \alpha(r) &= b + \tilde{b} [6\Gamma(r) - 1] = b - \tilde{b} + (2b + \mathbf{c}) \Gamma(r), \\ \Theta(r, x) &= \Theta_\kappa(r, x) = \mathbf{H}'_I(r, x) + \frac{\alpha(r)}{b} - \frac{1}{r}, \end{aligned}$$

which allows us to write (72) as

$$(73) \quad \dot{F} = \frac{\kappa}{2} F'' + \mathbf{H}_I F' + b \Theta F.$$

We will establish (73) for  $F$ . We note that  $F(r, x)$  is  $C^1$  in  $r$  and  $C^2$  in  $x$ . Indeed, in the previous section we showed the same for  $\tilde{\Psi}(r, x)$ , and it is easy to show that  $m^*(r)$  is continuous in  $r$  and hence  $\beta(r)$  is differentiable. Hence we can use Itô's formula freely. Before proceeding, let us show that this will also imply the result for  $\hat{F}$ . Let  $X_t^{(r)}, 0 \leq t \leq r$ , denote a solution to the SDE

$$(74) \quad dX_t^{(r)} = \mathbf{H}_I(r - t, X_t^{(r)}) dt + \sqrt{\kappa} dB_t.$$

Then (73) and the Feynman-Kac formula implies that for  $r > t > 0$ ,

$$F(r, x) = \mathbb{E}^x \left[ F(r - t, X_t^{(r)}) \exp \left\{ b \int_0^t \Theta(r - s, X_s^{(r)}) ds \right\} \right],$$

where  $\mathbb{E}^x$  denotes expectations assuming  $X_0^{(r)} = x$ . (We do not need to consider the delicate case  $t = r$  so the conditions for the Feynman-Kac formula are easily verified.) Using this and (71), we can see that

$$\hat{F}(r, x) = \mathbb{E}^x \left[ \hat{F}(r - t, X_t^{(r)}) \exp \left\{ b \int_0^t \Theta(r - s, X_s^{(r)}) ds \right\} \right],$$

and by invoking the Feynman-Kac theorem again, we see that  $\hat{F}$  also satisfies (73).

To prove the proposition for  $F$  we compare radial  $SLE_\kappa$  (from 1 to 0 in  $\mathbb{D}$ ) and annulus  $SLE_\kappa$  (from 1 to  $e^{-r+i\theta}$  in  $A_r$ ) for  $\kappa = 2/a \leq 4$ . These measures, restricted to an initial segment of the path which has not reached  $C_r$ , are absolutely continuous.

It is useful to view radial  $SLE_\kappa$  raised onto the covering space  $\mathbb{H}$  as we now describe. We describe radial  $SLE_\kappa$  as a periodic function on  $\mathbb{H}$ . We return to the radial Loewner equation (17) which we write as

$$(75) \quad \partial G_t(z) = \frac{a}{2} \cot_2(G_t(z) - U_t), \quad G_0(z) = z,$$

and view as an equation on  $\mathbb{H}$ . Here  $U_t$  is a standard Brownian motion with  $U_0 = 0$  and  $\cot_2(z) = \cot(z/2)$ . There is a corresponding curve  $\gamma$  in  $\mathbb{H}$  such that with probability one, for all  $t$ ,  $\gamma_t \cap \tilde{\gamma}_t = \emptyset$ . Let  $\eta_t = \psi \circ \gamma_t$  and define  $\tilde{g}_t$  by

$$\tilde{g}_t(e^{iz}) = e^{iG_t(z)}.$$

Then  $\tilde{g}_t$  is the unique conformal transformation of  $\mathbb{D} \setminus \eta_t$  onto  $\mathbb{D}$  with  $\tilde{g}_t(0) = 0$ ,  $\tilde{g}'_t(0) > 0$ . In fact,  $\tilde{g}'_t(0) = e^{at/2}$ . Radial  $SLE$  is usually described in terms of the differential equation for  $\tilde{g}_t$ .

We now relate the equation (75) to the annulus Loewner equation described in Section 3.9. We fix an “initial radius”  $r$ . As in that section, we define  $r(t)$  and  $h_t$  by saying that

$$h_t : S_r \setminus \hat{\gamma}_t \rightarrow S_{r(t)}$$

is a conformal transformation satisfying  $h_t(z+2\pi) = h_t(z)+2\pi$  with  $h_t(\pm\infty) = \pm\infty$  and  $h_t(\gamma(t)) = U_t$ . Recall that

$$\partial_t h_t(z) = 2\dot{r}(t) \mathcal{H}_{r(t)}(h_t(z) - U_t).$$

We define  $\Phi_t$  by

$$h_t = \Phi_t \circ G_t,$$

and define  $\tilde{h}_t, \tilde{\Phi}_t$  by

$$\tilde{h}_t(e^{iz}) = e^{ih_t(z)}, \quad \tilde{\Phi}_t(e^{iz}) = e^{i\Phi_t(z)},$$

so that  $\tilde{h}_t = \tilde{\Phi}_t \circ \tilde{g}_t$ . Note that  $\tilde{h}_t$  is the unique conformal transformation of  $A_r \setminus \eta_t$  onto  $A_{r(t)}$  with  $\tilde{h}_t(\eta(t)) = e^{iU_t}$ . Also, for real  $x$ ,

$$|\tilde{\Phi}'_t(e^{ix})| = \Phi'_t(x).$$

We note that (12) implies that for  $r(t) \geq 2$  and  $x \in \mathbb{R}$ ,

$$|\Phi'_t(x)| = 1 + O(e^{-r(t)}), \quad |\Phi''_t(x)| = O(e^{-r(t)}).$$

As in that section, we let

$$\sigma_s = \inf\{t : r(t) = s\}, \quad h_s^* = h_{\sigma_s},$$

and we set

$$\tilde{h}_s^* = \tilde{h}_{\sigma_s}, \quad \tilde{\Phi}_s^* = \tilde{\Phi}_{\sigma_s}, \quad \tilde{g}_s^* = \tilde{g}_{\sigma_s}.$$

**Lemma 7.2.** *Under the assumptions above,*

$$\partial_t |\tilde{\Phi}'_t(1)| \big|_{t=0} = \partial_t \Phi'_t(0) \big|_{t=0} = a \left[ \Gamma(r) - \frac{1}{2r} \right],$$

$$\partial_s |(\phi_{r-s}^*)'(1)| \big|_{s=0} = 2\Gamma(r) - \frac{1}{r}.$$

Here  $\Gamma(r)$  is as defined in (20).

*Proof.* Note that

$$\frac{a}{2} \cot_2(z) = a \left[ \frac{1}{z} - \frac{z}{12} \right] + O(|z|^3).$$

Recall that  $\dot{r}(0) = -a/2$  and from (40) we have

$$-a\mathcal{H}_r(z) = a \left[ \frac{1}{z} + z \left( \Gamma(r) - \frac{1}{12} - \frac{1}{2r} \right) \right] + O(|z|^3),$$

Therefore, the first result follows from (43) and the second from  $\tilde{\phi}_t = \tilde{\phi}_{r(t)}^*$ .  $\square$

Let  $\mu_1, \mu_2, \mu_3$  denote  $\mu_{\mathbb{D}}(1, -1)$ ,  $\mu_{\mathbb{D}}(1, 0)$ , and  $\nu_{A_r}(1, x)$ , respectively, and let  $w = e^{-r+ix}$ . Let  $z_t = e^{iU_t} = \tilde{g}_t(\eta(t))$ ,  $\zeta_t = \tilde{g}_t(-1)$ ,  $w_t = \tilde{g}_t(w)$ ,  $x_t = \arg w_t$ , where  $x_t$  is chosen to be continuous in  $t$  with  $x_0 = x$ . If  $t < \tau_r$ , these three measures are absolutely continuous with respect to each other and we can write down the Radon-Nikodym derivatives. Recall from Section 3.8 that

$$\frac{d\mu_2}{d\mu_1}(\eta_t) = \frac{\tilde{g}'_t(0)^{\tilde{b}} \Psi_{\mathbb{D}}(z_t, 0)}{|\tilde{g}'_t(-1)|^{\tilde{b}} \Psi_{\mathbb{D}}(z_t, \zeta_t)} = \frac{\tilde{g}'_t(0)^{\tilde{b}}}{|\tilde{g}'_t(-1)|^{\tilde{b}} \Psi_{\mathbb{D}}(z_t, \zeta_t)}.$$

Using similar reasoning for annulus *SLE* with respect to chordal *SLE*, we get

$$\frac{d\mu_3}{d\mu_1}(\eta_t) = \frac{|\tilde{g}'_t(w)|^{\tilde{b}} |\nu_{\tilde{g}_t(A_r)}(z_t, x_t)| \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\mathbb{D}_r, \eta_t) \right\}}{|\tilde{g}'_t(-1)|^{\tilde{b}} \Psi_{\mathbb{D}}(z_t, \zeta_t)}.$$

We have not actually defined the measure  $\nu_{\tilde{g}_t(A_r)}(z_t, x_t)$ , so let us describe it now. Since  $\tilde{g}_t(A_t)$  is a conformal annulus whose outer boundary is the unit circle, we can define  $\nu_{\tilde{g}_t(A_r)}(z_t, x_t)$  in the same way that  $\nu_{A_r}(1, x)$  was defined. In other words, it is annulus *SLE* between  $z_t$  and  $w_t$  in the conformal annulus  $\tilde{g}_t(A_r)$  restricted to curves of a particular winding number. The choice of winding number is determined by continuity in  $t$ .

Let

$$M_t = \frac{d\mu_3}{d\mu_2}(\eta_t) = \tilde{g}'_t(0)^{-\tilde{b}} |\tilde{g}'_t(w)|^{\tilde{b}} |\nu_{\tilde{g}_t(A_r)}(z_t, x_t)| \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\mathbb{D}_r, \eta_t) \right\}.$$

We see that  $M_t$  is a local martingale for radial *SLE* $_{\kappa}$ . Let  $\tilde{h}_t = \tilde{\phi}_t \circ \tilde{g}_t$ . Conformal covariance implies that

$$|\nu_{\tilde{g}_t(A_r)}(z_t, x_t)| = |\tilde{\phi}'_t(e^{iU_t})|^{\tilde{b}} |\tilde{\phi}'_t(\tilde{g}_t(w))|^{\tilde{b}} \Psi_{A_{r(t)}}(e^{iU_t}, \tilde{\phi}_t(\tilde{g}_t(w))).$$

Therefore,

$$M_t = \tilde{g}'_t(0)^{-\tilde{b}} |\tilde{\phi}'_t(e^{iU_t})|^{\tilde{b}} \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\mathbb{D}_r, \eta_t) \right\} |\tilde{h}'_t(w)|^{\tilde{b}} F(r(t), R_t),$$

where

$$R_t = \operatorname{Re}[h_t(z) - U_t].$$

We have shown the following.

**Proposition 7.3.** *If  $U_t$  is a standard Brownian motion, then*

$$M_t = J(t) F(r(t), R_t),$$

*is a local martingale where*

$$J(t) = \tilde{g}'_t(0)^{-\tilde{b}} |\tilde{\phi}'_t(e^{iU_t})|^{\tilde{b}} \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\mathbb{D}_r, \eta_t) \right\} |\tilde{h}'_t(w)|^{\tilde{b}},$$

*and  $R_t = \operatorname{Re}[h_t(x + ir) - U_t]$ .*

Using this proposition, we can write down a differential equation for  $F(s, x)$ . It is convenient to write the local martingale in the annulus parametrization. Let  $U_s^* = U_{\sigma_{r-s}}$ . Then  $U_s^*$  is a martingale with quadratic variation  $\sigma_{r-s}$ . Let  $R_s^* = h_{r-s}^*(z) - U_s^*$ . Then

$$dR_s^* = \partial_s h_{r-s}^*(z) dt + dU_s^*$$

Note that

$$\partial_s h_{r-s}^*(z) = \mathbf{H}_I(r, x), \quad \partial_s \sigma_{r-s} |_{s=r} = 2/a = \kappa.$$

The last proposition becomes the following.

**Proposition 7.4.** *For fixed  $r > 0$ , if  $R_s^* = h_{r-s}^*(z) - U_s^*$  and*

$$M_s^* = J^*(r-s) F(r-s, R_s^*),$$

where

$$J^*(s) = (\tilde{g}_s^*)'(0)^{-\tilde{b}} |(\tilde{\phi}_s^*)'(e^{iU_{\sigma_s}})|^b \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\mathbb{D}_s, \eta_{\sigma_s}) \right\} |(\tilde{h}_s^*)'(w)|^b,$$

then  $M_s^*$  is a martingale.

If we write dots for  $r$ -derivatives, then by considering the martingale at time  $s = 0$  and using Itô's formula, we get the equation

$$\dot{F} = \frac{\kappa}{2} F'' + \mathbf{H}_I F' - \dot{J} F,$$

where

$$-\dot{J}(r) = \partial_s J(r-s) |_{s=0}.$$

All that remains for proving Proposition 7.1 is to calculate  $-\dot{J}(r)$ .

**Lemma 7.5.**

$$-\dot{J}(r) = \alpha(r) + b \mathbf{H}_I'(r, x) - \frac{b}{r}.$$

*Proof.* We have parametrized radial  $SLE_{\kappa}$  such that

$$\partial_t \tilde{g}_t'(0) = (a/2) g_t'(0),$$

and hence

$$\begin{aligned} \partial_t \log [\tilde{g}_t'(0)^{-\tilde{b}}] |_{t=0} &= -\frac{a\tilde{b}}{2} = \frac{b(1-a)}{4}, \\ \partial_s \log [(\tilde{g}_{r-s}^*)'(0)^{-\tilde{b}}] |_{s=0} &= -\tilde{b}. \end{aligned}$$

The relationship between the Brownian loop measure and the bubble measure implies

$$\partial_s \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\mathbb{D}_r, \eta_{\sigma_{r-s}}) |_{s=0} = \frac{2}{a} \partial_t \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\mathbb{D}_r, \eta_t) |_{t=0} = \mathbf{c} \Gamma_{\mathbb{D}}(1, A_r) = \mathbf{c} \Gamma(r).$$

Lemma 7.2 shows that

$$\partial_s \log |(\tilde{\phi}_{r-s}^*)'(U_s^*)|^b |_{s=0} = -\frac{b}{r} + 2b\Gamma(r).$$

Recall that if  $z = x + ir$ ,  $w = e^{iz} = e^{-r+ix}$ ,  $\tilde{h}_s^*(w) = e^{ih_s^*(z)}$ , and hence

$$|(\tilde{h}_{r-s}^*)'(w)| = e^r e^{-\text{Im}[h_{r-s}^*(z)]} |(h_{r-s}^*)'(z)| = e^s |(h_{r-s}^*)'(z)|.$$

Therefore, using (42), we have

$$\partial_s \log |(\tilde{h}_{r-s}^*)'(w)|^b |_{s=0} = b + b \mathbf{H}_I'(r, x).$$

Adding all the terms, gives

$$b - \tilde{b} + (\mathbf{c} + 2b) \Gamma(r) + b\mathbf{H}'_I(r, x) - \frac{b}{r} = \alpha(r) + b\mathbf{H}'_I(r, x) - \frac{b}{r}.$$

□

**7.2. Comparing annulus SLE with radial SLE large  $r$ .** We now have an essentially complete description of annulus  $SLE_\kappa$ . In our framework, this is a measure  $\mu_{A_r}(1, e^{-r+ix})$  of total mass  $\hat{F}(r, x)$ . In the next subsection, we will prove the following.

**Theorem 7.6.** *There exist  $c_*, q \in (0, \infty)$  such that uniformly in  $x$ ,*

$$\hat{F}(r, x) = c_* r^{\mathbf{c}/2} e^{(b-\tilde{b})r} [1 + O(e^{-qr})], \quad r \rightarrow \infty.$$

Let  $\mu_2 = \mu_{\mathbb{D}}(1, 0)$  as before and let  $\mu_4 = \mu_{A_r}(1, e^{-r+ix})$  with corresponding probability measure  $\mu_4^\#$ . Suppose  $t$  is sufficiently small so that a curve starting at the unit disk cannot reach  $C_r$  by time  $t$ . Then, similarly to the previous section, if  $w = e^{-r+ix}$  and  $\zeta_t = \tilde{g}_t(\gamma(t))$ , we can write

$$\frac{d\mu_4}{d\mu_2}(\eta_t) = \frac{|\tilde{g}'_t(w)|^b}{\tilde{g}'_t(0)^{\tilde{b}}} \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\mathbb{D}_r, \eta_t) \right\} |\mu_{\tilde{g}_t(A_r \setminus \eta_t)}(\zeta_t, \tilde{g}_t(w))|.$$

**Proposition 7.7.** *There exists  $q > 0$  such that uniformly over  $t > 0$ ,  $r \geq \frac{ta}{2} + 2$ , and all initial segments  $\gamma_t$ ,*

$$\frac{d\mu_4}{d\mu_2}(\eta_t) = c_* e^{r(b-\tilde{b})} r^{\mathbf{c}/2} [1 + O(e^{-qu})],$$

where  $u = r - \frac{ta}{2}$ . In particular, there exists  $c < \infty$  such that

$$\left| \frac{d\mu_4^\#}{d\mu_2^\#}(\eta_t) - 1 \right| \leq c e^{-qu}.$$

*Proof.* Let  $\phi_t : \tilde{g}_t(A_r \setminus \eta_t) \rightarrow A_s$  be a conformal transformation sending  $C_0$  to  $C_0$  and let  $h_t = \phi_t \circ \tilde{g}_t$ . Using conformal covariance, we write

$$\frac{d\mu_4}{d\mu_2}(\eta_t) = \frac{|h'_t(w)|^b |\phi'_t(\zeta_t)|^b}{\tilde{g}'_t(0)^{\tilde{b}}} \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\mathbb{D}_r, \eta_t) \right\} |\mu_{A_s}(h_t(w), \phi_t(\zeta_t))|.$$

Suppose  $t$  is given,  $r \geq \frac{ta}{2} + 2$  and let  $u = r - \frac{ta}{2}$ . Recall that in our normalization  $\tilde{g}'_t(0) = e^{at/2}$ . Using the deterministic estimates from Lemma ??, we get

$$|h'_t(w)|^b = e^{atb/2} [1 + O(e^{-u})],$$

$$|\phi'_t(\zeta_t)|^b = 1 + O(e^{-u}),$$

$$\exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\mathbb{D}_r, \eta_t) \right\} = (r/u)^{\mathbf{c}/2} [1 + O(e^{-u})],$$

$$s = u + O(e^{-u}),$$

$$|\mu_{A_s}(h_t(w), \phi_t(\zeta_t))| = c_* u^{\mathbf{c}/2} e^{(b-\tilde{b})u} [1 + O(e^{-u})].$$

Combining these estimates gives the first equality and since the dominant factor does not depend on the initial segment, the second equality follows. □

**7.3. Proof of Theorem 7.6.** Let

$$\begin{aligned}\lambda(r) &= r^b \exp \left\{ - \int_1^r \alpha(s) ds \right\}, \\ K_1(r, x) &= \lambda(r) F(r, x), \\ K(r, x) &= \lambda(r) \Psi_{A_r}(1, e^{-r+ix}) = \lambda(r) \hat{F}(r, x) = \sum_{k \in \mathbb{Z}} F(r, x + 2\pi k).\end{aligned}$$

Proposition 3.5 gives

$$\alpha(r) = b - \tilde{b} + (2b + \mathbf{c}) \Gamma(r) = b - \tilde{b} + \frac{2b + \mathbf{c} + O(e^{-r})}{2r},$$

and hence

$$\lambda(r) = \lambda_\infty r^{-\mathbf{c}/2} e^{(\tilde{b}-b)r} [1 + O(r^{-1}e^{-r})].$$

Therefore, to prove Theorem 7.6, it suffices to show that there exists  $K_\infty \in (0, \infty)$  and  $c < \infty$  such that

$$|K(r, x) - K_\infty| \leq c e^{-r}.$$

Since

$$\dot{\lambda}(r) = \lambda(r) \left[ \frac{b}{r} - \alpha(r) \right],$$

it follows from Proposition 7.1 that  $K_1, K$  satisfy

$$\begin{aligned}\dot{K}_1 &= \frac{\kappa}{2} K_1'' + \mathbf{H}_I K_1' + b \mathbf{H}_I' K_1, \\ (76) \quad \dot{K} &= \frac{\kappa}{2} K'' + \mathbf{H}_I K' + b \mathbf{H}_I' K.\end{aligned}$$

The Feynman-Kac representation tells us that if  $r > t > 0$ ,

$$(77) \quad K(r, x) = \mathbb{E}^x \left[ K(r-t, X_t^{(r)}) \exp \left\{ \int_0^t \mathbf{J}(r-s, X_s^{(r)}) ds \right\} \right],$$

where  $X_t^{(r)}$  satisfies (74). Recall that

$$(78) \quad |\mathbf{H}_I(r, x)|, |\mathbf{J}(r, x)| \leq c_0 e^{-r}, \quad r \geq 1,$$

which implies

$$(79) \quad \left| \int_0^{r-t} \mathbf{H}_I'(z, X_s^{(r)}) ds \right| \leq c e^{-t}, \quad \exp \left\{ b \int_0^{r-1} \mathbf{H}_I'(z, X_s^{(r)}) ds \right\} \asymp 1,$$

and for  $r \geq 1$ ,

$$K(r, x) \asymp \mathbb{E}^x [K(X_{r-1})] \leq c \mathbb{E}^x [\exp \{-2bX_{r-1}\}],$$

where  $X_s = X_s^{(r)}$ .

---

♣ Those experienced with PDEs can probably skip the rest of this section. Since  $|\mathbf{H}_I| + |\mathbf{H}_I'| = O(e^{-r})$ , for large  $r$  the equation (76) is well approximated by the standard heat equation  $\dot{K} = \frac{\kappa}{2} K''$ . One just needs to keep track of the error terms. I have taken a probabilistic approach using coupling, but this is just personal preference.

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We will use standard coupling techniques to analyze the equation. Here is the basic estimate. We write  $x \equiv y$  if  $(y-x)/2\pi \in \mathbb{Z}$ .



**Lemma 7.8.** *There exist  $u > 0, c < \infty$  such that the following holds. Suppose  $r \geq 2$  and  $X_t = X_t^{(r)}, Z_t = Z_t^{(r)}$  are independent solutions to (74) with  $X_0 = x, Z_0 = y$  with  $x \leq y < x + 2\pi$ . Let*

$$T = \inf\{t : X_t \equiv Z_t\}.$$

*Then,*

$$\mathbb{P}\{T \geq t\} \leq c e^{-ut},$$

*and if  $t \leq 1$ ,*

$$\mathbb{P}\{T \geq t^2\} \leq c t^{-1} (y - x).$$

*If we define*

$$Y_t = \begin{cases} Z_t & t < T \\ Z_T + (X_t - X_T) & t \geq T \end{cases}$$

*Then  $Y_t$  satisfies (74) with  $Y_0 = y$  and  $Y_t \equiv X_t$  for  $t \geq T$ .*

**Proposition 7.9.**

- *There exist  $0 < c_1 < c_2 < \infty$  such that*

$$(80) \quad c_1 \leq K(r, x) \leq c_2, \quad r \geq 1, x \in \mathbb{R}.$$

- *There exists  $K_\infty \in (0, \infty)$  and  $u > 0$  and  $c < \infty$  such that*

$$|K(r, x) - K_\infty| \leq c e^{-ur}.$$

*Proof.* For fixed  $r$ ,  $x \leq y \leq x + 2\pi$ , let  $X_t, Y_t, T$  be as in Lemma 7.8 and let  $m_-(r), m_+(r)$  be the minimum and maximum, respectively, of  $K(r, x)$  for  $0 \leq x \leq 2\pi$ . From (77) and (79), we see that  $c_1 m_-(1) \leq K(r, x) \leq c_2 m_+(1)$ . Using (79),

$$K(r, x) = \mathbb{E}^x [F(r/2, K_{r/2})] [1 + O(e^{-r/2})].$$

This gives (80). Combining this with the coupling, we see that

$$K(r, x) = K(r, y) [1 + O(e^{-ur})].$$

□

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