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QUANTITATIVE MARKED LENGTH SPECTRUM RIGIDITY

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(M, g) closed negatively curved Riemannian manifold

- **Marked length spectrum** $(\mathcal{L}_g)$

  function whose values are lengths of closed geodesics

- **Rigidity**

  $\mathcal{L}_g = \mathcal{L}_{g_0} \implies g = g_0$

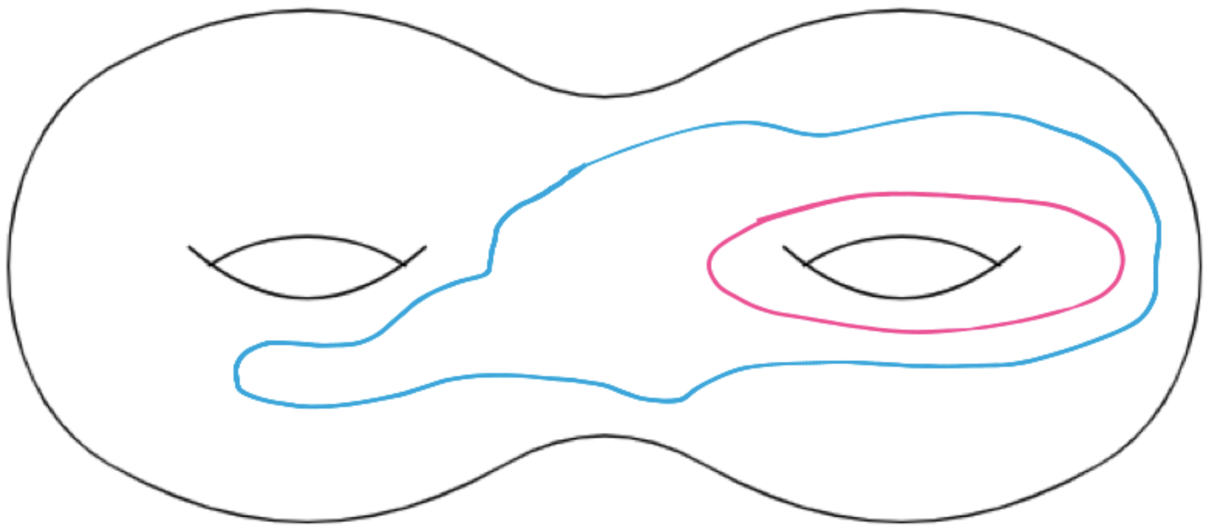
- **Quantitative rigidity**

  $\mathcal{L}_g \approx \mathcal{L}_{g_0} \implies g \approx g_0$

THE MARKED LENGTH SPECTRUM

Setting: (M, g) closed negatively curved Riemannian manifold

Fact: Every closed curve in M is freely homotopic to a unique closed geodesic



Definition:

$\mathcal{L}_g$: conjugacy class in $\pi_1(M)$ $\longmapsto$ length of closed geodesic

MARKED LENGTH SPECTRUM RIGIDITY

Question: What does $\mathcal{L}_g$ tell us about g?

Conjecture (Burns–Katok '85):

If (M, g) and (N, g_0) have $\pi_1(M) \cong \pi_1(N)$ and $\mathcal{L}_g = \mathcal{L}_{g_0}$, then g is isometric to g_0.

KNOWN ANSWERS

$$\mathcal{L}_g = \mathcal{L}_{g_0} \implies g = g_0$$

- **dimension 2** (Otal '90, Croke '91)
- **dimension at least 3, g_0 locally symmetric** (Hamenstädt '97, Besson–Courtois–Gallot '95)
- **locally** (Guillarmou–Lefeuvre '18)

KEY QUESTION

$$\mathcal{L}_g \approx \mathcal{L}_{g_0} \implies g \approx g_0$$

$$1 - \varepsilon \leq \frac{\mathcal{L}_g}{\mathcal{L}_{g_0}} \leq 1 + \varepsilon$$

SOME ANSWERS

- **Local case** (Guillarmou–Lefeuvre '18)

  Estimate $\|g - g_0\|_{H^{-\frac{1}{2}}(M)}$ in terms of $\mathcal{L}_g/\mathcal{L}_{g_0} - 1$,

  requires $\|g - g_0\|_{C^k}$ small

- **Surfaces**

  constant curvature (Thurston '98)

  best Lipschitz constant for $\varphi : (M, g) \to (N, g_0)$ homotopic to f $= \sup_{\gamma \in \pi_1(M)} \frac{\mathcal{L}_{g_0}(f_*(\gamma))}{\mathcal{L}_g(\gamma)}$

  variable curvature (B '22)

  Lipschitz constant depends continuously on $\mathcal{L}_g/\mathcal{L}_{g_0}$ near 1

LOCALLY SYMMETRIC SPACES, DIMENSION AT LEAST 3

Theorem (B '22)

If (N, g_0) is locally symmetric with $\dim(N) \geq 3$ and (M, g) satisfies $-b^2 \leq \sec(M) < 0$ and

$$1 - \varepsilon \leq \frac{\mathcal{L}_{g_0}}{\mathcal{L}_g} \leq 1 + \varepsilon,$$

then there is $F : M \to N$ and $C = C(n, \pi_1(N), b)$ with

$$1 - C\varepsilon^{1/8(n+1)} + h.o.t. \leq \|dF\| \leq 1 + C\varepsilon^{1/8(n+1)} + h.o.t.$$

LOCALLY SYMMETRIC SPACES, DIMENSION AT LEAST 3

- **Same volume** (Hamenstädt '97)

If the geodesic flow of (N, g_0) has C^1 Anosov splitting, then $\mathcal{L}_g = \mathcal{L}_{g_0} \implies \mathrm{vol}_g(M) = \mathrm{vol}_{g_0}(N)$

- **Isometry** (Besson–Courtois–Gallot '95)

If (N, g_0) is locally symmetric and $\dim(N) \geq 3$ then

$$\begin{matrix} \mathrm{vol}_g(M) = \mathrm{vol}_{g_0}(N) \\ h(g) = h(g_0) \end{matrix} \implies (M, g) \text{ isometric to } (N, g_0)$$

VOLUME ESTIMATE

Theorem (B '22)

If the geodesic flow of (N, g_0) has C^{1+a} Anosov splitting and (M, g) satisfies

$$1 - \varepsilon \leq \frac{\mathcal{L}_{g_0}}{\mathcal{L}_g} \leq 1 + \varepsilon,$$

then there is C depending on $(\tilde{N}, \tilde{g}_0)$ such that

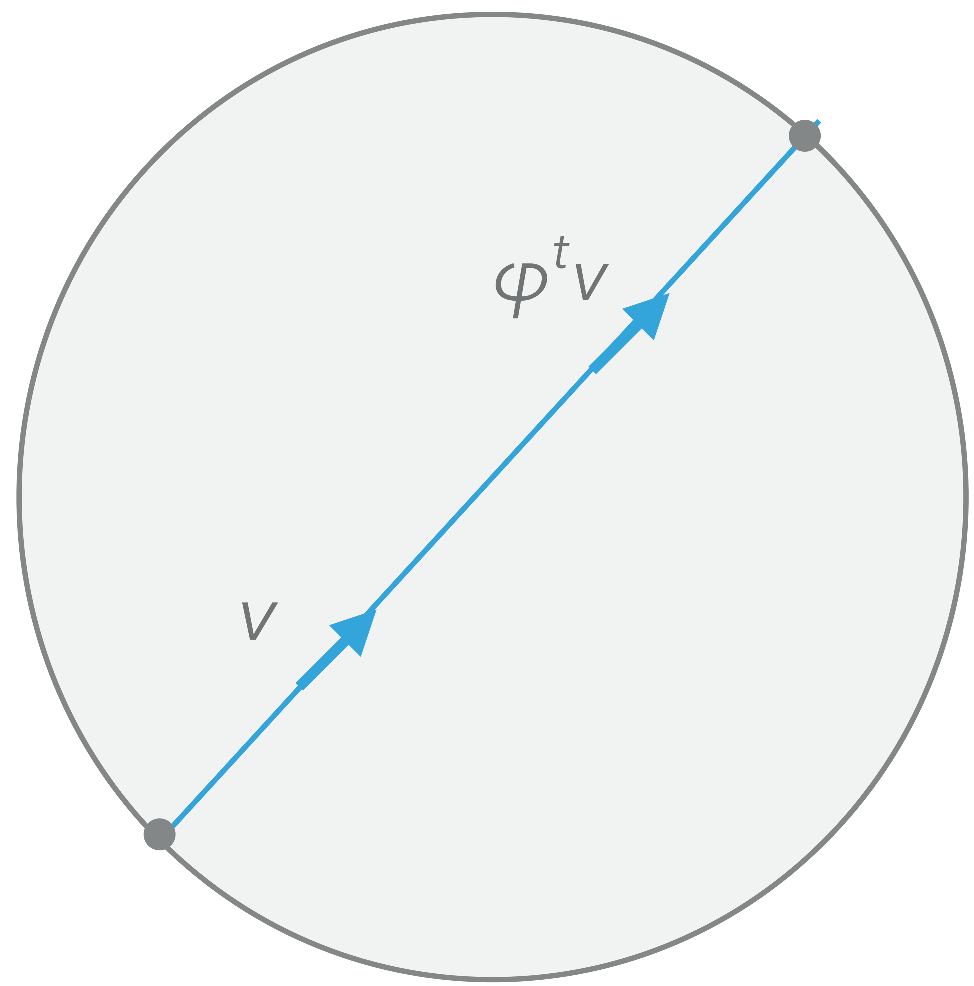
$$(1 - C\varepsilon^a)(1 - \varepsilon)^n \leq \frac{\mathrm{vol}_{g_0}(N)}{\mathrm{vol}_g(M)} \leq (1 + C\varepsilon^a)(1 + \varepsilon)^n.$$

For N locally symmetric, $(1 \pm C\varepsilon^a)$ becomes $(1 \pm C(n)\varepsilon^2)$

VOLUME ESTIMATE

$\mathrm{vol}_g(M)$ is determined by $\mu(T^1 M)$,
where μ is Liouville measure

$$\mathcal{G}\tilde{M} = T^1\tilde{M}/\sim$$



$$\mathcal{G}\tilde{M} \cong \partial\tilde{M} \times \partial\tilde{M} \setminus \Delta$$

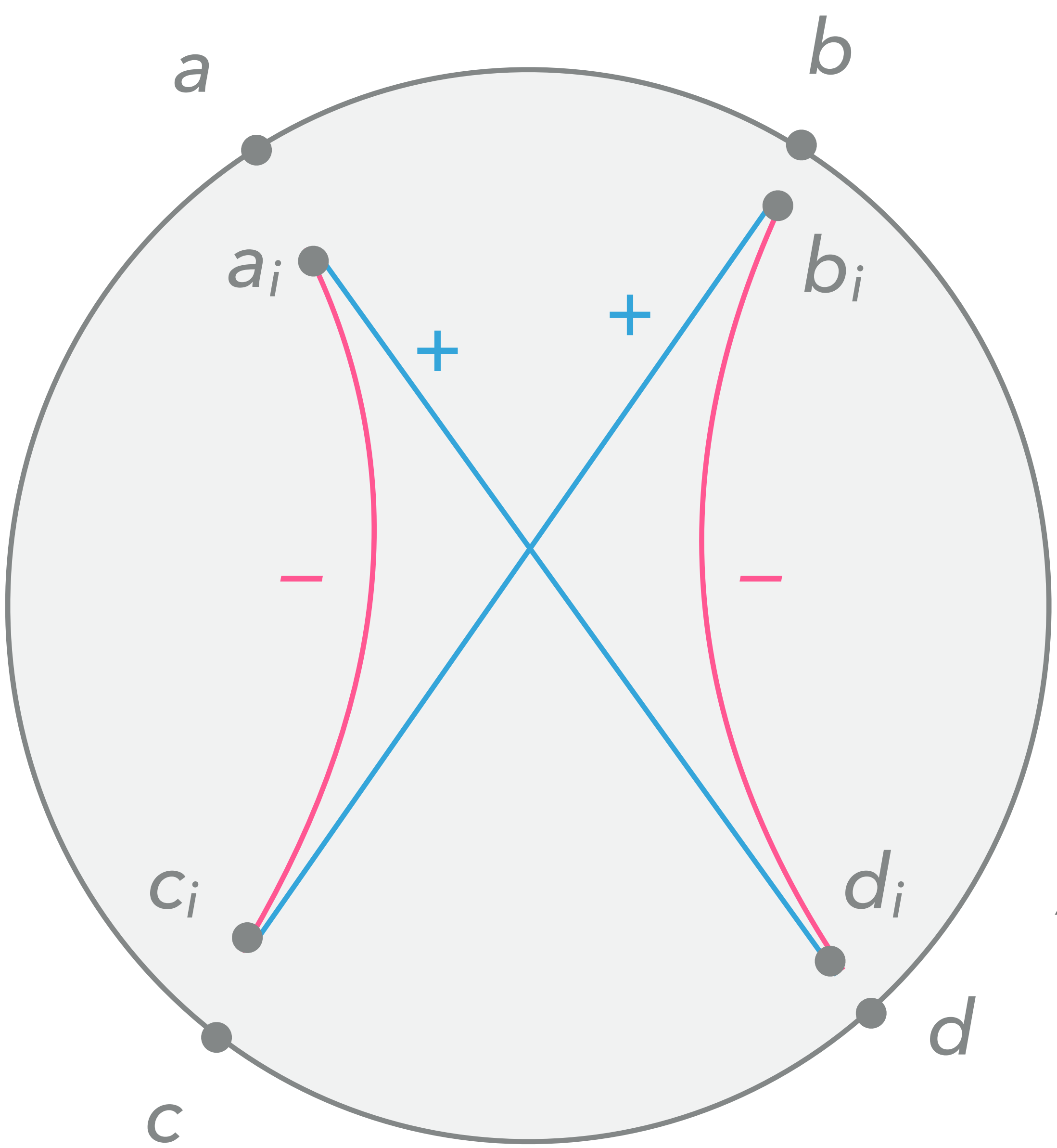
$$d\mu = d\lambda \times dt$$

Liouville current on $\mathcal{G}\tilde{M}$

Lebesgue measure on orbits

VOLUME ESTIMATE $d\mu = d\lambda \times dt$

cross-ratio $[a,b,c,d]$



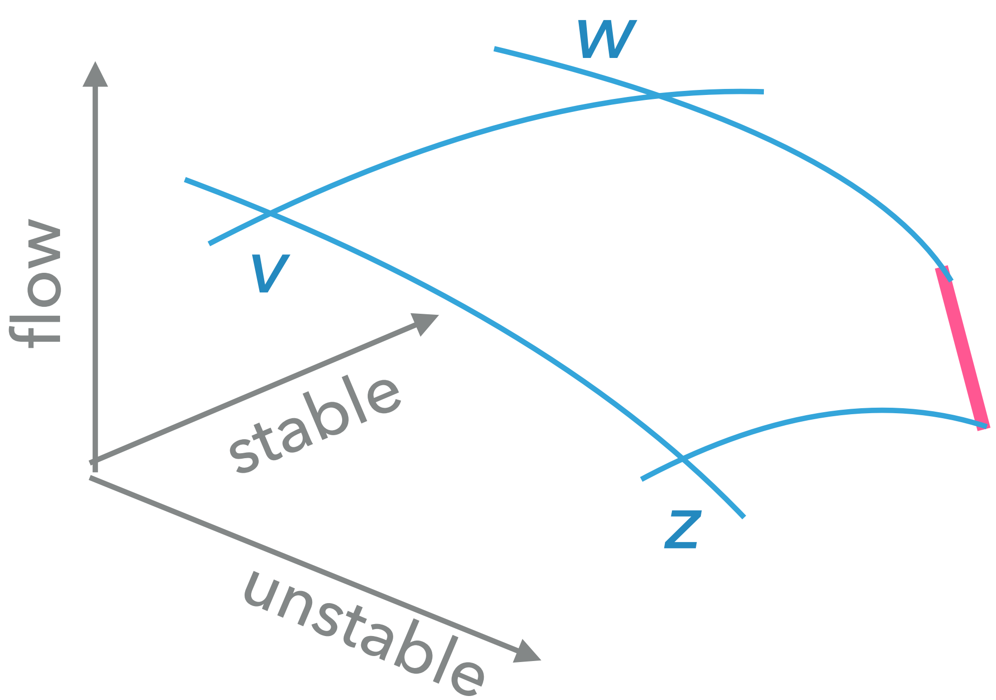
$\mathcal{L}_g = \mathcal{L}_{g_0} \implies [\,]_g = [\,]_{g_0}$ (Otal)

$\approx$ $\approx$ (B)

$\lambda([a,b] \times [c,d]) = \frac{1}{2}[a,b,c,d]$ (Otal)

VOLUME ESTIMATE $d\mu = d\lambda \times dt$

Hamenstädt relates λ and $[\cdot,\cdot,\cdot,\cdot]^{n-1}$

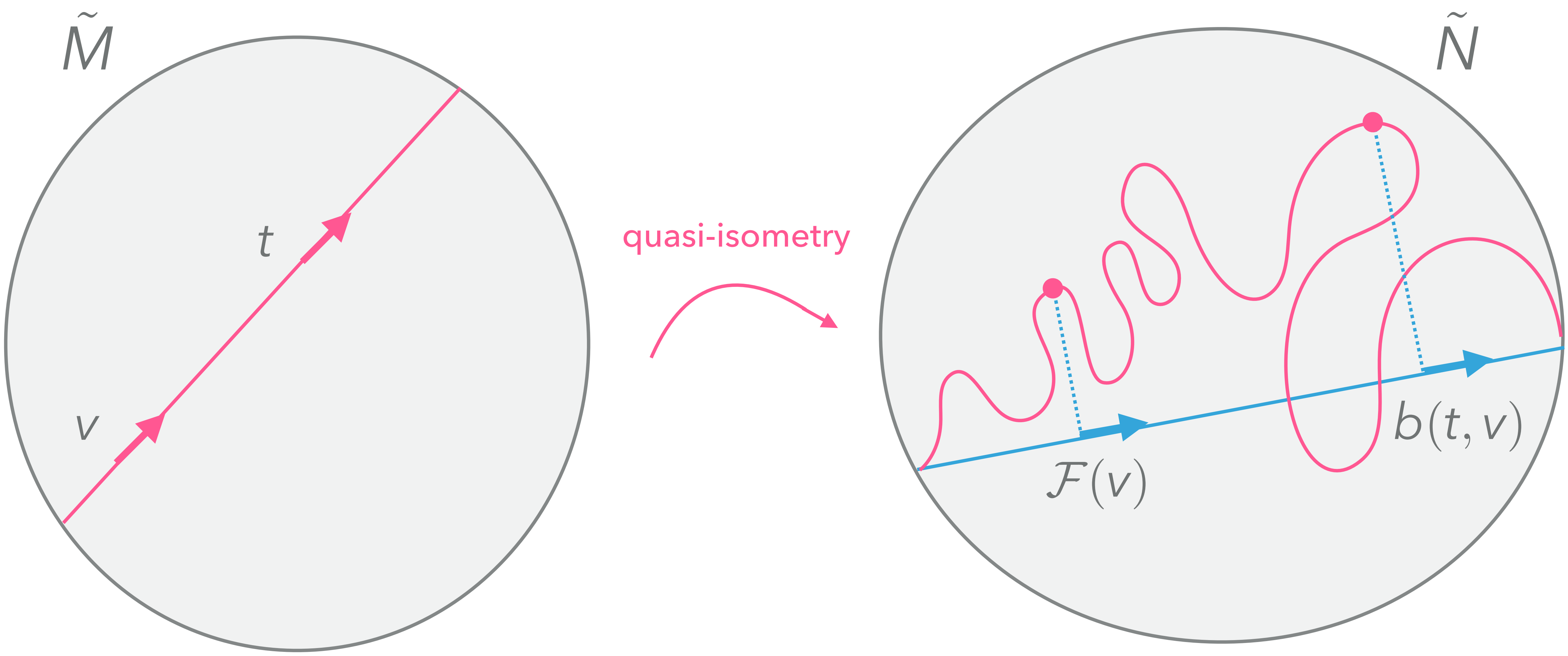


$$\sigma(w,z) = [\pi(v), \pi(z), \pi(-w), \pi(-v)]$$

$$(1 - C\varepsilon^a)(1-\varepsilon)^{n-1} \leq \frac{\lambda_g}{\lambda_{g_0}} \leq (1 + C\varepsilon^a)(1+\varepsilon)^{n-1}$$

VOLUME ESTIMATE $d\mu = d\lambda \times dt$

Orbit equivalence $\mathcal{F}(\varphi^t v) = \psi^{b(t,v)} \mathcal{F}(v)$ (Gromov)



Replace $b(t,v)$ with $a_c(t,v) = \frac{1}{c}\int_t^{t+c} b(s,v)\,ds$