Quantitative marked length spectrum rigidity

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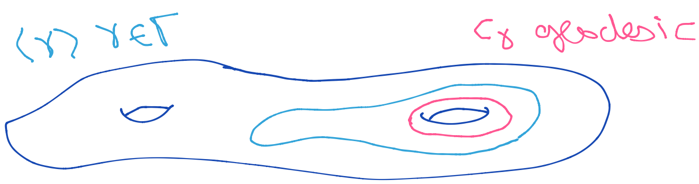
Setting: (M,g) compact Riemannian manifold (w/o bdry)
$\sec_g(M) < 0 \qquad n = \dim(M)$, $\Gamma = \pi_1(M)$.

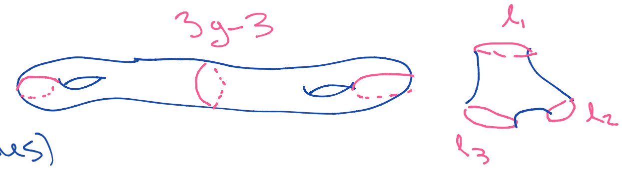


Definition: the marked length spectrum of (M,g)

$$\mathcal{L}_g : \begin{matrix}\text{conj classes} \\ \text{in } \Gamma\end{matrix} \longrightarrow \mathbb{R}$$
$$\langle\gamma\rangle \longmapsto l_g(c_\gamma)$$

Conjecture (Burns-Katok '85): "marked length spectrum rigidity"
(M,g), (N,g_0) $\mathcal{L}_g = \mathcal{L}_{g_0}$ $\Longrightarrow$ M & N are isometric.
$\pi_1(M) \cong \pi_1(N)$

Known answers:



- hyperbolic surfaces
  ($9g-9$ thm, g = genus)

- surfaces (Otal '90, Croke '91)

- $n \geq 3$, (N, g_0) locally symmetric (Hamenstädt '99, Besson–Courtois–Gallot '95)

- local rigidity (Guillarmou–Lefeuvre '18):
  $\exists K = K(n)$ $\exists \delta = \delta(g_0) > 0$ s.t. if $\|g - g_0\|_{C^K} < \delta$ & $\mathcal{L}_g = \mathcal{L}_{g_0}$
  then g is isometric to g_0.

Question: What if $\mathcal{L}_g \approx \mathcal{L}_{g_0}$? Is $g \approx g_0$?

Hypothesis: $1 - \varepsilon \leq \dfrac{\mathcal{L}_g(\gamma)}{\mathcal{L}_{g_0}(\gamma)} \leq 1 + \varepsilon$ QUANTITATIVE m.l.s. rigidity

Previous results:

- local rigidity (Guillarmou–Lefeuvre '18):
  estimate for $\|g - g_0\|_{H^{-\frac{1}{2}}(M)}$ in terms of $\mathcal{L}_g / \mathcal{L}_{g_0} - 1$,
  requires $\|g - g_0\|_{C^K} < \delta$.

- hyperbolic surfaces (Thurston '98):
  $\varphi : (M, g) \to (N, g_0)$ hom. to f best $\mathrm{Lip}(\varphi) = \sup_{\gamma \in \Gamma} \dfrac{\mathcal{L}_{g_0}(f_* \gamma)}{\mathcal{L}_g(\gamma)} \approx 1$

New results (B '22):

- surfaces

Thm 1: best $\mathrm{Lip}(\varphi)$ depends continuously on ε near $\varepsilon = 0$, uniformly continuous among surfaces of "bounded geometry"

- $n \geq 3$, (N, g_0) locally symmetric.

$\boxed{\text{Thm 2:}}$ (M,g) s.t. $1-\varepsilon \leq \mathcal{L}_g/\mathcal{L}_{g_0} \leq 1+\varepsilon$. let Λ s.t. $|\sec_g(M)| \leq \Lambda^2$.

$\exists C = C(n, \Gamma, \Lambda)$ $\exists F: M \to N$

$\|dF_p\|$ btwn $1 \pm C\varepsilon^{1/8(n+1)}$ + h.o.t.

What happens when $\varepsilon = 0$? $\varepsilon > 0$ $(\mathcal{L}_g \approxeq \mathcal{L}_{g_0})$

↳ Hamenstädt: $\mathcal{L}_g \approxeq \mathcal{L}_{g_0} \Rightarrow vol\, g(M) \approxeq vol_{g_0}(N)$

only need N has C^1 Anosov splitting

Thm3

↳ Besson–Courtois–Gallot: $n \geq 3$, (N, g_0) loc. symm.

$vol_g(M) \approxeq vol_{g_0}(N)$, $h(g) \approxeq h(g_0)$ $\Longrightarrow$ M & N are almost isometric.

Thm2

Thm 3: Assume (N, g_0) has ~~$C^{1+\alpha}$ Anosov splitting~~ locally symmetric. $(0<\alpha<1)$

(M,g) s.t. $1-\varepsilon \leq \mathcal{L}_g/\mathcal{L}_{g_0} \leq 1+\varepsilon$.

$\exists C = C(\tilde{N}, \tilde{g}_0)$ s.t.

$$\cancel{(1-C\varepsilon^{\alpha})}(1-\varepsilon)^n \leq \frac{vol(M)}{vol(N)} \leq \cancel{(1+C\varepsilon^{\alpha})}(1+\varepsilon)^n.$$

$(1-C(n)\varepsilon^2)$ $\qquad$ $(1+C(n)\varepsilon^2)$

Volume Estimate (thm 3) $\mathcal{L}_g \approx \mathcal{L}_{g_0} \Rightarrow vol_g(M) \approx vol_{g_0}(N)$.

μ - Liouville measure on T^1M

$$\mu(T^1M) = vol_g(M)\, Leb(S^{n-1}).$$

comes from α flow-inv. contact form on T^1M

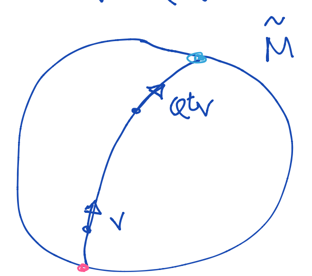
$\alpha \wedge (d\alpha)^{n-1}$ volume form ⇝ gives μ.

$T^1\tilde{M}$ locally $G\tilde{M} \times \mathbb{R}$

$$d\mu = \underbrace{d\lambda}_{\substack{\text{Liouville} \\ \text{current} \\ \text{on } G\tilde{M}}} \times \underbrace{dt}_{\text{Leb. on } \mathbb{R}}$$

$G\tilde{M} = T^1\tilde{M}/\sim$

$v \sim \varphi^t v$



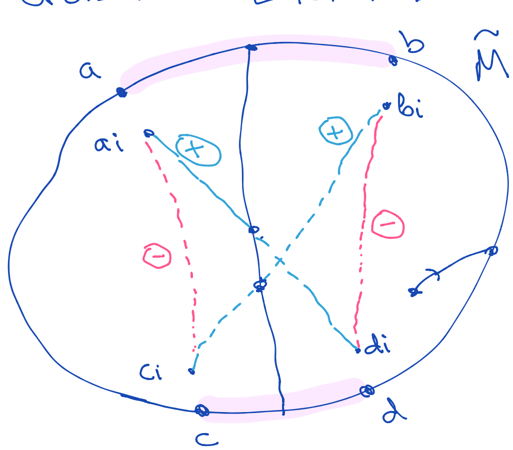
$G\tilde{M} = \partial\tilde{M} \times \partial\tilde{M} \setminus \Delta$

Controlling $d\lambda$: $d\mu = d\lambda \times dt$

marked length spectrum ⇝ cross-ratio ⇝ Liouville current

$\mathcal{L}_g$ $\qquad [\cdot,\cdot,\cdot,\cdot]_g \qquad d\lambda_g$

cross-ratio $[a,b,c,d]$

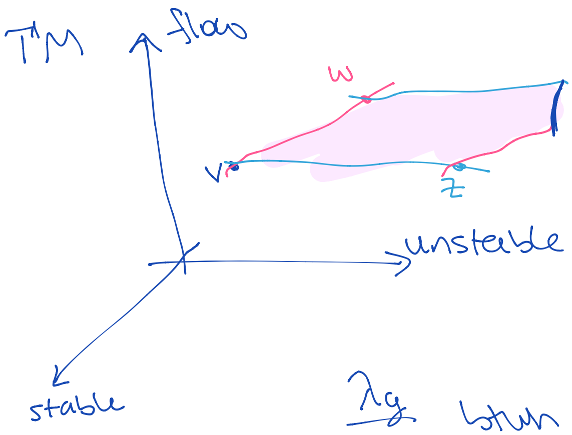


$\underset{\approx}{\mathcal{L}_g} = \mathcal{L}_{g_0} \Rightarrow [\ ,\ ,\ ]_g \underset{\approx}{=} [\cdot\ \cdot\ \cdot]_{g_0}$ (Otal) B

$n=2$: (Otal)

$$\lambda([a,b]\times[c,d]) = \frac{1}{2}[a,b,c,d]$$

$n \geq 3$: λ related to $[\cdot,\cdot,\cdot,\cdot]^{n-1}$ (Hamenstädt)



$$\sigma(w,z) = [\pi(v), \pi(-w), \pi(-v), \pi(z)]$$
$$\| $$
$$\int d\alpha$$

$d\lambda \rightsquigarrow (d\alpha)^{n-1}$

$\updownarrow$

$[\ ,\ldots]$

$$\frac{\lambda_g}{\lambda_{g_0}} \text{ within } (1 \pm C\varepsilon^{\alpha})(1\pm\varepsilon)^{n-1}$$

<u>Controlling dt</u> $\quad d\mu = d\lambda \times dt$

φ^t, ψ^t geodesic flows on T^1M, T^1N

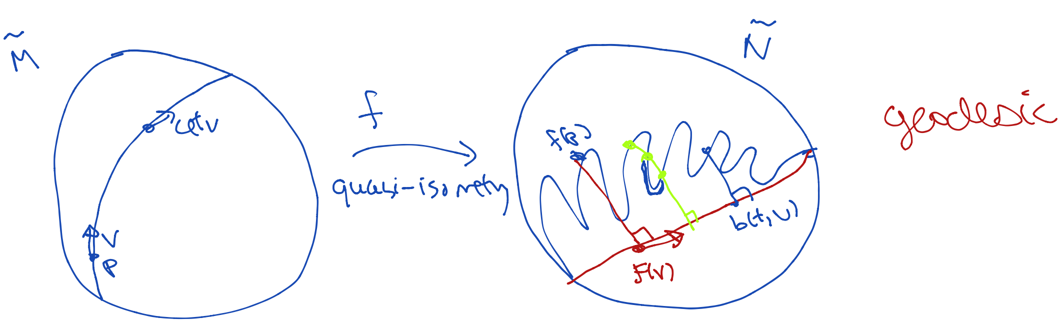
<u>Fact:</u> $\mathcal{L}_g = \mathcal{L}_{g_0} \Longleftrightarrow \varphi^t$ <u>conjugate</u> to ψ^t,

i.e. $\exists F: T^1M \to T^1N$ homeo s.t.

$$F(\varphi^t v) = \psi^t F(v)$$

In general, still <u>orbit-equivalent</u> (Gromov) $\quad \pi_1(M) \cong \pi_1(N)$

$$F(\varphi^t v) = \psi^{b(t,v)} F(v)$$



replace $b(t,v)$ with $a_c(t,v) = \frac{1}{c}\int_t^{t+c} b(s,v)\,ds$ for sufficiently large c.

$1-\varepsilon \le \frac{L_g}{L_{g_0}} \le 1+\varepsilon$ means $v = a^\tau v$ $\tau > 0$ $1-\varepsilon \le \frac{b(\tau,v)}{\tau} \le 1+\varepsilon$

speed $\left.\frac{d}{dt}\right|_{t=0} a_c(t,v) = \frac{b(c,v)}{c}$

<u>idea</u>: if c is very large then close to 1 on sets of "large measure" of $v \in T^1M$

$\mu(T^1M)$

<u>BCG map (Thm 2):</u>

Recall: $n \ge 3$, (N, g_0) locally symmetric.

$Vol_g(M) \approx Vol_{g_0}(N)$, $h(g) \approx h(g_0)$

want to construct $F: M \to N$ (almost) isometry

<u>BCG construction:</u>

$f: \tilde{M} \to \tilde{N}$, $f: \partial\tilde{M} \to \partial\tilde{N}$

$p \in \tilde{M}$, μ_p Patterson-Sullivan measure on $\partial\tilde{M}$

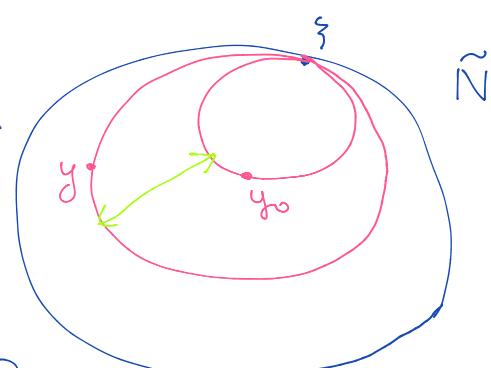
$F: \tilde{M} \to \tilde{N}$

$p \longmapsto bar(f_*\mu_p)$

$F(p)$ minimizes of

$y \longmapsto \int_{\partial\tilde{N}} B^{\xi}_{y_0}(y)\, d(f_*\mu_p)(\xi)$

$y \in \tilde{N}$,



Proof that F is an almost isometry:

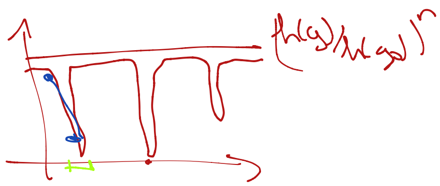
1) $|\mathrm{Jac}F(p)| \leq \dots \leq \left(\frac{h(g)}{h(g_0)}\right)^n$ $\quad n \geq 3$, (N, g_0) loc. symm.

$\frac{1}{n+1}$ 2) $\mathrm{vol}(N) \leq \int_M |\mathrm{Jac}F(p)| \leq \left(\frac{h(g)}{h(g_0)}\right)^n \mathrm{vol}(M)$

$\mathrm{vol}(N) \approxeq \mathrm{vol}(M)$, $h(g) \approxeq h(g_0)$ $\Longrightarrow$ almost equality holds in 2) $\underset{\text{hard}}{\Longrightarrow}$ almost equality in 1)

$\frac{1}{8}$ 3) show dF_p is a scalar matrix

analogous (showing eq. holds in $\det A \leq \left(\frac{\mathrm{trace}(A)}{n}\right)^n$)



strategy: show $p \mapsto |\mathrm{Jac}F(p)|$ is L-Lipschitz

don't want L to depend on (M, g)

Dalbo-Kim