The Minkowski content measure for the Liouville quantum gravity metric

Jinwoo Sung

University of Chicago

(Joint work with Ewain Gwynne)

Quantum surface vs. smooth surface

Definition (Formal)

Let $\gamma \in (0,2)$ be a constant and $U \subset \mathbb{C}$ be a simply connected domain. A **γ-LQG surface** is a two-dimensional Riemannian manifold (U, g) with the random metric tensor

$$g = e^{\gamma h}(dx^2 + dy^2)$$

where h is an instance of the Gaussian free field in U.

- We can parameterize any simply connected smooth surface with **isothermal coordinates**: the metric tensor takes the form

  $$e^{f(z)}(dx^2 + dy^2)$$

  where f is a smooth function.
- Area of a set A: $\int_A e^{f(z)}\, d^2 z$
- Length of a path P: $\int_0^1 e^{f(P(t))/2} |P'(t)| dt$
- Distance between points z, w: $\inf_{P: z \to w} \operatorname{len}(P)$

Mollifications of the whole-plane GFF

Definition (Formal)

A **whole-plane GFF** h is a centered Gaussian process on $\mathbb{C}$ with

$$\mathrm{Cov}(h(z), h(w)) = G(z, w) := \log \frac{(|z| \vee 1)(|w| \vee 1)}{|z - w|}.$$

- A whole-plane GFF is defined rigorously as a random distribution.
- Circle average: $h_\varepsilon(z) = (h, \lambda_{z,\varepsilon})$
  - $\lambda_{z,\varepsilon}$ is the uniform probability measure on $\partial B_\varepsilon(z)$.
  - A whole-plane GFF has normalization $h_1(0) = 0$.
- Heat kernel average: $h^*_\varepsilon(z) = (h * p_{\varepsilon^2/2})(z)$
  - $p_{\varepsilon^2/2}(dw) = \frac{1}{\pi\varepsilon^2} \exp(-\frac{|z-w|^2}{\varepsilon^2}) dw$

LQG measure (area)

For a smooth Riemannian metric $e^f(dx^2 + dy^2)$, the area measure is $e^{f(z)}\, d^2z$.

Definition (Duplantier–Sheffield, 2008)

The γ-**LQG measure** w.r.t. GFF h is the a.s. weak limit

$$\mu_h(dz) := \lim_{\varepsilon\to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)}\, d^2z.$$

- A special case of Gaussian multiplicative chaos (Kahane, 1985)
  - $\mu_h(dz) = \lim_{\varepsilon\to 0} \varepsilon^{\gamma^2/2} e^{\gamma h^*_\varepsilon(z)}\, d^2z$ in probability. (Shamov, 2014)
- Nontrivial when $\gamma \in (0, 2)$.
  - No point mass, $\mu_h(U) > 0$ a.s. for every nonempty open U.
  - Mutually singular w.r.t. the Lebesgue measure.
- Zero measure when $\gamma \geq 2$.

LQG metric (distance)

For a smooth Riemannian metric $e^f(dx^2+dy^2)$, $d(z,w) = \inf_{P:z\to w}\int_0^1 e^{f(P(t))/2}|P'(t)|\,dt$.

Definition (Gwynne–Miller, 2019)

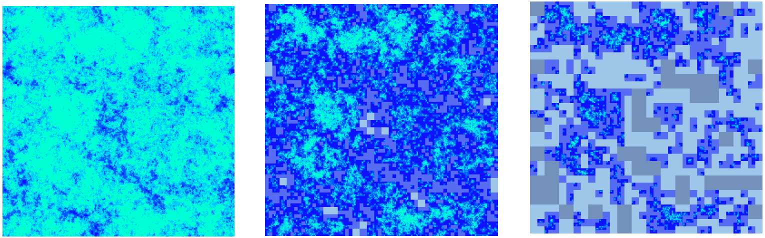
The γ-**LQG metric** w.r.t. GFF h is the limit

$$D_h(z,w) := \lim_{\varepsilon\to 0} \mathfrak{a}_\varepsilon^{-1}\left[\inf_{P:z\to w}\int_0^1 e^{\xi h_\varepsilon^*(P(t))}|P'(t)|dt\right]$$

in probability w.r.t. the local uniform topology on $\mathbb{C}\times\mathbb{C}$.

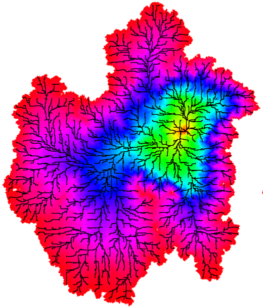
- $\xi := \gamma/d_\gamma$, where $d_\gamma > 2$ is the "fractal dimension of γ-LQG."
  (Ding–Gwynne, 2018; Ding–Zeitouni–Zhang, 2018)
- $\mathfrak{a}_\varepsilon$: Normalization factor chosen so that the limiting sequence is tight.
  (Ding–Dubédat–Dunlap–Falconet, 2019)
- For $\xi < 2/d_2$, induces the Euclidean topology.
- For $\xi > 2/d_2$, D_h exists but with singular points. (Ding–Gwynne, 2021)

Simulations of LQG measure and metric (by J. Miller)

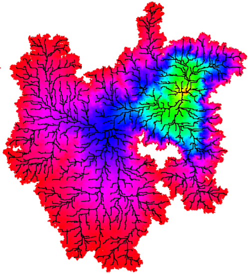


LQG measure

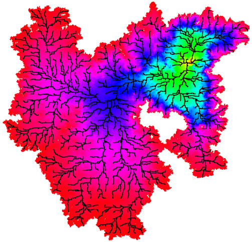
All sub-squares have approximately the same γ-LQG measure. They are colored according to their Euclidean size.



$\gamma = 0.75$



$\gamma = 1.25$



$\gamma = 1.75$

LQG metric

Simulations of γ-LQG metric balls. The colors indicate distances to the center of the ball. The black lines are geodesics to the center.

LQG measure vs. LQG metric

- LQG measure: $\mu_h(dz) = \lim_{\varepsilon\to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)}\, d^2 z$
- LQG metric: $D_h(z,w) = \lim_{\varepsilon\to 0} \mathfrak{a}_\varepsilon^{-1} \left[\inf_{P:z\to w} \int_0^1 e^{\frac{\gamma}{d_\gamma} h_\varepsilon^*(P(t))} |P'(t)| dt\right]$

Theorem (Berestycki–Sheffield–Sun, 2014)

The GFF h is almost surely determined by μ_h.

Question (Gwynne and Miller, 2019)

Does the LQG metric a.s. determine the LQG measure?

More concretely, can the LQG measure be recovered as some sort of Minkowski content measure w.r.t. the LQG metric?

Answer for $\gamma = \sqrt{8/3}$

Theorem (Miller–Sheffield, 2016)

The Brownian map is equivalent to the γ-LQG sphere for $\gamma = \sqrt{8/3}$.

Theorem (Le Gall, 2021)

Let $H(r) = r^4 \log\log(1/r)$. There exists a constant $\kappa > 0$ such that, almost surely for every Borel subset A of the Brownian map,

$$\mathrm{Vol}(A) = \kappa \lim_{\varepsilon \to 0} \inf \left\{ \sum_{i \in I} H(\mathrm{diam}(U_i)) : A \subset \bigcup_{i \in I} U_i,\ \mathrm{diam}(U_i) < \varepsilon\ \forall i \right\}.$$

That is, the volume measure of the Brownian map is a constant multiple of the Hausdorff measure w.r.t. the gauge function H.

- d_γ is the Hausdorff dimension of the γ-LQG metric. (Gwynne–Pfeffer, 2019)
- $d_{\sqrt{8/3}} = 4$ is the only known value of d_γ.

Minkowski dimension of the LQG metric

- $N_\varepsilon(A)$: number of D_h-balls of radius ε required to cover $A \subset \mathbb{C}$.
- The D_h-**Minkowski dimension** of A is δ if $N_\varepsilon(A) = \varepsilon^{-\delta+o(1)}$.

Theorem (Ang–Falconet–Sun, 2020)

For any compact set $K \subset \mathbb{C}$ and $\zeta > 0$, we a.s. have

$$\sup_{\varepsilon\in(0,1)} \sup_{z\in K} \frac{\mu_h(\mathcal{B}_\varepsilon(z; D_h))}{\varepsilon^{d_\gamma-\zeta}} < \infty \quad \text{and} \quad \inf_{\varepsilon\in(0,1)} \inf_{z\in K} \frac{\mu_h(\mathcal{B}_\varepsilon(z; D_h))}{\varepsilon^{d_\gamma+\zeta}} > 0.$$

Consequently, the **Minkowski dimension** of γ-LQG is d_γ.

Minkowski content on the LQG metric space

- $N_\varepsilon(A)$: number of D_h-balls of radius ε required to cover $A \subset \mathbb{C}$.
- The D_h-**Minkowski dimension** of A is δ if $N_\varepsilon(A) = \varepsilon^{-\delta+o(1)}$.
- The **Minkowski content** of a δ-dimensional set A is the constant c s.t. $N_\varepsilon(A) \sim c\varepsilon^{-\delta}$.

Theorem (Gwynne–S., 2022)

Let $\gamma \in (0,2)$. There exists a deterministic sequence $\{\mathfrak{b}_\varepsilon\}_{\varepsilon>0}$ depending only on γ such for every random bounded Borel set $A \subset \mathbb{C}$ with $\mu_h(\partial A) = 0$ a.s.,

$$\lim_{\varepsilon\to 0} \mathfrak{b}_\varepsilon^{-1} N_\varepsilon(A) = \mu_h(A) \quad \text{in probability.}$$

Consequently, D_h a.s. determines μ_h.

Our choice of $\{\mathfrak{b}_\varepsilon\}_{\varepsilon>0}$ satisfies:

- There exists $c > 1$ such that $c^{-1}\varepsilon^{-d_\gamma} \le \mathfrak{b}_\varepsilon \le c\varepsilon^{-d_\gamma}$ for all $\varepsilon \in (0,1)$.
- $\varepsilon \to \varepsilon^{d_\gamma}\mathfrak{b}_\varepsilon$ is a slowly varying function. That is, $\lim_{\varepsilon\to 0} \mathfrak{b}_{r\varepsilon}/\mathfrak{b}_\varepsilon = r^{-d_\gamma}$ for every $r > 0$.

LQG metric determines the conformal structure

Theorem (Ang–Falconet–Sun, 2020)

Let $\gamma \in (0,2)$. The whole-plane GFF h is almost surely determined by the random pointed metric measure space $(\mathbb{C}, 0, D_h, \mu_h)$, up to rotation and scaling of $\mathbb{C}$.

- That is, the map $(\mathbb{C}, h, 0) \mapsto (\mathbb{C}, 0, D_h, \mu_h)$ has a measurable inverse.
  - Generalization of the convergence of the Tutte embedding of the Poisson–Voronoi tessellation of the Brownian map to $\sqrt{8/3}$-LQG. (Gwynne–Miller–Sheffield, 2018)
- Our theorem says that we can use the Minkowski content to construct a measurable inverse of the natural projection $(\mathbb{C}, 0, D_h, \mu_h) \mapsto (\mathbb{C}, 0, D_h)$.

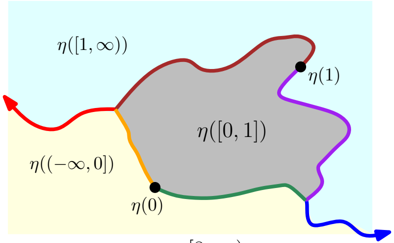
Corollary

The pointed metric space $(\mathbb{C}, 0, D_h)$ almost surely determines its parameterization on $\mathbb{C}$ and the associated GFF h modulo rotation and scaling of $\mathbb{C}$.

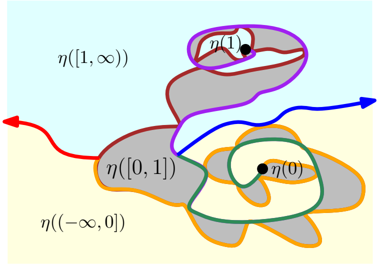
- This is a generalization of the equivalence between $\sqrt{8/3}$-LQG and the Brownian map.

The normalization constant

- The γ**-quantum cone** is the LQG surface $(\mathbb{C}, h^\gamma, 0)$ obtained by:
  - Sampling a point from the γ-LQG measure μ_h;
  - Re-centering at the sampled point, then zooming in.
- h^γ agrees with $h - \gamma \log|\cdot|$ when restricted to $\mathbb{D}$.
- Let η be the whole-plane **space-filling SLE**$_\kappa$ curve from ∞ to ∞, sampled independently of the quantum cone h^γ.
  - For $\kappa \geq 8$, η agrees locally with ordinary SLE$_\kappa$.
  - For $\kappa \in (4, 8)$, η is obtained by recursively filling in bubbles in an ordinary SLE$_\kappa$.
- Parameterize η s.t. $\eta(0) = 0$, $\mu_{h^\gamma}(\eta([s,t])) = t - s$.



$\kappa \in [8, \infty)$



$\kappa \in (4, 8)$

Definition

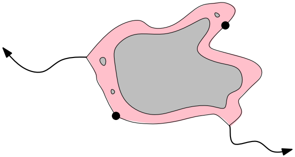
$$\mathfrak{b}_\varepsilon := \mathbb{E}[N_\varepsilon(\eta([0,1]); D_{h^\gamma})] \quad \text{where} \quad \kappa = 16/\gamma^2.$$

Proof overview

Step 1. Minkowski content approximations of SLE segments are tight.

- For each $p \geq 1$, there is a constant $C_p < \infty$ such that for every fixed $s < t$,

$$\sup_{0<\varepsilon<(t-s)^{1/d_\gamma}} \mathbb{E}[(\varepsilon^{d_\gamma} N_\varepsilon(\eta[s,t]))^p] < C_p |t-s|^p.$$



$\kappa \in [8, \infty)$

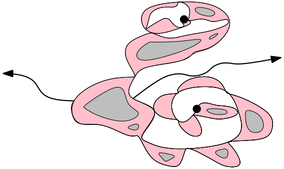
- Consequently, we can pick a subsequence $\varepsilon_n \to 0$ such that

$$(\varepsilon_n^{d_\gamma} N_{\varepsilon_n}(\eta[s,t]) : s < t \text{ rational}) \xrightarrow{d} (X_{[s,t]} : s < t \text{ rational}).$$

Step 2. Minkowski contents of SLE segments are additive.

- Let $N_\varepsilon^\circ(\eta[s,t]) = N_\varepsilon(\{z \in \eta[s,t] : D_{h^\gamma}(z, \partial\eta[s,t]) \geq \varepsilon\})$.
- We prove that $\lim_{\varepsilon\to 0} |N_\varepsilon(\eta[s,t]) - N_\varepsilon^\circ(\eta[s,t])| = 0$ a.s. Therefore,

$$(\varepsilon_n^{d_\gamma} N_{\varepsilon_n}^\circ(\eta[s,t]) : s < t \text{ rational}) \xrightarrow{d} (X_{[s,t]} : s < t \text{ rational}).$$



$\kappa \in (4, 8)$

- Consequently, $X_{[r,s]} + X_{[s,t]} = X_{[r,t]}$ a.s. for each rational $r < s < t$.

Application of the mating-of-trees theorem

- The internal metric in $U \subset \mathbb{C}$ is defined as

$$D^U_{h^\gamma}(z,w) := \inf_{U \supset P : z \to w} \mathrm{length}(P; D_{h^\gamma}).$$

- For $s < t$, let $U_{s,t} := \mathrm{int}(\eta[s,t])$. Consider the curve-decorated metric measure space

$$\mathcal{S}_{[s,t]} := (U_{s,t}, D^{U_{s,t}}_{h^\gamma}, \mu_{h^\gamma}|_{U_{s,t}}, \eta(\cdot - s)|_{[0,t-s]}).$$

Theorem (Duplantier–Miller–Sheffield, 2014)

1. **Stationarity.** For each fixed $s < t$, we have $\mathcal{S}_{[s,t]} \overset{d}{=} \mathcal{S}_{[0,t-s]}$.
2. **Independence.** For each fixed $s_0 < t_0 \le s_1 < t_1 \le \cdots \le s_k < t_k$, $\mathcal{S}_{[s_0,t_0]}, \mathcal{S}_{[s_1,t_1]}, \dots, \mathcal{S}_{[s_k,t_k]}$ are independent.

- **Key observation:** $N^\circ_\varepsilon(\eta[s,t])$ is measurable w.r.t. $\mathcal{S}_{[s,t]}$.
- For each fixed $s < t$, we have $X_{[s,t]} \overset{d}{=} X_{[0,t-s]}$.
- For $s_0 < t_0 \le \cdots \le s_k < t_k$, we have that $X_{[s_0,t_0]}, \dots, X_{[s_k,t_k]}$ are independent.

Proof overview

Step 3. Minkowski contents of SLE blocks are LQG measures.

- Define $Y_t := X_{[0,t]}$ for $t \geq 0$ and $Y_t := X_{[t,0]}$ for $t < 0$. Then $X_{[s,t]} = Y_t - Y_s$ a.s.
- $\{Y_t\}_{t\in\mathbb{Q}}$ a.s. extends to a continuous process $\{Y_t\}_{t\in\mathbb{R}}$ by the Kolmogorov continuity thm.
- $\{Y_t\}_{t\in\mathbb{R}}$ satisfies the following properties.
  - Stationary and independent increments
  - Continuous paths
  - Positive increments: $Y_s < Y_t$ a.s. whenever $s < t$.

Lemma

There exists a constant $c > 0$ depending only on γ such that for every $s < t$,

$$\lim_{\varepsilon\to 0} \mathbb{P}\{\varepsilon^{d_\gamma} N_\varepsilon(\eta[s,t]) > c(t-s)\} = 1.$$

- The only process satisfying these conditions is $Y_t = \mathfrak{c}t$ where $\mathfrak{c} > 0$ is deterministic.
- We replace the normalization constant from ε^{-d_γ} to $\mathfrak{b}_\varepsilon^{-1}$ so that $\mathfrak{c} = 1$.

Proof overview

Proposition

For every $s < t$, $\lim_{\varepsilon \to 0} \mathfrak{b}_\varepsilon^{-1} N_\varepsilon(\eta[s,t]) = t - s$ in probability.

Step 4. Extend to other bounded Borel sets with thin boundary.

- Approximate such a set A from inside and outside with small space-filling SLE blocks.
- To use the squeeze theorem, we need

$$\mu_{h^\gamma}(\partial A) = \lim_{\varepsilon \to 0} \mu_{h^\gamma}(\mathcal{B}_\varepsilon(\partial A)) = 0.$$

Step 5. Extend to other GFF variants by absolute continuity.

Outlook

1. Further applications of the mating-of-theory to the LQG metric?
   - Quantum boundary length as Minkowski content
   - Need an observable of the LQG metric which is determined locally.

Outlook

2. SLE$_\kappa$-decorated γ-LQG where $\kappa \neq 16/\gamma^2$
   - $(\mathbb{C}, 0, D_{h^\gamma}, \mu_{h^\gamma}, \eta) \stackrel{d}{=} (\mathbb{C}, 0, s^{1/d_\gamma} D_{h^\gamma}, s\mu_{h^\gamma}, \eta(s\cdot)) \stackrel{d}{=} (\mathbb{C}, \eta(t), D_{h^\gamma}, \mu_{h^\gamma}, \eta(\cdot + t))$
   - Tightness of Minkowski content approximations follow from
$$\mathbb{E}[(\mathrm{diam}(\eta([0,1]); D_{h^\gamma}))^p] < \infty \quad \text{for all } p \in \mathbb{R}.$$
   - Its proof uses standard LQG estimates but it does not use the mating-of-trees theory.

Theorem (Gwynne–S., 2022)

Almost surely, η parameterized by μ_h on the metric space $(\mathbb{C}, D_{h^\gamma})$ is locally Hölder continuous with any exponent $< 1/d_\gamma$ and is not locally Hölder continuous with any exponent $> 1/d_\gamma$.

   - Used in the study of meandric permutons (Borga–Gwynne–Sun, 2022)
   - E.g., the conjectured scaling limit of uniform meandric permutations: a γ-LQG surface decorated with two independent SLE$_8$ curves with $\gamma = \sqrt{\frac{1}{3}(17 - \sqrt{145})}$.