Quasi-invariance for SLE welding measures

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Abstract

A large class of Jordan curves on the Riemann sphere can be encoded by circle homeomorphisms via conformal welding, among which we consider the welding homeomorphism of the random SLE loops and the Weil–Petersson class of quasicircles. It is known from the work of Carfagnini and Wang [CW24] that the Onsager–Machlup action functional of SLE loop measures — the Loewner energy — coincides with the Kähler potential of the unique right-invariant Kähler metric on the group of Weil–Petersson circle homeomorphisms. This identity suggests that the group structure given by the composition shall play a prominent role in the law of SLE welding, which is so far little understood.

In this paper, we show a Cameron–Martin type result for random weldings arising from Gaussian multiplicative chaos, especially the SLE welding measures, with respect to the natural group action by Weil–Petersson circle homeomorphisms. More precisely, we show that these welding measures are quasi-invariant when pre- or post-composing the random welding by a fixed Weil–Petersson circle homeomorphism. Our proof is based on the characterization of the composition action in terms of Hilbert–Schmidt operators on the Cameron–Martin space of the log-correlated Gaussian field and the description of the SLE welding as the welding of two independent Liouville quantum gravity (LQG) disks.

Contents

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1 Introduction

1.1 Background and motivation

The Schramm–Loewner evolution (SLE_κ) loop measure is a one-parameter family of σ-finite measures indexed by $\kappa \in (0,8)$ on the space of non-self-crossing loops on the Riemann sphere. For $\kappa \in (0,4]$, the SLE_κ loop measure is supported on Jordan curves. SLE curves first arose as interfaces in the scaling limits of critical lattice models [Sch00, SW01, LSW04, Smi06, SS09, CDCH$^+$14, ACSW24] and the loop version of SLE was constructed and studied in [KS07, Wer08, KW16, BD16, Zha21].

The group of Weil–Petersson homeomorphisms, denoted $\mathrm{WP}(\mathbb{S}^1)$, is a subgroup of quasisymmetric circle homeomorphisms introduced in [Cui00, TT06]. The most straightforward characterization of $\mathrm{WP}(\mathbb{S}^1)$ is that $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ if and only if φ is absolutely continuous and $\log|\varphi'|$ belongs to the fractional Sobolev space $H^{1/2}(\mathbb{S}^1)$ [She18]. Takhtajan and Teo [TT06] showed that the Weil–Petersson Teichmüller space $T_0(1)$, which is the quotient space of $\mathrm{WP}(\mathbb{S}^1)$ up to post-composition by Möbius transformations, carries an essentially unique Kähler structure that is right-invariant (namely, invariant under post-composition by elements in $\mathrm{WP}(\mathbb{S}^1)$).

A close relationship between these two objects was first observed by Yilin Wang [Wan19b], who showed the equivalence between the Loewner energy — the large deviation rate function for SLE_κ as $\kappa \to 0$ [Wan19a, RW21, PW24] — and the universal Liouville action — the unique right-invariant Kähler potential on the Weil–Petersson Teichmüller space $T_0(1)$ found by Takhtajan and Teo [TT06]. Recently, the Loewner energy was further proven to be the Onsager–Machlup functional of the SLE_κ loop measure for $\kappa \in (0,4]$ in the work of Carfagnini and Wang [CW24], thus extending the connection beyond the semiclassical regime. Hence, it is natural to wonder how the Kähler and group structures on the Weil–Petersson Teichmüller space manifest on the SLE_κ loop measure. In this work, we concentrate on the latter part of the question: i.e., quasi-invariance under the composition of circle homeomorphisms given by conformal welding.

The Loewner energy of a Jordan curve is defined as the Dirichlet energy of its driving function [Wan19a, RW21, SW24]. The driving function of an SLE_κ loop is $\sqrt{\kappa}$ times the two-sided Brownian motion, whose Cameron–Martin space consists of functions with finite Dirichlet energy. Thus, the loops with finite Loewner energy, which are Weil–Petersson quasicircles as identified in [Wan19b], can be viewed as the Cameron–Martin space of SLE_κ loop measures for the "addition" of driving functions. Our result shows that despite the nonlinearity of the composition of welding homeomorphisms as opposed to the addition of driving functions, $\mathrm{WP}(\mathbb{S}^1)$ behaves as the Cameron–Martin space of the random welding homeomorphism corresponding to SLE_κ loops under the natural group action given by composition.

1.2 Main result

To describe our main result, we give a brief overview of conformal welding. Given an oriented Jordan curve η on the Riemann sphere $\hat{\mathbb{C}}$, let $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ denote any conformal maps from the unit disk $\mathbb{D} = \{z \in \hat{\mathbb{C}} : |z| < 1\}$ and the outer unit disk $\mathbb{D}^* := \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ onto the bounded and unbounded components Ω and Ω^* of $\hat{\mathbb{C}} \setminus \eta$, respectively. A classic result of Carathéodory states that f and g extend to homeomorphisms on the closed disks $\overline{\mathbb{D}}$ and $\overline{\mathbb{D}^*}$, respectively. The map $\psi := (g^{-1} \circ f)|_{\mathbb{S}^1}$ is then an orientation-preserving homeomorphism on the unit circle $\mathbb{S}^1$ to itself. We call any homeomorphism that arises in this way a *welding*.

Note that for any Jordan curve η and a Möbius map $\omega \in \mathrm{M\ddot{o}b}(\hat{\mathbb{C}})$ of the Riemann sphere $\hat{\mathbb{C}}$, the image $\omega(\eta)$ has the same welding as η. On the other hand, we may pre-compose them by Möbius maps fixing $\mathbb{S}^1$, the space of which we denote $\mathrm{M\ddot{o}b}(\mathbb{S}^1)$. Hence, conformal welding is better described in terms of the correspondence

$$\mathrm{M\ddot{o}b}(\hat{\mathbb{C}}) \backslash \{\text{Oriented Jordan curves}\} \to \mathrm{M\ddot{o}b}(\mathbb{S}^1) \backslash \mathrm{Homeo}_+(\mathbb{S}^1)/\mathrm{M\ddot{o}b}(\mathbb{S}^1). \tag{1.1}$$

This correspondence is neither injective nor onto; see [Bis07] and the references therein. (We note a recent article [Rod25] that describes every circle homeomorphism as a composition of two welding homeomorphisms.) The map (1.1) is injective when restricting to *conformally removable curves* (see Section 4.1). For example, quasicircles (images of the unit circle under quasiconformal homeomorphisms of the Riemann sphere $\hat{\mathbb{C}}$) are conformally removable [BA56], and SLE_κ curves are almost surely conformally removable for $\kappa \in (0, \kappa_0)$ where $\kappa_0 \in (4, 8)$ [JS00, RS05, KMS22, KMS23a]. However, there are no known geometric or analytic characterizations that are equivalent to conformal removability [Bis20].

In this work, we will only consider conformally removable curves, whose weldings comprise the space $\mathrm{RM}(\mathbb{S}^1)$. We endow $\mathrm{RM}(\mathbb{S}^1)$ with the topology of joint compact convergence for the Riemann maps f and g (i.e., Carathéodory topology for both components of $\hat{\mathbb{C}} \setminus \eta$). Let us first make the following observation, which allows us to consider the group of quasisymmetric homeomorphisms acting measurably on the random weldings corresponding to SLE_κ loops.

Proposition 1.1 (See Propositions 4.3 and 4.6)**.** *The pre- and post-compositions of conformally removable weldings by quasisymmetric homeomorphisms are continuous.*

Our main result concerns random weldings corresponding SLE_κ loops when $\kappa \in (0, 4)$. We focus on a specific realization of the one-to-one correspondence between weldings and loops, described in Definition 4.7. At this time, we note that the SLE_κ loop measure restricted to loops that separate 0 from ∞ induces a probability measure on the stabilizers of 1 in $\mathrm{RM}(\mathbb{S}^1)$, which we denote as $\mathrm{SLE}_\kappa^{\mathrm{weld}}$.

Theorem 1.2. *For $\kappa \in (0, 4)$, sample ψ_κ from $\mathrm{SLE}_\kappa^{\mathrm{weld}}$. We have the following for any fixed $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ with $\varphi(1) = 1$.*

- *The law of $\psi_\kappa \circ \varphi$ is mutually absolutely continuous with respect to* $\mathrm{SLE}_\kappa^{\mathrm{weld}}$.
- *The law of $\varphi^{-1} \circ \psi_\kappa$ is mutually absolutely continuous with respect to* $\mathrm{SLE}_\kappa^{\mathrm{weld}}$ *if the log-ratio* $\log|(\varphi(\cdot) - \varphi(1))/(\cdot - 1)|$ *belongs to* $H^{1/2}(\mathbb{S}^1)$.

We expect that if φ is a quasisymmetric circle homeomorphism which is not in $\mathrm{WP}(\mathbb{S}^1)$, then neither $\psi_\kappa \circ \varphi$ nor $\varphi^{-1} \circ \psi_\kappa$ has a law equivalent to $\mathrm{SLE}_\kappa^{\mathrm{weld}}$.

As far as the authors are aware, before this work, only analytic deformations of SLE loops (hence, equivalently, of their welding homeomorphisms) have been studied, within the framework of conformal restriction in particular. This is the first time a non-smooth deformation of SLE is considered. As communicated to us by the authors, Baverez and Jego [BJ25] have an independent work studying the SLE welding measure under composition by an analytic circle diffeomorphism, and they compute the Radon–Nikodym derivatives under such deformation using an approach different from but complementary to ours. We will comment on related literature in Section 1.4.

Remark 1.3. For each $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, the circle homeomorphism $\hat{\varphi}(\cdot) := \varphi(z_0 \cdot)/\varphi(z_0)$ obtained by conjugating φ with the rotation $z \mapsto z_0 z$ satisfies $\log|(\hat{\varphi}(\cdot) - \hat{\varphi}(1))/(\cdot - 1)| \in H^{1/2}(\mathbb{S}^1)$ for a.e. $z_0 \in \mathbb{S}^1$, as shown in Lemma 2.10. Instead of normalizing weldings corresponding to SLE_κ loops by considering those that stabilize $1 \in \mathbb{S}^1$, if we consider the post-composition of ψ_k by a uniform rotation of $\mathbb{S}^1$, then we obtain a version of Theorem 1.2 that is symmetric for pre- and post-compositions by elements of $\mathrm{WP}(\mathbb{S}^1)$: see Corollary 4.12.

1.3 Key ingredients of the proof

Our strategy for the proof of Theorem 1.2 is as follows. We first show that the log-correlated Gaussian field (LGF) on the unit circle is quasi-invariant under pullbacks by Weil–Petersson homeomorphisms. Then, we "lift" this characterization to that of the Gaussian multiplicative chaos (GMC). Finally, we obtain our main theorem by identifying the law of SLE welding in terms of compositions of two homomorphisms defined using GMC measures. Here, we explain this proof outline in further detail.

Given an integrable function h on $\mathbb{S}^1$, we define $\pi_0 h$ to be the projection to its "mean-zero part." That is, $\pi_0 h = h - c_0$ where c_0 is the average of h on $\mathbb{S}^1$. The LGF on the unit circle is defined as the formal sum

$$h(\cdot) = \sqrt{2} \sum_{n \geq 1} \xi_n h_n(\cdot), \tag{1.2}$$

where ξ_n is a sequence of i.i.d. standard Gaussian random variables and h_n is an orthonormal basis of $\mathcal{H}_0 := \pi_0 H^{1/2}(\mathbb{S}^1)$. Here, we write $\sqrt{2}$ to demonstrate that the covariance of $h(x)$ and $h(y)$ is $-2\log|x - y|$.

Let $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$ be an orientation preserving homeomorphism. Then, φ induces a pullback operator $\Pi(\varphi)$ given by

$$\Pi(\varphi)(f) := \pi_0(f \circ \varphi) \quad \text{for } f \in \mathcal{H}_0. \tag{1.3}$$

The operator was introduced in [NS95] and [Par97]; the former showed that $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ if and only if $\Pi(\varphi)$ is a bounded operator from $\mathcal{H}_0$ onto $\mathcal{H}_0$. We extend $\Pi(\varphi)$ to act on the LGF h using the expansion (1.2) and by linearity.

Theorem 1.4. *Let h be an LGF on the unit circle and suppose that $\varphi \in \mathrm{QS}(\mathbb{S}^1)$. Then, the law of $\Pi(\varphi)(h)$ is mutually absolutely continuous with LGF if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$. Otherwise, they are mutually singular.*

The proof of Theorem 1.4 is based on the Feldman–Hájek theorem, which gives a necessary and sufficient condition for classifying two infinite dimensional Gaussian measures on a locally convex space as either mutually absolutely continuous or mutually singular. In our context, this depends on whether $\Pi(\varphi)\Pi(\varphi)^* - I$ is Hilbert–Schmidt. We show that this condition holds if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ (see Lemma 2.7) based on previous results [Sch81, TT06, HS12] which identified certain operators associated with $\Pi(\varphi)$ to be Hilbert–Schmidt.

Our next step is to "lift" Theorem 1.4 to that for the Gaussian multiplicative chaos (GMC) measure $\mathcal{M}_h^\gamma$ defined from LGF h heuristically as $\exp(\frac{\gamma}{2}h)\,\mathrm{d}\theta$ for $\gamma \in (0, 2]$. See Section 3 for the precise definition and its basic properties. We define the normalized GMC measure as $\widehat{\mathcal{M}}_h := \mathcal{M}_h/\mathcal{M}_h(\mathbb{S}^1)$ such that the total measure on the circle is equal to 1.

Proposition 1.5. *Let $\mathcal{M}_h = \mathcal{M}_h^\gamma$ be the GMC measure corresponding to an LGF h for $\gamma \in (0, 2]$. If $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, then the following coordinate change rule holds for its pull-back: almost surely,*

$$\varphi^* \mathcal{M}_h = \mathcal{M}_{h\circ\varphi + Q\log|\varphi'|} \tag{1.4}$$

where $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$. Furthermore, for $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$, the law of the normalized pull-back measure $\varphi^*\widehat{\mathcal{M}}_h$ is absolutely continuous with respect to the law of $\widehat{\mathcal{M}}_h$ if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$.

The coordinate change formula (1.4) was proven to hold for 2D log-correlated field in [DS11]. This proof can be applied straightforwardly for the 1D LGF when φ is a diffeomorphism of $\mathbb{S}^1$ (Lemma 3.8). We prove the first part of Proposition 1.5 by approximating an arbitrary $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ with diffeomorphisms and showing that the corresponding GMC measures converge (Lemma 3.9). To show the second part of Proposition 1.5, we use that the GMC measure almost surely determines the LGF as proved in [BSS23] (for the $\gamma = 2$ case, in [Vih24]).

The final step of our proof of Theorem 1.2 is based on the theory of conformal welding of Liouville quantum gravity (LQG) surfaces. In particular, Ang, Holden, and Sun [AHS23] proved that the conformal welding of two independent LQG disks gives the SLE_κ loop measure. We use the following translation of this result in terms of the law of the welding homeomorphism.

If h is an LGF or a variant thereof, let

$$\phi_h^\gamma(z) := \exp(2\pi\mathfrak{i} \cdot \widehat{\mathcal{M}}_h^\gamma([1, z])),$$

where $[1, z] \subset \mathbb{S}^1$ denotes the arc running counterclockwise from 1 to z.

Lemma 1.6. *Let $\gamma \in (0, 2)$ and $\kappa = \gamma^2 \in (0, 4)$. If h_1 and h_2 are independent LGFs on the unit circle, then $\mathrm{SLE}_\kappa^{\mathrm{weld}}$ is mutually absolutely continuous with respect to the law of $(\phi^\gamma_{h_2 - \gamma\log|\cdot - 1|})^{-1} \circ \phi^\gamma_{h_1}$.*

Lemma 1.6 is a short version of Lemma 4.8. We will obtain Lemma 4.8 using conformal welding of quantum disks when each disk has one interior marked point, as established in [ACSW24] based on the works [She16, AHS23]. Theorem 1.2 follows from combining Proposition 1.5 with Lemma 1.6. The conformal welding result for quantum disks in [AHS23] is expected to hold for $\gamma = 2$, in which case Theorem 1.2 would extend to $\kappa = 4$.

1.4 Comments and related literature

The SLE_κ loop measure can be defined for $\kappa \in (0,8)$ [Zha21]. When $\kappa \in (4,8)$, SLE_κ is not simple [RS05]. However, it was recently proved in [ACSW24] that the conformal welding of generalized quantum disks (which have the topology of infinitely many disks concatenated into a tree-like shape) gives the SLE_κ loop for $\kappa \in (4,8)$. It would be interesting to consider if there is a family of homeomorphisms on the boundary of a generalized quantum disk that gives an analgous statement to Theorem 1.2.

Several recent works have considered conformal deformations of SLE loops. The conformal restriction covariance of chordal SLE was outlined by Lawler, Schramm, and Werner in [LSW03]. Its loop version was postulated by Kontsevich and Suhov [KS07] and proved by Zhan in [Zha21]. The work [SW24] computed the first variation of the driving function of a Loewner chain under quasiconformal deformations away from the curve, leading to the alternative proof of Loewner energy as a Kähler potential on the Weil–Petersson Teichmüller space. Based on this result, Gordina, Qian, and Wang [GQW24] used the SLE_κ loop measure for $\kappa \in (0,4]$ to construct a natural representation of the Virasoro algebra of central charge $c \le 1$. An independent work of Baverez and Jego [BJ24] developed the conformal field theory for the SLE_κ loop measure and proved its characterization as a Malliavin–Kontsevich–Suhov measure for $\kappa \in (0,4]$. On the one hand, these results provide evidence for the idea that the Weil–Petersson Teichmüller space should be the Cameron–Martin space of SLE loop measures. On the other hand, deformations considered in these works are limited to those which are analytic in a neighborhood of the Jordan curve. In this work, we consider the full class of Weil–Petersson quasisymmetric homeomorphisms acting on SLE loops for the first time.

Our proof of Theorem 1.2 can be adapted to show quasi-invariance for other weldings derived from GMC measures that have appeared in the literature. The work [AJKS11] showed, using geometric function theory methods without reference to LQG theory, that ϕ_h^γ is a welding for $\gamma \in (0,2)$. The same work also showed that $\phi_{h_2}^{\gamma_2} \circ (\phi_{h_1}^{\gamma_1})^{-1}$, where h_1, h_2 are independent LGFs and $\gamma_1, \gamma_2 \in (0,2)$, is a welding. Using a similar approach, the articles [BK23, KMS23b] showed that $(\phi_{h_2}^{\gamma_2})^{-1} \circ \phi_{h_1}^{\gamma_1}$, which is more alike the random homeomorphism in Lemma 1.6, is a welding for small $\gamma_1, \gamma_2 > 0$. In fact, 1.6 shows that the Jordan curve solving the welding problem for $(\phi_{h_2}^{\gamma})^{-1} \circ \phi_{h_1}^{\gamma}$ is locally mutually absolutely continuous with the chordal SLE_{γ^2} curve if we remove a neighborhood of the root (the image of $1 \in \mathbb{S}^1$) of the loop. We give the analogous quasi-invariance results for these random weldings in Corollary 4.13.

This paper is organized as follows. In Section 2, we introduce the groups of circle homeomorphisms that are of interest in this work and describe properties of the associated pull-back operator $\Pi(\varphi)$ on the function space $\mathcal{H}_0$. In Section 3, we introduce the LGF and GMC and prove their quasi-invariance under pullbacks by Weil–Petersson homeomorphisms. Section 4 completes the proof of the quasi-invariance of SLE welding, with the necessary development of the following concepts: conformally removable weldings and the continuity of the composition action on it by the quasisymmetric group, the SLE loop measure and the corresponding SLE welding measure, and the conformal welding of quantum disks.

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2 Pullback operator associated with the Weil–Petersson class

In this section, we introduce the group of quasisymmetric circle homeomorphisms and the associated pullback operators on a function space of the unit circle and the log-ratio. The first result of this section is Lemma 2.7, where we find various correspondences between subgroups of circle homeomorphisms and the properties of pullback operators. The second result is Lemma 2.10, where we prove that the log-ratio is in the same space as the log-derivative.

2.1 Quasisymmetric homeomorphisms and the Weil–Petersson class

Let $\mathbb{S}^1$ denote the unit circle, which we identify with the boundary of the unit disk $\mathbb{D}$ in the complex plane. The set of orientation-preserving homeomorphisms of $\mathbb{S}^1$, which we denote $\mathrm{Homeo}_+(\mathbb{S}^1)$, has a natural group structure with the composition of functions as the group action. In this subsection, we introduce various subgroups of it. Some trivial examples are $\mathrm{Diff}_+(\mathbb{S}^1)$, $\mathrm{Möb}(\mathbb{S}^1)$, and $\mathrm{Rot}(\mathbb{S}^1)$, which consist of smooth diffeomorphisms, Möbius transformations, and rotations, respectively.

We shall pay special attention to the subgroup of quasisymmetric homeomorphisms. We briefly recall its definition and basic properties for the reader who is unfamiliar with the concept and direct to, e.g., [Leh87] for further details.

Definition 2.1. We say that $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$ is *quasisymmetric* if there exists some $C_0 > 0$ such that

$$\frac{1}{C_0} \le \left| \frac{\varphi(e^{\mathrm{i}(\theta+t)}) - \varphi(e^{\mathrm{i}\theta})}{\varphi(e^{\mathrm{i}\theta}) - \varphi(e^{\mathrm{i}(\theta-t)})} \right| \le C_0 \tag{2.1}$$

for all $\theta \in \mathbb{R}$ and $t \in (0, 2\pi)$. Let $\mathrm{QS}(\mathbb{S}^1)$ denote the group of orientation preserving quasisymmetric homeomorphisms of the unit circle $\mathbb{S}^1$.

Beurling and Ahlfors [BA56] proved that $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$ is quasisymmetric if and only if there exists some quasiconformal homeomorphism ω of $\mathbb{D}$ onto itself that extends continuously to φ on $\mathbb{S}^1 = \partial\mathbb{D}$. That is, the *Beltrami coefficient*

$$\mu_\omega = \partial_{\bar{z}}\omega / \partial_z \omega$$

of ω is defined almost everywhere on $\mathbb{D}$ and satisfies $\|\mu_\omega\|_{L^\infty(\mathbb{D})} := \sup_{z\in\mathbb{D}} |\mu_\omega(z)| < 1$. Heuristically speaking, ω maps small circles centered at $z \in \mathbb{D}$ to ellipses with eccentricity $K_\omega(z)$, where the function

$$K_\omega = \frac{1 + |\mu_\omega|}{1 - |\mu_\omega|},$$

is called the *dilatation* of ω. The Teichmüller distance between two quasisymmetric homeomorphisms is defined as

$$\tau_1(\varphi_1,\varphi_2)=\inf\left\{\frac{1}{2}\log\frac{1+\|\frac{\mu_1-\mu_2}{1-\bar{\mu}_1\mu_2}\|_{L^\infty(\mathbb{D})}}{1-\|\frac{\mu_1-\mu_2}{1-\bar{\mu}_1\mu_2}\|_{L^\infty(\mathbb{D})}}\middle|\varphi_1=\omega_{\mu_1}|_{\mathbb{S}^1},\varphi_2=\omega_{\mu_2}|_{\mathbb{S}^1}\right\}.$$

The Teichmüller distance induces the natural topology on $\mathrm{QS}(\mathbb{S}^1)$. Let us further introduce two special subgroups of $\mathrm{QS}(\mathbb{S}^1)$.

Definition 2.2. We say that $\varphi\in\mathrm{QS}(\mathbb{S}^1)$ is *symmetric* if φ can be extended to a quasiconformal map ω on $\mathbb{D}$ whose Beltrami coefficient μ_ω satisfies $\mu_\omega(z)\to 0$ as $|z|\to 1$. Let $\mathrm{S}(\mathbb{S}^1)$ denote the sets of all symmetric orientation preserving homeomorphisms of $\mathbb{S}^1$.

The following subgroup, first introduced in [Cui00], is our protagonist.

Definition 2.3. We say that $\varphi\in\mathrm{QS}(\mathbb{S}^1)$ belongs to the *Weil–Petersson class* if φ has a quasiconformal extension ω to the unit disk whose Beltrami coefficient μ_ω satisfies

$$\int_{\mathbb{D}}|\mu_\omega(z)|^2\,(1-|z|^2)^{-2}\,\mathrm{d}A(z)<\infty, \tag{2.2}$$

where A denotes the area measure. Equivalently, φ is absolutely continuous (with respect to the arc-length measure) and $\log|\varphi'|\in H^{1/2}(\mathbb{S}^1)$. Here, $H^{1/2}(\mathbb{S}^1)$ is the fractional Sobolev space consisting of functions $f:\mathbb{S}^1\to\mathbb{R}$ satisfying

$$\iint_{\mathbb{S}^1\times\mathbb{S}^1}\left|\frac{f(x)-f(y)}{x-y}\right|^2\mathrm{d}x\,\mathrm{d}y<\infty. \tag{2.3}$$

Let $\mathrm{WP}(\mathbb{S}^1)$ denote the set of all quasisymmetric homeomorphisms of the unit circle that belong to the Weil–Petersson class.

The equivalence between the two definitions above is due to Yuliang Shen [She18]. Moreover, it is proved there that the above two metric induces the same topology. The following is an equivalent definition of $H^{1/2}(\mathbb{S}^1)$.

$$H^{1/2}(\mathbb{S}^1)=\left\{f:\mathbb{S}^1\to\mathbb{R}\;\middle|\;f(e^{\mathrm{i}\theta})=c_0+\sum_{n\in\mathbb{Z}\setminus\{0\}}c_n\frac{e^{\mathrm{i}n\theta}}{\sqrt{|n|}},\ \sum_{n=1}^{\infty}|c_n|^2<\infty\right\}.$$

We discuss this space in further detail in the next subsection. There is a further multitude of equivalent definitions for the Weil–Petersson class related to various parts of mathematics: see [Bis24] for a compilation.

Here is the relationship between the groups of circle homeomorphisms we have considered in this section so far.

$$\mathrm{Diff}_+(\mathbb{S}^1)\subsetneqq\mathrm{WP}(\mathbb{S}^1)\subsetneqq\mathrm{S}(\mathbb{S}^1)\subsetneqq\mathrm{QS}(\mathbb{S}^1)\subsetneqq\mathrm{RM}(\mathbb{S}^1)\subsetneqq\mathrm{Homeo}_+(\mathbb{S}^1).$$

Remark 2.4. The universal Teichmüller space $T(1)$ and the Weil–Petersson Teichmüller space $T_0(1)$ can be represented by $\mathrm{Möb}(\mathbb{S}^1)\backslash\mathrm{QS}(\mathbb{S}^1)$ and $\mathrm{Möb}(\mathbb{S}^1)\backslash\mathrm{WP}(\mathbb{S}^1)$, respectively. Identifying these coset spaces with the subgroup of homeomorphisms that fix $-1,-\mathrm{i}$, and 1, the group structure on these spaces is given by composition. As hinted in the introduction, taking the quotient under left actions by $\mathrm{Möb}(\mathbb{S}^1)$ corresponds to considering the equivalence class of quasicircles on $\hat{\mathbb{C}}$ modulo Möbius transformations that fix 0. The norm (2.2) is inherited from the Weil–Petersson metric, which gives the universal Teichmüller space a Hilbert structure [TT06].

In our work, we shall be mostly interested in the Weil–Petersson Teichmüller curve $\mathcal{T}_0(1) \simeq \mathrm{Rot}(\mathbb{S}^1)\backslash \mathrm{WP}(\mathbb{S}^1)$, which we identify with the cosets of homeomorphisms that fix 1. By conformal welding, these homeomorphisms can be identified with quasicircles in $\mathbb{C}$ that disconnect 0 from ∞ modulo Möbius transformations that fix 0 and ∞. See [TT06, Sec. I.1] for further details.

2.2 Sobolev spaces on the unit disk and its boundary

In this subsection, we firstly introduce the symplectic Hilbert space $\mathcal{H}_0$ consisting of elements of the fractional Sobolev space $H^{1/2}(\mathbb{S}^1)$ with zero mean. Looking ahead, the significance of the space $\mathcal{H}_0$ is that it is the Cameron–Martin space of the the log-correlated Gaussian field on $\mathbb{S}^1$ (see Section 3). Then we consider its compexification and the Poisson integral. Finally, we mention other Sobolev spaces.

Let $\mathcal{H}_0$ denote the real Hilbert space

$$\mathcal{H}_0 := \left\{ f : \mathbb{S}^1 \to \mathbb{R} \;\middle|\; f(e^{\mathrm{i}\theta}) = \sum_{n\neq 0} c_n \frac{e^{\mathrm{i}n\theta}}{\sqrt{|n|}} \text{ with } c_{-n} = \overline{c_n},\ \sum_{n=1}^{\infty} |c_n|^2 < \infty \right\} \tag{2.4}$$

with the inner product

$$\left\langle \sum_{n\neq 0} c_n \frac{e^{\mathrm{i}n\theta}}{\sqrt{|n|}}, \sum_{n\neq 0} d_n \frac{e^{\mathrm{i}n\theta}}{\sqrt{|n|}} \right\rangle = \sum_{n\neq 0} c_n \overline{d_n}. \tag{2.5}$$

We consider the canonical symplectic form Θ on $\mathcal{H}_0$ introduced in [NS95] as

$$\Theta(f,g) = \frac{1}{2\pi}\int_{\mathbb{S}^1} f\ \mathrm{d}g = -\mathrm{i}\sum_{n=1}^{\infty}(c_n \bar{d}_n - c_{-n}\bar{d}_{-n}). \tag{2.6}$$

for $f(e^{\mathrm{i}\theta}) = \sum_{n\neq 0} c_n \frac{e^{\mathrm{i}n\theta}}{\sqrt{|n|}}$ and $g(e^{\mathrm{i}\theta}) = \sum_{n\neq 0} d_n \frac{e^{\mathrm{i}n\theta}}{\sqrt{|n|}}$. Let us denote the group of bounded symplectomorphisms of $\mathcal{H}_0$ as $\mathrm{Sp}(\mathcal{H}_0)$.

By complex linearity, the symplectic form Θ defined in (2.6) extends to the complexification

$$\mathcal{H}_0^{\mathbb{C}} := \left\{ f : \mathbb{S}^1 \to \mathbb{C} \;\middle|\; f(e^{\mathrm{i}\theta}) = \sum_{n\neq 0} c_n \frac{e^{\mathrm{i}n\theta}}{\sqrt{|n|}},\ \sum_{n\neq 0} |c_n|^2 < \infty \right\}. \tag{2.7}$$

With respect to Θ, the Hilbert space $\mathcal{H}_0^{\mathbb{C}}$ has a canonical decomposition into two closed isotropic subspaces

$$\mathcal{H}_0^{\mathbb{C}} = W_+ \oplus W_-, \tag{2.8}$$

where

$$W_+ = \left\{ f : \mathbb{S}^1 \to \mathbb{C} \;\middle|\; f(e^{\mathrm{i}\theta}) = \sum_{n=1}^{\infty} a_n \frac{e^{\mathrm{i}n\theta}}{\sqrt{n}},\ \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}, \tag{2.9}$$

$$W_- = \left\{ f : \mathbb{S}^1 \to \mathbb{C} \;\middle|\; f(e^{\mathrm{i}\theta}) = \sum_{n=1}^{\infty} b_n \frac{e^{-\mathrm{i}n\theta}}{\sqrt{n}},\ \sum_{n=1}^{\infty} |b_n|^2 < \infty \right\}. \tag{2.10}$$

Let

$$\left\{ e_n = \frac{e^{\mathrm{i}n\theta}}{\sqrt{n}} \right\}_{n\geq 1} \text{ and } \left\{ f_n = \frac{e^{-\mathrm{i}n\theta}}{\sqrt{n}} \right\}_{n\geq 1} \tag{2.11}$$

denote the standard bases of the subspaces W_+ and W_-, respectively. Under these bases, W_+ and W_- are naturally isomorphic to $\ell^2(\mathbb{C})$.

Each element of $\mathrm{Sp}(\mathcal{H}_0)$, the group of bounded symplectomorphism of $\mathcal{H}_0$, extends to $\mathcal{H}_0^{\mathbb{C}}$ again by complex linearity. In the basis $\{e_n\}_{n\geq 1}$ and $\{f_n\}_{n\geq 1}$, they can be represented by the matrices

$$\begin{pmatrix} M & N \\ \bar{N} & \bar{M} \end{pmatrix} \text{ where } MM^* - NN^* = I, MN^t = NM^t. \tag{2.12}$$

Above, M^* denotes the adjoint matrix of M and M^t denotes the transpose matrix of M.

Let us now consider the relationship between $\mathcal{H}_0$ and the Dirichlet class of functions on the unit disk $\mathbb{D}$.

- The space $\mathcal{H}_0$ is naturally isomorphic to the real Hilbert space $\mathcal{D}_0$ of harmonic functions F on the unit disk $\mathbb{D}$ with $F(0) = 0$ and finite Dirichlet energy. That is,

  $$\mathcal{D}_0 = \left\{ F : \mathbb{D} \to \mathbb{R} \;\middle|\; F(z) = \sum_{n>0} c_n \frac{z^n}{\sqrt{n}} + c_{-n} \frac{\bar{z}^n}{\sqrt{n}}, c_{-n} = \overline{c_n}, \sum_{n=1}^{\infty} |c_n|^2 < \infty \right\}. \tag{2.13}$$

  If $F \in \mathcal{D}_0$, then it is straightforward to check that the trace $f := F|_{\mathbb{S}^1}$ on $\mathbb{S}^1$ is an element of $\mathcal{H}_0$ with

  $$\|f\|_{\mathcal{H}_0}^2 = \frac{1}{2\pi} \int_{\mathbb{D}} |\nabla F|^2 < \infty. \tag{2.14}$$

  On the other hand, let $\mathrm{P}(f)$ denote the Poisson integral of an integrable function f on the unit circle $\mathbb{S}^1$: i.e.,

  $$\mathrm{P}(f)(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{S}^1} \mathrm{Re}\, \frac{w+z}{w-z} \frac{f(w)}{w} \,\mathrm{d}w, \text{ for } z \in \mathbb{D}. \tag{2.15}$$

  Then, for each $f \in \mathcal{H}_0$, we have $\mathrm{P}(f) \in \mathcal{D}_0$ with $f = \mathrm{P}(f)|_{\mathbb{S}^1}$. Clearly, this isomorphism between $\mathcal{H}_0$ and $\mathcal{D}_0$ given by the trace operator and the Poisson integral extends naturally to that between their complexifications.

- Let $H^1(\mathbb{D})$ (resp. $H_0^1(\mathbb{D})$) be the real Hilbert space given by the completion of the space of smooth functions on $\mathbb{D}$ (resp. with compact support) with respect to the Dirichlet inner product. Then, we have the decomposition

  $$H^1(\mathbb{D}) = H_0^1(\mathbb{D}) \oplus \mathcal{D}_0 \oplus \mathbb{R} \tag{2.16}$$

  as a direct sum with respect to the Dirichlet inner product. We will see later that, from the decomposition above, there exists a decomposition of the Neumann Gaussian free field into the sum of independent Dirichlet Gaussian free field and the "harmonic" Gaussian field on $\mathbb{D}$ (see Remark 3.1).

We conclude this subsection by giving a quick overview of fractional Sobolev spaces of general index $s \in \mathbb{R}$ on the unit disk $\mathbb{D}$ and its boundary $\mathbb{S}^1$.

First, let us consider the space $H_0^s(\mathbb{D})$ with zero boundary conditions. Let $\{g_n\}_{n\geq 1}$ be a sequence of eigenfunctions of the Laplacian $-\Delta$ on $\mathbb{D}$ with Dirichlet boundary conditions which are mutually orthogonal with respect to the Dirichlet inner product. Let $\{\lambda_n\}_{n\geq 1}$ be the corresponding eigenvalues. That is, g_n and λ_n satisfy

$$\begin{cases} -\Delta g_n = \lambda_n g_n & \text{in } \mathbb{D} \\ g_n = 0 & \text{on } \partial\mathbb{D} \end{cases} \tag{2.17}$$

for each n. The eigenvalues $\{\lambda_n\}_{n\geq 1}$ are positive and satisfy $\lambda_n \to \infty$ as $n \to \infty$.

For each $s \in \mathbb{R}$, we define $H_0^s(\mathbb{D})$ as the real Hilbert space obtained by taking the completion of the set of smooth, compactly supported real-valued functions on the unit disk with respect to the inner product

$$\langle f, g\rangle_s = \sum_{n\geq 1} \lambda_n^s \langle f, g_n\rangle_{L^2} \langle g, g_n\rangle_{L^2}. \tag{2.18}$$

When $s = 1$, this inner product is the standard Dirichlet inner product on $\mathbb{D}$.

We define the Sobolev space $H^s(\mathbb{D})$ with free boundary conditions analogously using the eigenfunctions of $-\Delta$ on $\mathbb{D}$ with Neumann boundary conditions. For the eigenvalue 0 corresponding to the constant function, we let $0^s := 1$ for all $s \in \mathbb{R}$ in (2.18).

Similarly, we can define the Sobolev space $H^s(\mathbb{S}^1)$ of functions on $\mathbb{S}^1$ replacing the operator $-\Delta$ by $-\partial_{\theta\theta}$. Then, $\mathcal{H}_0$ agrees with the closed subspace $H^{1/2}(\mathbb{S}^1)/\mathbb{R}$ of functions $f \in \mathcal{H}$ with $\int_{\mathbb{S}^1} f = 0$ ("mean zero"). In other words, $\mathcal{H}$ consists of functions in $\mathcal{H}_0$ plus a constant. Let us denote the natural projection $H^{1/2}(\mathbb{S}^1) \to \mathcal{H}_0$ as

$$\pi_0(f) := f - \frac{1}{2\pi}\int_{\mathbb{S}^1} f. \tag{2.19}$$

The equivalence between this definition of $H^{1/2}(\mathbb{S}^1)$ and the condition (2.3) can be found in introductory texts on fractional Sobolev spaces.

2.3 The pullback operator

In this subsection, we introduce the pullback operator as a right group action of $\mathrm{Homeo}_+(\mathbb{S}^1)$ on $\mathcal{H}_0$. Then we prove a key relationship (Lemma 2.7) with the Weil–Petersson class.

Definition 2.5. Given a orientation preserving homeomorphism $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$, we define the pullback operator $\Pi(\varphi)$ on $\mathcal{H}_0^{\mathbb{C}}$ as

$$\Pi(\varphi)(f) := \pi_0(f \circ \varphi) = f \circ \varphi - \frac{1}{2\pi}\int_{\mathbb{S}^1} f \circ \varphi \, \mathrm{d}\theta. \tag{2.20}$$

Let us denote the space of bounded linear operators from $\mathcal{H}_0^{\mathbb{C}}$ to itself as $\mathscr{B}(\mathcal{H}_0^{\mathbb{C}})$. Nag and Sullivan [NS95] proved that $\Pi(\varphi) \in \mathscr{B}(\mathcal{H}_0^{\mathbb{C}})$ if and only if $\varphi \in \mathrm{QS}(\mathbb{S}^1)$, and that the assignment $\Pi : \mathrm{QS}(\mathbb{S}^1) \to \mathscr{B}(\mathcal{H}_0^{\mathbb{C}})$ defines a right group action on $\mathcal{H}_0^{\mathbb{C}}$ by symplectomorphisms. In the basis $\{e_n\}_{n\geq 1}$ and

$\{f_n\}_{n\geq 1}$ given in (2.11), the symplectomorphism $\Pi(\varphi)$ for $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ can be represented by the matrix of the form (2.12), whose entries are given by

$$M_{mn}(\varphi) = \frac{1}{2\pi}\sqrt{\frac{m}{n}}\int_{\mathbb{S}^1}\left(\varphi(e^{\mathrm{i}\theta})\right)^n e^{-\mathrm{i}m\theta}\mathrm{d}\theta, \tag{2.21}$$

$$N_{mn}(\varphi) = \frac{1}{2\pi}\sqrt{\frac{m}{n}}\int_{\mathbb{S}^1}\left(\varphi(e^{\mathrm{i}\theta})\right)^{-n} e^{-\mathrm{i}m\theta}\mathrm{d}\theta. \tag{2.22}$$

We note that the operator $\Pi(\varphi)$ preserves the subspaces W_+ and W_- (i.e., $\Pi(\varphi)$ belongs to the unitary subgroup $\mathrm{U}(\mathcal{H}_0)$ of $\mathrm{Sp}(\mathcal{H}_0)$ consisting of bounded symplectomorphisms with $N = 0$) if and only if $\varphi \in \mathrm{Möb}(\mathbb{S}^1)$.

We now give the key result of this section, which will be used in Section 3 to identify the quasi-invariance of the log-correlated Gaussian field on $\mathbb{S}^1$ under pullbacks by $\mathrm{Homeo}_+(\mathbb{S}^1)$. First, let us recall the definition of a Hilbert–Schmidt operator.

Definition 2.6. For any bounded linear operator T from a Hilbert space H to itself, we define the *Hilbert–Schmidt norm* $\|T\|_{\mathrm{HS}}$ by

$$\|T\|_{\mathrm{HS}}^2 := \sum_{\mathrm{a}\in\mathcal{A}} \|Te_{\mathrm{a}}\|_H^2 = \mathrm{Tr}(T^*T), \tag{2.23}$$

where $\{e_{\mathrm{a}}, \mathrm{a} \in \mathcal{A}\}$ is any orthonormal basis of H and T^* is the adjoint operator of T. For a self-adjoint bounded linear operator A, the trace $\mathrm{Tr}(A)$ of A is defined by the sum of all its eigenvalues if the series is absolutely summable and set to be $+\infty$ otherwise. We say T is *Hilbert–Schmidt* if $\|T\|_{\mathrm{HS}} < \infty$. The collection of all Hilbert–Schmidt operators on H forms a Hilbert space $\mathrm{HS}(H)$ with respect to the norm (2.23).

Note that if $H^{\mathbb{C}}$ is the complexification of H, then the complex extension of T belongs to $\mathrm{HS}(H^{\mathbb{C}})$ if and only if $T \in \mathrm{HS}(H)$. If $T \in \mathrm{HS}(H)$ and $S \in \mathscr{B}(H)$, then T^t, T^*, ST, and TS belong to $\mathrm{HS}(H)$.

Lemma 2.7. *Assume $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ and let $\Pi(\varphi) \in \mathscr{B}(\mathcal{H}_0)$ be the pullback operator defined by (2.20). Then, $\Pi(\varphi)\Pi(\varphi)^* - I$ is Hilbert–Schmidt if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$.*

Proof. Recall the matrices M and N associated with $\Pi(\varphi)$ as given by (2.21)–(2.22). Since $\Pi(\varphi) \in \mathrm{Sp}(\mathcal{H}_0)$, we have

$$\begin{pmatrix} M & N \\ \bar{N} & \bar{M}\end{pmatrix}\begin{pmatrix} M^* & -N^t \\ -N^* & M^t\end{pmatrix} = I.$$

as in (2.12). Thus,

$$\Pi(\varphi)\Pi(\varphi)^* - I = \begin{pmatrix} M & N \\ \bar{N} & \bar{M}\end{pmatrix}\begin{pmatrix} M^* & N^t \\ N^* & M^t\end{pmatrix} - I = 2\Pi(\varphi)\begin{pmatrix} & N^t \\ N^* & \end{pmatrix}. \tag{2.24}$$

In [HS12, Thm. 2.2], it was shown (up to the isomorphism described in the previous subsection) that N is Hilbert–Schmidt if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$. Combined with the fact that $\Pi(\varphi)$ and $\Pi(\varphi)^{-1} = \Pi(\varphi^{-1})$ are bounded for $\varphi \in \mathrm{QS}(\mathbb{S}^1)$, we obtain the desired result. ☐

Remark 2.8. For the same reason, the operator $\Pi(\varphi)\Pi(\varphi)^* - I$ is compact if and only if $\varphi \in \mathrm{S}(\mathbb{S}^1)$.

In fact, we will need that all characterizations of $\mathrm{WP}(\mathbb{S}^1)$ that we have discussed so far induce the same topology, which we check by retracing the proofs of equivalences among the various definitions.

Lemma 2.9. *If $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, then there exists a sequence $\varphi_n \in \mathrm{Diff}_+(\mathbb{S}^1)$ such that $\sup_{x\in\mathbb{S}^1} |\varphi_n(x)-\varphi(x)|$, $\|\Pi(\varphi_n \circ \varphi^{-1})\Pi(\varphi_n \circ \varphi^{-1})^* - I\|_{\mathrm{HS}}$, $\|\log|(\varphi_n \circ \varphi^{-1})'|\|_{H^{1/2}(\mathbb{S}^1)}$ and $\|\Pi(\varphi_n)u - \Pi(\varphi)u\|_{H^{1/2}(\mathbb{S}^1)}$ for any $u \in H^{1/2}(\mathbb{S}^1)$ all converge to 0 as $n \to \infty$.*

Proof. It is shown in [She18, Thm. 1.4] that the $H^{1/2}(\mathbb{S}^1)$ norm for $\log|\varphi'|$ induces the same topology as the Weil–Petersson metric. It is shown in [She18, Prop. 4.1] that the topology for φ induced by the Teichmüller distance is finer than the strong operator topology for $\Pi(\varphi)$. From the proof of [HS12, Thm. 4.2], when the dilatation is bounded, this is equivalent to the Hilbert–Schmidt norm for the matrix N with entries (2.22) and therefore the Hilbert–Schmidt norm for the matrix $\Pi(\varphi)\Pi(\varphi)^* - I$ by (2.24). The uniform convergence follows from the argument using the normal family. More precisely, up to subsequence, we can choose φ_n to be the boundary of the quasiconformal self-homeomorphism ω_n of $\mathbb{D}$ that agrees with φ at $1, \mathfrak{i}, -1$ with Beltrami coefficient

$$\mu_n(z) = \mu(z)\mathbf{1}_{\{|z|<1-1/n\}}$$

where μ is the Beltrami coefficient of the Douady–Earle extension ω of φ to $\mathbb{D}$. ☐

2.4 Estimates on the log-ratio

Now, we aim to informally replace the derivative with the difference quotient, but we actually use conformal welding. Looking ahead, it corresponds to adding one boundary marked point on the quantum disk, see Corollary 3.10. Let $H^{1/2}(\mathbb{S}^1, \mathbb{C})$ denote the complexification of $H^{1/2}(\mathbb{S}^1)$.

Lemma 2.10. *If $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, then for a.e. $z_0 \in \mathbb{S}^1$,*

$$u_\varphi(\cdot, z_0) := \log \frac{\varphi(\cdot) - \varphi(z_0)}{\cdot - z_0} \in H^{1/2}(\mathbb{S}^1, \mathbb{C}). \tag{2.25}$$

Moreover, the function $z_0 \mapsto \|u_\varphi(\cdot, z_0)\|_{H^{1/2}(\mathbb{S}^1,\mathbb{C})}$ is L^2-integrable.

Proof. Recall from the introduction the conformal welding decomposition for $\varphi \in \mathrm{WP}(\mathbb{S}^1)$: there exist quasiconformal maps f and g on $\hat{\mathbb{C}}$, conformal on $\mathbb{D}$ and $\mathbb{D}^*$, respectively, such that $\varphi = (g^{-1} \circ f)|_{\mathbb{S}^1}$, which is also a direct corollary of Lemma 4.1 when ψ is the identity map. Since

$$u_\varphi(\cdot, z_0) = \log \frac{f(\cdot) - f(z_0)}{\cdot - z_0} - \log \frac{g\circ\varphi(\cdot) - g\circ\varphi(z_0)}{\varphi(\cdot) - \varphi(z_0)} =: u_f(\cdot, z_0) - u_g(\varphi(\cdot), \varphi(z_0))$$

and $H^{1/2}(\mathbb{S}^1, \mathbb{C})$ is invariant under pullback by a quasisymmetric map, we only need to show that $u_f = u_f(\cdot, z_0)$ and $u_g = u_g(\cdot, \varphi(z_0))$ belong to $H^{1/2}(\mathbb{S}^1, \mathbb{C})$ for almost every $z_0 \in \mathbb{S}^1$.

It is shown in [TT06, Chap 2, Lemma 2.5] that $u_f(z, w) = \sum_{n,m=0}^\infty c_{n,m} z^n w^m$ with

$$\sum_{n,m=1}^\infty |nm|\,|c_{n,m}|^2, \sum_{n=1}^\infty |n|\,|c_{n,0}|^2, \sum_{m=1}^\infty |m|\,|c_{0,m}|^2 < \infty,$$

which is equivalent to the associated Grunsky operator being Hilbert–Schmidt. Define $F_n(e^{\mathrm{i}\psi}) := \sum_{m=0}^{\infty} c_{n,m} e^{\mathrm{i}m\psi}$ such that $\|F_n\|^2_{H^{1/2}(\mathbb{S}^1,\mathbb{C})} = \sum_{m=1}^{\infty} |m|\, |c_{n,m}|^2$, then

$$\|u_f(\cdot, z_0)\|^2_{H^{1/2}(\mathbb{S}^1,\mathbb{C})} = \sum_{n=1}^{\infty} |n|\, |F_n(z_0)|^2 =: F(z_0).$$

It follows that

$$\|F\|_{L^1(\mathbb{S}^1)} = \sum_{n=1}^{\infty} |n|\, \|F_n\|^2_{L^2(\mathbb{S}^1)} \le \sum_{n=1}^{\infty} |n|\, (\|F_n\|^2_{H^{1/2}(\mathbb{S}^1,\mathbb{C})} + |c_{n,0}|^2) < \infty.$$

The case for g is similar. □

Remark 2.11. When $\varphi \in C^3(\mathbb{S}^1) \supsetneq \mathrm{WP}(\mathbb{S}^1)$ is three times continuously differentiable, the proof is straightforward at every point.

Remark 2.12. For the same reason, plus controlling the term $c_{n,m}$ when $nm = 0$, we can show that if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, then

$$u_\varphi(e^{\mathrm{i}\theta}, e^{\mathrm{i}\psi}) = \sum_{n,m=-\infty}^{\infty} c_{n,m} e^{\mathrm{i}(n\theta + m\psi)}, \tag{2.26}$$

$$\text{where } \sum_{n,m=-\infty}^{\infty} |nm|\, |c_{n,m}|^2, \ \sum_{n=-\infty}^{\infty} |n|\, |c_{n,0}|^2, \ \sum_{m=-\infty}^{\infty} |m|\, |c_{0,m}|^2 < \infty. \tag{2.27}$$

It is nothing but the condition for the covariance functions of two equivalent LGFs. So we conjecture that (2.26) and (2.27) hold if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ without assuming $\varphi \in \mathrm{QS}(\mathbb{S}^1)$.

3 Quasi-invariance of Gaussian fields and boundary measures

In this section, we show that the law of the log-correlated Gaussian field on $\mathbb{S}^1$ is quasi-invariant under the pullback by $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ if and only if φ is of the Weil–Petersson class. We then "lift" this characterization to that for the Gaussian multiplicative chaos on $\mathbb{S}^1$.

3.1 Preliminaries of Log-correlated Gaussian field on the unit circle

In this subsection, we survey the definition and the basic properties of the log-correlated Gaussian field on the unit circle. This is a well-researched object, and we do not aim to be comprehensive in our introduction; we direct the reader to surveys such as [DRSV17] for further information.

The *(mean-zero) log-correlated Gaussian field* on $\mathbb{S}^1$ (denoted **LGF**) is a random generalized function on $\mathbb{S}^1$ which has a centered Gaussian law with Cameron–Martin space $\mathcal{H}_0$ and the Cameron–Martin norm $\|\cdot\|_{\mathcal{H}_0}$ That is,

$$h = \sum_{n\ge 1} \xi_n h_n, \tag{3.1}$$

where $\{h_n\}_{n\ge1}$ is an orthonormal basis of $\mathcal{H}_0$ comprised of continuous functions and $\{\xi_n\}_{n\ge1}$ is an i.i.d. sequence of standard normal random variables, is an instance of LGF on $\mathbb{S}^1$. For concreteness, we can take

$$h_{2n-1}(e^{\mathrm{i}\theta}) := \frac{e_n + f_n}{\sqrt{2}}(e^{\mathrm{i}\theta}) = \sqrt{\frac{2}{n}}\cos(n\theta), \quad h_{2n}(e^{\mathrm{i}\theta}) := \frac{e_n - f_n}{\sqrt{2}\mathrm{i}}(e^{\mathrm{i}\theta}) = \sqrt{\frac{2}{n}}\sin(n\theta), \tag{3.2}$$

where $\{e_n, f_n\}_{n\geq 1}$ is the standard basis for $\mathcal{H}_0^{\mathbb{C}}$ introduced in (2.11). However, the definition (3.1) is independent of the choice of the orthonormal basis $\{h_n\}_{n\geq 1}$ for $\mathcal{H}_0$.

The sum (3.1) can be seen to converge almost surely in $H^s(\mathbb{S}^1)$ for any $s < 0$. Equivalently, we may consider h in terms of the centered Gaussian process $\{\langle h, f\rangle\}_{f\in\mathcal{H}_0}$ where $\langle\, ,\, \rangle$ is the inner product (2.5) on $\mathcal{H}_0$. That is,

$$\left\langle h, \sum_{n\geq 1} c_n h_n \right\rangle := \sum_{n\geq 1} \xi_n c_n \tag{3.3}$$

for $\sum_{n\geq 1} c_n h_n \in \mathcal{H}_0$. Note that for $f \in \mathcal{H}_0$, we have $\mathrm{Var}(\langle h, f\rangle) = \|f\|^2_{\mathcal{H}_0}$. More generally, we consider

$$(h, \rho) := \sum_{n\geq 1} \xi_n \int_{\mathbb{S}^1} h_n(e^{\mathrm{i}\theta})\, \mathrm{d}\rho(e^{\mathrm{i}\theta}) \tag{3.4}$$

with signed Borel measures ρ on $\mathbb{S}^1$ for which the above sum is almost surely absolutely convergent. Then, such pairings (h, ρ) form a continuous version of the centered Gaussian process satisfying

$$\mathrm{Cov}((h, \rho), (h, \tilde{\rho})) = \iint_{\mathbb{S}^1\times\mathbb{S}^1} G(e^{\mathrm{i}\theta}, e^{\mathrm{i}\tilde{\theta}})\, \mathrm{d}\rho(e^{\mathrm{i}\theta})\, \mathrm{d}\tilde{\rho}(e^{\mathrm{i}\tilde{\theta}}), \tag{3.5}$$

where

$$G(e^{\mathrm{i}\theta}, e^{\mathrm{i}\tilde{\theta}}) := \sum_{n\geq 1} h_n(e^{\mathrm{i}\theta}) h_n(e^{\mathrm{i}\tilde{\theta}}) = -2\log|e^{\mathrm{i}\theta} - e^{\mathrm{i}\tilde{\theta}}| \tag{3.6}$$

is the *covariance kernel* of the LGF h. Note that for $f \in \mathcal{H}_0$, we have

$$\langle h, f\rangle = \left(h, \frac{(-\partial_{\theta\theta})^{1/2} f(e^{\mathrm{i}\theta})}{2\pi}\, \mathrm{d}\theta \right), \tag{3.7}$$

where $(-\partial_{\theta\theta})^{1/2}$ is the linear operator from $\mathcal{H}_0$ to $L^2(\mathbb{S}^1)$ which maps $\cos(n\theta)$ to $n\cos(n\theta)$ and $\sin(n\theta)$ to $n\sin(n\theta)$ for each positive integer n.

This definition of the LGF on $\mathbb{S}^1$ is similar to that of the *Gaussian free field* (GFF) on the unit disk $\mathbb{D}$, which is given by the sum (3.1) where $\{h_n\}_{n\geq 1}$ is chosen to be a sequence of functions on $\mathbb{D}$. The pairing of GFF with signed Borel measures ρ on $\mathbb{D}$ is defined analogously as in (3.4).

- The *zero boundary (Dirichlet) GFF* on $\mathbb{D}$ is given by choosing $\{h_n\}_{n\geq 1}$ to be any orthonormal basis of $H^1_0(\mathbb{D})$. The corresponding covariance kernel is given by the Dirichlet Green's function

$$G^{\mathrm{zero}}(z, w) = -\log|(z - w)/(1 - z\bar{w})|. \tag{3.8}$$

  We use Γ to denote a zero boundary GFF.

- We obtain the *harmonic Gaussian field* on $\mathbb{D}$ by choosing $\{h_n\}_{n\geq 1}$ to be any orthonormal basis of $\mathcal{D}_0$. The covariance kernel is given by

$$G(z, w) = -2\log|1 - z\bar{w}|, \tag{3.9}$$

  which agrees with (3.6) for $z, w \in \mathbb{S}^1$. By the isomorphism between $\mathcal{D}_0$ and $\mathcal{H}_0$ given in Section 2, we can identify it as the "harmonic extension" of an LGF to $\mathbb{D}$. We will use h to denote both an LGF on $\mathbb{S}^1$ and its harmonic extension on $\mathbb{D}$.

- The *free boundary (Neumann) GFF on* $\mathbb{D}$ (with mean zero on $\mathbb{S}^1$) is obtained by choosing $\{h_n\}_{n\geq 1}$ to be any orthonormal basis of $H^1(\mathbb{D})/\mathbb{R}$. The corresponding covariance kernel is given by the Neumann Green's function

$$G^{\text{free}}(z,w) = -\log|(z-w)(1-z\bar{w})|. \tag{3.10}$$

We use $\overline{\Gamma}$ to denote a free boundary GFF.

Remark 3.1. By the decomposition (2.16), if Γ and h are independent zero boundary GFF and harmonic Gaussian field on $\mathbb{D}$, respectively, then $\Gamma + h$ has the law of a free boundary GFF on $\mathbb{D}$. This can also be seen from the identify $G^{\text{zero}} + G = G^{\text{free}}$ satisfied by the covariance kernels (3.8), (3.9), and (3.10). We emphasize that the multiplicative factor of 2 in (3.9) (hence also in (3.6)) is necessary for this relationship to hold.

3.2 Quasi-invariance of Log-correlated Gaussian field

In this subsection, our goal is to culminate in the quasi-invariance result for pullbacks with respect to Weil-Petersson homeomorphisms (Theorem 3.3).

Given $\varphi \in \text{Homeo}_+(\mathbb{S}^1)$, we define the pullback of an LGF $h = \sum_{n\geq 1} \xi_n h_n$ formally as

$$h \circ \varphi := \sum_{n\geq 1} \xi_n (h_n \circ \varphi). \tag{3.11}$$

This can be rigorously considered as the centered Gaussian process

$$(h \circ \varphi, \rho) := (h, \varphi_* \rho) \tag{3.12}$$

indexed by signed Borel measures ρ on $\mathbb{S}^1$ for which the right-hand side of (3.12) is well-defined. Here, $\varphi_*\rho$ is the pushforward of ρ under the homeomorphism φ. Note that the covariance kernel of $h \circ \varphi$ is

$$G_\varphi(e^{\mathrm{i}\theta}, e^{\mathrm{i}\tilde{\theta}}) := G(e^{\mathrm{i}\theta}, e^{\mathrm{i}\tilde{\theta}}) = -2\log|\varphi(e^{\mathrm{i}\theta}) - \varphi(e^{\mathrm{i}\tilde{\theta}})| \tag{3.13}$$

since

$$\begin{aligned}
\mathrm{Cov}\left((h\circ\varphi,\rho),(h\circ\varphi,\tilde{\rho})\right) &= \mathrm{Cov}\left((h,\varphi_*\rho),(h,\varphi_*\tilde{\rho})\right) \\
&= \iint_{\mathbb{S}^1\times\mathbb{S}^1} G(e^{\mathrm{i}\theta}, e^{\mathrm{i}\tilde{\theta}})\, d(\varphi_*\rho)(e^{\mathrm{i}\theta})\, \mathrm{d}(\varphi_*\tilde{\rho})(e^{\mathrm{i}\tilde{\theta}}) \\
&= \iint_{\mathbb{S}^1\times\mathbb{S}^1} G(e^{\mathrm{i}\theta}, e^{\mathrm{i}\tilde{\theta}})\, d\rho(e^{\mathrm{i}\theta})\, \mathrm{d}\tilde{\rho}(e^{\mathrm{i}\tilde{\theta}}).
\end{aligned}$$

If $\varphi \in \mathrm{QS}(\mathbb{S}^1)$, then we define the mean-zero part of the pullback $h \circ \varphi$ of the LGF $h = \sum_{n\geq 1} \xi_n h_n$ formally as

$$\Pi(\varphi)(h) = \pi_0(h\circ\varphi) := \sum_{n\geq 1} \xi_n \Pi(\varphi)(h_n). \tag{3.14}$$

For quasisymmetric φ, since $\Pi(\varphi)$ is a bounded linear operator on $\mathcal{H}_0$, the (3.14) defines $\Pi(\varphi)(h)$ as a centered Gaussian field on $\mathbb{S}^1$ with the Cameron–Martin space $\mathcal{H}_0$ and the *covariance operator* $\Pi(\varphi)\Pi(\varphi)^*$. That is, $\{\langle \Pi(\varphi)(h), f\rangle = \sum_{n\geq 1} \xi_n \langle \Pi(\varphi)(h_n), f\rangle\}_{f\in\mathcal{H}_0}$ is a centered Gaussian process with

$$\mathrm{Cov}(\langle \Pi(\varphi)(h), f\rangle, \langle \Pi(\varphi)(h), g\rangle) = \langle \Pi(\varphi)^* f, \Pi(\varphi)^* g\rangle = \langle f, \Pi(\varphi)\Pi(\varphi)^* g\rangle. \tag{3.15}$$

Remark 3.2. The relationship between $h \circ \varphi$ and $\Pi(\varphi)(h)$ is the following: if

$$\iint_{\mathbb{S}^1 \times \mathbb{S}^1} G_\varphi(e^{\mathrm{i}\theta}, e^{\mathrm{i}\tilde{\theta}}) \, \mathrm{d}\theta \, \mathrm{d}\tilde{\theta} < \infty \tag{3.16}$$

so that $\fint_{\mathbb{S}^1} h \circ \varphi := (h \circ \varphi, (2\pi)^{-1} \, \mathrm{d}\theta)$ is an a.s. finite random variable, then

$$(\Pi(\varphi)(h), \rho) = (h \circ \varphi, \rho) - \rho(\mathbb{S}^1) \fint_{\mathbb{S}^1} h \circ \varphi \tag{3.17}$$

a.s. for signed Borel measures ρ on $\mathbb{S}^1$ for which $\iint_{\mathbb{S}^1 \times \mathbb{S}^1} |G_\varphi(e^{\mathrm{i}\theta}, e^{\mathrm{i}\tilde{\theta}})| \, \mathrm{d}\rho(e^{\mathrm{i}\theta}) \, \mathrm{d}\rho(e^{\mathrm{i}\tilde{\theta}}) < \infty$. This can be seen directly from the decompositions (3.11) and (3.14) for $h \circ \varphi$ and $\Pi(\varphi)(h)$, respectively.

One sufficient condition for (3.16) is for φ to be a diffeomorphism, since then

$$u_\varphi(e^{\mathrm{i}\theta}, e^{\mathrm{i}\tilde{\theta}}) := \log \left| \frac{\varphi(e^{\mathrm{i}\theta}) - \varphi(e^{\mathrm{i}\tilde{\theta}})}{e^{\mathrm{i}\theta} - e^{\mathrm{i}\tilde{\theta}}} \right| \tag{3.18}$$

is a continuous function on $\mathbb{S}^1 \times \mathbb{S}^1$ with $u_\varphi(e^{\mathrm{i}\theta}, e^{\mathrm{i}\theta}) = \log |\varphi'(e^{\mathrm{i}\theta})|$. Hence, $G_\varphi = G - 2u_\varphi$ is integrable with respect to the arc-length measure on $\mathbb{S}^1 \times \mathbb{S}^1$.

More generally, it suffices for φ^{-1} to be absolutely continuous with respect to the arc-length measure and satisfy $|(\varphi^{-1})'| \in H^{-1/2}(\mathbb{S}^1)$. Then,

$$\begin{aligned}
\iint_{\mathbb{S}^1 \times \mathbb{S}^1} G_\varphi(e^{\mathrm{i}\theta}, e^{\mathrm{i}\tilde{\theta}}) \, \mathrm{d}\theta \, \mathrm{d}\tilde{\theta} &= \iint_{\mathbb{S}^1 \times \mathbb{S}^1} \left(\sum_{n \geq 1} (h_n \circ \varphi)(e^{\mathrm{i}\theta}) \, (h_n \circ \varphi)(e^{\mathrm{i}\tilde{\theta}}) \right) \mathrm{d}\theta \, \mathrm{d}\tilde{\theta} \\
&= \sum_{n \geq 1} \left| \int_{\mathbb{S}^1} h_n(\varphi(e^{\mathrm{i}\theta})) \, \mathrm{d}\theta \right|^2 = \sum_{n \geq 1} \left| \int_{\mathbb{S}^1} h_n(e^{\mathrm{i}\theta}) \, |(\varphi^{-1})'|(e^{\mathrm{i}\theta}) \, \mathrm{d}\theta \right|^2 \\
&= \| |(\varphi^{-1})'| \|^2_{H^{-1/2}(\mathbb{S}^1)/\mathbb{R}} < \infty.
\end{aligned} \tag{3.19}$$

In particular, if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, then $|(\varphi^{-1})'| \in L^2(\mathbb{S}^1)$; this follows from the standard argument using the VMO space that $\log |\varphi'| \in H^{1/2}(\mathbb{S}^1)$ implies $|\varphi'|^p = \exp(p \log |\varphi'|) \in L^1(\mathbb{S}^1)$ for any $p \geq 1$. Hence, we have (3.16) in this case.

We now give our first main result, which identifies $\mathrm{WP}(\mathbb{S}^1)$ as the class of quasisymmetric circle homeomorphisms φ for which $\Pi(\varphi)(h)$ and h have equivalent laws.

Theorem 3.3. *Suppose $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ and f is a fixed real-valued function on $\mathbb{S}^1$. Let $\Pi(\varphi)(h)$ be the Gaussian field given in (3.14), where h is an LGF on $\mathbb{S}^1$. Then, the law of the random field $\Pi(\varphi)(h) + f$ is equivalent to that of an LGF on $\mathbb{S}^1$ if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ and $f \in \mathcal{H}_0$. Otherwise, the two laws are mutually singular.*

Proof. The Feldman–Hájek theorem states that the laws of h and $\Pi(\varphi)(h) + f$ are either equivalent or mutually singular, and the former holds if and only if

- The Cameron–Martin space for the law of $\Pi(\varphi)(h) + f$ is $\mathcal{H}_0$;
- The difference in the means, f, lies in the common Cameron–Martin space $\mathcal{H}_0$;
- The difference in the covariance operators, $\Pi(\varphi)\Pi(\varphi)^* - I$, is a Hilbert–Schmidt operator on the common Cameron–Martin space $\mathcal{H}_0$.

See, e.g., [DPZ14, Thm. 2.25]. By Lemma 2.7, these conditions are satisfied if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ and $f \in \mathcal{H}_0$.

Recalling from Definition 2.3 that $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$ is in the Weil–Petersson class if and only if $\log|\varphi'| \in H^{1/2}(\mathbb{S}^1)$, we immediately obtain the following corollary from Theorem 3.3. ☐

Corollary 3.4. *Let $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ and Q be a real constant. Furthermore, let u_φ be the function on $\mathbb{S}^1 \times \mathbb{S}^1$ given in (3.18) and let $z_0 \in \mathbb{S}^1$, $\alpha \in \mathbb{R}$ be fixed. If h is an LGF on $\mathbb{S}^1$, then the law of the random field*

$$\Pi(\varphi)(h) + \pi_0\left(Q\log|\varphi'| + \alpha u_\varphi(\cdot, z_0)\right)$$

is equivalent to that of h if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ and either of the following holds: $\alpha = 0$ or $u_\varphi(\cdot, z_0) \in H^{1/2}(\mathbb{S}^1)$. Otherwise, the two laws are mutually singular.

3.3 Preliminaries of Gaussian multiplicative chaos

Now we shift our attention to the Gaussian multiplicative chaos (GMC) measure on the unit circle. Given $\gamma \in (0, 2)$, the γ-GMC measure with respect to an LGF h on $\mathbb{S}^1$ is defined formally as

$$\mathcal{M}_h^\gamma(\mathrm{d}\theta) = e^{\frac{\gamma}{2}h(e^{\mathrm{i}\theta}) - \frac{\gamma^2}{8}\mathbb{E}[h(e^{\mathrm{i}\theta})^2]}\,\mathrm{d}\theta. \tag{3.20}$$

We omit the superscript γ and write $\mathcal{M}_h = \mathcal{M}_h^\gamma$ if there is no confusion. A rigorous study of GMC was initiated by Kahane [Kah85] with seminal contributions by Robert and Vargas [RV10], Duplantier and Sheffield [DS11], and Shamov [Sha16]. The GMC measure $\mathcal{M}_h$ is defined rigorously via renormalization. The following two equivalent approaches almost surely give the same measure as shown in [Ber17].

- Let $\{h_k\}_{k\geq 1}$ be a sequence of continuous functions on $\mathbb{S}^1$ forming an orthonormal basis for $\mathcal{H}_0$. Given an LGF $h = \sum_{k\geq 1}\xi_k h_k$, we define $\mathcal{M}_h^\gamma$ as the almost sure weak limit of the measures

$$\exp\left(\frac{\gamma}{2}\sum_{k=1}^{n}\xi_k h_k(e^{\mathrm{i}\theta}) - \frac{\gamma^2}{8}\sum_{k=1}^{n}\left(h_k(e^{\mathrm{i}\theta})\right)^2\right)\mathrm{d}\theta \tag{3.21}$$

  as $n \to \infty$.

- Let σ be a fixed nonnegative Radon measure on the interval $(-\pi, \pi)$ with unit mass and $\sup_{x\in(-\pi,\pi)}\int_{-\pi}^{\pi}\log_+(1/|x-y|)\,\sigma(\mathrm{d}y) < \infty$. For $e^{\mathrm{i}\theta} \in \mathbb{S}^1$ and $\varepsilon \in (0,1)$, define $\sigma_{e^{\mathrm{i}\theta},\varepsilon}$ to be the pushforward of σ under the map $x \mapsto e^{\mathrm{i}(\theta+\varepsilon x)}$, and denote $h_\varepsilon(e^{\mathrm{i}\theta}) := (h, \sigma_{e^{\mathrm{i}\theta},\varepsilon})$. Then, we define $\mathcal{M}_h^\gamma$ as the weak limit as $\varepsilon \to 0$ of

$$\exp\left(\frac{\gamma}{2}h_\varepsilon(e^{\mathrm{i}\theta}) + \frac{\gamma^2}{4}\iint_{\mathbb{S}^1\times\mathbb{S}^1}\log|x-y|\,\sigma_{e^{\mathrm{i}\theta},\varepsilon}(\mathrm{d}x)\,\sigma_{e^{\mathrm{i}\theta},\varepsilon}(\mathrm{d}y)\right)\mathrm{d}\theta \tag{3.22}$$

  in probability.

  For instance, we can choose σ to be the uniform probability measure on $[-1, 1]$, in which case $h_\varepsilon(e^{\mathrm{i}\theta})$ will be average value of the LGF h on the closed interval from $e^{\mathrm{i}(\theta-\varepsilon)}$ to $e^{\mathrm{i}(\theta+\varepsilon)}$ on $\mathbb{S}^1$. However, the limit (3.22) does not depend on the choice of the measure σ.

Since the GMC measure $\mathcal{M}_h$ is almost surely determined by the LGF h, we can define the measure $\mathcal{M}_h$ using the limits (3.21) or (3.22) when h is a random field on $\mathbb{S}^1$ whose law is absolutely continuous with respect to LGF. When h is a random field on $\mathbb{S}^1$ with a decomposition $h = c + \tilde{h}$ where c is a real-valued random variable and the law of $\tilde{h}$ is absolutely continuous with respect to LGF, then we define

$$\mathcal{M}_h^\gamma := e^{(\gamma/2)c}\mathcal{M}_{\tilde{h}}^\gamma. \tag{3.23}$$

Conversely, the GMC measure $\mathcal{M}_h^\gamma$ almost surely determines the LGF h [BSS23, Thm. 1.4].

Remark 3.5. When $\gamma = 2$, the limits (3.21) and (3.22) give the zero measure almost surely. However, a slightly modified limit gives a nontrivial measure $\mathcal{M}_h^{\mathrm{crit}}$ called the *critical Gaussian multiplicative chaos* [DRSV14a,DRSV14b]. As described in [APS19, Section 4.1.2] (also see [Pow20] and [PS24]), we can also obtain the critical GMC measure through the weak limit $\frac{1}{2(2-\gamma)}\mathcal{M}_h^\gamma \to \mathcal{M}_h^{\mathrm{crit}}$ in probability as $\gamma \to 2^-$. We also have that the critical GMC measure $\mathcal{M}_h^{\mathrm{crit}}$ almost surely determines the LGF h [Vih24].

Remark 3.6. Our multiplicative factor of $\gamma/2$ in (3.21)–(3.22) as well as the factor of 2 in (3.6) differs from some of the recent works on the GMC on the circle, such as [CN19, CGVV24]. In particular, they are chosen to agree with the literature on Liouville quantum gravity [DS11]. In the LQG theory, the quantum boundary length of a two-dimensional domain D is defined as the GMC measure on ∂D with respect to the free boundary GFF on D. More precisely, assume $D \subset \mathbb{H}$ and $\partial D \cap \mathbb{R}$ is a nonempty interval. Given free boundary GFF $\overline{\Gamma} = \Gamma + h$ where Γ is a zero boundary GFF and h is an independent harmonic Gaussian field h on D (recall Remark 3.1), we define the γ-LQG boundary length $\nu_{\overline{\Gamma}}$ on $\partial D \cap \mathbb{R}$ as the almost sure weak limit

$$\lim_{\varepsilon\to 0} \varepsilon^{\gamma^2/4} e^{\frac{\gamma}{2}\overline{\Gamma}_\varepsilon(x)}\,\mathrm{d}x, \tag{3.24}$$

where $\overline{\Gamma}_\varepsilon(x)$ is the average value of $\overline{\Gamma}$ on the semicircle $\partial B_\varepsilon(x) \cap \mathbb{H}$. By the equivalence between the GMC defined using (semi)circle averages and the Karhunen–Loève expansion of GFF as explained in [Ber17], we have that

$$\lim_{\varepsilon\to 0} e^{\frac{\gamma}{2}\overline{\Gamma}_\varepsilon(x) - \frac{\gamma^2}{8}\mathbb{E}[\overline{\Gamma}_\varepsilon(x)^2]}\,\mathrm{d}x = \lim_{\varepsilon\to 0} e^{\frac{\gamma}{2}h_\varepsilon(x) - \frac{\gamma^2}{8}\mathbb{E}[h_\varepsilon(x)^2]}\,\mathrm{d}x \tag{3.25}$$

and thus

$$\nu_{\overline{\Gamma}} = \lim_{\varepsilon\to 0} \varepsilon^{\gamma^2/4} e^{\frac{\gamma}{2}h_\varepsilon(x) - \frac{\gamma^2}{8}\mathbb{E}[\Gamma_\varepsilon(x)^2]}\,\mathrm{d}x = e^{-(\gamma^2/8)C}\nu_h. \tag{3.26}$$

Here,

$$\begin{aligned} C = \lim_{\varepsilon\to 0}\mathbb{E}[\Gamma_\varepsilon(x)^2] &= \lim_{\varepsilon\to 0}\frac{1}{\pi^2}\iint_{[0,\pi]^2}\big(-\log|\varepsilon e^{\mathrm{i}\theta} - \varepsilon e^{\mathrm{i}\tilde{\theta}}| + \log|\varepsilon e^{\mathrm{i}\theta} - \varepsilon e^{-\mathrm{i}\tilde{\theta}}|\big)\mathrm{d}\theta\,\mathrm{d}\tilde{\theta} \\ &= \frac{2}{\pi^2}\iint_{[0,\pi]^2}\log|e^{\mathrm{i}\theta} - e^{-\mathrm{i}\tilde{\theta}}|\,\mathrm{d}\theta\,\mathrm{d}\tilde{\theta} = -\frac{7}{\pi^2}\zeta(3) \approx -0.85, \end{aligned} \tag{3.27}$$

where $\zeta(3)$ is Apéry's constant. By the isomorphism between $\mathcal{H}_0$ and the harmonic extension $\mathcal{D}_0$, the relation (3.26) holds with ν_h, the boundary LQG measure on $\partial\mathbb{D}$ defined using the harmonic field h in the unit disk, replaced by $\mathcal{M}_h$, the GMC measure on $\mathbb{S}^1$ defined using the LGF h on the unit circle. In particular, if $\overline{\Gamma}$ is a free boundary GFF on $\mathbb{D}$ and h is its trace on $\mathbb{S}^1$ (equivalently, the harmonic part of $\overline{\Gamma}$ on $\mathbb{D}$), then the normalized measures $(\nu_{\overline{\Gamma}}(\mathbb{S}^1))^{-1}\nu_{\overline{\Gamma}}$ and $(\mathcal{M}_h(\mathbb{S}^1))^{-1}\mathcal{M}_h$ agree almost surely.

3.4 Quasi-invariance of Gaussian multiplicative chaos

Given an LGF h on $\mathbb{S}^1$, define the corresponding *normalized GMC measure* as

$$\widehat{\mathcal{M}}_h^\gamma := \frac{1}{\mathcal{M}_h(\mathbb{S}^1)}\mathcal{M}_h^\gamma. \tag{3.28}$$

Our goal for this subsection is to show the following quasi-invariance result for normalized GMC measures.

Proposition 3.7. *Suppose $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$. Let h be the LGF on $\mathbb{S}^1$ and $\widehat{\mathcal{M}}_h = \widehat{\mathcal{M}}_h^\gamma$ be the corresponding normalized GMC measure (3.28) for $\gamma \in (0, 2]$. Then, the law of the pullback $\varphi^*\widehat{\mathcal{M}}_h$ is absolutely continuous with respect to the law of $\widehat{\mathcal{M}}_h$ if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$. Moreover, if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, then we almost surely have*

$$\varphi^*\mathcal{M}_h^\gamma = \mathcal{M}^\gamma_{h\circ\varphi+Q\log|\varphi'|} \tag{3.29}$$

where $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$.

It is well-known that given a 2D log-correlated Gaussian field Γ, the corresponding LQG area measure μ_Γ almost surely satisfies the conformal coordinate change rule

$$\psi^*\mu_\Gamma = \mu_{\Gamma\circ\psi+Q\log|\psi'|} \tag{3.30}$$

if ψ is a conformal map and $Q = 2/\gamma + \gamma/2$ [DS11, Prop. 2.1]. In showing (3.29), we extend this result to non-smooth and even non-C^1 homeomorphisms φ on $\mathbb{S}^1$. To begin our proof of Proposition 3.7, we verify that the coordinate change formula holds when φ is a diffeomorphism.

Lemma 3.8. *Let h be an LGF on the unit circle $\mathbb{S}^1$ and suppose φ is an orientation-preserving C^1 diffeomorphism of $\mathbb{S}^1$. Then, for every $\gamma \in (0, 2]$ with $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$, we almost surely have the coordinate change rule (3.29).*

We note that there exist C^1 diffeomorphisms φ of $\mathbb{S}^1$ that does not belong to the Weil–Petersson class, in which case the law of the mean-zero part of $h \circ \varphi + Q \log|\varphi'|$ is singular to that of LGF on $\mathbb{S}^1$. Nevertheless, this is a Gaussian field whose covariance kernel $G_\varphi(x, y)$ is of the form $-\log|x - y| + u_\varphi(x, y)$ where u_φ is a continuous function on $\mathbb{S}^1 \times \mathbb{S}^1$ (recall Remark 3.2). Hence, as proved in [Ber17], the GMC measure $\mathcal{M}^\gamma_{h\circ\varphi+Q\log|\varphi'|}$ is well-defined via the limit (3.22) where the mollified field $h_\varepsilon(e^{\mathrm{i}\theta}) = (h, \sigma_{e^{\mathrm{i}\theta},\varepsilon})$ is replaced with $(h \circ \varphi, \sigma_{e^{\mathrm{i}\theta},\varepsilon}) + \int_{\mathbb{S}^1} Q \log|\varphi'| \,\mathrm{d}\sigma_{e^{\mathrm{i}\theta},\varepsilon}$.

Proof of Lemma 3.8. Let us first assume that $\gamma \in (0, 2)$. Consider the GMC measure given by the following weak limit in probability:

$$\widetilde{\mathcal{M}}^\gamma_{h\circ\varphi} := \lim_{\varepsilon\to 0} \exp\left(\frac{\gamma}{2}(h \circ \varphi)_\varepsilon(e^{\mathrm{i}\theta}) - \frac{\gamma^2}{8}\mathrm{Var}((h \circ \varphi)_\varepsilon(e^{\mathrm{i}\theta}))\right) \mathrm{d}\theta,$$

where

$$(h \circ \varphi)_\varepsilon(e^{\mathrm{i}\theta}) := \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} (h \circ \varphi)(e^{\mathrm{i}(\theta+t)}) \,\mathrm{d}t.$$

Let us define $h_\varepsilon(e^{\mathrm{i}\theta})$ similarly, so that $\mathrm{Var}(h_\varepsilon(e^{\mathrm{i}\theta})) = (2\varepsilon)^{-2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} -2\log|e^{\mathrm{i}\theta} - e^{\mathrm{i}\tilde{\theta}}|\,\mathrm{d}\theta\,\mathrm{d}\tilde{\theta}$. Then,

$$\mathrm{Var}((h\circ\varphi)_\varepsilon(e^{\mathrm{i}\theta})) - \mathrm{Var}(h_\varepsilon(e^{\mathrm{i}\theta})) = \frac{1}{4\varepsilon^2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} -2\log\left|\frac{\varphi(e^{\mathrm{i}(\theta+t)}) - \varphi(e^{\mathrm{i}(\theta+s)})}{e^{\mathrm{i}(\theta+t)} - e^{\mathrm{i}(\theta+s)}}\right|\mathrm{d}s\,\mathrm{d}t$$

converges uniformly to $-2\log|\varphi'(e^{\mathrm{i}\theta})|$ as $\varepsilon \to 0$. Hence, we have

$$\frac{\mathrm{d}\widetilde{\mathcal{M}}^\gamma_{h\circ\varphi}}{\mathrm{d}\mathcal{M}^\gamma_{h\circ\varphi}}(e^{\mathrm{i}\theta}) = |\varphi'(e^{\mathrm{i}\theta})|^{\gamma^2/4}. \tag{3.31}$$

Let us choose $\{h_n\}_{n\geq 1}$ to be the orthonormal basis (3.2) for $\mathcal{H}_0$. Suppose we have the decomposition $h = \sum_{n\geq 1}\xi_n h_n$, where $\{\xi_n\}_{n\geq 1}$ are i.i.d. standard normal random variables. Then, since $h\circ\varphi = \sum_{n\geq 1}\xi_n(h_n\circ\varphi)$ is a Karhunen–Loève expansion where each $h_n\circ\varphi$ are continuous on $\mathbb{S}^1$, we have from [Ber17, Sec. 5] that for any interval $A\subset\mathbb{S}^1$,

$$\begin{aligned}
\mathcal{M}^\gamma_h(\varphi(A)) &= \lim_{N\to\infty}\int_{\varphi(A)} e^{\sum_{n=1}^N\left(\frac{\gamma}{2}\xi_n h_n(e^{\mathrm{i}\theta}) - \frac{\gamma^2}{8}[h_n(e^{\mathrm{i}\theta})]^2\right)}\,\mathrm{d}\theta \\
&= \lim_{N\to\infty}\int_A e^{\sum_{n=1}^N\left(\frac{\gamma}{2}\xi_n (h_n\circ\varphi)(e^{\mathrm{i}\theta}) - \frac{\gamma^2}{8}[(h_n\circ\varphi)(e^{\mathrm{i}\theta})]^2\right)}|\varphi'(e^{\mathrm{i}\theta})|\,\mathrm{d}\theta \\
&= \int_A |\varphi'(e^{\mathrm{i}\theta})|\,\mathrm{d}\widetilde{\mathcal{M}}^\gamma_{h\circ\varphi}(e^{\mathrm{i}\theta}) = \int_A |\varphi'(e^{\mathrm{i}\theta})|^{1+\gamma^2/4}\,\mathrm{d}\mathcal{M}^\gamma_{h\circ\varphi}(e^{\mathrm{i}\theta}) \\
&= \mathcal{M}^\gamma_{h\circ\varphi+Q\log|\varphi'|}(A)
\end{aligned} \tag{3.32}$$

almost surely. For the last equality, we used the identity $\mathrm{d}\mathcal{M}^\gamma_{h+f}/\mathrm{d}\mathcal{M}^\gamma_h = e^{(\gamma/2)f}$ for any continuous f on $\mathbb{S}^1$, which can be checked from the definition (3.22) of the GMC measure. The equality for critical GMC holds by taking the limit as $\gamma\to 2^-$ as in Remark 3.5. ☐

To extend Lemma 3.8 to the Weil–Petersson class, we approximate φ by a sequence of diffeomorphisms which are continuous in the topology induced by the Weil–Petersson metric as in Lemma 2.9. We now show that the GMC measure on $\mathbb{S}^1$ is almost surely continuous under these approximations. To our knowledge, this is the first time that the GMC measure is shown to be continuous with respect to the random field. The continuity with respect to the base measure and the parameter γ has been shown in 2D in [PS24, Prop 4.1].

Lemma 3.9. *Let $A\in\mathscr{B}(\mathcal{H}_0)$ be invertible in $\mathscr{B}(\mathcal{H}_0)$ and suppose AA^*-I is Hilbert–Schmidt. Suppose $\{A_n\}_{n\geq 1}$ is a sequence in $\mathscr{B}(\mathcal{H}_0)$ such that A_n converges to A in the weak operator topology and $\|(A_nA^{-1})(A_nA^{-1})^*-I\|_{\mathrm{HS}}\to 0$. Let $\{\beta_n\}_{n\geq 1}$ be a sequence in $\mathcal{H}_0$ which converges to $\beta\in\mathcal{H}_0$. Let h be an LGF on $\mathbb{S}^1$. Then, for $\gamma\in(0,2)$, the Gaussian multiplicative chaos measures $\mathcal{M}_{A_nh+\beta_n}$ converges weakly to $\mathcal{M}_{Ah+\beta}$ in probability.*

Proof. Assume first that $A=I$ and $\beta=0$. In this case, as outlined in [Ber17, Sec. 6], it suffices to show that $\mathcal{M}_{A_nh+\beta}(S)\to\mathcal{M}_h(S)$ in probability for each fixed interval $S\subseteq\mathbb{S}^1$. The proof proceeds by considering the approximations (3.21) of $X:=\mathcal{M}_h(S)$ using the orthonormal basis $\{h_k\}_{k\in\mathbb{N}}$ for $\mathcal{H}_0$ given in (3.2). That is, let

$$X_m := \int_S \exp\left(\frac{\gamma}{2}\sum_{k=1}^m\langle h,h_k\rangle h_k(e^{\mathrm{i}\theta}) - \frac{\gamma^2}{8}\sum_{k=1}^m(h_k(e^{\mathrm{i}\theta}))^2\right)\mathrm{d}\theta. \tag{3.33}$$

We know from [Ber17, Eq. (5.1)] that $X_m = \mathbb{E}[X|\mathcal{F}_m]$ where $\mathcal{F}_m$ is the σ-algebra generated by $\{\langle h, h_k\rangle\}_{1\le k\le m}$, whence $X_m \to X$ in $L^1(\mathbb{P})$. We consider the analogous approximations

$$X_m^{(n)} := \int_S \exp\left(\frac{\gamma}{2}\sum_{k=1}^m \langle A_n h + \beta_n, h_k\rangle h_k(e^{\mathrm{i}\theta}) - \frac{\gamma^2}{8}\sum_{k=1}^m (h_k(e^{\mathrm{i}\theta}))^2\right)\mathrm{d}\theta, \tag{3.34}$$

which converges in probability to $X^{(n)} := \mathcal{M}_{A_n h+\beta_n}(S)$ as $n\to\infty$ since the law of $A_n h + \beta_n$ is absolutely continuous with respect to that of h by the Feldman–Hájek theorem (cf. Theorem 3.3). We now claim the following.

- For each $m \in \mathbb{N}$, as $n \to \infty$,

$$X_m^{(n)} \to X_m \quad \text{in probability.} \tag{3.35}$$

- There exists a positive integer n_0 such that for any $\delta > 0$, we have

$$\lim_{m\to\infty}\sup_{n\ge n_0} \mathbb{P}(|X_m^{(n)} - X^{(n)}| > \delta) = 0. \tag{3.36}$$

Then, for any given $\delta > 0$, we can choose sufficiently large m and then n_0 such that

$$\mathbb{P}(|X^{(n)} - X_m^{(n)}| > \delta) + \mathbb{P}(|X_m^{(n)} - X_m| > \delta) + \mathbb{P}(|X_m - X| > \delta) < \delta.$$

That is, $X^{(n)} \to X$ in probability as $n \to \infty$.

Let us now show (3.35). Note from (3.2) that $|h_k(e^{\mathrm{i}\theta})| \le \sqrt{2}$ for every k and θ. Hence,

$$\begin{aligned}\mathbb{E}\left[\sup_{e^{\mathrm{i}\theta}\in\mathbb{S}^1}\left|\sum_{k=1}^m \langle (A_n - I)h + \beta_n, h_k\rangle h_k(e^{\mathrm{i}\theta})\right|\right] &\le \sqrt{2}\sum_{k=1}^m \left(\mathbb{E}\left[|\langle (A_n - I)h, h_k\rangle|\right] + |\langle \beta_n, h_k\rangle|\right)\\ &\le \sqrt{2}\sum_{k=1}^m \|(A_n - I)^* h_k\|_{\mathcal{H}_0} + 2\sqrt{m}\|\beta_n\|_{\mathcal{H}_0}.\end{aligned}$$

Note that

$$\begin{aligned}\|(A_n - I)^* h_k\|_{\mathcal{H}_0}^2 &= |\langle h_k, (A_n A_n^* - I)h_k\rangle + \langle h_k, (2I - A_n - A_n^*)h_k\rangle|\\ &\le \|A_n A_n^* - I\|_{\mathrm{HS}} + 2|\langle h_k, (A_n - I)h_k\rangle|,\end{aligned}$$

which tends to 0 as $n \to \infty$ by assumption. Combining the above two inequalities, we see that given any sequence of positive integers, we can find a further sequence $\{n_j\}_{j\ge1}$ such that $\sup_{e^{\mathrm{i}\theta}\in\mathbb{S}^1} |\sum_{k=1}^m \langle (A_{n_j} - I)h + \beta_{n_j}, h_k\rangle - \langle h, h_k\rangle h_k(e^{\mathrm{i}\theta})| \to 0$ almost surely as $j \to \infty$. As we have

$$\inf_{e^{\mathrm{i}\theta}\in S} e^{\frac{\gamma}{2}\sum_{k=1}^m \langle (A_{n_j}-I)h+\beta_{n_j}, h_k\rangle h_k(e^{\mathrm{i}\theta})} \le \frac{X_m^{(n_j)}}{X_m} \le \sup_{e^{\mathrm{i}\theta}\in S} e^{\frac{\gamma}{2}\sum_{k=1}^m \langle (A_{n_j}-I)h+\beta_{n_j}, h_k\rangle h_k(e^{\mathrm{i}\theta})}$$

from the definitions of $X_m^{(n)}$ and X_m, we conclude that $X_m^{(n_j)} \to X_m$ almost surely as $j \to \infty$ and thus $X_m^{(n)} \to X_m$ in probability as $n \to \infty$.

Let us now show (3.36). For this, we will first check that if ρ_n is the Radon–Nikodym derivative of the law of $A_n h + \beta_n$ with respect to that of h, then $\sup_{n\ge n_0} \mathbb{E}[(\rho_n)^2] < \infty$ for sufficiently large

n_0.[1] Let us fix $A \in \mathscr{B}(\mathcal{H}_0)$ with $\|AA^* - I\|_{\mathrm{HS}}^2 < 1/2$ and $\beta \in \mathcal{H}_0$. Choose $\{\tilde{h}_n\}_{n\ge1}$ be an orthonormal eigenbasis for $\mathcal{H}_0$ with respect to $AA^* - I$, with $(AA^* - I)\tilde{h}_k = a_k\tilde{h}_k$ for each k. Observe that $\{\langle h, \tilde{h}_k\rangle\}_{k\ge1}$ are i.i.d. standard normal random variables, whereas $\{\langle Ah + \beta, \tilde{h}_k\rangle\}_{k\ge1}$ are independent Gaussian with mean $\langle \beta, \tilde{h}_k\rangle =: b_k$ and variance $\langle \tilde{h}_k, AA^*\tilde{h}_k\rangle = (1 + a_k)$. Hence, the Radon–Nikodym derivative ρ of the law of $Ah + \beta$ with respect to that of h is given by

$$\rho\left(\sum_{k\ge1} x_k\tilde{h}_k\right) = \prod_{k\ge1} \frac{(2\pi(1+a_k))^{-1/2}\exp(-(x_k - b_k)^2/2(1+a_k))}{(2\pi)^{-1/2}\exp(-x_k^2/2)} =: \prod_{k\ge1} F_k(x_k).$$

A straightforward calculation gives us that if Z is a standard normal random variable, then

$$\mathbb{E}[\rho^2] = \prod_{k\ge1} \mathbb{E}[(F_k(Z))^2] = \prod_{k\ge1} \frac{1}{(1-a_k^2)^{1/2}} \exp\left(\frac{b_k^2}{1-a_k^2}\right).$$

Now, since $\sum_{k\ge1} a_k^2 = \|AA^* - I\|_{\mathrm{HS}}^2 < 1/2$, we have $a_k^2 < 1/2$ for all k. Hence,

$$\log \mathbb{E}[\rho^2] \le \sum_{k\ge1} a_k^2 + 2\sum_{k\ge1} b_k^2 = \|AA^* - I\|_{\mathrm{HS}}^2 + 2\|\beta\|_{\mathcal{H}_0}^2.$$

Since $\|A_nA_n^* - I\|_{\mathrm{HS}} \to 0$ and $\|\beta_n\|_{\mathcal{H}_0} \to 0$ as $n \to \infty$, we conclude that $\mathbb{E}[\rho_n^2] \to 1$ as $n \to \infty$. Fix n_0 to be a positive integer such that $\sup_{n\ge n_0} \mathbb{E}[(\rho_n)^2] < \infty$. Now, given any $\delta > 0$, we have

$$\begin{aligned}\mathbb{P}(|X_m^{(n)} - X^{(n)}| > \delta) = \mathbb{E}[\rho_n \mathbf{1}_{\{|X_m - X| > \delta\}}] &\le K\mathbb{P}(|X_m - X| > \delta) + \mathbb{E}[\rho_n \mathbf{1}_{\{\rho_n > K\}}] \\ &\le \frac{K}{\delta}\mathbb{E}|X_m - X| + \sqrt{\frac{\mathbb{E}[(\rho_n)^2]}{K}}.\end{aligned}$$

Since $X_m \to X$ in $L^1(\mathbb{P})$, by first choosing sufficiently large K and then choosing large m, we can make $\sup_{n\ge n_0} \mathbb{P}(|X_m^{(n)} - X^{(n)}| > \delta)$ arbitrarily small. This completes the proof of (3.36) and therefore the lemma in the case $A = I$ and $\beta = 0$.

For general $A \in \mathscr{B}(\mathcal{H}_0)$ such that $\|AA^* - I\|_{\mathrm{HS}} < \infty$ and $\beta \in \mathcal{H}_0$, recall that the law of $\tilde{h} := Ah + \beta$ is absolutely continuous with respect to that of h by the Feldman–Hájek theorem. Let $\tilde{A}_n := A_nA^{-1} \in \mathscr{B}(\mathcal{H}_0)$ and $\tilde{\beta}_n := \beta_n - A_nA^{-1}\beta \in \mathcal{H}_0$, so that $A_nh + \beta_n = \tilde{A}_n\tilde{h} + \tilde{\beta}_n$. Then, $\tilde{A}_n \to I$ in the weak operator topology, $\|\tilde{A}_n\tilde{A}_n^* - I\|_{\mathrm{HS}} \to 0$, and $\|\tilde{\beta}_n\|_{\mathcal{H}_0} \to 0$ by the assumptions of the lemma, so we conclude that $\mathcal{M}_{\tilde{A}_n\tilde{h}+\tilde{\beta}_n} = \mathcal{M}_{A_nh+\beta_n}$ converges weakly to $\mathcal{M}_{\tilde{h}} = \mathcal{M}_{Ah+\beta}$ in probability. □

We are now ready for a proof of Proposition 3.7. We divide this into two parts, first showing that if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, then the laws of $\widehat{\mathcal{M}}_h$ and $\widehat{\mathcal{M}}_{h\circ\varphi+Q\log|\varphi'|}$ are absolutely continuous by checking that the coordinate change formula (3.29) holds.

Proof of Sufficiency. Let us first consider $\gamma \in (0, 2)$. Suppose $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ and let $\varphi_n \in \mathrm{Diff}_+(\mathbb{S}^1)$ be a sequence of approximations to φ given in Lemma 2.9. Let $A_n = \Pi(\varphi_n)$ and $\beta_n = \pi_0(Q\log|\varphi_n'|)$ for each n, and similarly let $A = \Pi(\varphi)$ and $\beta = \pi_0(Q\log|\varphi'|)$. Then, we have $A_n \to A$ in the strong operator topology and $\|(A_nA^{-1})(A_nA^{-1})^* - I\|_{\mathrm{HS}} \to 0$ by our choice of approximations. Also, since

[1]The same calculations can be used to show that $\sup_{n\ge1} \mathbb{E}[(\rho_n)^p] < \infty$ for sufficiently small $p > 1$, or, for any given $p > 1$, we have $\sup_{n\ge n_0} \mathbb{E}[(\rho_n)^p] < \infty$ for sufficiently large n_0.

$\log|(\varphi_n\circ\varphi^{-1})'| = (\log|\varphi_n'|-\log|\varphi'|)\circ\varphi^{-1}$, we have $\|\beta_n-\beta\|_{\mathcal{H}_0} \leq Q\|\Pi(\varphi)\|_{\mathscr{B}(\mathcal{H}_0)}\|\log|(\varphi_n\circ\varphi^{-1})|'\|_{\mathcal{H}_0}$ tending to 0 as $n\to\infty$.

We have from Lemma 3.8 that the coordinate change formula

$$\varphi_n^*\mathcal{M}_h = \mathcal{M}_{h\circ\varphi_n+Q\log|\varphi_n'|} = e^{\frac{\gamma}{4\pi}\int_{\mathbb{S}^1}(h\circ\varphi_n+Q\log|\varphi_n'|)}\mathcal{M}_{A_nh+\beta_n}$$

holds almost surely for every n. By Lemma 3.9, $\mathcal{M}_{A_nh+\beta_n}$ converges weakly in probability to $\mathcal{M}_{Ah+\beta}$. Since $\sum_k\left|\int_{\mathbb{S}^1}(h_k\circ\varphi_n - h_k\circ\varphi)\right|^2 \leq C\|\varphi_n'-\varphi'\|^2_{H^{-1/2}(\mathbb{S}^1)} \to 0$ as $n\to\infty$, we have $\int_{\mathbb{S}^1} h\circ\varphi_n \to \int_{\mathbb{S}^1} h\circ\varphi$ almost surely. We also have $\int_{\mathbb{S}^1}\log|\varphi_n'| \to \int_{\mathbb{S}^1}\log|\varphi'|$, so

$$\varphi_n^*\mathcal{M}_h \to e^{\frac{\gamma}{4\pi}\int_{\mathbb{S}^1}(h\circ\varphi+Q\log|\varphi'|)}\mathcal{M}_{Ah+\beta} = \mathcal{M}_{h\circ\varphi+Q\log|\varphi'|} \tag{3.37}$$

weakly in probability as $n\to\infty$.

On the other hand, since $\|\varphi-\varphi_n\|_\infty \to 0$, for every $F\in C(\mathbb{S}^1)$, we have

$$\lim_{n\to\infty}\int_{\mathbb{S}^1} F\,\mathrm{d}(\varphi_n^*\mathcal{M}_h) = \lim_{n\to\infty}\int_{\mathbb{S}^1}(F\circ\varphi_n)\,\mathrm{d}\mathcal{M}_h = \int_{\mathbb{S}^1}(F\circ\varphi)\,\mathrm{d}\mathcal{M}_h = \int_{\mathbb{S}^1} F\,\mathrm{d}(\varphi^*\mathcal{M}_h)$$

almost surely. That is, $\varphi_n^*\mathcal{M}_h$ converges almost surely in the weak topology to $\varphi^*\mathcal{M}_h$. We thus conclude that $\varphi^*\mathcal{M}_h = \mathcal{M}_{h\circ\varphi+Q\log|\varphi'|}$ almost surely. As described in Remark 3.5, we have $\frac{1}{2(2-\gamma)}\mathcal{M}_h^\gamma \to \mathcal{M}_h^{\mathrm{crit}}$ weakly in probability as $\gamma\to 2^-$. Hence, the coordinate change rule (3.29) holds for the critical case $\gamma = 2$ as well.

To conclude the proof, observe that

$$\varphi^*\widehat{\mathcal{M}}_h = \frac{1}{\mathcal{M}_h(\mathbb{S}^1)}\varphi^*\mathcal{M}_h = \frac{1}{\varphi^*\mathcal{M}_h(\mathbb{S}^1)}\varphi^*\mathcal{M}_h = \frac{1}{\mathcal{M}_{Ah+\beta}(\mathbb{S}^1)}\mathcal{M}_{Ah+\beta} = \widehat{\mathcal{M}}_{Ah+\beta} \tag{3.38}$$

almost surely and the law of $Ah+\beta$ is equivalent to that of h by Theorem 3.3. Therefore, the law of $\varphi^*\widehat{\mathcal{M}}_h$ is equivalent to that of $\widehat{\mathcal{M}}_h$. □

Proof of necessity. Let $\varphi\in\mathrm{Homeo}_+(\mathbb{S}^1)$ and assume that the law of the pull-back $\varphi^*\widehat{\mathcal{M}}_h$ is absolutely continuous with respect to that of $\widehat{\mathcal{M}}_h$. Then, since the GMC almost surely determines the LGF [BSS23, Vih24], there exists a random mean-zero distribution $\tilde{h}$ on $\mathbb{S}^1$ with a law absolutely continuous respect to LGF on $\mathbb{S}^1$ such that $\widehat{\mathcal{M}}_{\tilde{h}} = \varphi^*\widehat{\mathcal{M}}_h$ almost surely.

Choose a sequence of smooth $\varphi_n\in\mathrm{Diff}_+(\mathbb{S}^1)$ which converges uniformly to φ (e.g., by convolution). Since $\widehat{\mathcal{M}}_{h\circ\varphi_n+Q\log|\varphi_n'|} = \varphi_n^*\widehat{\mathcal{M}}_h \to \varphi^*\widehat{\mathcal{M}}_h$ almost surely, we have $\Pi(\varphi_n)h + \pi_0(Q\log|\varphi_n'|) \to \tilde{h}$ in law. In particular, $\tilde{h}$ is a Gaussian field whose law is absolutely continuous with respect to that of LGF on $\mathbb{S}^1$. Then, by the Feldman–Hájek theorem, there exists some $Q\in\mathscr{B}(\mathcal{H}_0)$ with $Q-I$ Hilbert–Schmidt such that for any $f,g\in\mathcal{H}_0$,

$$\begin{aligned}\lim_{n\to\infty}\langle\Pi(\varphi_n)^*f,\Pi(\varphi_n)^*g\rangle &= \lim_{n\to\infty}\mathrm{Cov}(\langle\Pi(\varphi_n)h,f\rangle,\langle\Pi(\varphi_n)h,g\rangle)\\ &= \mathrm{Cov}(\langle\tilde{h},f\rangle,\langle\tilde{h},g\rangle) = \langle f,Qg\rangle\end{aligned} \tag{3.39}$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathcal{H}_0$. That is, $\Pi(\varphi_n)\Pi(\varphi_n)^*$ converges in the weak operator topology to Q. In particular, for any $f\in\mathcal{H}_0$, we have

$$\lim_{n\to\infty}\|\Pi(\varphi_n)^*f\|^2_{\mathcal{H}_0} = \langle f,Qf\rangle \leq \|Q\|_{\mathscr{B}(\mathcal{H}_0)}\|f\|^2_{\mathcal{H}_0}.$$

Then, by the uniform boundedness principle,

$$\sup_n \|\Pi(\varphi_n)\|_{\mathscr{B}(H)} = \sup_n \|\Pi(\varphi_n)^*\|_{\mathscr{B}(H)} < \infty.$$

For any $f \in \mathcal{H}_0$ with $\|f\|_{\mathcal{H}_0} \le 1$, by Fatou's lemma,

$$\begin{aligned}\|\Pi(\varphi)f\|_{\mathcal{H}_0}^2 &= \iint_{\mathbb{S}^1\times\mathbb{S}^1} \frac{|f(\varphi(e^{\mathrm{i}\theta})) - f(\varphi(e^{\mathrm{i}\psi}))|^2}{|e^{\mathrm{i}\theta} - e^{\mathrm{i}\psi}|^2}\,\mathrm{d}\theta\,\mathrm{d}\psi \\ &\le \varliminf_{n\to\infty} \iint_{\mathbb{S}^1\times\mathbb{S}^1} \frac{|f(\varphi_n(e^{\mathrm{i}\theta})) - f(\varphi_n(e^{\mathrm{i}\psi}))|^2}{|e^{\mathrm{i}\theta} - e^{\mathrm{i}\psi}|^2}\,\mathrm{d}\theta\,\mathrm{d}\psi = \varliminf_{n\to\infty} \|\Pi(\varphi_n)f\|_{\mathcal{H}_0}^2 \\ &\le \varliminf_{n\to\infty} \|\Pi(\varphi_n)\|_{\mathscr{B}(\mathcal{H}_0)}^2 < \infty.\end{aligned}$$

Therefore, $\Pi(\varphi) \in \mathscr{B}(\mathcal{H}_0)$ and $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ follows by Lemma 2.7.

We see from the formulas (2.21) and (2.22) that $\Pi(\varphi_n) \to \Pi(\varphi)$ entrywise with respect to the basis (2.11) for $\mathcal{H}_0^{\mathbb{C}}$. Hence, $Q = \Pi(\varphi)\Pi(\varphi)^*$ and $\Pi(\varphi)\Pi(\varphi)^* - I$ is Hilbert–Schmidt. We conclude $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ using Lemma 2.7. ☐

In the Liouville theory, we often consider Gaussian multiplicative chaos measures with respect to a log-correlated Gaussian field plus a logarithmic singularity (cf. Lemma 1.6). Below, we give an analog of Propositon 3.7 for such fields. Recall from (2.25) the log-ratio

$$u_\varphi(z,w) = \log\frac{\varphi(z) - \varphi(w)}{z - w}.$$

Corollary 3.10. *Fix $\gamma \in (0,2]$. Let h be the LGF on $\mathbb{S}^1$ and $\mathfrak{h}(z) := -\alpha \log|z-1|$ where α is a positive constant. Suppose $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$ fixes 1. Then, the law of the normalized pullback measure $\varphi^*\widehat{\mathcal{M}}_{h+\mathfrak{h}}$ is absolutely continuous with respect to the law of $\widehat{\mathcal{M}}_{h+\mathfrak{h}}$ if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ and $u_\varphi(\cdot,1) \in H^{1/2}(\mathbb{S}^1)$.*

Proof. Suppose $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ and $u_\varphi(\cdot,1) \in H^{1/2}$, in which case $\operatorname{Re} u_\varphi(\cdot,1) = \log|(\varphi(\cdot)-1)/(\cdot-1)| \in H^{1/2}$. For each fixed C, since the law of $h + (\mathfrak{h}\wedge C)$ is absolutely continuous to that of h, we almost surely have $\varphi^*\mathcal{M}_{h+(\mathfrak{h}\wedge C)} = \mathcal{M}_{h\circ\varphi+(\mathfrak{h}\circ\varphi\wedge C)+Q\log|\varphi'|}$ by Proposition 3.7. Letting $C \to \infty$, since GMC is a local functional of the field, we obtain $\varphi^*\mathcal{M}_{h+\mathfrak{h}} = \mathcal{M}_{h\circ\varphi+\mathfrak{h}\circ\varphi+Q\log|\varphi'|}$ almost surely. Since $\mathfrak{h}\circ\varphi - \mathfrak{h} + Q\log|\varphi'| \in H^{1/2}$ by our assumption that $\operatorname{Re} u_\varphi(\cdot,1) \in H^{1/2}$, the law of $\Pi(\varphi)h + \pi_0(\mathfrak{h}\circ\varphi + Q\log|\varphi'|)$ is absolutely continuous with respect to that of $h + \pi_0(\mathfrak{h})$ by Theorem 3.3. This implies that the law of $\varphi^*\widehat{\mathcal{M}}_{h+\mathfrak{h}}$ is absolutely continuous with respect to that of $\widehat{\mathcal{M}}_{h+\mathfrak{h}}$.

On the other hand, suppose that the law of $\varphi^*\widehat{\mathcal{M}}_{h+\mathfrak{h}}$ is absolutely continuous with respect to that of $\widehat{\mathcal{M}}_{h+\mathfrak{h}}$ for some $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$. Pick $\varphi_n \in \mathrm{Diff}_+(\mathbb{S}^1)$ which converges uniformly to φ. As in the proof of Proposition 3.7, there exists a random distribution $\tilde{h}$ on $\mathbb{S}^1$ whose law is mutually absolutely continuous with respect to that of the LGF on $\mathbb{S}^1$ such that $\tilde{h}_n := \Pi(\varphi_n)h + \pi_0(\mathfrak{h}\circ\varphi_n - \mathfrak{h} + Q\log|\varphi_n'|)$ converges in law to $\tilde{h}$ as $n \to \infty$. Noting that $\mathrm{Cov}(\langle\tilde{h}_n, f\rangle, \langle\tilde{h}_n, g\rangle) = \langle\Pi(\varphi_n)^*f, \Pi(\varphi_k)^*g\rangle$ for any $f, g \in \mathcal{H}_0$, a proof analogous to that of Proposition 3.7 gives $\varphi \in \mathrm{WP}(\mathbb{S}^1)$. Then, $\tilde{h}_n$ converges in law to $\Pi(\varphi)h + \pi_0(-\alpha\operatorname{Re} u_\varphi(\cdot,1) + Q\log|\varphi'|)$, and the absolute continuity of its law with respect to LGF implies $u_\varphi(\cdot,1) \in H^{1/2}(\mathbb{S}^1)$. ☐

4 Quasi-invariance of SLE welding

In this section, we introduce the space $\mathrm{RM}(\mathbb{S}^1)$ of conformally removable weldings. We endow it with a topology so that composition gives a continuous group action of quasisymmetric circle homeomorphisms $\mathrm{QS}(\mathbb{S}^1)$ on $\mathrm{RM}(\mathbb{S}^1)$ (Lemma 4.1). This is due to the definition of quasi-invariance. For a measure $\mathscr{P}$ on a measurable space $(S, \mathcal{F})$, let G be a group that acts (left or right) measurably on S. We say that $\mathscr{P}$ is *quasi-invariant* under the action of G if for each $g \in G$ and $A \in \mathcal{F}$, we have $\mathscr{P}(A) = 0$ if and only if $\mathscr{P}(gA) = 0$. That is, the pull-back of $\mathscr{P}$ by g is mutually absolutely continuous with respect to $\mathscr{P}$. We refer the reader to [Kha09] for more background. In particular, we show the measurability of the group action of Weil–Petersson homeomorphisms on SLE weldings.

After that, we introduce the SLE loop shape measure and find the law of the corresponding welding homeomorphism (see Lemma 4.8), which leads to the proof of Theorem 1.2.

4.1 Conformal welding of Jordan curves

We say that a compact set K is *(quasi)conformally removable* if any homeomorphism of $\hat{\mathbb{C}}$ that is (quasi)conformal off K is (quasi)conformal on $\hat{\mathbb{C}}$. An easy application of the measurable Riemann mapping theorem shows that K is conformally removable if and only if it is quasiconformally removable. We refer the reader to the survey [You15] for an in-depth look at conformal removability.

It is well known that if a Jordan curve η is conformally removable, then the corresponding welding homeomorphism is unique up to pre- and post-compositions by $\mathrm{Möb}(\mathbb{S}^1)$. Given an oriented Jordan curve η, let $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ be any conformal maps onto the components of $\hat{\mathbb{C}} \setminus \eta$ on the left and the right of the curve, respectively. Suppose $\tilde{f} : \mathbb{D} \to \tilde{\Omega}$ and $\tilde{g} : \mathbb{D} \to \tilde{\Omega}^*$ is another pair of conformal maps such that $\tilde{\eta} := \hat{\mathbb{C}} \setminus (\tilde{\Omega} \cup \tilde{\Omega}^*)$ is a Jordan curve and $(\tilde{g}^{-1} \circ \tilde{f})|_{\mathbb{S}^1} = (g^{-1} \circ f)|_{\mathbb{S}^1}$. Then, we can define a homeomorphism of $\hat{\mathbb{C}}$ given by $\tilde{f} \circ f^{-1}$ on $\overline{\Omega}$ and $\tilde{g} \circ g^{-1}$ on $\overline{\Omega^*}$. Since this homeomorphism is conformal on $\hat{\mathbb{C}} \setminus \eta$, it must be conformal on all of $\hat{\mathbb{C}}$. Hence, $\tilde{\eta}$ is an image of η under a Möbius transformation as desired. However, there is no analytic characterization of weldings corresponding to conformally removable curves [Bis20].

Recall that For a compact set K in $\hat{\mathbb{C}}$, *Hausdorff dimension* $d_H(K)$ is defined to be

$$\inf\{\epsilon \geq 0 | \inf_{D \in \mathscr{D}} \sum_{\mathrm{a} \in \mathcal{A}} r_{\mathrm{a}}^{\epsilon} = 0\},$$

where $\mathscr{D}$ is the family of finite coverings $D = \{D_{\mathrm{a}}\}_{\mathrm{a} \in \mathcal{A}}$ indexed by a set $\mathcal{A}$ of K by discs D_{a} with radius r_{a}.

Lemma 4.1. *Let* $\mathrm{RM}(\mathbb{S}^1)$ *denote the space of orientation-preserving circle homeomorphisms that are weldings of conformally removable Jordan curves. If* $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ *and* $\psi \in \mathrm{RM}(\mathbb{S}^1)$*, then* $\varphi^{-1} \circ \psi$ *are* $\psi \circ \varphi$ *are in* $\mathrm{RM}(\mathbb{S}^1)$*. If we further assume that* $\varphi \in \mathrm{S}(\mathbb{S}^1)$*, then the three curves that solve the welding problem for* ψ*,* $\psi \circ \varphi$ *and* $\varphi^{-1} \circ \psi$ *share the same Hausdorff dimension.*

Proof. Given $\varphi \in \mathrm{QS}(\mathbb{S}^1)$, choose a quasiconformal extension ω of φ^{-1} to $\hat{\mathbb{C}}$ whose Beltrami coefficient $\mu = \partial_{\bar{z}}\omega / \partial_z \omega$ satisifies the symmetry condition $\mu(z) = (z/\bar{z})^2 \bar{\mu}(1/\bar{z})$. For a conformally removable Jordan curve η solving the welding problem for ψ, let $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ denote the conformal maps onto the bounded and unbounded complementary components of η, respectively. Let $\tilde{\omega}$ be a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ that fixes 0, 1 and ∞, whose Beltrami coefficient is given by

$$\tilde{\mu} = \begin{cases} \left(\mu f'/\bar{f}'\right) \circ f^{-1} & \text{in } \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

We define as above so that $\tilde{f} = \tilde{\omega} \circ f \circ \omega^{-1} : \mathbb{D} \to \tilde{\omega}(\Omega)$ and $\tilde{g} = \tilde{\omega} \circ g : \mathbb{D}^* \to \tilde{\omega}(\Omega^*)$ are exactly the conformal maps corresponding to the Jordan curve $\tilde{\eta} = \tilde{\omega}(\eta)$, whose welding homeomorphism is $\tilde{\psi} := (\tilde{g}^{-1} \circ \tilde{f})|_{\mathbb{S}^1} = \psi \circ \varphi$. To see that $\tilde{\eta}$ is quasiconformally removable, note that given a homeomorphism F of $\hat{\mathbb{C}}$ which is quasiconformal off $\tilde{\eta}$, we have that $\tilde{F} = F \circ \tilde{\omega}$ is a homeomorphism of $\hat{\mathbb{C}}$ which is quasiconformal off η. Since η is (quasi)conformally removable, F is quasiconformal on $\hat{\mathbb{C}}$ and it follows that $\tilde{F}$ is quasiconformal on $\hat{\mathbb{C}}$. This shows that if $\psi \in \mathrm{RM}(\mathbb{S}^1)$ and $\varphi \in \mathrm{QS}(\mathbb{S}^1)$, then $\psi \circ \varphi \in \mathrm{RM}(\mathbb{S}^1)$.

Let us further assume that $\varphi \in \mathrm{S}(\mathbb{S}^1)$. Let d and $\tilde{d}$ denote the Hausdorff dimensions of η and $\tilde{\eta} = \tilde{\omega}(\eta)$, respectively. From classical distortion estimates for quasiconforaml maps [Ast94], we have

$$(1/d - 1/2)/K \leq 1/\tilde{d} - 1/2 \leq K(1/d - 1/2)$$

where K is the supremum of dilatation of $\tilde{\omega}$ near η. This is asymptotically equal to 1, as $\tilde{\omega}$ is asymptotically conformal near η. More precisely, we have $K_r \to 1$ as $r \uparrow 1$, where K_r is the supremum of dilatation of $\tilde{\omega}$ restricted on $\mathbb{C} \backslash f(r\mathbb{D})$. Therefore, we have $\tilde{d} = d$.

The analogous results for $\varphi^{-1} \circ \psi$ follow in a similar manner, taking $\tilde{\omega}$ to be a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ where its Beltrami coefficient agrees with the pullback of μ by g in Ω^* and 0 otherwise. □

Remark 4.2. We may replace the Hausdorff dimension in Lemma 4.1 by the upper Minkowski dimension, the packing dimension, Assouad dimension, etc., as they share the similar dilatation-dependent distortion bounds. See [CGT23] and the discussion therein.

Now we define the post-composition $R : \mathrm{RM}(\mathbb{S}^1) \times \mathrm{QS}(\mathbb{S}^1) \to \mathrm{RM}(\mathbb{S}^1)$ by $R(\psi, \varphi) = \psi \circ \varphi$ and the pre-composition $L : \mathrm{RM}(\mathbb{S}^1) \times \mathrm{QS}(\mathbb{S}^1) \to \mathrm{RM}(\mathbb{S}^1)$ by $R(\psi, \varphi) = \varphi^{-1} \circ \psi$. Recall that we the topology on $\mathrm{QS}(\mathbb{S}^1)$ is induced from the Teichmüller distance. We define the topology on $\mathrm{RM}(\mathbb{S}^1)$ by the Carathéodory topology on the two conformal maps via conformal welding. More precisely, we say $\psi_n \to \psi$ if there exist welding solutions $\psi_n = g_n^{-1} \circ f_n$ and $\psi = g^{-1} \circ f$ such that $f_n \to f$ and $g_n \to g$ uniformly on compact subsets.

Proposition 4.3. *The post-composition and pre-composition are continuous.*

Proof. Suppose $\psi_n \to \psi$ and $\varphi_n \to \varphi$ in $\mathrm{RM}(\mathbb{S}^1)$ and $\mathrm{QS}(\mathbb{S}^1)$, respectively. Let $f_n \to f$ and $g_n \to g$ uniformly on compact subsets be the conformal maps corresponding to ψ_n and ψ. Let μ_n and μ be the Beltrami coefficient of the quasiconformal extension ω_n and ω of φ_n^{-1} and φ^{-1}, respectively. We may choose μ_n so that $\mu_n \to \mu$ in the L^∞ norm. Define $\tilde{\mu}_n$ as in (4.1) using μ_n and f_n instead.

Observe in the proof of Lemma 4.1 that if $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$ and $\mu_n \to \mu$ almost everywhere, then $\tilde{\mu}_n \to \tilde{\mu}$ almost everywhere on $\hat{\mathbb{C}}$. Hence, if we choose ω_n and $\tilde{\omega}_n$ to fix $0, 1, \infty$, then $\omega_n^{-1} \to \omega^{-1}$ and $\tilde{\omega}_n \to \tilde{\omega}$ uniformly with respect to the spherical metric [Leh87, Thm 4.6]. Thus, $\tilde{f}_n = \tilde{\omega}_n \circ f_n \circ \omega_n^{-1} \to \tilde{\omega} \circ f \circ \omega^{-1} = \tilde{f}$ and $\tilde{g}_n = \tilde{\omega}_n \circ g_n \to \tilde{\omega} \circ g = \tilde{g}$ uniformly on compact subsets.

The analogous results for $\varphi^{-1} \circ \psi$ follow in a similar manner as before. □

As we will see in the next subsection, from the perspective of the SLE_κ loop measure, it is more appropriate to consider a correspondence between the following spaces.

- The space of orientation-presering circle homeomorphisms which arise as weldings of conformally removable Jordan curves that fix 1, denoted $\mathbb{S}^1\backslash \mathrm{RM}(\mathbb{S}^1)$.
- The space of conformally removable Jordan curves η on $\hat{\mathbb{C}}$ separating 0 from ∞ with a unit conformal radius of $\hat{\mathbb{C}} \setminus \eta$ viewed from 0, denoted $\mathcal{J}_\#$.

Definition 4.4. Given $\eta \in \mathcal{J}_\#$, let Ω and Ω^* be the bounded and unbounded components of $\hat{\mathbb{C}} \setminus \eta$, respectively. Consider the unique pair of Riemann maps $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ satisfying $f(0) = 0$, $f'(0) = 1$, $g(\infty) = \infty$, and $f(1) = g(1)$. The corresponding element of $\mathbb{S}^1\backslash \mathrm{RM}(\mathbb{S}^1)$ is given by $(g^{-1} \circ f)|_{\mathbb{S}^1}$.

Note that this correspondence is one-to-one since the only map $\omega \in \mathrm{M\ddot{o}b}(\hat{\mathbb{C}})$ satisfying $\omega(0) = 0$, $\omega'(0) = 1$, and $\omega(\infty) = \infty$ is the identity. We endow $\mathcal{J}_\#$ with the Carathéodory topology for Ω and Ω^*, which is equivalent to the topology of local uniform convergence for f and g. We endow the topology on $\mathbb{S}^1\backslash \mathrm{RM}(\mathbb{S}^1)$ that makes the above correspondence a homeomorphism.

Remark 4.5. Given a function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$, let $\mathcal{J}_\omega$ be a subset of $\mathcal{J}_\#$ consisting of curves for which the Riemann map f and g as in Definition 4.4 admit ω as a modulus of continuity on the $\mathbb{S}^1$. Then, the topology on $\mathcal{J}_\omega$ inherited from $\mathcal{J}_\#$ is equivalent to that of uniform convergence of the Riemann map f and g, and hence that induced by the Hausdorff distance on compact subsets of $\hat{\mathbb{C}}$.

Take $\omega(r) = r^\alpha$ for some fixed $\alpha \in (0, 1)$. Then, it is a classic fact that $\mathcal{J}_\omega$ includes all K-quasicircles in $\mathcal{J}_\#$ with $K < 1/\alpha$. Moreover, for $\kappa_\alpha \in (0, 4)$ depending on α, the SLE_κ shape measure is supported in $\mathcal{J}_\omega$ for $\kappa \in (0, \kappa_\alpha)$. (See the next subsection.)

The correspondence in Definition 4.4, when restricted to quasisymmetric circle homeomorphisms in $\mathbb{S}^1\backslash \mathrm{RM}(\mathbb{S}^1)$ and quasicircles in $\mathcal{J}_\#$, agrees with the models of the universal Teichmüller curve $\mathcal{T}(1)$ as discussed in [TT06, Sec. I.1.2]. Let us consider $\mathbb{S}^1\backslash \mathrm{QS}(\mathbb{S}^1)$ as a right topological group [Ber73] equipped with the topology inherited from $\mathcal{T}(1)$: i.e., the one induced from the Teichmüller distance.

Proposition 4.6. *The post-composition and pre-composition are continuous restricted on* $\mathbb{S}^1\backslash \mathrm{RM}(\mathbb{S}^1) \times \mathbb{S}^1\backslash \mathrm{QS}(\mathbb{S}^1)$.

Proof. We only need to take care of the normalization. Note that if $\tilde{f}_n$ and $\tilde{g}_n$ are the conformal maps corresponding to $\psi_n \circ \varphi_n$ as in Definition 4.4, then they differ from $\tilde{\omega}_n \circ f \circ \omega_n^{-1}$ and $\tilde{\omega}_n \circ g$ by a post-composition by $z \mapsto c_n z$ where $1/c_n = (\tilde{\omega}_n \circ f \circ \omega_n^{-1})'(0)$. Since $(\tilde{\omega}_n \circ f \circ \omega_n^{-1})'(0) \to (\tilde{\omega} \circ f \circ \omega^{-1})'(0)$, we conclude that $\tilde{f}_n \to \tilde{f}$ and $\tilde{g}_n \to \tilde{g}$ uniformly on compact subsets. The analogous results for $\varphi^{-1} \circ \psi$ follow in a similar manner as before. $\square$

Some homeomorphisms are not proven to be weldings, where we consider the measure on homeomorphisms and endow the topology of uniform convergence. Then the post-composition and pre-composition are still continuous.

4.2 SLE loop welding measure

For $\kappa \in (0,4]$, the SLE_κ loop measure μ^κ is a σ-finite measure on the space of Jordan curves on $\widehat{\mathbb{C}}$ satisfying the following properties as defined by Kontsevich and Suhov [KS07].

- Generalized restriction property: For any simply connected domain $D \subsetneq \hat{\mathbb{C}}$, we define μ_D^κ by

$$\frac{\mathrm{d}\mu_D^\kappa}{\mathrm{d}\mu^\kappa}(\cdot) = \mathbf{1}_{\{\cdot \subset D\}} \exp\left(c(\kappa)\Lambda^*(\cdot, D^c)/2\right)$$

  where $c(\kappa) = (6-\kappa)(3\kappa-8)/2\kappa$ is the central charge of SLE_κ and $\Lambda^*(\eta, D^c)$ is the size of the normalized Brownian loop measure for loops hitting both η and D^c.
- Conformal invariance: If $f : D \to D'$ is a conformal map between two subdomains of $\widehat{\mathbb{C}}$, then the pushforward measure $f_*\mu_D^\kappa$ is agrees with $\mu_{D'}^\kappa$.

For each $\kappa \in (0,4]$, the SLE_κ loop measure μ^κ exists and is unique up to a multiplicative constant. This was proved by Werner [Wer08] for $\kappa = 8/3$; the measure μ^κ for the entire range of $\kappa \in (0,4]$ was constucted by Zhan [Zha21] and was proved to be unique by Bav4rez and Jego [BJ24].

From the perspective of conformal welding, it is natural to consider the *shape measure* of the SLE_κ loops. First, consider the restriction $\mu^\kappa_{\mathbb{C}\setminus\{0\}}$ of the SLE_κ loop measure μ^κ to the loops that separate 0 and ∞. Given a Jordan curve η separating 0 and ∞, let $\mathrm{CR}(\eta, 0)$ denote the conformal radius $|f'(0)|$ of the bounded component D_η of $\widehat{\mathbb{C}} \setminus \eta$ viewed from 0 where $f : \mathbb{D} \to D_\eta$ is a conformal map with $f(0) = 0$. Then, we define $\mu^\kappa_\#$ to be the conditional law[2] of $\mu^\kappa_{\mathbb{C}\setminus\{0\}}$ on the set of loops η for which $\mathrm{CR}(\eta, 0) = 1$. We may choose the multiplicative constant for the SLE_κ loop measure μ^κ such that the shape measure $\mu^\kappa_\#$ is a probability measure; we assume this henceforth.

Definition 4.7. For $\kappa \in (0,4]$, we define $\mathrm{SLE}_\kappa^{\mathrm{weld}}$ to be the probability measure on $\mathbb{S}^1 \setminus \mathrm{RM}(\mathbb{S}^1)$ given by the pushforward of the SLE_κ loop shape measure $\mu^\kappa_\#$ under the measurable one-to-one correspondence $\mathcal{J}_\#^{0,\infty} \to \mathbb{S}^1 \setminus \mathrm{RM}(\mathbb{S}^1)$ induced by conformal welding.

4.3 Conformal welding of quantum disks

Fix $\gamma \in (0,2)$ and let $\kappa = \gamma^2$. Building upon Sheffield's quantum zipper [She16], Ang, Holden, and Sun proved in [AHS23] that conformally welding two independent γ-LQG disks gives a quantum sphere decorated with an independent SLE_κ loop. We give a translation of this result in terms of the welding homeomorphism of two independent LGFs on $\mathbb{S}^1$.

Recall that given an LGF h on $\mathbb{S}^1$, we let $\widehat{\mathcal{M}}_h$ to be the γ-GMC measure $\mathcal{M}_h = \mathcal{M}_h^\gamma$ corresponding to h normalized to have unit mass. Define $\phi_h = \phi_h^\gamma \in \mathrm{Homeo}_+(\mathbb{S}^1)$ by

$$\phi_h^\gamma(z) := \exp(2\pi \mathrm{i} \cdot \widehat{\mathcal{M}}_h^\gamma([1,z])),$$

for each $z \in \mathbb{S}^1$ where $[1,z] \subset \mathbb{S}^1$ denotes the arc running counterclockwise from 1 to z. That is, the pushforward of the normalized GMC measure $\widehat{\mathcal{M}}_h$ under the homeomorphism ϕ_h is the uniform (arc-length) measure on $\mathbb{S}^1$. The following is the main result of this subsection.

[2] One way to give this law is to consider the restriction of $\mu^\kappa_{\mathbb{C}\setminus\{0\}}$ to loops η with $1/2 \leq \mathrm{CR}(\eta,0) \leq 1$, which is a finite measure since it must wind around the disk $\frac{1}{8}\mathbb{D}$ and intersect the unit circle $\partial\mathbb{D}$ [Zha21]. Scaling each loop η sampled under this measure using the map $z \mapsto z/\mathrm{CR}(\eta,0)$ gives a loop sampled from $\mu^\kappa_\#$ by the conformal invariance of the SLE_κ loop measure.

Lemma 4.8. *Let $\gamma \in (0,2)$ and $\kappa = \gamma^2 \in (0,4)$. Suppose h_1, h_2 are independent LGFs on $\mathbb{S}^1$ and α is a uniform rotation of $\mathbb{S}^1$ independent from h_1 and h_2. Let $\mathcal{W}^\kappa$ denote the law of $(\phi^\gamma_{h_2})^{-1} \circ \alpha \circ \phi^\gamma_{h_1} \in \mathrm{Homeo}_+(\mathbb{S}^1)$. Then, the pushforward of $\mathcal{W}^\kappa$ under the natural projection $\mathrm{Homeo}_+(\mathbb{S}^1) \to \mathbb{S}^1 \backslash \mathrm{Homeo}_+(\mathbb{S}^1)$ is equivalent to $\mathrm{SLE}^{\mathrm{weld}}_\kappa$. Furthermore, if we identify $\mathbb{S}^1 \backslash \mathrm{Homeo}_+(\mathbb{S}^1)$ with the stabilizers of 1 in $\mathrm{Homeo}_+(\mathbb{S}^1)$, then $\mathrm{SLE}^{\mathrm{weld}}_\kappa$ is equivalent to the law of $(\phi^\gamma_{h_2 - \gamma \log|\cdot - 1|})^{-1} \circ \phi^\gamma_{h_1}$.*

Note that $(\phi^\gamma_{h_2})^{-1} \circ \alpha \circ \phi^\gamma_{h_1}$ is the circle homeomorphism corresponding to the conformal welding of two disks with boundary length measures $\widehat{\mathcal{M}}_{h_1}$ and $\widehat{\mathcal{M}}_{h_2}$ where we make a random shift for the point on the second disk glued to 1 on the first disk according to the boundary length measure $\widehat{\mathcal{M}}_{h_2}$. This result immediately implies Theorem 1.2.

Proof of Theorem 1.2. Combine Lemma 4.8 with Proposition 3.7. □

We will obtain Lemma 4.8 using the conformal welding of independent quantum disks each with one interior marked point as established in [ACSW24]. To state this result, let us recall some basic definitions from the LQG literature. A γ-LQG disk with one marked point is defined as the equivalence class $(D, \Gamma, z)/\sim_\gamma$ of the tuple of a domain D conformally equivalent to the unit disk, a random distribution h, and a point $z \in D$, where $(D, \Gamma, z) \sim_\gamma (\tilde{D}, \tilde{\Gamma}, \tilde{z})$ if there exists a conformal map $f : D \to \tilde{D}$ such that

$$\Gamma = \tilde{\Gamma} \circ f + Q \log |f'| \tag{4.2}$$

for $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and $\tilde{z} = f(z)$. We define a γ-LQG sphere with two marked points and a Jordan curve similarly as an equivalence class $(D, \Gamma, \eta, z_1, z_2)/\sim_\gamma$ up to conformal transformations where the field Γ satisfies the coordinate change rule (4.2).

We now define Liouville field on $\mathbb{D}$. While the Liouville theory on simply connected domains with boundary was stated originally on the unit disk $\mathbb{D}$ [HRV18], subsequent works such as [Rem20, RZ22, ARS23] used the upper half-plane $\mathbb{H}$ as the standard domain. Nevertheless, Proposition 3.4 and Proposition 3.7 of [HRV18] imply that the (infinite) law we give here agrees with the definitions of Liouville fields on $\mathbb{H}$ up to a γ-dependent multiplicative factor. See [RZ22, Section 5] for further details.

Definition 4.9 ([ARS23, Lem. 4.4–4.5])**.** Let $P_\mathbb{D}$ be the law of the free-boundary GFF on $\mathbb{D}$ normalized to have zero mean on the boundary $\partial\mathbb{D}$. Let $P_\mathbb{C}$ denote the law of the whole-plane GFF with zero mean on the unit circle.[3]

1. For $\ell > 0$, define $\mathrm{LF}^{(\gamma,0)}_\mathbb{D}(\ell)$ to be the law of the field $\Gamma^\ell := \hat{\Gamma} + \frac{2}{\gamma} \log \frac{\ell}{\nu_{\hat{\Gamma}}(\partial\mathbb{D})}$, where

$$\hat{\Gamma} := \Gamma - \gamma \log|\cdot| \tag{4.3}$$

for the field Γ sampled under the reweighted measure $\ell^{-4/\gamma^2}(\nu_{\hat{\Gamma}}(\partial\mathbb{D}))^{4/\gamma^2 - 1} P_\mathbb{D}$. Let $\mathcal{M}^{\mathrm{disk}}_{1,0}(\gamma; \ell)$ be the measure on quantum surfaces $(\mathbb{D}, \Gamma^\ell, 0)/\sim_\gamma$, where Γ^ℓ is sampled from $\mathrm{LF}^{(\gamma,0)}_\mathbb{D}(\ell)$. This is a finite measure on quantum disks with boundary length ℓ and one interior marked point. Define

$$\mathrm{LF}^{(\gamma,0)}_\mathbb{D} = \int_0^\infty \mathrm{LF}^{(\gamma,0)}_\mathbb{D}(\ell)\, d\ell \quad \text{and} \quad \mathcal{M}^{\mathrm{disk}}_{1,0}(\gamma) = \int_0^\infty \mathcal{M}^{\mathrm{disk}}_{1,0}(\gamma; \ell)\, d\ell. \tag{4.4}$$

[3] This is a centered Gaussian field with covariance kernel $G_\mathbb{C}(z,w) = -\log|z-w| + \log|z|_+ + \log|w|_+$.

2. For $\ell > 0$, define $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0),(\gamma,1)}(\ell)$ to be the law of the field $\Gamma^\ell := \hat{\Gamma} + \frac{2}{\gamma}\log\frac{\ell}{\nu_{\hat{\Gamma}}(\partial\mathbb{D})}$, where

$$\hat{\Gamma} := \Gamma - \gamma\log|\cdot| - \gamma\log|\cdot - 1| \tag{4.5}$$

for the field Γ sampled under the reweighted measure $\ell^{-4/\gamma^2+1}(\nu_{\hat{\Gamma}}(\partial\mathbb{D}))^{4/\gamma^2}P_{\mathbb{D}}$.

3. Let $\mathrm{LF}_{\mathbb{C}}^{(\gamma,0),(\gamma,\infty)}$ denote the law of the field $\Gamma - \gamma\log|\cdot| - 2(Q-\gamma)\log|\cdot|_+ + c$ where (Γ, c) is sampled from the law $P_{\mathbb{C}} \times [e^{2(\gamma-Q)c}dc]$ and $|z|_+ = \max\{1, |z|\}$.

We note that $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0)}(\ell)$ and $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0),(\gamma,1)}(\ell)$ are finite measures for each $\ell > 0$ [HRV18, Cor. 3.10]. See also [RZ22] for explicit formulas for their total masses. The following lemma clarifies the relation between these two measures, which follows from combining Definition 2.10, Proposition 3.4, and Proposition 3.9 of [ARS23] and Proposition 2.21 of [ACSW24].

Lemma 4.10. *Given a finite measure λ, let $\lambda^\#$ denote the this law normalized to be a probability measure. Fix $\ell > 0$.*

1. *Sample a field Γ on $\mathbb{D}$ from $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0)}(\ell)^\#$, then sample $e^{\mathrm{i}\theta} \in \partial\mathbb{D}$ from the normalized boundary length measure $\hat{\nu}_{\Gamma^\ell} := \nu_{\Gamma^\ell}/\nu_{\Gamma^\ell}(\partial\mathbb{D})$. Then, $\Gamma^\ell(e^{\mathrm{i}\theta}\cdot)$ has the law $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0),(\gamma,1)}(\ell)^\#$.*
2. *Sample a field Γ on $\mathbb{D}$ from $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0),(\gamma,1)}(\ell)^\#$ and independently sample a uniform boundary point $e^{\mathrm{i}\theta} \in \partial\mathbb{D}$ (from the normalized arc-length measure on $\partial\mathbb{D}$). Then, $\Gamma^\ell(e^{\mathrm{i}\theta}\cdot)$ has the law $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0)}(\ell)^\#$.*

Given quantum disks $\mathcal{D}_1$ and $\mathcal{D}_2$, let $\mathrm{Weld}(\mathcal{D}_1, \mathcal{D}_2)$ denote the *uniform conformal welding* of $\mathcal{D}_1$ and $\mathcal{D}_2$ as described in [ACSW24, Sec. 2.5]. That is, $\mathrm{Weld}(\mathcal{D}_1, \mathcal{D}_2)$ is a probability measure on the quantum sphere obtained by conformally welding the boundaries of $\mathcal{D}_1$ and $\mathcal{D}_2$ starting from matching the points sampled independently on each boundary from the normalized boundary length measures. Put otherwise, $\mathrm{Weld}(\mathcal{D}_1, \mathcal{D}_2)$ is the law of the quantum sphere $(\widehat{\mathbb{C}}, \Gamma, \eta, 0, \infty)/\sim_\gamma$ where, if D_1 and D_2 are the bounded and unbounded components of $\widehat{\mathbb{C}}\setminus\eta$, then $(\mathcal{D}_1, \mathcal{D}_2) = ((D_1, \Gamma|_{D_1}, 0)/\sim_\gamma, (D_2, \Gamma|_{D_2}, \infty)/\sim_\gamma)$ and the quantum boundary length measures of $\mathcal{D}_1$ and $\mathcal{D}_2$ agree on η, which we call the quantum length measure of η. Moreover, if we sample $p \in \eta$ uniformly from the quantum length measure normalized to be a probability measure, then the joint law of $(D_1, \Gamma|_{D_1}, 0, p)/\sim_\gamma$ and $(D_2, \Gamma|_{D_2}, 0, p)/\sim_\gamma$ is that of $\mathcal{D}_1$ and $\mathcal{D}_2$ each with a boundary point sampled independently of each other from the normalized quantum boundary length measure. Let us define the law

$$\mathrm{Weld}(\mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;\ell), \mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;\ell)) = \int \mathrm{Weld}(\mathcal{D}_1, \mathcal{D}_2)\, d[\mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;\ell) \times \mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;\ell)](\mathcal{D}_1, \mathcal{D}_2). \tag{4.6}$$

We now state the conformal welding theorem of [AHS23] for the loops that separate 0 and ∞, and translate it to Lemma 4.8.

Lemma 4.11 ([ACSW24, Lemma 7.7][4]). *Let $\kappa = \gamma^2 \in (0, 4)$. If (Γ, η) is sampled from $\mathrm{LF}_{\mathbb{C}}^{(\gamma,0),(\gamma,\infty)} \times \mu_\#^\kappa$, then the law of the loop-decorated quantum surface $(\widehat{\mathbb{C}}, \Gamma, \eta, 0, \infty)/\sim_\gamma$ is equal to $\int_0^\infty \ell \cdot \mathrm{Weld}(\mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;\ell), \mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;\ell))\,\mathrm{d}\ell$ up to a γ-dependent multiplicative constant.*

[4] In place of $\mathbb{C}$ and $\mu_\#^\kappa$, [ACSW24] uses the infinite cylinder $\mathcal{C} = \mathbb{R}\times\mathbb{S}^1$ and the law of SLE_κ loops on $\mathcal{C}$ separating $\pm\infty$ that touch $\{0\}\times\mathbb{S}^1$ from the left but do not cross it, respectively. By the conformal invariance of SLE_κ loops and the translation invariance of $\mathrm{LF}_{\mathcal{C}}^{(\gamma,\pm\infty)}$ [AHS24, Theorem 2.13], our statement is equivalent to [ACSW24, Lemma 7.7].

Proof of Lemma 4.8. Consider the disintegration

$$\mathrm{LF}_{\mathbb{C}}^{(\gamma,0),(\gamma,\infty)} \times \mu_{\#}^{\kappa} = \int_0^{\infty} [\mathrm{LF}_{\mathbb{C}}^{(\gamma,0),(\gamma,\infty)} \times \mu_{\#}^{\kappa}](\ell)\, \mathrm{d}\ell$$

where $[\mathrm{LF}_{\mathbb{C}}^{(\gamma,0),(\gamma,\infty)} \times \mu_{\#}^{\kappa}](\ell)$ is a measure on the pair (Γ, η) such that the quantum length of η with respect to Γ, which we denote $\nu_\Gamma(\eta)$, equals ℓ. From Definition 4.9, we see that a version of $[\mathrm{LF}_{\mathbb{C}}^{(\gamma,0),(\gamma,\infty)} \times \mu_{\#}^{\kappa}](\ell)$ is given by $(\Gamma + \frac{2}{\gamma}\log(\ell/\nu_\Gamma(\eta)), \eta)$ where (Γ, η) is sampled from the restriction of

$$\frac{\ell^{-4/\gamma^2}}{\int_1^\infty t^{-4/\gamma^2}\mathrm{d}t}\mathrm{LF}_{\mathbb{C}}^{(\gamma,0),(\gamma,\infty)} \times \mu_{\#}^{\kappa}$$

to the event $\{\nu_\Gamma(\eta) \geq 1\}$. In particular, note that the marginal law of η under $[\mathrm{LF}_{\mathbb{C}}^{(\gamma,0),(\gamma,\infty)} \times \mu_{\#}^{\kappa}](\ell)$ is equivalent to $\mu_{\#}^{\kappa}$.

Fix $\ell = 1$. Recall that the quantum disk $(\mathbb{D}, \Gamma, 0)/\sim_\gamma$ with Γ sampled from $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0)}(1)^{\#}$ has the law $\mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;1)^{\#}$. Define $\phi_\Gamma(z) = \exp(2\pi \mathrm{i} \hat{\nu}_\Gamma([1,z]))$, where $[1,z]$ denotes the arc from 1 to z counterclockwise. Suppose Γ_1, Γ_2 are independent fields sampled from $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0)}(1)^{\#}$. Observe that if α_1, α_2 are independent samples from the uniform measure on the rotation group of $\mathbb{S}^1$ which are further independent from Γ_1, Γ_2, then $(\alpha_2 \circ \phi_{\Gamma_2|_{\partial\mathbb{D}}})^{-1} \circ (\alpha_1 \circ \phi_{\Gamma_1|_{\partial\mathbb{D}}})$ is the homeomorphism corresponding to the uniform conformal welding of quantum disks $\mathcal{D}_1 = (\mathbb{D}, \Gamma_1, 0)/\sim_\gamma$ and $\mathcal{D}_2 = (\mathbb{D}, \Gamma_2, 0)/\sim_\gamma$. That is, if η is the Jordan curve which corresponds to the welding homeomorphism $(\alpha_2 \circ \phi_{\Gamma_2|_{\partial\mathbb{D}}})^{-1} \circ (\alpha_1 \circ \phi_{\Gamma_1|_{\partial\mathbb{D}}})$, then there is a random field Γ such that $(\widehat{\mathbb{C}}, \Gamma, \eta, 0, \infty)/\sim_\gamma$ has the law $\mathrm{Weld}(\mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;1)^{\#}, \mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;1)^{\#})$. Hence, by Lemma 4.11, η has the law $\mu_{\#}^{\kappa}$ when considered up to rotations of $\widehat{\mathbb{C}}$ around 0. By the invariance of the law $\mu_{\#}^{\kappa}$ under such rotations, if we choose $\alpha_0 \in \partial\mathbb{D}$ uniformly yet again independently from $\Gamma_1, \Gamma_2, \alpha_1, \alpha_2$, then $\alpha_0 \cdot \eta$ has the law $\mu_{\#}^{\kappa}$. Observe that if we denote the rotation $z \mapsto \alpha_0 \cdot z$ also as α_0, then the welding homeomorphism corresponding to $\alpha_0 \cdot \eta$ is given by $(\alpha_2 \circ \phi_{\Gamma_2|_{\partial\mathbb{D}}})^{-1} \circ (\alpha_1 \circ \phi_{\Gamma_1|_{\partial\mathbb{D}}}) \circ \alpha_0^{-1}$ when both are considered as random elements of $\mathbb{S}^1 \backslash \mathrm{Homeo}_+(\mathbb{S}^1)$. From Definition 4.9, we see that if Γ is a sample from $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0)}(1)^{\#}$, then the law of $\Gamma|_{\partial\mathbb{D}}$ is equivalent to that of $h - \frac{2}{\gamma}\nu_h(\mathbb{S}^1)$ where h an LGF on $\mathbb{S}^1$. That is, the law of $\phi_{\Gamma|_{\partial\mathbb{D}}}$ is equivalent to that of $\phi_{h - \frac{2}{\gamma}\nu_h(\mathbb{S}^1)} = \phi_h$.

Thus, to paraphrase our conclusion so far, if h_1, h_2 are LGFs on $\mathbb{S}^1$ and $\alpha_0, \alpha_1, \alpha_2$ are uniform random variables on $\mathbb{S}^1$, all mutually independent, the pushforward of the law of $\phi_{h_2}^{-1} \circ (\alpha_2^{-1} \circ \alpha_1) \circ \phi_{h_1} \circ \alpha_0^{-1}$ under the projection $\mathrm{Homeo}_+(\mathbb{S}^1) \to \mathbb{S}^1 \backslash \mathrm{Homeo}_+(\mathbb{S}^1)$ is equivalent to $\mathrm{SLE}_\kappa^{\mathrm{weld}}$. Since the law of LGF on $\mathbb{S}^1$ is invariant under fixed rotations of $\mathbb{S}^1$ and α_0 is independent of h_1, we have that $\phi_{h_1}(\alpha_0^{-1}(1))^{-1} \cdot \phi_{h_1} \circ \alpha_0^{-1}$ agrees in law with ϕ_{h_1}. Moreover, $\alpha_2^{-1} \circ \alpha_1$ is a uniform rotation of $\mathbb{S}^1$ independent from h_1, h_2, and α_0, so we obtain the first part of the lemma.

For the second part of the lemma, we observe that $e^{\mathrm{i}\Theta} := (\alpha_2 \circ \phi_{\Gamma_2|_{\partial\mathbb{D}}})^{-1} \circ (\alpha_1 \circ \phi_{\Gamma_1|_{\partial\mathbb{D}}})(1) = (\phi_{\Gamma_2|_{\partial\mathbb{D}}})^{-1} \circ (\alpha_2^{-1} \circ \alpha_1)(1)$ is, conditioned on Γ_2, a point sampled uniformly from the boundary length measure ν_{Γ_2} normalized to be a probability measure. By Lemma 4.10, we see that $e^{-i\Theta} \cdot (\phi_{\Gamma_2|_{\partial\mathbb{D}}})^{-1} \circ (\alpha_2^{-1} \circ \alpha_1)$ agrees in law with $\phi_{\tilde{\Gamma}_2}^{-1}$ where $\tilde{\Gamma}_2$ is sampled from $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0),(\gamma,1)}(1)^{\#}$. Since the law of $\tilde{\Gamma}_2|_{\partial\mathbb{D}}$ is equivalent to that of $h_2 - \gamma\log|\cdot - 1|$ from Definition 4.9, the second part of the lemma follows as in the previous paragraph. □

4.4 Equivalence for other weldings

We end with a few remarks on generalizations of Theorem 1.2. From Lemma 2.10, for each $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ and almost every $z_0 \in \mathbb{S}^1$, we have that

$$\hat{\varphi}(e^{\mathrm{i}\theta}) := \frac{1}{\varphi(z_0)}\varphi(z_0 e^{\mathrm{i}\theta}) \tag{4.7}$$

satisfies

$$\operatorname{Re} u_{\hat{\varphi}}(\cdot, 1) = \log\left|\frac{\hat{\varphi}(\cdot) - \hat{\varphi}(1)}{\cdot - 1}\right| \in H^{1/2}(\mathbb{S}^1). \tag{4.8}$$

Hence, the asymmetry in Theorem 1.2 can be seen to be associated with our choice of stabilizers of 1 as the representatives of $\mathbb{S}^1 \backslash \mathrm{RM}(\mathbb{S}^1)$. We record a symmetric version of Theorem 1.2 without the "special" point 1.

Corollary 4.12. *For $\kappa \in (0,4)$, sample ψ_κ from $\mathrm{SLE}_\kappa^{\mathrm{weld}}$ and z_0 from a probability measure that is mutually absolutely continuous with respect to the arc-length measure on the unit circle. Define $\hat{\psi}_\kappa(\cdot) := z_0\psi_\kappa(\cdot)$. Then, for any $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, the laws of $\hat{\psi}_\kappa \circ \varphi$ and $\varphi^{-1} \circ \hat{\psi}_\kappa$ are mutually absolutely continuous with respect to that of $\hat{\psi}_\kappa$.*

The homeomorphism ϕ_h^γ giving the normalized GMC measures $\widehat{\mathcal{M}}_h^\gamma$ of intervals was first introduced in [AJKS11], which showed using analytic methods that if h_1 and h_2 are independent LGFs on $\mathbb{S}^1$ and $\gamma_1, \gamma_2 \geq 0$ are sufficiently small, then $(\phi_{h_1}^{\gamma_1}) \circ (\phi_{h_2}^{\gamma_2})^{-1}$ is almost surely a welding. (Let ϕ_h^γ be the identity map if $\gamma = 0$.) The works [BK23] or [KMS23b] showed that $(\phi_{h_1}^{\gamma_1})^{-1} \circ (\phi_{h_2}^{\gamma_2})$ is almost surely a welding as well for sufficiently small γ_1, γ_2. A notable feature of these works is that the constants γ_1 and γ_2 can be distinct; however, the law of the Jordan curves that solve the welding problem for these homeomorphisms were not identified. Our analysis leads to the following quasi-invariance results for these random homeomorphisms.

Corollary 4.13. *Let $\varphi, \tilde{\varphi} \in \mathrm{Homeo}_+(\mathbb{S}^1)$ fix 1. For independent LGFs $h, \tilde{h}$ on $\mathbb{S}^1$ and $\gamma, \tilde{\gamma} \in (0, 2]$, we have:*

- *The law of $\phi_h^\gamma \circ \varphi$ is mutually absolutely continuous with respect to the law of ϕ_h^γ if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$.*
- *The law of $\varphi \circ (\phi_h^\gamma)^{-1}$ is mutually absolutely continuous with respect to the law of $(\phi_h^\gamma)^{-1}$ if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$.*
- *The law of $\varphi \circ (\phi_h^\gamma)^{-1} \circ \phi_{\tilde{h}}^{\tilde{\gamma}} \circ \tilde{\varphi}$ is mutually absolutely continuous with respect to the law of $(\phi_h^\gamma)^{-1} \circ \phi_{\tilde{h}}^{\tilde{\gamma}}$ if φ, $\tilde{\varphi} \in \mathrm{WP}(\mathbb{S}^1)$.*

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