On simple singularities and Weyl monodromy actions in mixed characteristic - Draft 11.22.23

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Abstract

We study the ramification on the cohomology of a smooth proper surface $X$ in mixed characteristic, in the particular case where $X$ degenerates to a surface over $\mathbb{F}_p$ with rational double point singularities. We find that the associated monodromy action of inertia depends on a formal affine neighborhood of the singularity, and under sufficient restrictions on characteristic $p$, it is tamely ramified and generated by a distinguished element of an appropriate Weyl group related to the singularity. Along the way we extend to mixed characteristic some results of Brieskorn and Slodowy concerning simultaneous resolutions of surface singularities. We also compare our Weyl group actions to certain Springer representations constructed by Borho and MacPherson, via the notion of relative perversity as developed by Hansen and Scholze.

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1. Introduction

Let $K$ be a complete discrete valuation field with valuation ring $\mathcal{O}_K$ of mixed characteristic $(0, p)$. The celebrated Néron-Ogg-Shafarevich theorem, first proven for elliptic curves and then generalized to abelian varieties by Serre–Tate ([ST68]), states that an abelian variety $X$ over $K$ has good reduction, meaning that there exists a smooth proper model $\mathcal{X} \to \text{Spec}(\mathcal{O}_K)$ with generic fiber $X$, if and only if the natural action of $G_K = \text{Gal}(\overline{K}/K)$ on the étale cohomology group $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ for $\ell \neq p$ is unramified, i.e. it restricts to a trivial action of inertia $I_K$. Here $I_K$ is defined as the kernel of the surjection $G_K \to G_k$, where $G_k$ is the absolute Galois group of the residue field $k$.

1.1 Good reduction beyond abelian varieties. For a general smooth proper variety $X$ over $K$ there is a subtler relationship between good reduction and ramification, i.e. the nontriviality of the monodromy action of $I_K$ on $H^n(X_{\overline{K}}, \mathbb{Q}_\ell)$. A necessary condition for good reduction is that $H^n(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for all $n$ and $\ell \neq p$, however the converse fails even for $X$ a smooth curve of genus $g \geq 2$. Oda has established that if $X$ is a smooth proper curve of genus $g \geq 2$, it admits good reduction if and only if a certain $G_K$-action on the pro-$\ell$ completion of the geometric étale fundamental group $\pi_1^{\text{ét}}(X_{\overline{K}})$ is unramified ([Oda95]). More recently, Liedtke–Matsumoto have shown that a K3 surface $X$ with unramified $H^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ admits good reduction after a finite unramified base extension, under the stricter assumption that $X$ admits a potentially semistable model ([LM18]).

1.2 Bad reduction of surfaces and singularities. In this article, we will consider degenerations (models) of a smooth proper surface $X/K$. More specifically, we focus on Galois representation $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ and relate the types of singularities that may appear on such degenerations to the ramification of $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$. Any smooth surface $X$ admits an integral proper flat model $\mathcal{X} \to \text{Spec}(\mathcal{O}_K)$: by Nagata’s compactification theorem ([Sta18 Tag0F3T]) there exists a proper scheme $\mathcal{X} \to \text{Spec}(\mathcal{O}_K)$ with an open immersion
$X \hookrightarrow \mathcal{X}$, and up to normalizing $\mathcal{X}$ and taking the closure of $X$ in $\mathcal{X}$ we get that $\mathcal{X}$ is reduced and dominates $\text{Spec}(\mathcal{O}_K)$, so $\mathcal{X}$ is integral, proper and flat over $\text{Spec}(\mathcal{O}_K)$.

One can then ask how ‘badly’ singular the special fiber $\mathcal{X}_k$ is. When $\mathcal{X}$ is (strictly) semistable, i.e. $\mathcal{X}_k$ is a simple normal-crossings divisor in $\mathcal{X}$, Rapoport-Zink have shown ([RZ82]) that the nearby cycles spectral sequence abutting to $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ gives rise to a weight filtration on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$. Since the weight-monodromy conjecture holds for surfaces in mixed characteristic ([loc. cit., Satz 2.13]), the weight filtration coincides up to shift with the monodromy filtration induced from the $I_K$-action. This allows one to determine the full monodromy action on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ by computing the $I_K$-action on the graded pieces of the weight filtration. These graded pieces are related to the cohomology of special fiber $\mathcal{X}_k$, which admits a nice combinatorial description by the snc property.

For general semistable schemes, a cornerstore theorem concerning their ramification is the unipotency of the $I_K$-action ([SGA 7 I]).

In a somewhat orthogonal direction, one can suppose instead that the proper smooth surface $X$ admits an integral model $\mathcal{X}$ with $\mathcal{X}_k$ having isolated singularities of a certain kind. Typically $\mathcal{X}$ will not be semistable, and so the monodromy action of $I_K$ on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is at worst quasi-unipotent ([SGA 7 I], Exposé I). One can ask whether $X$ admits a potentially semistable or even smooth model, and how the monodromy depends on the type of singularities of $\mathcal{X}_k$. The interest in surface singularities over $k \simeq \mathbb{F}_p$ here is twofold. Firstly, there has been recent progress on our understanding of positive and mixed characteristic singularities; see [MS21] for a ring-theoretic approach using perfectoid techniques. Secondly, the relationship with monodromy actions has not been explored a lot beyond varieties acquiring ordinary double points, which have been classically studied in [SGA 7 II].

A recent result of D. Kim ([Kim20]) investigates the monodromy action on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ related to an integral model $\mathcal{X}$ acquiring ordinary double points on the special fiber. Via an explicit calculation of a suitable semistable model of $X$ and the Rapoport-Zink spectral sequence of [RZ82], Kim showed that the monodromy action factors through $\text{Gal}(L/K) \simeq \mathbb{Z}/2$, the Galois group of the unique ramified quadratic extension $L$ of $K$. Thus $I_K$ acts trivially on $H^2(\overline{X}_K, \mathbb{Q}_\ell)$ or through a nontrivial quadratic character, dependent on a formal affine neighborhood of the singularity in $\mathcal{X}$.

### 1.3 Overview of our work.

In this paper, we establish a generalization of [Kim20] to a natural class of surface singularities over $\overline{k} = \overline{\mathbb{F}}_p$, that of rational double points. Such singularities have the benefit of being amenable to mixed-characteristic extensions, while still being ‘mildly’ singular with easily computable minimal resolutions. Instead of finding a semistable model of $X$, we investigate the possible formal affine neighborhoods of the singularities of $\mathcal{X}_k$ in $\mathcal{X}$ via their explicit miniversal deformations. These miniversal deformation equations are determined by $\mathcal{X}_K$, so throughout this paper (unless stated otherwise) we assume $K = \overline{K}$ is the completion of its maximal unramified extension. In particular the special fiber $\mathcal{X}_k$ and its singularity live over $\overline{k} = k$.

We adapt classical results of Tjurina, Brieskorn and Slodowy ([Bri70], [Slo80b], [Tyu70]) regarding so-called simultaneous resolutions of singularities to the mixed characteristic
setting, yielding that $X$ admits a smooth model $\tilde{X}$ after a finite base-change $L/K$. Results of Artin ([Art74]) show that $\tilde{X}$ exists at worst in the category of algebraic spaces, and its fibes are (algebraic) surfaces. Dependent on a restriction on the characteristic $p$ (see Definition 2.3) and on a formal affine neighborhood of each singularity, we can make the monodromy $I_K$-action precise:

**Theorem 1.1.** Suppose $(K, \mathcal{O}_K, k)$ is the data of a complete DVR of mixed characteristic $(0, p)$ with $p$ sufficiently good, and let $X/K$ be a smooth proper surface with an integral model $\mathcal{X}$ over $\mathcal{O}_K$ so that $\mathcal{X}_k$ has a unique rational double point. Let $E$ be the exceptional divisor of the minimal resolution of $\mathcal{X}_k$ with dual graph isomorphic to a Dynkin diagram of type $A_n, D_n$ or $E_n$. If $W$ is the Weyl group of the Dynkin diagram, then

1. The monodromy $I_K$-action on $H^2_{\text{ét}}(X_K, \mathbb{Q}_\ell)$ factors through a cyclic subgroup $\langle w \rangle$ of $W$, dependent up to conjugacy on a formal affine neighborhood of the singularity.

2. The Weyl element $w$ acts on $H^2_{\text{ét}}(X_K, \mathbb{Q}_\ell)$ via a Springer $W$-representation, and $X$ achieves good reduction after a ramified base-change of degree $\text{ord}(w)$.

3. In the case of $A_n$-singularities, for every Weyl conjugacy class there exists an element $w$ in the class and a model $\mathcal{X}$ degenerating to $A_n$-singular surface $\mathcal{X}_k$ so that $w$ acts as the monodromy operator on $H^2_{\text{ét}}(\mathcal{X}_K, \mathbb{Q}_\ell)$.

In particular we recover the results of [Kim20], which in this terminology deal with $A_1$-singularities (ordinary double points). The main novelty here is to bridge the gap between the characteristic zero and characteristic $p$ cases of monodromy actions on cohomology, since for big enough $p$ the classification of rational double points over $\mathbb{C}$ and $\mathbb{F}_p$ is the same. In particular we find that, if the action of monodromy is tame, then the monodromy operator acts in the same way as the complex monodromy operator for such singularities. As far as we know, there has not been a thorough investigation of the relationship between conjugacy classes of Weyl groups and degenerations to (rational double point or otherwise) singularities, even over $\mathbb{C}$. We hope to further explore this relationship in the future.

**1.4 Outline of proof.** The main tools for the proof of Theorem 1.1 are a theorem of Berkovich ([Ber96]) and the Grothendieck–Springer resolution. Via Berkovich’s argument we may relate the nearby cycles of special fiber $\mathcal{X}_k$ to the “formal” nearby cycles of the completion of $\mathcal{X}_k$ at the singularity, and show that the monodromy action on $H^2(\mathcal{X}_K, \mathbb{Q}_\ell)$ depends on a formal neighborhood of the singularity. We may then ‘embed’ the local picture into the miniversal deformation of the singularity and use a mixed-characteristic incarnation of the Grothendieck-Springer resolution.

Over $\mathbb{C}$, the Grothendieck–Springer resolution (or Grothendieck alteration) $\pi : \tilde{g} \to g$ furnishes a connection between simple Lie algebras $g$ and simple surface singularities, which are exactly the (complex) RDPs. This connection was studied by Brieskorn in the ’70s, following a conjecture of Grothendieck ([Bri70]), and full details were written up in ([Slo80b]). The Grothendieck alteration may be thought of as an enhancement of simultaneous resolutions of surface singularities on the geometric side, and as a generalization of
the Springer resolution $\widetilde{N} \to N$ on the representation-theoretic side. Concretely, one may realize RDPs as generic points in the subregular nilpotent orbit of the nilpotent cone $N$ of $\mathfrak{g}$. In turn, $\mathfrak{g}$ is connected to the singularity via its Dynkin diagram, which is isomorphic to the dual graph of the exceptional divisor in the minimal resolution of the singularity. These considerations still make sense for Chevalley algebras over $\text{Spec}(\mathcal{O}_K)$.

After a ramified base-change via the Weyl cover, we may simultaneously resolve all singularities appearing on the nilpotent cone, and in particular we obtain a resolution of the singularity in our model $\mathcal{X}$ by pulling back along an appropriate base-change on $\text{Spec}(\mathcal{O}_K)$. Using a recent notion of relative perversity from Hansen–Scholze ([HS21]), we may describe the associated monodromy $W$-action of the Weyl cover as an action on the relatively perverse sheaf $R\pi_*\mathbb{Q}_\ell[\dim \mathfrak{g}]$, which ends up being the Springer $W$-action as constructed by Borho–MacPherson ([BM83]). Along the way we also derive an $\ell$-adic instance of the Springer correspondence and view Springer theory as the study of nearby cycles over a larger mixed-characteristic base. This gives a $p$-adic picture analogous to the interaction between Springer theory and nearby cycles in the complex setting. The upshot is that, in sufficiently large characteristic $p$, we may describe explicitly the resulting ramification in the cohomology of $\mathcal{X}$ by factoring the $I_K$-action through a restriction of the Springer $W$-representation associated with the singularity of $\mathcal{X}_k$. In the case of $A_n$-singularities we can furthermore check by hand that the monodromy $I_K$-action can lie in any Weyl conjugacy class.

### 1.5 Connections with other work.

Our results are parallel to results of Shepherd-Barron ([SB01], [SB21]), who extended the Grothendieck-Springer resolution and some related results of Brieskorn to “good” characteristic, via a different method and in the context of groups instead of Lie algebras. Shepherd-Barron also extended arguments of Artin ([Art74]) regarding simultaneous resolutions to show that one also gets so-called Weyl covers for Brieskorn’s resolutions in all characteristics. In small characteristic, however, he notes that “one does not have a formula for the action of any reflection of $W$” on the relevant cohomology groups ([SE21], Introduction). Our approach is instead a natural extension of the methods described in ([Slo80b], [Slo80a]), and at the cost of restricting the characteristic we may describe what the corresponding $W$-action must be.

### 1.6 Organization of paper.

Section 2 contains some background on rational double points, simultaneous resolutions and (miniversal) deformations of isolated singularities in the mixed-characteristic setting. Section 3 covers the necessary Lie-theoretic notions including the description of the nilpotent cone as a fiber of the adjoint quotient and the Grothendieck-Springer resolution, in the context of Chevalley algebras over a mixed-characteristic DVR. Section 4 extends results of Slodowy ([Slo80b]) regarding the construction of suitable transverse slices to nilpotent orbits in Lie algebras. Finally Section 5 presents the main argument, involving tools from the study of perverse sheaves and nearby cycles to determine our desired monodromy action in terms of certain Springer representations of the Weyl group.
1.7 Notations and conventions. $\mathcal{O}_K$ will denote the complete mixed-characteristic DVR $\mathcal{W}(\mathbb{F}_p)$, whose residue field is $\mathbb{F}_p$ (of characteristic $p > 0$) and whose fraction field is $\mathcal{W}(\mathbb{F}_p)$. When appropriate, we denote $\mathcal{O}_K$ by $\mathcal{W}(k)$, with uniformizer $\pi$, residue field $k$ and fraction field $K$. The corresponding (strictly) henselian trait is $\mathcal{S} = \text{Spec}(\mathcal{O}_K)$ with generic fiber $\eta = \text{Spec}(K)$ and special fiber $s = \text{Spec}(k)$. In the other direction, $K$ may be written as $k(\eta)$ and $k$ may be written as $k(s)$. Geometric fibers are denoted with a bar, e.g. $\overline{\eta}$ or $\overline{s}$, and base-changes under generic and special fibers are denoted with a subscript, e.g. $g_\eta$ or $g_s$. $X//G$ means the categorical quotient of $X$ by an action of $G$ on $X$; when $X$ is affine, $X//G = \text{Spec}(X^G)$ is the GIT quotient.

The ring of dual numbers is denoted $k[[\varepsilon]] = k[[x]]/(x^2)$. Cox($g$) denotes the Coxeter number of the Lie or Chevalley algebra $g$. All cohomologies $H^i_{\text{ét}}(X, -)$ will be regarded as étale unless otherwise noted (and we will drop the “ét” notation).

Given a complete discrete valuation field $K$, we denote by $K^{\text{unr}}$, resp. $K^{\text{tr}}$ its maximal unramified, resp. tamely ramified extension in the choice of a separable closure $\overline{K}$. When no confusion arises we denote an algebraic closure of $K$ also by $\overline{K}$. The inertia subgroup $I \subset \text{Gal}(\overline{K}/K^{\text{unr}})$ is defined as $\text{Gal}(\overline{K}/K^{\text{tr}})$.

2. Rational double points and their miniversal deformations

This section gives some relevant background on rational double points and simultaneous resolutions. Sections 2.3-2.5 describe explicitly the miniversal deformations of these singularities. While the results may be known to the experts, the associated deformation problems here have a mixed-characteristic base $W(k)$ instead of $k$, so for completeness we develop the mixed characteristic case here.

2.1 Rational double points. Let $X$ be an algebraic surface over an algebraically closed field $k$. A rational singularity $x \in X(k)$ is a normal singularity (i.e. $\mathcal{O}_{X,x}$ normal) for which there exists a resolution $f : \tilde{X} \to X$ satisfying $\bigcap f_*\mathcal{O}_{\tilde{X}} = 0$ for $i \geq 1$.

By Zariski’s Main Theorem, the reduced exceptional divisor $E = f^{-1}(x)_{\text{red}}$ is a union of smooth rational curves $E_i$, elucidating the term ‘rational singularity’ (see [Bad01], Lemma 3.8). Rational double points are a particular class of rational singularities pinning down the self-intersections of exceptional divisors $E_i$:

**Definition 2.1.** A normal surface singularity $(X, x)$ is a rational double point (henceforth RDP) if its minimal resolution $f : \tilde{X} \to X$ has reduced exceptional divisor $E = \bigcup E_i$ so that all $E_i$ are smooth rational curves with self-intersection $E_i^2 = -2$.

There are various equivalent characterizations of RDPs; for example, they are surface singularities $(X, x)$ whose Zariski tangent space $m_x/m_x^2$ has dimension 3 and $\mathcal{O}_{X,x}$ has multiplicity 2 (hence the term ‘double points’). The tangent space dimension here implies RDPs are regularly embedded in codimension 1 and are hence hypersurface singularities. Since smooth points have local rings of multiplicity 1 and tangent spaces of dimension
2, this characterization makes apparent that RDPs are the ‘mildest’ surface singularities one can ask for.

Another property RDPs enjoy is that they are absolutely isolated, meaning they can be resolved after a finite number of blowups at points; in fact, each blowup yields an RDP of different type until they are all resolved (see [Sta18, Tag 0BGB]).

A crucial fact about RDPs is that in appropriate characteristic (see Definition 2.3 below) they are taut, i.e. completely determined up to isomorphism by the dual graph of their minimal resolution. This is classical over \( \mathbb{C} \), and has been extended to positive characteristic by [Art77]:

**Theorem 2.2.** Let \( X \) be a projective surface over \( k \).

(i) ([Bad01], Thm. 3.32, [Art66]) If \( E \) is a connected curve on \( X \) with smooth rational curve components \( E_i \) so that \( E_i^2 = -2 \), then the only possible dual graphs for \( E \) are the Dynkin diagrams \( A_n, D_n, E_6, E_7 \) and \( E_8 \).

(ii) ([Art77]) If \( x \in X(k) \) is an RDP and \( \text{char}(k) \) is very good (see Definition 2.3) then up to analytic isomorphism \( \hat{O}_{X,x} \cong k[[x,y,z]]/(f(x,y,z)) \), where \( f(x,y,z) \) and the dual graph of the minimal resolution of \( x \) are given by the table below:

<table>
<thead>
<tr>
<th>( f(x,y,z) )</th>
<th>Dual graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z^2 + x^2 + y^{n+1} )</td>
<td>( A_n )</td>
</tr>
<tr>
<td>( z^2 + x^2y + y^{n-1} )</td>
<td>( D_n )</td>
</tr>
<tr>
<td>( z^2 + x^3 + y^4 )</td>
<td>( E_6 )</td>
</tr>
<tr>
<td>( z^2 + x^3 + xy^3 )</td>
<td>( E_7 )</td>
</tr>
<tr>
<td>( z^2 + x^3 + y^5 )</td>
<td>( E_8 )</td>
</tr>
</tbody>
</table>

A polynomial \( f(x,y,z) \) defining an RDP is usually called a ‘normal form’. Since the notion of ‘good characteristic’ will be ubiquitous in this paper, we record the definition here.

**Definition 2.3.** Given a semisimple simply-laced Lie algebra \( g \) over \( k \) of characteristic \( p > 0 \), so that its Dynkin diagram has components \( A_n, D_n \) or \( E_n \), we say \( p \) is

(a) **good** with respect to \( g \) if \( p \neq 2 \) if \( g \) has any \( D_n \) components, \( p \neq 2, 3 \) if \( g \) has any \( E_6, E_7 \) components and \( p \neq 2, 3, 5 \) if \( g \) has any \( E_8 \) components (note there are no \( p \) restrictions for \( A_n \) components).

(b) **very good** with respect to \( g \) if it is good and for any \( A_n \) component we have \( p \nmid n + 1 \)

(c) **sufficiently good** with respect to \( g \) if \( p \nmid |W(g)| \), i.e. it does not divide the order of the associated Weyl group.

For simple Lie algebras it is automatic that sufficiently good \( \Rightarrow \) very good \( \Rightarrow \) good. We will revisit this definition in Remark 3.3 for a more intuitive explanation of these restrictions.
Theorem 2.2 identifies RDPs with the corresponding simple simply-laced Lie algebras of the specified Dynkin type, so we can refer to \( p \) being (very) good for the singularity, the Lie algebra or the Dynkin diagram interchangeably. Theorem 2.2 is no longer true when \( p \) is not good, and there is more than one equation describing an RDP with the same dual graph; see ([Art77]) for details.

Remark 2.4. We have the following byproduct of the proof of Theorem 2.2. We may resolve RDP \( x \in X(k) \) by iterated blowups along points, and at each step we get an RDP of different Dynkin type. All the Dynkin diagrams corresponding to each RDP appearing in the resolution process are subdiagrams of the Dynkin diagram of \( x \). Conversely, all subdiagrams of the Dynkin diagram of \( x \) correspond to these ‘intermediate’ RDPs during the iterated blowup process.

### 2.2 Simultaneous resolutions

We next describe a particularly rare notion of resolving singularities of schemes in families. In order to be consistent with [Art74], we enlarge the category of schemes to include separated algebraic spaces, though in practice we will only resolve schemes.

**Definition 2.5.** Let \( f : X \rightarrow S \) be a finite-type morphism of separated algebraic spaces. A *simultaneous resolution* is a commutative diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi} & X \\
\downarrow{\tilde{f}} & & \downarrow{f} \\
\tilde{S} & \xrightarrow{\psi} & S
\end{array}
\]

where \( \tilde{f} \) is smooth, \( \pi \) is proper, \( \psi \) is a finite surjection and for all geometric points \( \tilde{s} \rightarrow \tilde{S} \) with image \( s = \psi(\tilde{s}) \rightarrow S \), the induced morphism on the fibers

\[
\pi_s : \tilde{X}_{\tilde{s}} \rightarrow X_s \times_S \tilde{S}
\]

is a resolution of the singularities of \( X_{s,\text{red}} \). If \( S = \text{Spec}(k) \) is a point then we recover the usual notion of a resolution \( \pi : \tilde{X} \rightarrow X \) of algebraic spaces over \( k \) ([Sta18, Tag 0BHV]).

**Remark 2.6.** One generally needs to impose some assumptions on the fibers, e.g. geometrically reduced and excellent, in order for resolutions of singularities to exist in the first place. There are also explicit examples of projective maps \( \tilde{X} \rightarrow S \) whose simultaneous resolution \( \tilde{X} \) is not a scheme (see Remark 2.12), so in general one needs to consider algebraic spaces. In practice, however, \( f \) will be a morphism of henselianized (localized) schemes and the schematic fibers will be equipped with the reduced-induced subscheme structure, so we will not need to mention the above assumptions.

**Example 2.7.** We expound on why simultaneous resolutions rarely exist with the following examples. Assume \( f : X \rightarrow S \) is flat.

(i) If \( S \) is a smooth curve with closed point \( s \) so that \( X_s \) is a reduced singular curve then a simultaneous resolution does not exist; any resolution will introduce an exceptional locus \( E \) in \( \tilde{X} \) and \( E \cap \tilde{X}_s \) is a singular set in \( \tilde{X}_s \).
(ii) If $S$ is a smooth curve with closed point $s$ so that $X_s$ has dimension $\geq 3$ and is a complete intersection with (at worst) isolated singularities, then $X$ is $\mathbb{Q}$-factorial ([SGA 2], XI.3.13.(ii)) and hence it is well-known that $X$ admits no small resolutions.

Therefore the exceptional locus will be a divisor and $\tilde{X}_s$ cannot be smooth as in the previous example.

There are more examples of obstructions to constructing simultaneous resolutions and it is difficult to come up with sufficiency conditions for their existence. In light of this, it is a surprising theorem that, after a ramified base-change, simultaneous resolutions do exist for families of surfaces acquiring RDP singularities. The following was independently discovered by Brieskorn and Tjurina, then generalized by Brieskorn in the complex setting and by Artin in the algebraic setting:

**Theorem 2.8** ([Bri66], [Tyu70], [Art74]). Let $f: \mathcal{X} \rightarrow S$ be a flat morphism of schemes and $s$ a closed point of $S$ so that $X_s$ is a surface with a unique RDP $x \in X_s$. Let $\tilde{f}: \mathcal{X}(x) \rightarrow S(s)$ be the induced map of the associated henselianized schemes $\mathcal{X}(x)$ and $S(s)$. Then there exists a finite surjection $\psi: \tilde{S} \rightarrow S(s)$ of henselian schemes, branched over a Cartier divisor $\Delta \subset S(s)$, so that $\tilde{f}$ admits a simultaneous resolution $\tilde{f}: \tilde{X} \rightarrow \tilde{S}$ fitting into the following diagram:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi_1} & X(x) \times_S \tilde{S} \\
\downarrow \tilde{f} & & \downarrow \pi_2 \\
\tilde{S} & \xrightarrow{\psi} & S(s)
\end{array}
\]

Here $\tilde{f}$ is smooth and $\pi_1, \pi_2$ form the Stein factorization of proper map $\pi = \pi_2 \circ \pi_1$. Cartier divisor $\Delta$ is called the discriminant divisor (or the ramification locus) of map $\tilde{f}$.

**Remark 2.9.** In the complex-analytic category one can replace the henselizations with $\tilde{f}$ being a map of germs of singularities $(\mathcal{X}, x) \rightarrow (S, s)$.

**Remark 2.10.** Suppose $\mathcal{X}, S$ are complex schemes and $S$ is affine. Brieskorn’s theorems show that the Galois group of the finite cover $\tilde{S} \rightarrow S$ is the Weyl group $W$ of the Dynkin diagram associated to the RDP in $\mathcal{X}_s$ (cf. Theorem 2.2). Moreover, the pullback $\psi^* \Delta$ of the discriminant divisor is a hyperplane arrangement in affine space $\tilde{S}$, determined up to sign by the root system of the Dynkin diagram. These results do not follow from the algebraic methods of Artin ([Art74]), but later results of Shepherd-Barron ([SB21]) showed that a “suitable polarization” of Artin’s simultaneous resolution functor $\text{Res}_{X/S}$ yields that $\tilde{S} \rightarrow S$ is a so-called Weyl cover in the algebraic setting too.

**Example 2.11.** Let $S = \text{Spec}(O_K)$ be a strictly henselian trait with residue characteristic $p \neq 2$, uniformizer $\pi$, closed point $s = \text{Spec}(k)$ and generic point $\eta = \text{Spec}(K)$. Let $\mathcal{X} = V(x^2 + y^2 + z^2 - \pi^N) \subseteq \mathbb{A}^3_S$ ($N \geq 1$) be a threefold, flat over $S$, with singular fiber $\mathcal{X}_s$ over $k$; note the special fiber $\mathcal{X}_s = V(x^2 + y^2 + z^2) \subseteq \mathbb{A}^3_k$ has an ordinary double point at the origin.
Suppose $N = 2n$ is even. Blowing up $\mathcal{X}$ along ideals $(x, z \pm \pi^n)$ gives two different small resolutions $\mathcal{X}_- \to \mathcal{X}$; smallness here is a consequence of both ideals defining non-Cartier Weil divisors on $\mathcal{X}$. Both $\mathcal{X}_-, \mathcal{X}_+$ are smooth over $S$ with generic fibers $(\mathcal{X}_\eta)_\eta \simeq \mathcal{X}_\eta$ over $K$. Viewing $\mathcal{X}$ as a hyperplane section $V(\pi^N) \cap \mathcal{X}$ and setting $u = z - \pi^n$, the universal property of blowing up gives

$$(\mathcal{X}_+)_s \simeq B\ell_{(x,u)}(\mathcal{X}) \times_{\mathcal{X}} V(x^2 + y^2 + u^2) = B\ell_{(x,y,u)} V(x^2 + y^2 + u^2)$$

i.e. $(\mathcal{X}_+)_s \to \mathcal{X}_s$ is the minimal resolution of $\mathcal{X}_s$, red, since a single blowup resolves the ordinary double point. In this situation there exists a birational map $f : \mathcal{X}_- \to \mathcal{X}_+$ induced by $(x, z - \pi^n) \to (x, z + \pi^n)$; it is known as the Atiyah flop.

Suppose now $N = 2n + 1$ is odd. In this case $\mathcal{X}$ does not admit a simultaneous resolution over $S$ (e.g. when $N = 1$, the obstruction is the smoothness of total space $\mathcal{X}$). To repeat the arguments of the previous paragraph we pass to the unique ramified quadratic extension $L = K(\sqrt{\pi})$ of $K$ so that $\mathcal{X}_L = V(x^2 + y^2 + z^2 + \pi^2 N)$ now admits small resolutions by blowing up along ideals $(x, z \pm \pi^N)$. We thus have a simultaneous resolution after base-change $\text{Spec}(\mathcal{O}_L) \to \text{Spec}(\mathcal{O}_K)$, ramified over the closed point $s$.

Remark 2.12. Simultaneous resolutions need not exist in the category of schemes when one considers more ‘global’ contexts of simultaneously resolving projective families. As Artin’s example in ([Art74], p. 330) shows, if $\mathcal{X} \to \text{Spec}(\mathcal{O}_K)$ is a projective family so that the generic fiber $\mathcal{X}_{\eta}$ is a quartic K3 surface of geometric Picard rank 1 (such K3 surfaces exist; see [vL07], Thm. 3.1), and the special fiber $\mathcal{X}_s$ a nodal quartic, then localizing at the node we obtain a situation like that of Example 2.11 and neither $\mathcal{X}_-$ nor $\mathcal{X}_+$ are schemes. By ([Art74], Thm. 1) however, simultaneous resolutions of surfaces will be at worst algebraic spaces whose fibers are schemes, since this is true for any smooth 2-dimensional algebraic space.

A second important point is that simultaneous resolutions are generally non-unique, e.g. Example 2.11 yields two non-isomorphic resolutions $\mathcal{X}_-, \mathcal{X}_+$ related by a flop $\mathcal{X}_- \to \mathcal{X}_+$.

2.3 Formal deformations of singularities. In what follows, let $S = \text{Spec}(\mathcal{O}_K)$ be a complete DVR with algebraically closed residue field $k$ and uniformizer $\pi$. Let $\mathcal{X} \to S$ be a flat proper surface with special fiber $\mathcal{X}_k$ containing a unique RDP $x$. We may choose affine coordinates so that the local ring of the singularity has the form

$$R_0 = \mathcal{O}_{\mathcal{X}_k,x} \simeq k[[x, y, z]] / f(x, y, z)$$

with $f(x, y, z) = 0$ the normal form of the rational double point (2.2). We aim to describe the possibilities of what $\widehat{\mathcal{O}}_{\mathcal{X}_k,x}$, the completed local ring of the model $\mathcal{X}$ at point $x$ (viewed as a point in $\mathcal{X}$) can look like, by interpreting $\widehat{\mathcal{O}}_{\mathcal{X}_k,x}$ as an appropriate deformation of the singularity. We next clarify the notion of deformations we will use.

Notation 2.13. Write $V_0 = \text{Spec}(R_0)$ for the affine scheme of the singularity, and $\text{Art}_k$ resp. $\widehat{\text{Art}}_k$ for the category of artinian local, resp. complete noetherian local rings with
residue field $k$. Both types of rings become canonically $W(k)$-algebras via the unique lift of the natural surjection $W(k) \to k$. For $R$ in $\text{Art}_k$ we set
\[ R_n = R/m_R^{n+1}, \quad \overline{R} = R/\pi R, \quad \overline{R}_n = R/(\pi, m_R^{n+1}) \simeq \overline{R}/m_{\overline{R}}^{n+1} \]
so that $R_n$ is in $\text{Art}_k$ and $\overline{R}$ resp. $\overline{R}_n$ are complete noetherian local, resp. artinian local $k$-algebras; note $R \simeq \varprojlim R_n$ and $\overline{R} \simeq \varprojlim \overline{R}_n$. We will also call
\[ T_m(R) = m_R/m^2_R, \quad T_m(R)_{\text{red}} = m_R/(\pi, m^2_R) \simeq m_{\overline{R}}/m^2_{\overline{R}} \]
the cotangent, resp. reduced cotangent spaces of $R \in \{\text{Art}_k, \text{Art}_k\}$; they are both naturally $k$-vector spaces. To $V_0$ we associate two deformation functors
\[ \text{Def}_{V_0} : \text{Art}_k \to \text{Set}, \quad \widehat{\text{Def}}_{V_0} : \text{Art}_k \to \text{Set} \]
\[ R \in \{\text{Art}_k, \text{Art}_k\} \mapsto \{\text{isoclasses of deformations } V \to \text{Spec}(R)\} \]
The tangent space $\text{Def}_{V_0}(k[z])$ parametrizes so-called 1st-order deformations of $V_0$ and has a canonical $k$-vector space structure. For $R$ in $\widehat{\text{Art}}_k$ we will call a deformation $V \to \text{Spf}(R)$ of $V_0$ formal, with the understanding that it arises as an inverse limit of deformations of $V_0$ over artinian local rings. We give a preliminary lemma on what such deformations can look like.

**Lemma 2.14** ([Har10], Thm. 9.2). Let $V_0/k$ be an affine hypersurface singularity defined by polynomial $f(x, y, z)$.

(i) For any small extension $\phi : R_{i+1} \to R_i$ in $\text{Art}_k$ and deformation $V_i = \text{Spec}(R_i[x, y, z])$ of $V_0$ we have $I = (F_i)$ principal and any flat lift of $V_i$ to a deformation $V_{i+1} \to \text{Spec}(R_{i+1})$ is of the form
\[ \text{Spec}(R_{i+1}[x, y, z]/(F_{i+1})), \quad F_{i+1} \equiv F_i \mod (\ker \phi) \]

(ii) If $R \in \widehat{\text{Art}}_k$ then any formal deformation $V \to \text{Spf}(R)$ of $V_0$ is of the form
\[ \text{Spf}(R[x, y, z]/(F)), \quad F \equiv f \mod m_R \]

**Remark 2.15.** In plainer language, Lemma 2.14 says that deformations of affine hypersurfaces are still hypersurfaces; more generally, it is true that deformations of local complete intersections are also local complete intersections.

To the deformation theory of $V_0$ we may associate the Lichtenbaum-Schlessinger functors $T^i(V_0) = \text{Ext}^i(\Omega_{R_0}, R_0)$, where $T^0(V_0)$ is just the tangent module of $R_0$. It is known ([Vis97], Prop. 6.4) that $T^1(V_0)$ parametrizes first-order deformations of $V_0$ and $T^2(V_0)$ contains an obstruction space for their liftings. By the explicit description of $V_0$ we can be more precise:
Lemma 2.16. We have \( \text{Def}_{V_0}(k[\varepsilon]) \simeq \text{Ext}^1(\Omega_{R_0}, R_0) \simeq R_0/J \) where \( J = (f_x, f_y, f_z) \) is the Jacobian ideal, and \( \text{Ext}^2(\Omega_{R_0}, R_0) = 0 \). In particular, deformations of \( V_0 \) are unobstructed.

Proof. The first isomorphism is ([Ser07], Thm. 2.4.1 (iv)). The conormal sequence associated to \( V_0 \to A^3_k \) is

\[
0 \to (f)/(f)^2 \to \Omega_{A^3_k} \otimes R_0 \to \Omega_{R_0} \to 0
\]

which is exact on the left by ([Vis97], Lemma 4.7) and so dualizing we get exact sequence

\[
0 \to \Omega_{R_0}^\vee \to (\Omega_{A^3_k} \otimes R_0)^\vee \xrightarrow{d^\vee} ((f)/(f)^2)^\vee \xrightarrow{\partial} \text{Ext}^1(\Omega_{R_0}, R_0) \to 0 \tag{2.1}
\]

with \( \partial \) surjective as \( \Omega_{A^3_k} \otimes R_0 \) is free over \( R_0 \). The map \( (f)/(f)^2 \to \Omega_{A^3_k} \otimes R_0 \) is given by the Jacobian matrix \( f \mapsto df \) and \( (f)/(f)^2 \simeq R_0 \) is free of rank 1, hence the adjoint map \( d^\vee \) has image \( J = (f_x, f_y, f_z) \). So via the boundary map \( \partial \) we get \( \text{Ext}^1(\Omega_{R_0}, R_0) \simeq R_0/J \) and, as a byproduct of exact sequence (2.1), \( \text{Ext}^2(\Omega_{R_0}, R_0) = 0 \). \( \square \)

Definition 2.17. The \( k \)-vector space

\[
T^1(V_0) = R_0/J \simeq \frac{k[[x, y, z]]}{(f, f_x, f_y, f_z)}
\]

is called the Tjurina algebra of the singularity in \( V_0 \). It has finite dimension since \( V(f, f_x, f_y, f_z) \) is supported on the unique singular point - in general, for isolated singularities, the Tjurina algebra is finite-dimensional.

2.4 The Kodaira–Spencer map. This section follows ([Vis97], §6). Given a formal deformation \( \phi : V \to \text{Spf}(R) \) of \( V_0 \), we aim to attach to it a \( k \)-linear map \( T_m(R_0)^\vee_{\text{red}} \to T^1(V_0) \) defined in the following steps.

An element of \( T_m(R_0)^\vee_{\text{red}} \) corresponds to a map \( R \to k[\varepsilon] \) which, by \( W(k) \)-linearity and \( \varepsilon^2 = 0 \), factors through a map \( R \to \overline{R}_1 \); write \( \overline{\phi}_1 \) for the base-change of \( \phi \) along this map. Set

\[
\overline{\phi}_0 : V_0 \times_k \text{Spec}(\overline{R}_1) \longrightarrow \text{Spec}(\overline{R}_1)
\]

to be the trivial deformation of \( V_0 \) over \( \overline{R}_1 \) so that \( \overline{\phi}_0, \overline{\phi}_1 \in \text{Def}_{V_0}(\overline{R}_1) \). By Lemma 2.14, both \( \overline{\phi}_0, \overline{\phi}_1 \) embed as hypersurfaces in \( A^3_{\overline{R}_1} \), defined by ideals \( I_0, I_1 \subset \overline{R}_1[[x, y, z]] \) respectively. If \( I = (f) \) is the ideal of \( V_0 \) in \( k[[x, y, z]] \), we may lift a section \( F \in I \) to sections \( F_0 \in I_0, F_1 \in I_1 \). By virtue of the square-zero extension

\[
0 \to m_{\overline{R}_1} \longrightarrow \overline{R}_1 \longrightarrow k \longrightarrow 0
\]

the difference \( F_0 - F_1 \) lies in \( m_{\overline{R}_1} \otimes_k k[[x, y, z]] \), and its image \( [F_0 - F_1] \) under \( k[[x, y, z]] \to R_0 \) depends only on the choice of \( F \). Hence \( F \mapsto [F_0 - F_1] \) yields an element

\[
\nu \in \text{Hom}_{k[[x, y, z]]}(I, m_{\overline{R}_1} \otimes_k R_0) \simeq m_{\overline{R}_1} \otimes_k \text{Hom}_{R_0}(I/I^2, R_0) = m_{\overline{R}_1} \otimes_k ((f)/(f)^2)^\vee
\]

Homomorphism \( \nu \) is well-defined in general for any two deformations of \( V_0 \) over an artinian \( R \); see ([Vis97], Prop. 2.8) for more details. We will only be using it in the following definition.
Definition 2.18. The Kodaira–Spencer map is a $k$-linear map $\text{KS}_\phi : T_m(R)_\text{red}^\vee \to T^1(V_0)$ defined as the image of homomorphism $\nu$ described above under map

$$\text{id} \otimes \partial : m_{\mathcal{F}_1} \otimes_k ((f)/(f)^2)^\vee \to m_{\mathcal{F}_1} \otimes_k \text{Ext}^1(\Omega_{R_0}, R_0) \simeq \text{Hom}_k(T_m(R)_{\text{red}}^\vee, T^1(V_0))$$

where $\partial$ is the boundary map induced from the conormal exact sequence (2.1) of $V_0$, $\to A_3^3$.

By construction, map $\nu$ depends only on $\phi$, hence the Kodaira–Spencer morphism depends only on $\phi$ - it is well-defined by ([Vis97], Prop. 4.11) and satisfies various functorial properties (loc. cit., Prop. 6.10). We note the following:

Proposition 2.19 ([Vis97], Prop. 6.11). Let $\phi : V \to \text{Spf}(R)$ be a formal deformation of $V_0$ and for $\alpha \in (m_R/m_R^2)^\vee \simeq \text{Hom}(R, k[\varepsilon])$ let $f_{\alpha} : R \to k[\varepsilon]$ be the corresponding map. Then the induced 1st-order deformation $f_{\alpha}^*\phi : V_1 = V \times_R \text{Spec}(k[\varepsilon]) \to \text{Spec}(k[\varepsilon])$ defines a conormal exact sequence via $V_0 \hookrightarrow V_1$, and $\text{KS}_\phi(\alpha) \in T^1(V_0)$ is this extension class.

2.5 Miniversal deformations in mixed characteristic. We can now describe a class of deformations of $V_0$ from which all others are induced in a certain ‘minimal’ way. Recall that if $\phi : V \to \text{Spf}(R)$ is a formal deformation of $V_0$ and $h_R = \text{Hom}(R, -) : \text{Art}_k \to \text{Set}$ is the Hom functor of $R$, then $\phi$ induces a natural transformation

$$h_R \to \widehat{\text{Def}}_{V_0}$$

Definition 2.20.

(i) $\phi$ is versal when this natural transformation is formally smooth, i.e. if $B \to A$ in $\text{Art}_k$ then the map

$$h_R(B) \to h_R(A) \times_{\text{Def}_{V_0}(A)} \widehat{\text{Def}}_{V_0}(B)$$

induced from the obvious commutative diagram is a surjection.

(ii) $\phi$ is miniversal if it is versal and the Kodaira–Spencer map $\text{KS}_\phi$ is an isomorphism.

By Lemma 2.16 and the definitions in Notation 2.13 it is clear that this definition of miniversality coincides with the more common one (see e.g. [Har10], §15). We also note that versality is usually stated in terms of artinian local rings, but one can deduce versality in the above (more general) case from the artinian one ([Vis97], Lemma 7.3).

Proposition 2.21 (Schlessinger criteria, [Har10] Thms. 16.2, 18.1). $\text{Def}_{V_0}$ admits a miniversal deformation if and only if the tangent space $\text{Def}_{V_0}(k[\varepsilon])$ is finite-dimensional. In particular there exists a miniversal deformation when $V_0$ has isolated singularities.

Proposition 2.22. If $\phi : V \to \text{Spf}(R)$ is a formal deformation of $V_0$ such that $R$ is a power series algebra over $W(k)$ and the Kodaira-Spencer map $\text{KS}_\phi$ is an isomorphism, then $\phi$ is miniversal.
Proof. Suppose $R = W(k)[t_1, \cdots, t_n]$ and $\text{KS}_\phi$ is an isomorphism. By Proposition 2.21 there exists a miniversal formal deformation $\psi : W \to \text{Spf}(S)$, yielding by versality a homomorphism $f : S \to R$ and hence a map $df : T_m(S)_{\text{red}} \to T_m(R)_{\text{red}}$, dual to the differential map of reduced tangent spaces. Since $\text{KS}_\phi, \text{KS}_\psi$ are isomorphisms and $\text{KS}_\psi = \text{KS}_\phi \circ df$ by the functoriality of the Kodaira–Spencer construction ([Vis97], Prop. 6.10 (b)), $df$ is an isomorphism.

It remains to show $f$ is an isomorphism. We will use $df$ to construct a map $g : R \to S$ so that the dual differentials of (non-reduced) cotangent spaces

$$D(f \circ g) : m_R/m_R^2 \to m_R/m_R^2, \quad D(g \circ f) : m_S/m_S^2 \to m_S/m_S^2$$

are surjective, then by functoriality of differentials ([Vis97], Lemma 7.5) $f \circ g$ and $g \circ f$ will be isomorphisms so that in particular $f$ is an isomorphism.

To construct $g$: Observe $T_m(R)_{\text{red}} \simeq k[t_1, \cdots, t_n]/(t_1, \cdots, t_n)^2$ has generators $\bar{t}_1, \cdots, \bar{t}_n$. As $df$ is an isomorphism, we can pick basis $\{\bar{g}_1, \cdots, \bar{g}_n\}$ of $m_S/(\pi, m_S^2)$ so that $df(\bar{g}_i) = \bar{t}_i$ and lift them to a set of representatives $g_i \in m_S$. On artinian quotients we define

$$g_m : R_m = W(k)[t_1, \cdots, t_n]_{(\pi, t_1, \cdots, t_n)^m} \to S_m, \quad \bar{t}_i^{(m)} \mapsto \bar{g}_i^{(m)}$$

where $\bar{t}_i^{(m)}, \bar{g}_i^{(m)}$ denote the images of $t_i, g_i$ under $R \to R_m, S \to S_m$ respectively. Since $R \simeq \varprojlim R_m, S \simeq \varprojlim S_m$ this defines a homomorphism $g = \varprojlim g_m : R \to S$ mapping $t_i \mapsto \bar{g}_i$, which by construction induces a surjection $dg : m_R/m_R^2 \to m_S/m_S^2$. In turn $dg$ induces surjections $m_R^m/m_R^{m+1} \to m_S^m/m_S^{m+1}$ and as the source and target of this map are respectively the kernels of $R_{m+1} \to R_m$ and $S_{m+1} \to S_m$, we inductively get surjections $R_m \to S_m$; the base case is $R_1 \simeq k \oplus m_R/m_R^2 \to S_1 \simeq k \oplus m_S/m_S^2$. Since $R_m = R_m/\pi R_m$, by Nakayama’s lemma we get that the lifts $g_m : R_m \to S_m$ are surjective, and since $R_m, S_m$ are artinian local rings we get $g = \varprojlim g_m$ surjective, hence the dual differential $Dg : m_R/m_R^2 \to m_S/m_S^2$ is surjective. Repeating the exact same argument for $f : S \to R$ yields a surjection $Df : m_S/m_S^2 \to m_R/m_R^2$. Thus the maps in (2.2) are surjective and we are done.

We now come to the crux of this section: describing the miniversal deformations of $V_0$.

**Proposition 2.23** (see [Vis97], Example 7.17). Let $r = \dim_k T^1(V_0), R = W(k)[t_1, \cdots, t_r]$ and choose elements $g_1, \cdots, g_r$ in $W(k)[x, y, z]$ so that $\{\bar{g}_1, \cdots, \bar{g}_r\}$ forms a basis of $T^1(V_0)$. Consider the hypersurface $V = V(F) \subset \mathbb{A}^3_R$ defined by the vanishing of power series

$$F(x, y, z, t_1, \cdots, t_r) = f(x, y, z) + \sum_{i=1}^r t_ig_i(x, y, z) \in W(k)[x, y, z, t_1, \cdots, t_r]$$

Then $V \to \text{Spec}(R)$ induces a miniversal deformation of $V_0$.

**Proof.** In view of Proposition 2.22 it suffices to show the Kodaira–Spencer map $\text{KS}_\phi$ is an isomorphism; we may set $n = r$ in the proof of Proposition 2.22 since $n$ there was
defined to be the dimension of $T_m(R)_{\text{red}} \simeq T^1(V_0)$. Let $T_m(R)_{\text{red}} = \langle \tilde{t}_1, \cdots, \tilde{t}_r \rangle \simeq m_{\mathcal{R}_i}$ for $\tilde{t}_i$ the images of indeterminates $t_i \mod (\pi, m_{\mathcal{R}_i}^2)$ and by abuse of notation denote the (dual) basis of $T_m(R)_{\text{red}}$ also by $\tilde{t}_i$, then we claim $\text{KS}_\phi(\tilde{t}_i) = \tilde{g}_i$ so that $\text{KS}_\phi$ maps a basis to a basis and hence is a $k$-linear isomorphism.

Now Proposition 2.19 says that $\tilde{t}_i$ corresponds to a map $f: \overline{R}_1 \to k[\varepsilon]$ mapping $\tilde{t}_i \mapsto \varepsilon$, $\tilde{t}_j \neq i \mapsto 0$ via duality on $k$-vector space $m_{\mathcal{R}_i}$ and $\text{KS}_\phi(\tilde{t}_i)$ is the extension class in $T^1(V_0)$ of the pullback of deformation $\phi$ via $R \to \overline{R}_1 \to k[\varepsilon]$ to the 1st-order deformation $f^* \phi$, i.e.

$$\text{Spec} \left( \frac{W(k)[x, y, z, t_1, \cdots, t_r]}{f(x, y, z) + \sum_{i=1}^r t_ig_i(x, y, z)} \otimes k[\varepsilon] \right) \simeq \text{Spec} \left( \frac{k[\varepsilon, x, y, z]}{f(x, y, z) + \varepsilon g_i(x, y, z)} \right) f^* \phi \rightarrow \text{Spec}(k[\varepsilon])$$

where $\tilde{g}_i$ is the image of $g_i$ in $k[\varepsilon, x, y, z]$. To compute this extension class we trace through the definition of the Kodaira–Spencer map: we have two first-order deformations, $\overline{\phi}_1 = f^* \phi$ and the trivial deformation $\overline{\phi}_0 : V_0 \times_k \text{Spec}(k[\varepsilon]) \to \text{Spec}(k[\varepsilon])$, with corresponding lifts of $f \in (f) \subset k[x, y, z]$ being $f + \varepsilon \tilde{g}_i$ and $f$, viewed as elements of $k[\varepsilon, x, y, z]$. Then map $\nu$ in the definition of the Kodaira–Spencer map sends $f \mapsto [f + \varepsilon \tilde{g}_i - f] = [\varepsilon \tilde{g}_i] \in (\varepsilon) \otimes_k R_0$, hence in the quotient $(\varepsilon) \otimes_k R_0 \to R_0 \to R_0/J = T^1(V_0)$ (where $J$ is the Jacobian ideal) we get exactly the class $\tilde{g}_i$. Thus $\text{KS}_\phi(\tilde{t}_i) = \tilde{g}_i$. □

**Example 2.24.** Let $R_0$ be the local ring of an $A_{n-1}$ singularity $f(x, y, z) = x^2 + z^2 + y^n$. A basis of the Tjurina algebra in good characteristic (see Definition 2.3) is given by

$$\frac{k[x, y, z]}{(f, f_x, f_y, f_z)} \simeq \frac{k[y]}{y^{n-1}} \simeq \bigoplus_{i=0}^{n-2} k \cdot y^i$$

so that $T^1(\text{Spec}(R_0))$ has dimension $r = n - 1$ and we may choose $g_i(x, y, z) = y^{i-1}$. Then

$$F(x, y, z) = x^2 + z^2 + y^n + t_{n-1}y^{n-2} + \cdots + t_2y + t_1$$

is a miniversal deformation of $\text{Spec}(R_0)$ with base $W(k)[t_1, \cdots, t_{n-1}]$.

We return to the setting in the beginning of Section 2.3, namely $X/S$ is a flat proper surface with special fiber $X_k$ containing an RDP $x \in X_k(k)$.

**Corollary 2.25.** For $\hat{O}_{X,x}$ the completed local ring of $X$ at $x \to X_k \subset X$ we have

$$\hat{O}_{X,x} \simeq \frac{W(k)[x, y, z]}{F(x, y, z)}$$

for some polynomial $F(x, y, z)$ that is the pullback of (2.3) under $W(k)[t_1, \cdots, t_r] \to W(k)$ induced by versality. Here $f(x, y, z)$ is the normal form of singularity $x$, $r$ is the dimension of the Tjurina algebra of the singularity and the $t_i$ are specialized to elements in the maximal ideal $m_{W(k)} = (\pi)$ of $W(k)$.

**Proof.** This is immediate from Proposition 2.23 once we establish (2.4) for some power series $F(x, y, z) \in W(k)[x, y, z]$, which is the content of Lemma 2.14. □
Remark 2.26. While a priori we speak of miniversal deformations as maps of formal schemes \( V \to \text{Spf}(R) \), we can regard the miniversal deformation of Proposition 2.23 as an algebraic deformation over henselian scheme \( \text{Spec}(W(k)[t_1, \ldots, t_r]) \). This is more generally due to a theorem of Elkik which states that formal deformations of affine schemes with isolated singularities are algebraic ([Elk74]). Thus, by replacing \( X \) with an affine étale neighborhood of the singularity in Corollary 2.25, we may and do consider the (miniversal or otherwise) deformations of RDPs as usual scheme morphisms over a henselian base.

3. The geometry of the Grothendieck alteration

We now collect the necessary Lie-theoretic prerequisites in order to define miniversal deformations of RDP singularities in terms the adjoint quotient of Lie algebras. The main goal is to define and study the classical Grothendieck-Springer resolution over base \( \text{Spec}(\mathcal{O}_K) \) instead of \( \mathbb{C} \) (Section 3.5). To this end, Sections 3-3.3 extend the relevant notions of nilpotent, semisimple and (sub)regular elements to the setting of Chevalley algebras over \( \mathcal{O}_K \).

3.1 Chevalley bases and Chevalley algebras. Throughout this section we work over base \( S = \text{Spec}(\mathcal{O}_K) \), where \( \mathcal{O}_K \) is a mixed-characteristic complete DVR with algebraically closed residue field \( k \) and fraction field \( K \); the corresponding closed and generic points of \( S \) are respectively \( s \) and \( \eta \).

We recall the existence of Chevalley bases. Let \( g_\eta \) be a semisimple Lie algebra over algebraically closed field \( K \) of characteristic zero, \( h \) a fixed Cartan subalgebra and \( \Phi \) the corresponding root system with a basis \( \Delta \) of simple roots. Via the Cartan decomposition \( g_\eta = h \oplus \bigoplus_{\alpha \in \Phi} g_{\alpha, \eta} \)

one may choose \( \{e_\alpha \in g_{\alpha, \eta} \mid \alpha \in \Phi\} \) forming a \( \mathbb{Z} \)-basis for each 1-dimensional space \( g_{\alpha, \eta} \)

and \( \{h_\beta \in h \mid \beta \in \Delta\} \) fundamental coroots subject to certain compatibility relations ([Hum12], §25.2). Set \( \{e_\alpha, h_\beta\} \) forms a Chevalley basis with corresponding Chevalley \( \mathbb{Z} \)-algebra

\[
\mathfrak{g}_\mathbb{Z} = \bigoplus_{\beta \in \Delta} \mathbb{Z}h_\beta \oplus \bigoplus_{\alpha \in \Phi} \mathbb{Z}e_\alpha
\]

One can also construct an associated group scheme \( G \) over \( \mathbb{Z} \), the Chevalley group, playing the role of the Lie group of \( \mathfrak{g}_\mathbb{Z} \); ([Hum12], §25.4) discusses the adjoint case but we may always take \( G \) to be the simply-connected cover of the adjoint group. We may further base-change to \( \mathcal{O}_K \) so that \( \mathfrak{g} \) becomes a free \( \mathcal{O}_K \)-module and \( G \) is a group scheme over \( S \). We call \( \mathfrak{g} \) a Chevalley algebra of \( ADE \) type if \( \mathfrak{g}_\eta \) is simple of \( ADE \) type.

From now on we fix a simple, simply-connected, split Chevalley group scheme \( G/S \) and a torus and Borel \( T \to B \). We have \( B = T \rtimes \mathcal{R}_u(B) \) where \( \mathcal{R}_u(B) \) is the unipotent radical, a smooth normal subgroup \( S \)-scheme in \( B \), and so \( T \simeq B/\mathcal{R}_u(B) \). We write \( \mathfrak{g}, h, b \) and \( n_b \) respectively for the Chevalley algebras of \( G, T, B \) and \( \mathcal{R}_u(B) \), so that \( h, b, n_b \) form respectively a Cartan, a Borel and the nilradical of the Borel.
One still has a notion of the adjoint action of $G$ on $\mathfrak{g}$ ([Con14], §5.1) and hence the adjoint action of $T$ decomposes $\mathfrak{g}$ into weight spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \quad (3.1)$$

where each $\mathfrak{g}_\alpha$ is a rank 1 free $\mathcal{O}_K$-module and $\Phi$ consists of characters $\alpha : T \to \mathbb{G}_m$.

### 3.2 Root data and adjoint Weyl actions.

Retaining the assumptions and notations of Section 3.1, let $T/S$ be a maximal torus of split group scheme $G$ and let $r = \dim_S(T) = \text{rk}_S(\mathfrak{g})$ be the rank of the associated Chevalley algebra $\mathfrak{g}$. By the split hypothesis there exists a free $\mathbb{Z}$-module $X^*(T)$ of rank $r$ so that $T \cong \text{Hom}(X^*(T), \mathbb{G}_m) \otimes \mathbb{Z} \mathcal{O}_K$, i.e. $\mathfrak{h} \cong \text{Lie}(T) \cong X^*(T)^\vee \otimes \mathbb{Z} \mathcal{O}_K$.

Set $X_*(T) = X^*(T)^\vee$ for the $\mathbb{Z}$-dual. The Cartan decomposition of $\mathfrak{g}$ (see Equation (3.1)) yields a set $\Phi \subset X^*(T) \setminus \{0\}$ of roots $\alpha : T \to \mathbb{G}_m, \mathcal{O}_K$ and a corresponding set of coroots $\alpha^\vee \in \Phi^\vee \subset X_*(T) \setminus \{0\}$.

**Definition 3.1.** The 4-tuple $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ is a root datum for $(G, T)$ and the quotient $W = W_G(T) = N_G(T)/T$ is the Weyl group associated to $(G, T)$.

See ([Con14], Prop. 5.1.6) for a proof that the 4-tuple in Definition 3.1 satisfies the conditions of being a root datum; the Weyl group $W$ is identified with the usual Weyl group associated to $\Phi$ and is in particular generated by the set of reflections

$$\{ s_\alpha = \text{id} - \alpha^\vee \otimes \mathbb{Z} \alpha \mid \alpha \in \Phi \}$$

From this description it follows that $W$ is a finite and constant group scheme over $S$.

We record here two natural actions of the Weyl group on affine spaces, which will be useful for us later on.

**Definition 3.2.** Let $W$ be the Weyl group associated to $(G, T)$ and $\mathfrak{h} = \text{Lie}(T)$.

(i) The adjoint action $\text{Ad} : G \to \text{End}(\mathfrak{g})$ restricts to an action $N_G(T) \times \mathfrak{h} \to \mathfrak{h}$, since $\text{Ad}_g(h) \in \mathfrak{h}$ for $g \in N_G(T)(\mathcal{O}_K)$ and $h \in \mathfrak{h}(\mathcal{O}_K)$. In particular the adjoint $T$-action on $\mathfrak{h}$ is trivial since $T$ is abelian, and so the adjoint action descends to the adjoint Weyl action $W \to \text{End}(\mathfrak{h})$, $w \mapsto \text{Ad}_w$ for a lift $n_w \in N_G(T)$ of $w \in W$. As $G$ is simple, we may identify $\mathfrak{h} \cong \mathfrak{h}^\vee \cong X^*(T) \otimes \mathbb{Z} \mathcal{O}_K$ so that the adjoint Weyl action is identified with the natural reflection action of $W$ on $X^*(T) \otimes \mathbb{Z} \mathcal{O}_K$.

(ii) There natural action of $W$ on $G/T$ via $(gT) \cdot w = gn_wT$ for a lift $n_w \in N_G(T)$ is the right multiplication action of $W$. This also yields a natural action on $G/T \times S \mathfrak{h}$ via $(gT, h) \cdot w = (gn_wT, \text{Ad}_{n_w^{-1}}(h))$, which we will make use of later (see Proposition 3.17).
Remark 3.3 (Good primes and torsion primes). Given a reduced root system \( \Phi \subset X^*(T) \), a prime \( p \) is said to be torsion for a simply-connected Lie group \( G \) if there exists a \( \mathbb{Z} \)-closed root subsystem \( \Phi' \subseteq \Phi \) so that the quotient of \( \mathbb{Z} \)-lattices \( \mathbb{Z}\Phi'/\mathbb{Z}(\Phi') \) has \( p \)-torsion. An equivalent definition of a good prime is that there is no \( \mathbb{Z} \)-closed root subsystem \( \Phi' \subseteq \Phi \) so that \( \mathbb{Z}\Phi/\mathbb{Z}\Phi' \), so that the good primes are exactly the non-torsion primes (\cite{Slo80}, §3.12). Similarly, a prime \( p \) is very good if it is good and \( p \) does not divide \(|(\mathbb{Z}\Phi')^*/\mathbb{Z}\Phi|\). See (loc. cit., §3.6) for a table of the values of \(|(\mathbb{Z}\Phi')^*/\mathbb{Z}\Phi|\) in the case of \( g \) simple simply-laced; it turns out that the only good but not very good case of prime \( p \) occurs when \( p \mid n + 1 \) and \( g \) is of Type \( A_n \).

3.3 Nilpotent and semisimple elements of Chevalley algebras. We now come to the definition of the nilpotent scheme and nilpotent sections. Recall that when \( g \) is a Lie algebra over \( k \), the nilpotent variety \( \mathcal{N}_g \) is a reduced closed subscheme of \( g \) that is the Zariski closure of the nilpotent elements, and by Galois descent \( \mathcal{N}_g \) is also well-defined over non-algebraically closed fields.

Proposition 3.4 (\cite{Cot22b}, Thm. 4.12). For a Chevalley \( \mathcal{O}_K \)-algebra \( g \), there exists a unique closed \( S \)-scheme \( \mathcal{N}^{\text{sch}}_g \subset g \) that is reduced and \( (\mathcal{N}^{\text{sch}}_g)_\pi \simeq \mathcal{N}^{\text{sch}}_g \) for geometric points \( \pi \to S \). Here the right-hand side denotes the usual nilpotent variety over a field.

Proof. Existence is shown more generally in (\cite{Con14}, Thms. 4.6 and 4.12) so we only discuss uniqueness in our particular case. Suppose \( X,Y \) are closed reduced \( S \)-subschemes in \( A_S \) so that on geometric fibers \( Y_s \simeq X_s \hookrightarrow A^n_S \) and both \( X_s,Y_s \) are reduced, then we claim \( X \simeq Y \). The Zariski closure \( \overline{X}_\eta \) of \( X_\eta \) is reduced as \( X_\eta \) is an open reduced subscheme of \( X \), hence so is \( X' = \overline{X}_\eta \cup X_s \). Since both \( X',X \) are reduced closed subschemes of \( A^n_S \) with the same points, \( X' = X \). A similar argument for \( Y \) yields

\[
Y = \overline{Y}_\eta \cup Y_s \simeq \overline{X}_\eta \cup X_s = X
\]

Since nilpotent schemes behave well under base-change, we will refer to the nilpotent scheme \( \mathcal{N}^{\text{sch}}_g \) as \( \mathcal{N}_g \) for any Chevalley \( \mathcal{O}_K \)-algebra. We can also define:

Definition 3.5. An \( \mathcal{O}_K \)-valued section \( x \in g(\mathcal{O}_K) \) is fiberwise nilpotent if \( x_s \in \mathcal{N}_g(k) \) and \( x_\eta \in \mathcal{N}_g(\overline{K}) \). By a small refinement of the proof in Proposition 3.4, this is equivalent to \( x \) being an \( \mathcal{O}_K \)-valued section of \( \mathcal{N}_g \).

For \( x \in g(\mathcal{O}_K) \), we can define the centralizer \( C_G(x) \) as follows. Through the adjoint action \( \text{Ad} : G \times g \to g \) we obtain a functor \( C_G(x) \) so that on \( \mathcal{O}_K \)-algebras \( R \),

\[
C_G(x)(R) = \{ g \in G(R) \mid \text{Ad}_g(x) = x \}
\]

By (\cite{Cot22b}, Lemma 2.1), \( C_G(x) \) is represented by a closed \( S \)-subgroup scheme of \( G \), which we denote by \( C_G(x) \); over algebraically closed fields, the reduced scheme underlying \( C_G(x) \) is the usual centralizer.
Definition 3.6. A section $x \in \mathfrak{g}(\mathcal{O}_K)$ is regular if its centralizer subscheme $C_G(x)$ satisfies $\dim(C_G(x)_s) = \dim(C_G(x)_\pi) = r$ where $r = \text{rank}(G)$. By upper-semicontinuity of fiber dimension for group schemes, it suffices to have $\dim(C_G(x)_s) = r$ since $\dim(C_G(x)) \geq r$ for semisimple $G$ (see [Hum95], §1.6).

Definition 3.7. For a Lie algebra $\mathfrak{g}$ over an algebraically closed field we call $x \in \mathfrak{g}$ subregular if $\dim(C_G(x)_s) = r + 2$. For non-regular elements $x$ one has $\dim(C_G(x)_s) \geq r + 2$ so that subregular elements are the next “closest” to being regular; see ([Hum95], §4.11) for more details.

If $\mathfrak{g}$ is now a Chevalley $\mathcal{O}_K$-algebra, a nilpotent section $x \in \mathcal{N}_g(\mathcal{O}_K)$ is called fiberwise subregular if both $x_s \in \mathcal{N}_g(k)$, $x_\pi \in \mathcal{N}_g(K)$ are subregular nilpotent elements; if we do not require $x$ to be (fiberwise) nilpotent then subregularity in this setting may not make sense (cf. the following example).

Example 3.8. We discuss a couple of pathologies that may occur with these definitions.

(i) The subregular notion need not behave well in fibers. Let

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}_3(\mathcal{O}_K), \ \text{char}(k) > 3$$

then $x_\pi \in \mathfrak{sl}_3(k(\bar{\eta}))$ is regular nilpotent but special fiber $x_s \in \mathfrak{sl}_3(k(s))$ is subregular. In general we get that $x$ stays in the same (adjoint) nilpotent orbit if it satisfies a purity assumption in the sense of [Cot22a], i.e. constant centralizer dimension on the fibers - see (loc. cit., Prop. 5.10).

(ii) Regular semisimple elements (see Definition 3.9 below) can become nilpotent regular or non-regular elements. Let

$$x = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}, \ y = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} \text{ (in } \mathfrak{sl}_2(\mathcal{O}_K))$$

then $x_\pi, y_\pi$ are regular semisimple but $x_s$ is regular nilpotent and $y_s$ is subregular nilpotent. For $x$, even though $\dim_{\mathcal{O}_K}(C_G(x))$ is locally constant, centralizer $C_G(x)$ is not flat.

Definition 3.9 ([BC19], §4.1.5, SGA 3IV Exposé XIV.2). An element $x \in \mathfrak{g}(\mathcal{O}_K)$ is called fiberwise semisimple if $x_\pi \in \mathfrak{g}(k(\pi))$ is semisimple for all geometric points $\pi \to S$. An element $x \in \mathfrak{g}(\mathcal{O}_K)$ is called regular semisimple if it lies in some Cartan $\mathfrak{h} \subset \mathfrak{g}$ and for all geometric points $\pi$ we have

$$\mathfrak{h}_\pi = \bigcup_{n \geq 0} (\ker(\text{ad}^n_{x_\pi}))$$

This union may be thought of as the centralizer of $x$ so that it has fiberwise dimension equal to $\text{rank}(\mathfrak{g}_\pi)$. Regular semisimple elements form an open set $\mathfrak{g}^{rs} \subset \mathfrak{g}$. 

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By ([SGA 3], Exposé XIV.2), set $g^{rs}$ is $S$-fiberwise dense in $g$ and its construction commutes with base-change, whence it gives the usual notion of regular semisimple elements of Lie algebras over a field, i.e. $x \in g_k$ so that $C_G(x)^G$ is a maximal torus in $G_k$. This also makes the density of the semisimple locus in $g$ apparent.

Similar methods as in ([Cot22b], Lemma 2.1) yield that the functor of regular sections in $g$ is represented by an open subscheme $g^{reg} \subset g$ over $S$. In the case of good characteristic, ([BC19], Lemma 4.1.6) gives $g^{rs} \subset g^{reg}$ so that both are $S$-fiberwise dense in $g$.

### 3.4 The adjoint quotient

We retain the notations and assumptions of Section 3.1 for Chevalley group $G/S$ and Chevalley algebra $g$. The adjoint quotient morphism $\chi : g \to g/G$ is the natural categorical quotient induced from the adjoint action $\mathrm{Ad} : G \to \mathrm{End}(g)$, and the target of $\chi$ may be identified with $\mathrm{Spec}((\text{Sym}_G g^G)^G)$, the “adjoint $G$-invariants”. The associated adjoint Weyl action (Definition 3.2 (i)) on $h$ induces another categorical quotient map $h \to h/W$, and restriction $O_K[g]^G \to O_K[h]^W$ yields the Chevalley morphism $h/W \to g/G$. For the properties of the counterparts of these morphisms over algebraically closed fields, we refer to [Slo80b], §3.10, §3.12 and §3.14.

**Proposition 3.10** ([BC19], Thms 4.1.10, 4.1.14). For $G, g, h, W$ as above, characteristic $p$ very good for $G$ and $r = r_k(G)$, we have $h/W \simeq g/G \simeq A_r^O_K$ and the formation of categorical quotient $g/G$ commutes with base-change.

Note the conditions of both Theorems 4.1.10 and 4.1.14 of [BC19] are satisfied; $p$ being (very) good implies $G$ root-smooth in the terminology of (loc. cit.) and working étale-locally on $S$ is unnecessary since $O$ equals its strict henselization.

Before we investigate the adjoint quotient $\chi$ we need a preliminary lemma.

**Lemma 3.11** ([Ric17], 4.1.3). Let $f : X \to Y$ be a morphism of finite-type $S$-schemes with $X$ flat over $S$. If the base-changed morphism $f_s : X_s \to Y_s$ is smooth for the special fiber $s$ then $f$ is smooth.

**Proof.** From the fibral criterion of flatness ([Sta18, Tag 00MP]) we get $f$ flat, and the smooth locus $U \subseteq X$ is open dense, so if it contains all closed points of $X$ we get $U = X$ and $f$ smooth. As the residue fields are algebraically closed, we may without loss of generality reduce to showing $X \times_Y \mathrm{Spec}(k(y)) \to \mathrm{Spec}(k(y)) = s$ is smooth, which holds by the smoothness assumption on $f_s$ and ([Sta18, Tag 01V9](2)).

**Proposition 3.12.** Morphism $\chi : g \to g/G$ is a flat surjection, with irreducible normal (geometric) fibers of codimension $r$, and the nilpotent locus $N^g_0$ is exactly $\chi^{-1}(0)$. Over the regular locus $g^{reg}$, $\chi$ is a smooth surjection and $g^{reg}$ intersects each geometric fiber in a single open dense adjoint $G$-orbit. The geometric fibers meeting the regular semisimple locus $g^{rs}$ consist of exactly one adjoint $G$-orbit.

**Proof.** Most of the proof is in ([BC19], 4.1.18, 4.2.6) but in our case of $S = \mathrm{Spec}(O_K)$ we may be more specific. Since $g/G$ commutes with base-change (Proposition 3.10) we may pass to geometric fibers $s, \overline{s}$, whence the respective adjoint quotients $\chi_s, \chi_{\overline{s}}$ have
(geometric) irreducible fibers of codimension $r$ (Slo80b, §3.10(iv) and §3.14) Since $\mathfrak g/g$ is smooth and the fibers have the same dimension, we get $\chi$ flat by miracle flatness (see e.g. [Sta18 Tag 00R4]), in particular $\chi$ is a local complete intersection morphism ([Sta18 Tag 0E9K]) and so the fibers are Cohen-Macaulay.

Now regular locus $g^\text{reg}$ is $S$-fiberwise dense in $g$ by the existence of regular nilpotents in $g_T$ and $g_s$, hence generically smooth, hence reduced and so it is flat over $\mathcal O_K$ (e.g. [Liu02, 4.3.9]). By Lemma 3.11 to show $g^\text{reg} \to g/G$ is smooth surjective it suffices to do so for the base-changed morphism $g^\text{reg}_s \to g_s/G_s$ over special fiber $k(s)$, where it is known for good characteristics; concretely, we have that $g^\text{reg}_s \to g_s/G_s$ is flat by miracle flatness and every fiber is regular by (Slo80b, §3.10(vi) and §3.14). Surjectivity follows from a simple calculation on tangent spaces. Moreover by (Slo80b, §3.10(ii)) we have that $g^\text{reg}_s \cap \chi_s^{-1}(h_s)$ is open dense in fiber $\chi_s^{-1}(h_s)$ (where $h_s \in h_s/W(k(s))$ and similarly for the generic fiber $g_\emptyset$, so we get that $g^\text{reg}$ intersects $S$-fiber $\chi^{-1}(h)$ in a dense open subset ($h \in h/W(\mathcal O_K)$). In particular $\chi^{-1}(h)$ is generically smooth hence flat over $S$, so as the geometric fibers $\chi_s^{-1}(h_s), \chi_\emptyset^{-1}(h_\emptyset)$ are normal (Slo80b, §3.10(v) and §3.14), by (Mat89, Thm. 23.9) we get that $\chi^{-1}(h)$ is normal over $S$.

For $\mathcal N_\emptyset$, note we have already established on geometric points $\eta, s$ that $\mathcal N_\emptyset(K) = \chi^{-1}(0)(K)$ and $\mathcal N_\emptyset(k) = \chi^{-1}(0)(k)$, and moreover both $\mathcal N_\emptyset, \chi^{-1}(0)$ are reduced (the latter from above, the former from being reduced on geometric points and $S$-flatness), so $\mathcal N_\emptyset \simeq \chi^{-1}(0)$ as $S$-schemes. For the last part, note that $x \in g^{rs}(k)$ has adjoint orbit $A_d(x)$ closed (Slo80b, §3.10(iii) and §3.14) of codimension $r$, so if a geometric fiber $\chi_s^{-1}(h_s)$ intersects it, by density of regular $G$ orbits and equality of dimensions $\chi_s^{-1}(h_s)$ of just that particular adjoint $G$-orbit.

\begin{proposition} [BC19, 4.2.7] Quotient $h/\mathcal O_K$ finite flat and the natural $W$-action on $h^{rs}$ is free and $C_G(h) \simeq T$ for any $h \in h^{rs}(\mathcal O_K)$. Consequently, $h^{rs} \to h^{rs}/W$ is an étale $W$-torsor.
\end{proposition}

\begin{proof} Again the proof follows a simplified version of the argument in [BC19]. Finiteness is automatic as $W$ is finite, and by the fibral criterion of flatness and miracle flatness (as $h, h^{rs}/W$ are smooth; for the smoothness of the latter, see Remark (ii) of (Slo80b, §3.15)) we have that $h \to h^{rs}/W$ is flat. To show that $W$-action on $h^{rs}$ is free, it suffices to show that the natural map $W \times_S h^{rs} \to h^{rs} \times_S h^{rs}$, $(w, h) \mapsto (w(h), h)$ is a scheme monomorphism, and by (EGA IV, 17.2.6) it suffices to check so on closed points of $S$. So we can reduce to the algebraically closed field case, where by (Ric17, Lemma 2.3.3) we have that $W$ acts freely on $h^{rs}_s$ and moreover $C_{G_s}(h_s) \simeq T_s$ for $h_s \in h^{rs}_s(k(s))$. Thus we get $S$-fiberwise a torus in $G$ and so $C_G(h) \simeq T$ for all $h \in h^{rs}(\mathcal O_K)$.
\end{proof}

\begin{corollary} Morphism $h^{rs} \to h^{rs}/W$ is a Galois cover with Galois group $W$ acting transitively on the fibers.
\end{corollary}

3.5 Relative Grothendieck–Springer resolutions. We retain the notations and assumptions of Section 3.1 for Chevalley group $G/S$ and Chevalley algebra $g$. The free $\mathcal O_K$-module $b$ obtains a $B$-module structure via the adjoint action of $B$ on $b$ and we can
therefore form the following associated bundle

$$\tilde{g} := G \times^B b = G \times_S b//B, \quad b \cdot (g, x) = (g b^{-1}, \text{Ad}_b(x)) \text{ for } b \in B, g \in G, x \in b$$

(3.2)

with the induced $B$-action on $G \times_S b$ indicated on the right.

**Lemma 3.15.** $\tilde{g}$ is a smooth Zariski-locally trivial $G$-torsor over $G/B$ with fiber $b$.

**Proof.** Note that any two Borels $B_1, B_2$ in $G$ are conjugate étale-locally on $S$ ([Con14], 5.2.11) so $G/B$ exists as a smooth projective $S$-scheme ([Con14], 2.3.6) and its generic and special fibers are correspondingly the flag varieties of $G_0 \overline{\pi}$ and $G_\sigma$. In this case $\pi : G \to G/B$ makes $G$ into a $B$-torsor on $G/B$, inducing $\pi : \tilde{g} \to G/B$, $[g, x] \mapsto \pi(g)$ so that $\tilde{g}$ is a fiber bundle over $G/B$ with fiber $b$, equipped with a $G$-action from the (left) $G$-action on $G/B$ so that $\tilde{g}$ becomes $G$-equivariant.

We get that $\tilde{g}$ is a scheme by pulling an open affine cover $\{U_i\}$ of $G/B$ to an open affine cover $\tilde{\pi}^{-1}(U_i)$ of $\tilde{g}$, the affineness of $\tilde{\pi}$ being a consequence of ([Jan03], §5.14). Now we can construct Zariski-local sections $G/B \supset U_i \to G$ for the (a priori étale-locally trivial) $B$-torsor $\pi$. It suffices to construct a composition series of $S$-subschemes for $B$ so that successive quotients are $G_m$ or $G_a$, since $G_m$ and $G_a$-torsors are Zariski-locally trivial if and only if they’re étale-locally trivial ([SGA 1], XI.5.1). Since $B = T \rtimes \mathcal{R}_u(B)$, and $T \simeq G_m^r \mathcal{O}_K$ while $\mathcal{R}_u(B) \simeq \prod_{\alpha > 0} U_\alpha$ decomposes into “root groups” $U_\alpha \simeq G_a$ (i.e. so that Lie($U_\alpha) = g_a$) we can build our composition series as in ([Con14], 5.1.16).

Local triviality implies $G \times_{G/B} U_i \simeq B \times U_i$ and we can lift to Zariski-local sections for $\tilde{g}$ since $U_i \times_{G/B} \tilde{g} \simeq (U_i \times_{G/B} G) \times B \simeq U_i \times b$, and $\tilde{g}$ is smooth since $b$ is (see [Jan03], §5.16 for more details) - so $\tilde{g}$ is indeed a Zariski-locally trivial smooth $G$-torsor. \hfill \Box

We have a closed immersion $\tilde{g} \hookrightarrow G \times^B g \simeq G/B \times_{\mathcal{O}_K} g$ into a trivial $G$-torsor over $G/B$ (which can be checked on the fibers of $S$), yielding an equivalent description of $\tilde{g}$ as

$$\tilde{g} = \{(B', x) \in G/B \times_{\mathcal{O}_K} g \mid x \in \text{Lie}(B')\}$$

From this, we can define a dominant morphism $\pi : \tilde{g} \to G/B \times_S g \to g$ via projection to the second factor $[g, x] \mapsto (gb, \text{Ad}_g(x)) \mapsto \text{Ad}_g(x)$; it follows that $\pi$ is proper since $G/B$ is projective over $S$. It is evident then that forming $\tilde{g}$ commutes with base-change on $S$ and on geometric points $s, \tilde{s}, g$, and $\tilde{g}_s$ are the more commonly known Grothendieck–Springer resolutions as defined e.g. in ([Slo80a], §3.3). They are particular types of simultaneous resolutions (see Section 2.2).

**Lemma 3.16.** $\pi : \tilde{g} \to g$ is finite over the open dense locus $g^{\text{reg}} \subset g$.

**Proof.** As the formations of $\tilde{g}, \pi$ commute with base-change we may check this over the closed point $s$, where it suffices to show $x \in g^{\text{reg}}(k(s))$ if and only if $x$ is contained in finitely many Lie algebras of Borel subgroups $B_s \subset G_s$; this is found in ([Slo80b], §3.8, §3.14). We thus get quasi-finiteness, hence finiteness since the map on $g^{\text{reg}}$ is a base-change of the proper map $\pi$. See ([BC19], 4.2.3) for a more general version. \hfill \Box
Since $\pi$ is proper dominant and generically finite, we may call it the Grothendieck alteration. Another map of interest is $\tilde{\chi} : \tilde{g} \to h$, constructed as follows: given our choice of Borel $B$, $b/n_b = \text{Lie}(B/R_u(B))$ is the “universal Cartan” and $\tilde{\chi}$ is $(B', x) \mapsto x \mod n_b$. More precisely, there exists a short exact sequence of torsors

$$0 \to G \times^B n_b \to \tilde{g} \to G \times^B \text{Lie}(B/R_u(B)) \to 0$$

induced from $n_b \hookrightarrow b$, since all terms are locally trivial over $S$. Now $B$ acts trivially on $\text{Lie}(B/R_u(B))$ as the latter is an abelian subalgebra, so that

$$G \times^B \text{Lie}(B/R_u(B)) \simeq G/B \times \text{Lie}(B/R_u(B)) \simeq G/B \times h$$

and $\tilde{\chi}$ is the projection of $\tilde{g} \to G/B \times h$ onto the second factor. Note $\tilde{\chi}$ is flat by miracle flatness and the fibral criterion of flatness ([Sta18, Tag 00MP]), hence smooth by Lemma 3.11 as $\tilde{\chi}$ is identified with the smooth morphism $\tilde{g}_s \to h_s$ defined in ([Slo80a], §3.3).

**Proposition 3.17.** Keeping notations as above, the morphisms $\pi$ and $\tilde{\chi}$ fit into a commutative diagram

$$\begin{array}{ccc}
\tilde{g} & \xrightarrow{\pi} & g \\
\downarrow{\tilde{\chi}} & & \downarrow{\chi} \\
h & \longrightarrow & h/W
\end{array} \tag{3.3}$$

so that, $S$-fiberwise, $\pi_s$ and $\pi_{\tilde{\eta}}$ induce the Grothendieck-Springer resolutions on $g_s$ resp. $g_{\tilde{\eta}}$. In particular, restriction $\tilde{g}^{rs} = \pi^{-1}(g^{rs}) \to g^{rs}$ is a $W$-torsor, $\tilde{g}_{\tilde{\eta}}^{reg} = \pi^{-1}(g^{reg}) \simeq g^{reg} \times_{h/W} h$ and $\tilde{\chi} : \tilde{g}^{rs} \to h^{rs}$ is $W$-equivariant.

**Proof.** The commutativity of (3.3) follows if $b \to g \xrightarrow{\pi} g/G$ factors through

$$b \to b/n_b \simeq h \to h/W \simeq g/G$$

On the level of sections we want $O_K[g]G \to O_K[b]$ to factor through $O_K[b] \to O_K[b]$. Over an algebraically closed field this amounts to $\text{Ad}_g(h+n)$ having semisimple part conjugate to $h$ (for $h \in b, n \in n_b$); for the general version see ([BC19], §4.1.7).

We next prove the statements involving $\tilde{g}^{rs}, \tilde{g}^{reg}$ as follows. Define $X = G/T \times_S h^{rs}$, equipped with the $W$-action of Definition 3.2 (ii). We aim to show $\tilde{g}^{rs} \simeq X$ equipped with this $W$-action; by ([EGA IV], 17.9.5) it suffices to show the corresponding map over $k(s), k(\tilde{\eta})$ is an isomorphism, whence it holds by the proof of ([KW13], Thm. 9.1). Moreover $X/W \simeq g^{rs}$ over $k(s)$ and $k(\tilde{\eta})$, so this isomorphism extends over $S$ again by ([EGA IV], 17.9.5).

To show $W$ acts transitively on the $S$-fibers of $\tilde{g}^{rs} \to g^{rs}$, it suffices as before to pass to geometric fibers. Say $(g_1T, h_1), (g_2T, h_2) \in X$ mapping to $\text{Ad}_{g_1}(h_1) = \text{Ad}_{g_2}(h_2) \in g^{rs}$. Then $\text{Ad}_{g_2^{-1}g_1}(h_1) = h_2$ so $n_w = g^{-1}g_1 \in N_G(T)$ since $n_w$ conjugates the two centralizers

\footnote{Note the terminology of ([KW13]) differs from the standard one - what they call “regular” is regular semisimple in our terminology.}

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$C_G(h_1), C_G(h_2),$ which are both $T$ as $h_1, h_2 \in \mathfrak{h}^{rs}$. Letting $w = [n_w^{-1}] \in W$ we obtain
$(g_1T, h_1) \cdot w = (g_2T, h_2),$ establishing that $W$ acts freely transitively on the fibers. Thus
$\tilde{g}^{rs} \to g^{rs}$ is a $W$-torsor.

Now $g^{reg} \times_{g/W} \mathfrak{h}$ is smooth over $S$ since $g^{reg} \to \mathfrak{h}/W$ is (by Proposition 3.12), and its restriction $g^{reg} \times_S g^{rs}$ is furthermore a $W$-torsor over $g^{rs}$ by base-changing along $W$-torsor $g^{rs} = \mathfrak{h}^{rs}/W$. Then morphism $\tilde{g}^{rs} = \tilde{g}^{reg}|_{\mathfrak{g}^{rs}} \to g^{reg} \times_S g^{rs}$ is $W$-equivariant as both map to $\mathfrak{h}$ via projection, so we have an isomorphism of $W$-torsors over $g^{rs}$, which extends to all of $g^{reg}$ by uniqueness of normalizations; see ([BC19], 4.2.12) for more details. In light of the specified $W$-action on $\tilde{g}^{rs}$ again, we likewise obtain that $\tilde{\chi}_{\tilde{g}^{rs}}$ is $W$-equivariant. That the induced morphisms of $\pi$ on the geometric fibers correspond to the Grothendieck-Springer resolutions is now clear since formation of $\chi$ and $\mathfrak{h} \to \mathfrak{h}/W$ commutes with base-change on $S$, and the $k(s)$-version of (3.3) commutes by ([Slo80a], §3.3), yielding moreover that $\pi_s$ that is proper and fiberwise a resolution of the singularities of $g_s$ (loc. cit.).

4. Integral Slodowy slices

We now extend Slodowy’s construction of transverse slices ([Slo80b], §5) to our setting of Chevalley $O_K$-algebras, with the intention of proving a Grothendieck simultaneous resolution statement for slices (Section 4.6). Throughout, $G$ will be an affine group scheme, in particular either a Lie group over an algebraically closed field or a split, simple, simply-connected Chevalley group over $S = \text{Spec}(O_K)$. $T \subset B$ denote a fixed choice of torus and Borel, and the corresponding Lie algebras over $k$ or $S$ are $\mathfrak{h} \subset \mathfrak{b} \subset g$.

4.1 Remarks on Slodowy slices. We review the theory of transverse slices to $G$-orbits from ([Slo80b], §5) in a slightly more general setting.

Definition 4.1. Let $G$ act on an integral scheme $X$ over $S$. A **transverse Slodowy slice** to the orbit $G \cdot x$ of $x \in X$ is a locally closed subvariety $S \subseteq X$ so that $x \in S(O_K)$, the action morphism $\alpha : G \times S \to X, \alpha(g, s) = g \cdot s$ is smooth and the dimension of $S$ is minimal with respect to these two conditions.

Based on their definition, we can deduce some useful properties of Slodowy slices.

Lemma 4.2 ([Slo80b], §5.1 Lemma 3). Suppose $X$ is smooth and affine with the adjoint action of Chevalley group $G$ over $S$, $Y$ is another $S$-scheme with trivial $G$-action and $f : X \to Y$ is a $G$-invariant morphism. Let $x, y \in X(O_K)$ lie in the same $G$-orbit in $X$ and assume that centralizers $C_G(x), C_G(y)$ are smooth. Suppose $S_1, S_2$ are Slodowy slices transverse to the orbit at $x$ resp. $y$. Then $(S_1, x), (S_2, y)$ are étale-locally isomorphic over $Y$. In particular the henselizations of $S_1$ at $x$ and $S_2$ at $y$ are isomorphic.

Proof. We may assume $x = y$ (since $y = g \cdot x$ for some $g \in G$). As $C_G(x)$ is smooth, $\text{Lie}(C_G(x)) \subset g$ is identified with the Chevalley subalgebra of $\text{ad}_x$-invariants ([Con14], 2.2.4), inducing an $S$-splitting $g = \text{Lie}(C_G(x)) \oplus \mathfrak{g}_1$ for some complementary $O_K$-module $\mathfrak{g}_1$. Then the rest of the proof of ([Slo80b], §5.1, Lemma 3) goes through: choose a projection $p : G \to g$ étale at the identity (cf. loc. cit., §5.1 Lemma 1’s proof) and set...
$G_1 = p^{-1}(V), G_2 = \{g^{-1} \mid g \in G_1\}$ inside $G$. By construction, the induced action maps
$\mu_i : G_i \times S_i \to X$ are étale at $(1, x)$ and hence so are the base-changed maps arising from cartesian diagrams
\[
\begin{array}{c}
G_1 \times S_1 \xrightarrow{\mu_1} X \\
\downarrow \\
(G_1 \times S_1) \times_X S_2 \xrightarrow{\mu_1} S_2 \\
\downarrow \\
(G_2 \times S_2) \times_X S_1 \xrightarrow{\mu_2} S_1
\end{array}
\]

We furthermore have $(G_1 \times S_1 \times_X S_2) \simeq (G_2 \times S_2) \times_X S_1$ via $(g, s) \mapsto (g^{-1}, g \cdot s)$. Hence
we can choose a neighborhood of $(1, x)$ inside this space, with étale maps to $S_1, S_2$. 

Remark 4.3. The condition of centralizer $C_G(x)$ being smooth over $S$ is satisfied in the case $X = g$ and $x \in g(O_K)$ a fiberwise subregular nilpotent section (cf. Definition 3.6) - note that by ([Cot22a], Thm. 1.1), the centralizer satisfies all relevant ‘purity’ assumptions. 

Lemma 4.4 ([Slo80b], §5.1, Lemma 2). If $G$ acts on schemes $X, Y$ with respective action maps $\alpha_X, \alpha_Y$, $S \subset X$ is a locally closed integral subscheme and $f : X \to Y$ is a $G$-equivariant morphism, then the following diagram is cartesian and the top arrow is smooth if the bottom arrow is smooth:

\[
\begin{array}{c}
G \times (S \times_Y X) \xrightarrow{\alpha_X} X \\
\downarrow \text{id} \times f \\
G \times S \xrightarrow{\alpha_Y} Y
\end{array}
\]

4.2 Jacobson–Morozov in characteristic $p$. The aim of the next few sections is to construct a suitable Slodowy slice $S$ over $S$, transverse to a chosen $S$–fiberwise nilpotent element $x \in g$. The goal is to use slice $S$ to describe a miniversal deformation of RDP $x_s \in \mathcal{N} \cap S$ and a simultaneous resolution for slices (see Section 4.6). We first make some remarks on $\mathfrak{sl}_2$–representation theory in characteristic $p > 0$, following ([Slo80b], §7.1).

Definition 4.5. Let $k$ be algebraically closed of characteristic $p$ and fix a standard basis \{h, x, y\} for $\mathfrak{sl}_2(k)$. A representation $\rho : \mathfrak{sl}_2 \to \mathfrak{gl}_n$ is called good (in the sense of ([Slo80b], §7.1)) when $\rho(x)^{p-1} = \rho(y)^{p-1} = 0$; for $p = 0$ we posit that all $\mathfrak{sl}_2$–representations are good.

If $V$ is an $n$-dimensional ($n < p$) $k$-vector space with basis $\{v_1, \ldots, v_n\}$, we can define an irreducible $\mathfrak{sl}_2$–representation $\rho_n$ on $V$ as follows:

$\rho_n(x)v_1 = 0, \quad \rho_n(y)v_n = 0$

$\rho_n(x)v_{i+1} = i(n-i)v_i \mod p, \quad \rho_n(y)v_i = v_{i+1}, \quad \rho_n(h)v_i = (n-2i+1)v_i \mod p \text{ for } i \leq n-1$

Theorem 4.6 ([Slo80b], §7.1). Any good $\mathfrak{sl}_2$–representation $\rho$ is completely reducible and decomposes into a sum of good irreducible $\mathfrak{sl}_2$–representations of the above form. For each $n < p$ there is a unique $n$–dimensional good irreducible representation.
Denote by $V_n$ the unique $n$–dimensional good irreducible $\mathfrak{sl}_2$–representation. $V_n$ decomposes further into weight spaces $V_n(k)$, which are 1-dimensional eigenspaces for the action of $\rho_n(h)$ as multiplication by $k$ and $-n+1 \leq k \leq n-1$. Then the nilpotent endomorphisms $\rho_n(x)$, $\rho_n(y)$ respectively induce isomorphisms $V_n(k) \xrightarrow{\sim} V_n(k+2)$ and $V_n(k) \xrightarrow{\sim} V_n(k-2)$; they also respectively annihilate the highest weight space $V_n(n-1)$ and the lowest weight space $V_n(-n+1)$.

**Theorem 4.7** (Jacobson–Morozov, [ST18] Thm. 1.1). Let $G$ be a simple Lie group over $k$ with Lie algebra $\mathfrak{g}$. Suppose $p = 0$ or $p > \text{Cox}(\mathfrak{g})$. For each nilpotent $x \in \mathfrak{g}(k)$ there is a completion to an $\mathfrak{sl}_2$–triple $\{h, x, y\}$ coming from a faithful representation $\rho : \mathfrak{sl}_2 \to \mathfrak{g}$ with $x = \rho(x_0)$, $y = \rho(y_0)$, $h = \rho(h_0)$ for $\{h_0, x_0, y_0\}$ the standard basis of $\mathfrak{sl}_2(k)$. The triple $\{h, x, y\}$ is unique up to $C_G(x)$–conjugation and the composite representation $\text{ad} \circ \rho : \mathfrak{sl}_2 \to \mathfrak{gl}(\mathfrak{g})$ is a good representation.

**Remark 4.8.** According to [ST18], restriction $p > \text{Cox}(\mathfrak{g})$ is optimal for the uniqueness of $\mathfrak{sl}_2$–triple $\{h, x, y\}$ up to conjugation, and it improves previous bounds such as $4\text{Cox}(\mathfrak{g}) - 2$ appearing in [Slo80b].

### 4.3 SLodowy slices via Jacobson–Morozov

Let $G$ denote a split, simple, simply-connected Chevalley group of Type $A_n$ over $S$ with Chevalley algebra $\mathfrak{g}$. Consider the setting of Section 4.1 with $G$ acting on $X = \mathfrak{g}$ via the adjoint action and $x \in \mathfrak{g}(O_K)$ an $S$-fiberwise subregular nilpotent element; this condition ensures both generic and special fibers $x_T, x_s$ remain in the respective (unique) subregular nilpotent orbit of $\mathfrak{g}_T, \mathfrak{g}_s$. In view of Lemma 4.2 we may construct a suitable Slodowy slice at $x$, transverse to the (adjoint) subregular nilpotent orbits of $\mathfrak{g}_T, \mathfrak{g}_s$ and any other Slodowy slice at $x$ will be étale-locally isomorphic to it. Moreover this allows us to replace $x$ with another representative in its nilpotent orbit, so we may assume $x$ has a ‘standard’ form.

**Proposition 4.9.** Assume $p > \text{Cox}(\mathfrak{g})$. There is a choice of subregular nilpotent representative $x \in \mathfrak{g}(O_K)$ extending to a section of $\mathfrak{sl}_2$–triples $\{h, x, y\} \subset \mathfrak{g}(O_K)$ i.e. $\{h_T, x_T, y_T\}$ and $\{h_s, x_s, y_s\}$ are $\mathfrak{sl}_2$–triples respectively in $\mathfrak{g}_T$ and $\mathfrak{g}_s$.

**Proof.** It is clear by the restrictions (see Theorem 4.7) that fiberwise subregular $x$ induces unique $\mathfrak{sl}_2$–triples in Lie algebras $\mathfrak{g}_T, \mathfrak{g}_s$ up to the action of $\text{Lie}(C_G(x)) \subset \mathfrak{g}(O_K)$. Subregular nilpotent elements in Type $A_n$ have standard Levi form in characteristic zero, i.e. they are regular nilpotent in a Levi subalgebra $\mathfrak{l}$ corresponding to a parabolic $\mathfrak{p} \subseteq \mathfrak{g}$ determined by a subset of simple roots $I \subset \Delta$. In this case, if we denote the simple roots as $\Delta = \{\alpha_1, \cdots, \alpha_n\}$, the corresponding Levi and subregular representative are

$$\mathfrak{l} = h_\mathfrak{g} \oplus \bigoplus_{\alpha_i \in \mathfrak{g}_\mathfrak{g}_\mathfrak{g}} \mathfrak{g}_{\alpha_i, \mathfrak{g}} , \quad x_{\text{subreg}} = \sum_{i=1}^{n-1} e_{\alpha_i}$$

(4.1)

where Chevalley basis elements $e_{\alpha_i}$ are viewed as root vectors in $\mathfrak{g}_T$ (see Section 3.1). By the discussing preceding this proposition we may replace our subregular $x$ with the “standard” representative $x = x_{\text{subreg}}$ in Equation (4.1), which has this form because it is
a regular representative in \( \mathfrak{l} \) (see e.g. [Ric17], Lemma 3.1.1); as vectors \( e_{\alpha_i} \) form a \( \mathbb{Z} \)-basis for \( \mathfrak{g} \), clearly \( x \in \mathfrak{g}(\mathcal{O}_K) \) still.

We now complete \( x \) to an \( \mathfrak{sl}_2 \)-triple \( \{x, y, h\} \). By ([CM93], §3.6) an element \( h = \sum a_i h_{\alpha_i} \in \mathcal{h}_\Pi \) satisfies \( [h, e_{\alpha_i}] = d_i e_{\alpha_i} \) if and only if the weights of the Dynkin diagram corresponding to \( x \) are \( (d_1, \cdots, d_n) \). Up to relabeling we have \( d_i = 2 \) for \( i \neq n \), \( d_n = 0 \) (see loc. cit., 3.6.4). Therefore condition \([h, x] = 2x\) yields a system of equations

\[
\sum_{j=1}^{n} a_j \frac{\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_j, \alpha_j \rangle} = \sum_{j=1}^{n} a_j C_{j,i} = d_i, \quad 1 \leq i \leq n \tag{4.2}
\]

where \( C \) is the Cartan matrix of \( \mathfrak{g} \). This gives a unique \( h \) with \( \mathcal{O}_K \)-coefficients if and only if \( \det(C) = n + 1 \) is invertible in \( \mathcal{O}_K \), which is granted by the restriction \( p > \text{Cox}(\mathfrak{g}) = n + 1 \).

For \( y = \sum b_i e_{-\alpha_i} \), a simple calculation using the Chevalley relations yields

\[
\sum_{i=1}^{n-1} a_i h_{\alpha_i} = h = [x, y] = \left[ \sum_{i=1}^{n-1} e_{\alpha_i}, \sum_{i=1}^{n-1} b_i e_{-\alpha_i} \right] = \sum_{i=1}^{n-1} b_i h_{\alpha_i}
\]

so coefficients \( b_i = a_i \) are still in \( \mathcal{O}_K \). Since \([h, y] = -2y\) follows from \([h, x] = 2x\), we have constructed an \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g}(\mathcal{O}_K) \); one can check that \( a_i \mod p \neq 0 \) so that the mod \( p \) reductions \( \{\tilde{h}, \tilde{x}, \tilde{y}\} \) still form a subregular \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g}_s \).

Lemma 4.10. Fix an integral \( \mathfrak{sl}_2 \)-triple \( \{h, x, y\} \) as in Proposition 4.9. Then \( \mathcal{O}_K \)-module \( \mathfrak{g}/ad_x(\mathfrak{g}) \) is free of rank \( r + 2 \).

Proof. By assumption \( x \in \mathfrak{g}(\mathcal{O}_K) \), so \( \text{ad}_x(\mathfrak{g}) \subset \mathfrak{g} \) and quotient \( \mathfrak{g}/ad_x(\mathfrak{g}) \) is an \( \mathcal{O}_K \)-module. Consider \( \{h_\eta, x_\eta, y_\eta\} \) as an \( \mathfrak{sl}_2 \)-triple in the \( \overline{K} \)-Lie algebra \( \mathfrak{g}_\overline{K} \), then the dimension of the centralizer \( C_{g_\eta}(x_\eta) \), which is \( r + 2 \) by subregularity. So \( \mathfrak{g}/ad_x(\mathfrak{g}) \) is a rank \( r + 2 \) module, possibly with torsion.

Now consider the mod \( p \) \( \mathfrak{sl}_2 \)-triple \( \{h_s, x_s, y_s\} \) inside \( \mathfrak{g}_s \). Since \( p > \text{Cox}(\mathfrak{g}) \), \( \mathfrak{g}_s \) is a good \( \mathfrak{sl}_2 \)-representation decomposing into irreducible good \( \mathfrak{sl}_2 \)-representations \( V_{d_1}, \cdots, V_{d_r} \). Each irreducible representation \( V_{d_i} \) decomposes further into \( d_i \) 1–dimensional eigenspaces

\[
V_{d_i}(-d_i + 1), \cdots, V_{d_i}(d_i - 1)
\]

for the action of \( ad_{h_s} \) and by \( \mathfrak{sl}_2 \)-theory \( ad_x \) maps \( V_{d_i}(k) \) isomorphically to \( V_{d_i}(k + 2) \) (cf. Section 4.2). So

\[
\mathfrak{g}_s/ad_x(\mathfrak{g}) \cong \bigoplus_{i=1}^{r} V_{d_i}(-d_i + 1) \tag{4.3}
\]

is the direct sum of all lowest weight eigenspaces. Furthermore as \( ad_x \) annihilates all highest weight eigenspaces \( V_{d_i}(d_i - 1) \), the number of irreducible components of \( \mathfrak{sl}_2 \)-representation \( \mathfrak{g} \) (and hence the dimension of \( \mathfrak{g}_s/ad_{x_s}(\mathfrak{g}_s) \) equals \( r + 2 \), the dimension of the centralizer of subregular \( x_s \).

Since forming module quotients commutes with base–change, identification \( (\mathfrak{g}/ad_x(\mathfrak{g})) \otimes k \cong \mathfrak{g}_s/ad_{x_s}(\mathfrak{g}_s) \) implies equidimensionality of the fibers of \( \mathfrak{g}/ad_x(\mathfrak{g}) \), hence it is flat over \( \mathcal{O}_K \) and therefore it is a free rank \( r + 2 \) module. \( \square \)
Corollary 4.11. Retaining the notation of Lemma 4.10, if \( a \) denotes a free \( \mathcal{O}_K \)-submodule of \( g \) complementary to \( \text{ad}_x(g) \) then \( S = x + a \) is a Slodowy slice transverse at \( x \) fiberwise to the nilpotent orbits of \( x_s \) and \( x_\eta \).

Proof. The choice of such an \( a \) is possible via Lemma 4.10 since \( g/\text{ad}_x(g) \) is free. We observe in the special fiber \( \mathcal{S} := S \otimes k = x_s + a_s \) that \( a_s \) is a complement to \( \text{ad}_{x_s}(g_s) \) in \( g_s \) and hence isomorphic to \( \bigoplus_i V_{d_i}(-d_i + 1) \) (in the notation of the proof of Lemma 4.10). It forms a Slodowy slice in the traditional sense by \( \text{(Slo80a, §2.4)} \) and is transverse at \( x_s \) to the subregular orbit. The action map \( \mu : G \times S \to g \) is smooth, since by Lemma 3.11 it suffices to check smoothness on the special fiber, where \( \overline{\mu} : G_s \times \mathcal{S} \to g_s \) is precisely the smooth action map of Slodowy slice \( \mathcal{S} \). Since \( \dim(\mathcal{S}) = \dim(S) \) by flatness, we have checked that \( S \) satisfies all conditions of a transverse Slodowy slice. \( \square \)

We may identify \( a \simeq \ker(\text{ad}_y) \) so that \( S = x + \ker(\text{ad}_y) \) by Corollary 4.11 this is the standard formula for Slodowy slices.

4.4 Spaltenstein slices. We now consider Chevalley algebras of Type \( D_n \) or \( E_n \). In this case we may construct a transverse Slodowy slice which works for all good characteristics, thus improving on the restriction \( p > \text{Cox}(g) \). Following \( \text{Spa84} \), let \( G \) be a simple simply-connected split group scheme over \( S \) as before, with corresponding Chevalley algebra \( g \) not of type \( A_n \), and fix a maximal torus, Borel \( T \subset B \) in \( G \) and a fiberwise subregular nilpotent \( x \in g(\mathcal{O}_K) \).

It is known (cf. \( \text{CM93, §8.2 and §8.4} \)) in the simply-laced cases \( (D_n, E_n) \) that \( x \) is a distinguished nilpotent and so by the classification of nilpotent orbits in good characteristic there exists a unique distinguished parabolic \( P \subset G \) corresponding to the orbit of \( x \); for the notions of distinguished elements and subgroups we refer to \( \text{Spa84} \) and \( \text{(CM93, §8)} \). We may therefore define a 1–parameter subgroup \( \lambda : \mathbb{G}_{m,S} \longrightarrow T \) so that, with respect to a root basis \( \Delta \) of the group scheme \( G \) and root system \( \Phi_P \) of \( P \) we have

\[
\langle \lambda(t), \alpha \rangle = \begin{cases} 0, & \alpha \in \Delta, \ -\alpha \in \Phi_P \\ 2, & \alpha \in \Delta, \ -\alpha \not\in \Phi_P \end{cases}
\]

Theorem 4.12 (\( \text{Spa84} \)). Let \( \lambda : \mathbb{G}_{m,S} \longrightarrow T \) be constructed as above and let \( g = \bigoplus_i g(i) \) be the \( \mathbb{Z} \)-graded decomposition of \( g \) into eigenspaces for the induced \( \mathbb{G}_m \)-action

\[
g(i) = \{ x \in g(\mathcal{O}_K) \mid \text{Ad}_{\lambda(t)}(x) = t^i x \ \forall \ t \in \mathbb{G}_m(\mathcal{O}_K) \} \]

Then \( x \in g(2) \) and there exists an affine \( \mathbb{G}_m \)-stable subspace \( a \subset g \) complementary to \( [x, g] \), so that \( S = x + a \) is a Slodowy slice, transverse at \( x \) fiberwise to the nilpotent orbits of \( x_s \) and \( x_\eta \).

We will call such slices \( S \) Spaltenstein to differentiate them from the slices constructed in Corollary 4.11 via Jacobson–Morozov. The proof in \( \text{(loc. cit.)} \) carries over to \( \mathcal{O}_K \) since \( \text{rk}[x, g] = \text{dim}_k[x_s, g_s] \) as a consequence of \( \text{dim}_k C_{g_s}(x_s) = \text{dim}_\mathcal{O}_K C_G(x) \), and one concludes that \( [x, g] \) is a direct factor of \( g \), hence it is free as in Lemma 4.10. Furthermore
\[ \mathfrak{g}(i) \subset [x, \mathfrak{g}] \] for \( i > 0 \) so there exists an affine \( \mathbb{G}_m \)-stable complement \( \mathfrak{a} \subset \bigoplus_{i \leq 0} \mathfrak{g}(i) \) i.e. \( \mathfrak{a} = \mathfrak{g}/[x, \mathfrak{g}] \). The action map \( \mu : G \times S \to \mathfrak{g} \), \( (g, s) \mapsto \text{Ad}_g(s) \) for \( S = x + \mathfrak{a} \) is smooth in a neighborhood of \( (1, 0) \), so by virtue of the homogeneous \( \mathbb{G}_m \)-action it is smooth (see e.g. \cite{Slo80}, §7.4 Cor. 1).

**Remark 4.13.** We can also extend Proposition 4.9 to the \( D_n \) and \( E_n \) cases by writing down a suitable subregular representative \( x \in \mathfrak{g}(\mathcal{O}_K) \), (e.g. [CM93], §5. produces such an \( x \) for \( \mathfrak{g} \) of type \( D_n \)), and then finding \( h, y \) is a computational exercise in the vein of the proof of Proposition 4.9. We chose to consider Spaltenstein slices here so that we can relax the assumption \( p > \text{Cox}(\mathfrak{g}) \) to just \( p \) being a good prime.

### 4.5 \( \mathbb{G}_m \)-actions and \( \mathbb{G}_m \)-deformations.

Suppose we have constructed a suitable transverse slice \( S \) at a fixed fiberwise subregular nilpotent \( x \in \mathfrak{g}(\mathcal{O}_K) \); if \( \mathfrak{g} \) is of type \( A_n \), the slice is \( S = x + \ker(\text{ad}_x) \) for \( \{x, y, h = [x, y]\} \) forming an integral \( \mathfrak{sl}_2 \)-triple, while for the other simply–laced types \( S \) is a Spaltenstein slice constructed as in Section 4.4. The goal of this section is to discuss some \( \mathbb{G}_m \)-actions on slices \( S \) and relate them to the notion of \( \mathbb{G}_m \)-equivariant deformations.

We first define a 1–parameter subgroup \( \lambda : \mathbb{G}_m, S \to T \) when \( \mathfrak{g} \) is of type \( A_n \), similar to Section 4.4. Let \( x \) be a fiberwise subregular nilpotent section of \( \mathfrak{g} \) and complete it to an \( \mathfrak{sl}_2 \)-triple \( \{x, y, h\} \) over \( S \) (Section 4.3). Then \( \{x_s, y_s, h_s\} \) defines a good \( \mathfrak{sl}_2 \)-representation on \( \mathfrak{g}_s \) (Theorem 4.7). Let \( V_{d_i} \) be an irreducible summand of this representation, the unique irreducible good \( \mathfrak{sl}_2 \)-representation of dimension \( d_i \), and let \( \{v_1, \ldots, v_{d_i}\} \) be a basis. There exits a \( \mathbb{G}_m \)-action on \( V_{d_i} \) by linearly extending

\[ t \cdot v_k = t^{d_i-2k+1}v_k \quad t \in \mathbb{G}_m, \quad k \leq d_i \quad (4.4) \]

**Definition 4.14.** If \( \mathfrak{g} \simeq \bigoplus_{i} V_{d_i} \) is the decomposition of \( \mathfrak{g} \) into irreducible good \( \mathfrak{sl}_2 \)-representations, there is a uniquely defined \( \mathbb{G}_m \)-action \( \lambda : \mathbb{G}_m \to \text{Aut}(\mathfrak{g}) \) which operates on each summand \( V_{d_i} \) by the rule (4.4). It commutes with the Lie bracket of \( \mathfrak{g} \) and hence factors through a 1–parameter subgroup, still denoted \( \lambda \). We call \( \lambda \) a 1–parameter subgroup adapted to \( x_s \).

This decomposition holds over \( \mathcal{O}_K \) since \( \mathfrak{g}_\eta \simeq \mathfrak{g}_s \). Moreover, by abuse of notation we denote by \( V_{d_i} \) the \( \mathcal{O}_K \)-module generated by the \( \mathbb{Z} \)-basis \( \{v_1, \ldots, v_{d_i}\} \) for the \( k \)-vector space \( V_{d_i} \) in (4.4). Note that \( \lambda(\mathbb{G}_m) \subseteq T \) and the action of \( \text{ad}_{h_s} \) decomposes \( \mathfrak{g} \) into eigenspaces

\[ \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i), \quad \mathfrak{g}(i) = \{v \in \mathfrak{g} \mid \text{Ad}_{\lambda(t)}(v) = t^i v \ \forall \ t \in \mathbb{G}_m\} \]

which also function as eigenspaces for the \( \lambda \)-action (\cite{Slo80} §7.1, §7.3). In particular \( x \in \mathfrak{g}(2) \) and \( \lambda \) coincides with the 1–parameter subgroup constructed for Spaltenstein slices \( S \) (Theorem 4.12).

**Definition 4.15.** Let \( \lambda \) be as above and \( m : \mathbb{G}_m \to \text{Aut}(\mathfrak{g}) \) be the usual left–multiplication action.
(i) The action \( \mu : t \cdot v = m(t^2)\text{Ad}_{\lambda(t^{-1})}(v), \ v \in \mathfrak{g} \)

fixes the element \( x \) and preserves the Slodowy slice \( \mathcal{S} \). Moreover it is a contracting action on \( \mathcal{S} \), i.e. it extends to an action \( \mathbb{A}^\mathfrak{g}_2 \to \text{Aut}(\mathcal{S}) \) with \( 0 \cdot s = x \) for all \( s \in \mathcal{S} \).

(ii) Let \( \tilde{\mathcal{S}} \) be the preimage of \( \mathcal{S} \) under the Grothendieck alteration \( \pi \). Define an action on \( \tilde{\mathcal{S}} \) via \( \tilde{\mu} : t \cdot [g, v] = [\lambda(t^{-1})g, t^2v], \ t \in \mathbb{G}_m(\mathcal{O}_K), \ g \in G(\mathcal{O}_K), \ v \in \mathfrak{g}(\mathcal{O}_K) \)

This action preserves \( \tilde{\mathcal{S}} \).

By the definition of \( \mu, \tilde{\mu} \) we have that the Grothendieck alteration \( \pi : \tilde{\mathcal{S}} \to \mathcal{S} \) is \( \mathbb{G}_m, \mathcal{S} \)-equivariant, and a simple calculation yields that \( \tilde{\chi} : \tilde{\mathcal{S}} \to \mathfrak{h} \) is also \( \mathbb{G}_m, \mathcal{S} \)-equivariant when \( \mathfrak{h} \cong \mathbb{A}^\mathfrak{g}_2 \) is equipped with contracting \( \mathbb{G}_m, \mathcal{S} \)-action \( t \cdot h = t^2h \); in this case, the action contracts \( \mathfrak{h} \) to the origin.

We next equip \( \mathfrak{h}/W \) with a \( \mathbb{G}_m, \mathcal{S} \)-action so that the adjoint quotient \( \chi : \mathcal{S} \to \mathfrak{h}/W \) becomes \( \mathbb{G}_m, \mathcal{S} \)-equivariant. By ([Dem73], §6 Théorème 3 and Corollaire), if \( p \) is a good prime, then \( \text{Sym}(X^*(T))^W \) is a graded polynomial \( \mathcal{O}_K \)-algebra with homogeneous generators \( \chi_1, \cdots, \chi_r \) of (homogeneous) degrees \( d_1, \cdots, d_r \), and furthermore

\[
\text{Sym}(X^*(T) \otimes \mathbb{Z} k)^W \cong \text{Sym}(X^*(T))^W \otimes \mathbb{Z} k
\]

Thus the degrees \( d_i \) of the generators \( \chi_i \) are the same as their mod \( p \) versions, which are recorded in ([Slo80b], Table in p. 112). As \( T \) is split we also have

\[
\mathfrak{h} \cong \text{Spec}(\text{Sym}(X^*(T))), \ \mathfrak{h}/W \cong \text{Spec}(\text{Sym}(X^*(T))^W) \cong \text{Spec}(\mathcal{O}_K[\chi_1, \cdots, \chi_r])
\]

Define a \( \mathbb{G}_m, \mathcal{S} \)-action on \( \mathfrak{h}/W \) by linearly extending \( t \cdot \chi_i = t^{2d_i} \chi_i \). The following proposition carries over to the relative setting without change.

**Proposition 4.16** ([Slo80b], §7.4 Prop. 1). The adjoint quotient \( \chi : \mathcal{S} \to \mathfrak{h}/W \) is \( \mathbb{G}_m, \mathcal{S} \)-equivariant with respect to action \( \mu \) on \( \mathcal{S} \) and the above action on \( \mathfrak{h}/W \).

**Remark 4.17.** Note that the \( \mathbb{G}_m, \mathcal{S} \)-action \( \mu \) defined on \( \mathcal{S} \) has the opposite weights of the \( \mathbb{G}_m, \mathcal{S} \)-actions defined in ([Ric17], §4.3) and ([BC19], §4.2.4) for the Kostant slice \( \mathcal{S} \). Both constructions are essentially equivalent; we chose this formulation so as to get a contraction on \( \mathfrak{h} \) and stay consistent with the weight conventions discussed in ([Slo80b], §7.4) and ([Spr83], §2).

We now explain the notion of \( \mathbb{G}_m \)-deformations. Suppose \( X_0 \) is a singular hypersurface with a \( \mathbb{G}_m \)-action so that it is \( \mathbb{G}_m \)-equivariantly isomorphic to \( V(f) \) for some weighted–homogeneous polynomial \( f(x_1, \cdots, x_n) \). Here weighted homogeneity means that there exists a tuple \( (d, k_1, \cdots, k_n) \) so that for any monomial

\[
a_{i_1, \cdots, i_n} x_1^{i_1} \cdots x_n^{i_n}
\]

Actually, \( p \) non-torsion suffices; see Remark 3.3.
appearing in \( f \) we have \( \sum_j i_j k_j = d \). In this case we can study the \( \mathbb{G}_m \)-equivariant deformation theory of \( X_0 \simeq V(f) \) by replacing the objects and maps of the associated deformation functors in Section 2.3 with (formal) schemes equipped with a \( \mathbb{G}_{m,S} \)-action and \( \mathbb{G}_{m,S} \)-equivariant morphisms. In particular we may speak of \( \mathbb{G}_{m,S} \)-miniversal deformations.

It is not obvious that \( \mathbb{G}_{m,S} \)-miniversal deformations exist. The following theorem is based on the existence of general miniversal deformations and is proven by checking the definition of miniversality after equipping all objects with a \( \mathbb{G}_m \)-action.

**Theorem 4.18** ([Slo80b], §2.5 Thm). Suppose \( f \in k[x,y,z] \) is weighted–homogeneous and defines a hypersurface \( V(f) \) with isolated singularities in \( \mathbb{A}^3_k \). Then there exists a \( \mathbb{G}_{m,S} \)-miniversal deformation of \( V(f) \) over \( S = \text{Spec}(W(k)) \).

**Example 4.19.** Let us compare the \( \mathbb{G}_m \)-actions for a specific RDP surface. Polynomial \( f(x,y,z) = z^2 + x^2 + y^4 \) defines an \( A_3 \)-singularity (Theorem 2.2) and is weighted–homogeneous with weights \( (2,1,2) \). Its miniversal deformation is given by

\[
V(F) \longrightarrow \text{Spec}(\mathcal{O}_K[t_1,t_2,t_3]), \quad F(x,y,z,a,b,c) = z^2 + x^2 + y^4 + t_1y^2 + t_2y + t_3
\]

(see Example 2.24). Polynomial \( F \) is weighted–homogeneous with weights \( (2,1,2,2,3,4) \) and as a result base \( \text{Spec}(\mathcal{O}_K[t_1,t_2,t_3]) \) admits a \( \mathbb{G}_{m,S} \)-action with weights \( (2,3,4) \), while total space \( V(F) \) admits a \( \mathbb{G}_{m,S} \)-action with weights \( (2,1,2,2,3) \).

On the other hand, for the subregular orbit in \( \mathfrak{sl}_4 \) (the unique Lie algebra of type \( A_3 \)), one may compute its decomposition into good irreducible \( \mathfrak{sl}_2 \)-representations (see [Slo80b], §7.4 Example) as

\[
\mathfrak{sl}_4 \simeq V_4 \oplus V_2 \oplus V_2' \oplus V_2'' \oplus V_0
\]

where \( \mathfrak{sl}_2 \)-modules \( V_i \) have highest weight \( i \). By inspecting the weights of the \( \mu \)-action of Definition 4.15 we get weights \( (6,4,4,4,2) \); note that in both Sections 4.3 and 4.4 a basis for \( S \) (up to translation with \( x \)) is given by a choice of lowest weight vectors for each irreducible \( \mathfrak{sl}_2 \)-summand, and there is an identification \( S \simeq \text{Spec}(\mathcal{O}_K[x_1,\cdots,x_{r+2}]) \) by choosing \( x_i \) to be dual to the lowest–weight vectors. This is why we get the aforementioned weights (cf. [Slo80b], p. 110). For \( \mathfrak{h}/W \) we have that the homogeneous degrees of the fundamental generators are \( (2,3,4) \) (see [Slo80b], Table p. 112); this also follows directly from reading off the degrees of symmetric polynomials \( \sigma_2, \sigma_3, \sigma_4 \).

Note that if we double the weights of the \( \mathbb{G}_m \)-action on \( V(F) \), then up to reordering variables the miniversal deformation \( \phi : V(F) \to \text{Spec}(\mathcal{O}_K[t_1,t_2,t_3]) \) and \( \chi : S \to \mathfrak{h}/W \) are both \( \mathbb{G}_{m,S} \)-equivariant with the same \( \mathbb{G}_m \)-weights on source and target.

The previous example illustrates a more general principle, which we will clarify in Theorem 4.21 in the next section.

### 4.6 Grothendieck alterations for transverse slices.
We retain the assumptions of Section 4.5 and fix a Slodowy slice \( S \) at a fiberwise subregular nilpotent section \( x \) of \( \mathfrak{g} \)
(either by the Jacobson–Morozov method or the Spaltenstein method). Let \( \pi : \tilde{S} \rightarrow S \) be the restriction of the Grothendieck alteration on \( S \) and define \( \mathbb{G}_{m,S} \)-actions \( \mu, \tilde{\mu} \) on \( S, \tilde{S} \) as in Section 4.5.

**Proposition 4.20.** Consider the following commutative diagram induced by restricting diagram (3.3)

\[
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\pi} & S \\
\downarrow{\chi_{\text{res}}} & & \downarrow{\chi_{\text{res}}} \\
\tilde{h} & \xrightarrow{\psi} & h/W \\
\end{array}
\tag{4.5}
\]

\( \tilde{S} \) and its preimage \( \tilde{S} \) under the Grothendieck alteration. Then this diagram induces \( S \)-fiberwise a simultaneous resolution of the singularities of \( \chi : S \rightarrow h/W \).

**Proof.** Let \( \alpha : G \times S \rightarrow g, \tilde{\alpha} : G \times \tilde{S} \rightarrow \tilde{g} \) denote the respective restrictions of the adjoint action of \( G \) on \( g \) and \( \tilde{g} \). By \( G \)-invariance of the adjoint quotient \( \chi \) and its resolution \( \tilde{\chi} \), their restrictions to \( S, \tilde{S} \) yield commutative diagrams:

\[
\begin{array}{ccc}
G \times S & \xrightarrow{\alpha} & g \\
\downarrow{p_2} & & \downarrow{\chi} \\
S & \xrightarrow{\chi_{\text{res}}} & h/W \\
\end{array} \hspace{1cm} \begin{array}{ccc}
G \times \tilde{S} & \xrightarrow{\tilde{\alpha}} & \tilde{g} \\
\downarrow{p_2} & & \downarrow{\tilde{\chi}} \\
\tilde{S} & \xrightarrow{\tilde{\chi}_{\text{res}}} & \tilde{h} \\
\end{array}
\tag{4.6}
\]

As \( \chi \circ \alpha \) is flat and \( p_2 \) is flat and surjective, \( \chi_{\text{res}} \) is flat (\cite[Tag 02JZ]{Sta18}). Since \( \pi : \tilde{S} \rightarrow S \) comes from base-changing proper morphism \( \tilde{g} \rightarrow g \), it is also proper. Applying Lemma 4.4 to \( \pi : \tilde{g} \rightarrow g \) we get that \( \tilde{\alpha} : G \times \tilde{S} \rightarrow \tilde{g} \) smooth, so as \( p_2, \tilde{\chi} \) are smooth, it follows that \( \tilde{\chi}_{\text{res}} \) is smooth as well. We may therefore base–change to the special fiber by Lemma 3.11 (since formation of diagram (4.5) commutes with base–change) and check that we have a simultaneous resolution for \( \tilde{S} \rightarrow \tilde{h}_s/W \), where \( \tilde{S} = S \otimes k \) is a Slodowy slice over \( k \). In this setting, the statement is true by (\cite[Slo80b, §5.3 Corollary]{Slo80b}).

For ease of notation we will henceforth refer to \( \chi_{\text{res}}, \tilde{\chi}_{\text{res}} \) as \( \chi \) and \( \tilde{\chi} \) respectively, when no confusion can arise. We now relate Proposition 4.20 (i.e. diagram (4.5)) to the miniversal deformation of an RDP singularity and its minimal resolution. What follows is essentially the main theorem of \cite{Slo80b}; the study of miniversal deformations of minimal resolutions in this context is due to \cite{Pin77}.

**Theorem 4.21.** Let \( \chi : S \rightarrow h/W \) denote the localization of map \( \chi \) obtained by henselianizing \( S \) and \( h/W \) at \( x \) and \( 0 \) respectively. Let \( \tilde{S} \) be the preimage of \( S \) under the Grothendieck alteration and \( \tilde{\chi} : \tilde{S} \rightarrow \tilde{h} \) the associated map as in diagram (4.5). Assume \( p = \text{char}(k) > n + 1 \) if \( g \) is of Type \( A_n \), otherwise assume \( p \) is good for \( g \). Then \( \chi \) is a \( \mathbb{G}_{m,S} \)-miniversal deformation of the RDP \( x_s \in (\mathcal{N} \cap S)(k) \) and \( \tilde{\chi} \) is a miniversal deformation of the minimal resolution of \( x_s \).
Proof. Let $r$ be the rank of RDP $x_s$ and consider morphism $\chi$ first. By Proposition 2.23 there exists a miniversal (algebraic) deformation of $x_s$, which we denote as $\phi : V(F) \to \text{Spec}(R)$ for some $F \in \mathcal{O}_K[x, y, z, t_1, \ldots, t_r]$ and $R = \mathcal{O}_K[t_1, \ldots, t_r]$. By Theorem 4.18 there exists a $\mathbb{G}_m,S$–action on $V(F)$ and $\text{Spec}(R)$ making $\phi$ a $\mathbb{G}_m,S$–miniversal deformation.

We know that $\chi^{-1}(0) = N \cap S$ has an RDP singularity at $x_s$ and $\chi$ is $\mathbb{G}_m,S$–equivariant with respect to action $\mu$ on $S$ and the “Weyl exponents” action on $h/W$ (Proposition 4.16), hence by miniversality we get a $\mathbb{G}_m,S$–equivariant morphism $f : h/W \to \text{Spec}(R)$. It suffices to show $f$ is an isomorphism, and by the fibral isomorphism criterion ([EGA IV], Cor. 17.9.5) it suffices to show $f_s : h_s/W \to \text{Spec}(R \otimes k)$ is an isomorphism over geometric point $s = \text{Spec}(k)$. For $\tilde{\chi}$, the statement holds by ([Slo80b], §8.7 Thm) by comparing the $\mathbb{G}_m$–weights on each of $h_s/W$ and $\text{Spec}(R \otimes k)$. For $\chi$, the statement holds by ([SB01], Thm. 3.4).  

Remark 4.22. Shepherd-Barron in ([SB01], Thm. 4.6) has also shown the above theorem via different methods and in good characteristic; the proof similarly contains a passage from characteristic zero to positive characteristic. In particular we may assume that $p$ is good for Theorem 4.21 to be true.

Remark 4.23. The statement a priori concerns formal miniversal deformations, but since we are dealing with affine (isolated) singularities these deformations are algebraic (see Remark 2.26). We may therefore consider the henselianized versions of $\chi, \tilde{\chi}$ when we refer to the simultaneous resolution diagram (4.5) and wlog. write $h/W \simeq \text{Spec}(\mathcal{O}_K[t_1, \ldots, t_r])$.

5. The monodromy Weyl action

We now come to the central part of the article, the description of monodromy actions in terms of Weyl groups. We discuss the relevant $W$-actions in Section 5.6 along with the proof of the main theorem. In order to construct said $W$-actions, multiple tools from the theory of nearby cycles and (relative) perverse sheaves need to be combined, so we explain these concepts next.

5.1 Classical nearby cycles. Throughout this section we work in the small étale topos setting. Our base scheme is a strictly henselian trait $S = \text{Spec}(\mathcal{O}_K)$ with closed point $s = \text{Spec}(k)$ and generic point $\eta = \text{Spec}(K)$. We denote geometric points with a bar, e.g. $\bar{s}$, with the understanding that the underlying residue field is separably closed (e.g. $\bar{\eta} = \text{Spec}(K^{\text{sep}})$). Whenever appropriate we assume $\mathcal{O}_K$ is complete.

Given a finite-type $S$-scheme $X$, we denote by $D^b(X)$ the bounded derived category of $\mathbb{Q}_\ell$-sheaves on $X$, where $\ell \neq p = \text{char}(k)$. Most of the formalism below is usually developed first for finite coefficient rings $\mathbb{Z}/\ell^n$, but standard reductions via inverse limits and taking $(-) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ yield the same statements for $\mathbb{Q}_\ell$-coefficients, so we choose not to belabor this point.

Definition 5.1. Let $X \to S$ be a finite-type $S$-scheme with generic fiber $X_\eta$, geometric generic fiber $X_{\bar{\eta}}$ and special fiber $X_s$. Denote the respective inclusions by
(i) The nearby cycles functor is $R\Psi_X : D^b(X_\eta) \to D^b(X_s)$, $\mathcal{F} \mapsto i^* R\tilde{j}_*(\mathcal{F}_\eta)$, where $\mathcal{F}_\eta$ is the pullback of $\mathcal{F}$ to $X_\eta$. Complex $R\Psi_X(\mathcal{F})$ is naturally equipped with an action of inertia $I = \text{Gal}(\eta/\eta) = \text{Gal}_K$ (SGA 7 II, Exposé XIII §1.3).

(ii) For $\mathcal{F} \in D^b(X)$, adjunction map $\mathcal{F} \to R\tilde{j}_*(\mathcal{F}_\eta)$ gives an exact triangle

\[ i^* \mathcal{F} \xrightarrow{\phi} R\Psi_X(\mathcal{F}) \to R\Phi_X(\mathcal{F}) \xrightarrow{[1]} \] (5.1)

defining the vanishing cycles functor $R\Phi : D^b(X) \to D^b(X_s)$ as the cone of map $\phi$.

Nearby and vanishing cycles have various functorial properties. For example, if $f : X \to Y$ is a morphism of $S$-schemes inducing maps $f_\eta : X_\eta \to Y_\eta$ and $f_s : X_s \to Y_s$ between the geometric generic and special fibers, there are natural maps

\[ R\Psi_Y(Rf_\eta_* \mathcal{F}_\eta) \to Rf_s_* R\Psi_X(\mathcal{F}), \quad \mathcal{F} \in D^b(X) \] (5.2)

\[ f^*_s R\Psi_Y(\mathcal{F}) \to R\Psi_X(f^*_\eta \mathcal{F}_\eta), \quad \mathcal{F} \in D^b(Y) \] (5.3)

Map (5.2) is an isomorphism when $f$ is proper, and map (5.3) is an isomorphism when $f$ is smooth. Moreover the natural $I$-action extends to $R\Phi_X(\mathcal{F})$, making exact triangle (5.1) $I$-equivariant (SGA 7 II, Exposé XIII, 2.1.7.1, 2.1.7.2, 2.1.2.4).

Given an $S$-scheme $X$ with $\mathcal{F} \in D^b(X)$ and a point $x \in X_s$, let $X_{(x)}$ denote the strict henselization of $X$ at $x \to X$. Then the stalks of the nearby cycles are computed as

\[ (R\Psi_X(\mathcal{F}))_x \simeq R\Gamma(X_{(x)} \times_S \eta, \mathcal{F}_\eta) \]

We adopt the terminology of Illusie (cf. [Illusie 17], §1.3) in saying that $X_{(x)}$ represents an $\ell$-adic Milnor ball and generic fiber $X_{(x)} \times_S \eta$ represents an $\ell$-adic Milnor fiber, consistent with the classical fact that stalks of (complex) nearby cycles compute the cohomology of Milnor fibers.

**Example 5.2.** A more direct relationship with the classical Milnor fiber can be seen as follows. Assume $X$ is a flat relative surface over $S$, smooth outside an isolated rational singularity on the special fiber $x \to X_s$, then $R\Phi_X(\mathcal{F})$ is supported on the physical point $x$ and we obtain $(R^0\Phi_X(\mathcal{F}))_x \simeq (R^0\Psi_X(\mathcal{F}))_x \simeq 0$ and $(R^i\Phi_X(\mathcal{F}))_x \simeq (R^i\Psi_X(\mathcal{F}))_x$ for $i > 0$. In this case we have (SGA 7 II, Exposé XVI)

\[ (R^i\Phi_X(\mathcal{F}))_x \simeq \begin{cases} \Lambda^r, & i = 2 \\ 0, & i \neq 2 \end{cases} \]

where $r \geq 1$ is the dimension of the stalk as a $\mathbb{Q}_\ell$-vector space. This can be thought of as the $\ell$-adic analogue of the topological Milnor fiber being a bouquet of $n$-spheres, hence having only top cohomology.
Since we will only deal with nearby cycles of families $X \to S$ acquiring isolated rational singularities, it is worth re-emphasizing that $R^i\Psi_X\mathbb{Q}_\ell \simeq R^i\Phi_X\mathbb{Q}_\ell$ for $i > 0$. Hence any statements regarding stalks of nearby cycles should be compatible with analogous statements in the complex setting, where people usually consider so-called “vanishing homology”.

**Remark 5.3.** In the case that $x$ is a hypersurface singularity locally defined by the vanishing of a weighted–homogeneous polynomial $f(x, y, z)$ (such as the normal form of an RDP as defined in Theorem 2.2), we have that the dimension of the Tjurina algebra $\dim(T^1)$ (see Definition 2.17) equals the Milnor number $\mu = \text{length}(\mathcal{E}\text{xt}^1(\Omega^1_{X/S}, \mathcal{O}_X))$. If the $I$-action on $R\Psi_X\mathbb{Q}_\ell$ is tamely ramified, meaning the action factors through tame quotient $I \to I_t \simeq I/P$, we have $r = \mu = \dim(T^1)$. More generally the Deligne–Milnor conjecture states that $\dim(T^1) = r + \text{Sw}(R^2\Psi_X\mathbb{Q}_\ell)$, where the Swan conductor term measures the wild ramification of the nearby cycles ([SGA 7 II], Exposé XVI, Conj. 1.9).

We will see that under our restrictions on the characteristic $p$, for a surface family $X/S$ acquiring RDP singularities we have $R\Psi_X\mathbb{Q}_\ell \simeq (R\Psi_X\mathbb{Q}_\ell)^P$ (the wild inertia invariants) i.e. $R\Psi_X\mathbb{Q}_\ell$ are identified with the tame inertial nearby cycles $R\Psi_X\mathbb{Q}_\ell = t^rRj^u\mathbb{Q}_\ell$, where $j^u : X_{\eta^u} \hookrightarrow X$ is induced from inclusion $\eta^u = \text{Spec}(K^u) \to S$.

### 5.2 Nearby cycles on formal schemes.

In this section we assume $S = \text{Spec}(\mathcal{O}_K)$ is a complete local trait. In [Ber96], Berkovich constructs a nearby cycles functor variant for a class of formal schemes over $\text{Spf}(\mathcal{O}_K)$, which includes finite-type schemes $\mathcal{X}/S$ completed along a closed subscheme of the special fiber $Y \subseteq \mathcal{X}_s$ ([Ber96], §1). Denote by $\hat{\mathcal{X}}$ the formal completion $\hat{\mathcal{X}}_Y$. Its special fiber $\hat{\mathcal{X}}_s$ is identified with finite-type scheme $Y$, and the generic fiber $\hat{\mathcal{X}}_{\eta}$ is a rigid-analytic space over $K$.

There exists an equivalence between formal schemes étale over $\hat{\mathcal{X}}$ and formal schemes étale over $\hat{\mathcal{X}}_s$ ([Ber96], Prop. 2.1.(i)) and so composing the associated functor $\hat{\mathcal{X}}_s \to \hat{\mathcal{X}}$ with the generic fiber functor $\hat{\mathcal{X}} \to \hat{\mathcal{X}}_{\eta}$ induces a map of sites $\nu : (\hat{\mathcal{X}}_{\eta})_{\text{qét}} \to (\hat{\mathcal{X}}_s)_{\text{ét}}$. The source endows $\hat{\mathcal{X}}_{\eta}$ with its quasi-étale site, where the quasi-étale covers of analytic spaces are defined in the sense of ([Ber94], §3). There exists also a natural morphism of sites $\mu : (\hat{\mathcal{X}}_{\eta})_{\text{qét}} \to (\hat{\mathcal{X}}_{\eta})_{\text{ét}}$.

**Definition 5.4.** For an étale sheaf $\mathcal{F} \in D^b(\hat{\mathcal{X}}_{\eta})$, $R\Psi^\text{Ber}\mathcal{F}$ defines the Berkovich nearby cycles functor $R\Psi^\text{Ber}_{\hat{\mathcal{X}}} : D^b(\hat{\mathcal{X}}_{\eta}) \to D^b(\hat{\mathcal{X}}_s)$. It is a sheaf naturally equipped with a $\text{Gal}(\overline{\eta}/\eta) = \text{Gal}_K$-action ([Ber96], Rem. 2.6).

The crux of this construction is a comparison theorem with the algebraic nearby cycles defined on $\mathcal{X}/S$; the rough idea is that $R\Psi_X\mathbb{Q}_\ell|_Y$ depends only on the formal completion $\mathcal{X}$ along $Y$, hence on a formal neighborhood of $Y$. The comparison theorem below is stated for étale sheaves of torsion prime to $p$, though as we have remarked before the statement works for $\mathbb{Q}_\ell$ as well.

**Theorem 5.5** ([Ber96], Thm. 3.1, Cor. 3.5). Let $\mathcal{X}, Y, \mathcal{X}$ be as above. Let $\mathcal{F}$ be an étale constructible sheaf on $\mathcal{X}_s$ with torsion prime to $p$, and denote by $\tilde{\mathcal{F}}$ its pullback to $\hat{\mathcal{X}}_{\eta}$.
Suppose $Y/S$ is proper (e.g. finite). Then there exist canonical isomorphisms
\[ R^n\Psi_X^*F|_Y \cong R^n\Psi_{X,\text{Ber}}^*(\hat{F}), \quad \Gamma(Y, R\Psi_X^*F) \cong \Gamma(X_{\bar{\eta}}, F) \]
compatible with the action of $\text{Gal}(\bar{\eta}/\eta)$ on either side.

**Corollary 5.6.** Suppose $\mathcal{X}$ is a proper flat surface over $S$ with smooth generic fiber $\mathcal{X}_\eta$ and special fiber $\mathcal{X}_s$ having exactly one RDP $x \in \mathcal{X}_s(k)$. Then the $\text{Gal}(\bar{\eta}/\eta)$-action on $H^2_{\text{ét}}(\mathcal{X}_\eta, \mathbb{Q}_\ell)$ depends only on a formal affine neighborhood of $\bar{x}$ in $\mathcal{X}$.

**Proof.** Let
\[ \hat{\mathcal{O}}_{\mathcal{X}_s, x} \simeq k[[x, y, z]]/f(x, y, z) \]
be the completed local ring at the RDP singularity, $f$ the normal form describing the RDP. By Corollary 2.25
\[ \hat{\mathcal{O}}_{\mathcal{X}, \bar{x}} \simeq \mathcal{O}_K[[x, y, z]]/F(x, y, z), \quad F(x, y, z) \equiv f(x, y, z) \mod p \]
for some polynomial $F(x, y, z)$ that is the pullback of a miniversal equation of the RDP as in Corollary 2.25. Now by the comparison of Berkovich (Theorem 5.5), we obtain a canonical Galois-equivariant isomorphism
\[ (R^2\Psi_{\mathcal{X}}^*\mathbb{Q}_\ell)_{\bar{x}} \cong H^2((\text{Spf}(\hat{\mathcal{O}}_{\mathcal{X}_s, x}))_{\bar{\eta}}, \mathbb{Q}_\ell) \tag{5.4} \]
where the generic fiber of the formal completion of $\mathcal{X}$ along $\bar{x}$ is $\mathcal{X}_{\bar{\eta}} = \text{Spf}(\hat{\mathcal{O}}_{\mathcal{X}, \bar{x}})_{\bar{\eta}}$, a rigid-analytic variety over $\bar{\eta}$, and the right-hand side denotes Berkovich’s $\ell$-adic cohomology for analytic spaces (Ber96, §3). Now let $\mathcal{Y} \to S$ denote the affine relative surface
\[ \mathcal{Y} = \text{Spec}(\mathcal{O}_K[[x, y, z]]/F(x, y, z)), \quad \mathcal{Y}_s = \text{Spec}(k[[x, y, z]]/f(x, y, z)) \]
having the same RDP at $\bar{y} \to \mathcal{Y}_s$ as $\bar{x} \to \mathcal{X}_s$. We have $\hat{\mathcal{O}}_{\mathcal{Y}_{\bar{y}}, \bar{y}} \simeq \hat{\mathcal{O}}_{\mathcal{X}, \bar{x}}$ and so (5.4) gives $R^2\Psi_{\mathcal{X}}^*\mathbb{Q}_\ell)_{\bar{x}} \simeq (R^2\Psi_{\mathcal{Y}}^*\mathbb{Q}_\ell)_{\bar{y}}$ Galois-equivariantly. To compare this with the Galois action $H^2_{\text{ét}}(\mathcal{X}_\eta, \mathbb{Q}_\ell)$, we use the nearby cycles spectral sequence:
\[ E_2^{ij} = H^i(\mathcal{X}_s, R^j\Psi_{\mathcal{X}}^*\mathbb{Q}_\ell) \Rightarrow H^{i+j}(\mathcal{X}_s, R\Psi_{\mathcal{X}}^*\mathbb{Q}_\ell) \simeq H^{i+j}(\mathcal{X}_\eta, \mathbb{Q}_\ell) \]
Since $R^2\Psi_{\mathcal{X}}^*\mathbb{Q}_\ell$ is a skyscraper sheaf supported on $\bar{x}$, the $E_2$-page is
\[
\begin{array}{ccc}
H^0(\mathcal{X}_s, R^2\Psi_{\mathcal{X}}^*\mathbb{Q}_\ell) & 0 & 0 \\
0 & 0 & \rightarrow 0 \\
H^0(\mathcal{X}_s, \mathbb{Q}_\ell) & H^1(\mathcal{X}_s, \mathbb{Q}_\ell) & H^2(\mathcal{X}_s, \mathbb{Q}_\ell)
\end{array}
\]
hence the spectral sequence degenerates at $E_2$ and $E_2^{p,q} = gr_p H^{p+q}(\mathcal{X}_\eta, \mathbb{Q}_\ell)$ for the abutment filtration $F^\bullet$. Looking at the nontrivial graded pieces yields $E_0^{0,2} = H^2(\mathcal{X}_\eta, \mathbb{Q}_\ell)/E_2^{0,0}$ i.e. we obtain $H^2(\mathcal{X} \eta, \mathbb{Q}_\ell)$ and the induced monodromy $\text{Gal}(\overline{\eta}/\eta)$-action on the stalks of $R\Psi_X \mathbb{Q}_\ell$; the latter claim follows from the Galois-equivariance of the $E_2$-page of the spectral sequence. □

5.3 Nearby cycles on a Grothendieck topos. Since we will need to describe nearby cycles of $\ell$-adic sheaves over bases of dimension $> 1$, we collect here their general formalism and properties in amenable situations. In order to do this, we will need the language of oriented toposes; for a modern English reference, see ([Ill17], §1).

We retain the conventions of Section 5.1 and consider $S$-schemes $f : X \to S$, $g : Y \to S$. Bounded derived category of étale sheaves $\text{D}^b(X)$ is assumed to have coefficient ring $\mathbb{Z}/\ell^n$, but the statements below will also work for $\mathbb{Q}_\ell$-coefficients.

**Definition 5.7.** The left oriented 2-product topos $X \leftarrow S Y$ is the Grothendieck topos defined in a universal way by the data of 2-commutative diagram

![Diagram](5.5)

where $X, Y$, and $S$ denote the étale toposes associated to the schemes and $\tau : g \circ p_2 \to f \circ p_1$ is a 2-morphism. From these data we get a defining site for $X \leftarrow S Y$ with the following covering families: for maps $U \to V \leftarrow W$ étale over $X \to S \leftarrow Y$ put $\{U_i \to V \leftarrow W\}$ a covering of $U \to V \leftarrow W$ where $\{U_i\} \to U$ is a covering, and $\{U \to V \leftarrow W_i\}$ a covering of $U \to V \leftarrow W$ where $\{W_i\} \to W$ is a covering. The third type of families is given by coverings $\{U \to V' \leftarrow W\}$ of $U \to V \leftarrow W$ for which the induced square

![Diagram](37)

is cartesian; note the individual maps $V' \to V$, $W' \to W$ need not be coverings in this case ([Ill17], 1.1.1).

Maps between such oriented toposes $X' \leftarrow S' Y' \to X \leftarrow S Y$ are defined by the data of maps $\{X' \to X, S' \to S, Y' \to Y\}$ and appropriate 2-morphisms between these maps ([Ill17], 1.1.2).

A special case is is $(Y, g) = (S, \text{id})$, where the oriented product $X \leftarrow S$ is called the vanishing topos of $X/S$. The points of the vanishing topos consist of triples $(\overline{\pi}, \overline{\eta}, \text{sp})$ where $\overline{\pi} \to X, \overline{\eta} \to S$ are geometric points together with a specialization morphism $\text{sp} : \overline{\eta} \to S_{(f(x))}$.
Definition 5.8 (Nearby cycles over general bases).

(i) There exists a unique morphism $\Psi : X \rightarrow X \times_{S} S$ compatible with diagram (5.5),
by the universal property of products. The derived pushforward

$$R\Psi_{f} = R\Psi_{*} : D^{b}(X) \rightarrow D^{b}(X \times_{S} S)$$

is the nearby cycles functor relative to $f$.

(ii) Let $(\pi, \eta, \text{sp})$ be a point of $X \times_{S} S$ with $\pi$ over a geometric point $\pi \rightarrow S$, $\eta \in S$
and sp : $\eta \rightarrow S_{(\eta)}$ a fixed specialization. There exists a unique map $S_{(\eta)} \rightarrow S_{(\eta)}$
compatible with the specialization and map $\eta \rightarrow S_{(\eta)}$ ([Sta18, Tag 08HR]). We
therefore have natural inclusions

$$i_{s} : X_{\pi} \hookrightarrow X_{(\pi)} = X \times_{S} S_{(\pi)}, \quad j_{s}^{\ast} : X_{(\eta)} = X \times_{S} S_{(\eta)} \rightarrow X_{(\pi)}$$

and $R\Psi_{s}^{\ast} = i_{s}^{\ast} Rj_{s}^{\ast} : D^{b}(X_{(\eta)}) \rightarrow D^{b}(X_{\pi})$ is the sliced nearby cycles functor.

The two nearby cycles are related as follows: one identifies topos $X_{\pi} \times_{S} \eta$ as sheaves on $X_{\pi}$
together with a $\text{Gal}(\eta/\eta)$-action, and by functoriality we have a morphism of toposes

$$i_{(s, \eta)} : X_{\pi} = X_{\pi} \times_{S} \eta \rightarrow X_{\pi} \times_{S} S \xrightarrow{i_{s}} X \times_{S} S$$

Then $R\Psi_{s}^{\ast} = (i_{(s, \eta)})^{\ast} R\Psi_{f}$. See ([Ill17], §1.3 and §1.4) for details. We mention in passing
that one can also define a vanishing cycles functor $R\Phi_{f}$ in this generality, but we will not use it; the construction is given in (loc. cit., 1.2.4).

Nearby cycles in this generality still satisfy functorial properties. An important example
is the case of $S$–schemes $f : X \rightarrow S$ and $g : Y \rightarrow S$ and a map $h : X \rightarrow Y$ of $S$–schemes.
Then the induced commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\Psi_{f}} & X \times_{S} S \\
\downarrow{h} & & \downarrow{h} \\
Y & \xrightarrow{\Psi_{g}} & Y \times_{S} S
\end{array}
\xleftarrow{i_{X,(s,\eta)}} X_{\pi}
\xleftarrow{i_{Y,(s,\eta)}} Y_{\pi}
$$

yields $R\Psi_{g}(R\Phi_{*}F) \simeq R\Psi_{h} R\Psi_{f} F$ for $F \in D^{b}(X)$; furthermore, if $h$ is proper, then formation of $R\Phi_{h}$ commutes with base–change on $X$ and $S$, so that in particular diagram (5.6)
induces an isomorphism (see [Org06], Lemme 8.1.1)

$$R\Psi_{g,\eta}^{\ast}(R\Phi_{*}F) = (i_{Y,(s,\eta)})^{\ast} R\Psi_{g}(R\Phi_{*}F) \simeq R(h_{s})_{\ast} (i_{X,(s,\eta)})^{\ast} R\Psi_{f} F = R(h_{s})_{\ast} R\Psi_{s,\eta}^{\ast} F \quad (5.7)$$

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There are natural generalizations of \( \ell \)-adic Milnor fibers to this setting (cf. Section 5.1). Given a point \((\overline{\sigma}, \overline{\eta}, sp)\) of \( X \times_S S \) with \( \overline{\sigma} \) over \( \sigma \) and a sheaf \( \mathcal{F} \in \text{D}^b(X) \), the stalks of nearby cycles may be computed via ([SGA 4], Exposé VII, §5.8) as

\[
R\Psi^s_{\overline{\sigma}}(\mathcal{F})_{\overline{\sigma}} \simeq R\Psi^s_f(\mathcal{F})_{(\overline{\sigma}, \overline{\eta}, sp)} \simeq \Gamma(X_{(\overline{\sigma})} \times_{S_{(\overline{\eta})}} S_{(\overline{\eta})}, \mathcal{F}) \tag{5.8}
\]

The scheme \( X_{(\overline{\sigma})} \times_{S_{(\overline{\eta})}} S_{(\overline{\eta})} \) is called the Milnor tube at \((\overline{\sigma}, \overline{\eta}, sp)\); it contains Milnor fiber \( X_{(\overline{\sigma})} \times_S S_{(\overline{\eta})} \) as a closed subscheme.

**Example 5.9.** We relate these constructions to the classical nearby cycles. Let \( S \) be a strictly henselian trait with closed point \( \overline{\sigma} \), generic point \( \overline{\eta} \) and geometric generic point \( \eta \). Note \( S_{(\overline{\sigma})} = S \) and \( S_{(\overline{\eta})} = \eta \). For \( f : X \to S \) of finite type we have

\[
X \times_S S = (X_{(\overline{\eta})} \times_S S) \cup (X_s \times_S S) = X_{\overline{\eta}} \cup X_s \cup (X_s \times_S \overline{\eta})
\]

where the last (nontrivial) topos on the right is identified with sheaves on \( X_s \) together with a \( \text{Gal}(\overline{\eta}/\eta) \)-action. The classical nearby cycles \( R\Psi_X(\mathcal{F}) \) for \( \mathcal{F} \in \text{D}^b(X_{(\overline{\eta})}, \Lambda) \) are

\[
R\Psi_X(\mathcal{F})_{(\overline{\eta})} = R\Psi^s_f(\mathcal{F})_{|X_s \times_S \overline{\eta}}
\]

and are therefore identified with the sliced nearby cycles \( R\Psi^s_{(\overline{\eta})}(\mathcal{F}) \). The restriction map is induced from \( X_s \to X \) and \( \overline{\eta} \to S \). Moreover, for a geometric point \( \overline{\sigma} \to X_s \), the Milnor tube \( X_{(\overline{\sigma})} \times_S S_{(\overline{\eta})} \simeq X_{(\overline{\sigma})} \times_S \overline{\eta} \) is identified with the Milnor fiber.

In general, nearby cycles \( R\Psi^s_f \mathcal{F} \) need not be well-behaved; for example, it may not be constructible. Furthermore, base-changing via \( S' \to S \) yields cartesian squares

\[
\begin{align*}
X' & \xrightarrow{g} X \quad & X' & \xrightarrow{g} X \\
S' & \xrightarrow{f'} \quad & S' & \xrightarrow{g} X \times_S S
\end{align*}
\]

and the associated base-change map

\[
(\xrightarrow{g})^* R\Psi^s_f \mathcal{F} \longrightarrow R\Psi^s_{f'}(g^* \mathcal{F})
\]

is not always an isomorphism: see ([Ill17], 1.7(d)) for a classical example of Deligne which shows that for the origin blowup \( f : \mathbb{A}^2_k \to \mathbb{A}^2_k \), \( R\Psi^s_f \mathbb{Q}_\ell \) is not constructible and does not commute with base-change on \( \mathbb{A}^2_k \). Moreover, for any point in the exceptional divisor and a nonzero point on \( \mathbb{A}^2_k \), the associated Milnor tube is not of finite type.

Nevertheless, a special case where all the above assertions are true is the following.

**Theorem 5.10** (Deligne, [Ill17], 1.7(c)). Let \( f : X \to S \) be separated and finite-type, and \( \mathcal{F} \in \text{D}_c^b(X) \). Let \( Z \) be the complement of the largest open set \( U \subseteq X \) so that \( f|_U \) is universally locally acyclic over \( S \). If \( Z \to S \) is quasifinite, then \( R\Psi^s_f \mathcal{F} \) is constructible and its formation commutes with any base-change \( S' \to S \).
Remark 5.11. We call \((f, \mathcal{F}) \Psi\)-good in this case. It follows from the theorem that the sliced nearby cycles \(R\Psi_f(\mathcal{F})\) are also constructible and commute with base-change; in particular the cohomology of Milnor tubes as computed in \((5.8)\) restricts isomorphically to the cohomology of Milnor fibers:

\[
(R\Psi_f \mathcal{F})_{(\varpi, \eta, sp)} \sim \mathbb{R}\Gamma(X_{(\varpi)} \times_{S(\eta)} \eta, \mathcal{F}) \quad (5.9)
\]

5.4 Relative perverse sheaves and the Grothendieck alteration. We retain the conventions of Section 5.1. In this section we will discuss a recent notion of a relative perverse \(t\)-structure on \(D^b_c(X)\) for \(S\)-schemes \(X\), which is compatible (in a sense) with the absolute perverse \(t\)-structures on the geometric fibers i.e. \(D^b_c(X_{\eta})\) and \(D^b_c(X_s)\). As before we may assume the underlying coefficient field to be \(\mathbb{Q}_\ell\).

Following [HS21], we define a full subcategory of \(D^b(X)\)

\[
p^{/S}D(X)_{\leq 0} := \{ \mathcal{F} \in D^b(X) \mid \mathcal{F}|_{X_\varpi} \in p^D(X_\varpi)_{\leq 0} \text{ for all geometric points } \varpi \to S \} \quad (5.10)
\]

where \((p^D_{\leq 0}, p^D_{\geq 0})\) denotes the absolute perverse \(t\)-structure for schemes over fields. In the particular case of \(S = \text{Spec}(\mathcal{O}_K)\), one just restricts \(\mathcal{F}\) to \(X_\varpi\) and \(X_s\) and checks perversity in the usual absolute sense.

It can be shown that \((5.10)\) forms the connective part of a \(t\)-structure on \(D^b(X)\) and the goal is then to show that the coconnective part \(p^{/S}D(X)_{\geq 0}\) has the required description in analogy with \((5.10)\) and further induces a \(t\)-structure on the bounded derived category of constructible complexes \(D^b_c(X)\).

Theorem 5.12 ([HS21], Thm.6.1). Let \(f : X \to S\) be a finite-type \(S\)-scheme. The full subcategories \((p^{/S}D_{\leq 0}, p^{/S}D_{\geq 0})\) of \(D^b(X)\) define a unique \(t\)-structure on \(D^b(X)\), the relative perverse \(t\)-structure, so that

(i) A sheaf \(\mathcal{F} \in D^b(X)\) is in \(p^{/S}D(X)_{\leq 0}\), resp. \(p^{/S}D(X)_{\geq 0}\), if and only if \(\mathcal{F}|_{X_\varpi} \in p^D(X_\varpi)_{\leq 0}\) and \(\mathcal{F}|_{X_\varpi} \in p^D(X_\varpi)_{\geq 0}\), resp. \(\mathcal{F}|_{X_\varpi} \in p^D(X_\varpi)_{\geq 0}\) and \(\mathcal{F}|_{X_\varpi} \in p^D(X_\varpi)_{\leq 0}\).

(ii) For any morphism \(g : \tilde{S} \to S\) with induced base-change \(\tilde{g} : X_{\tilde{S}} \to X\), pullback functor \(\tilde{g}^* : D^b(X) \to D^b(X_{\tilde{S}})\) is \(t\)-exact with respect to the relative perverse \(t\)-structure, hence commutes with the associated truncations \((\tau_{\leq 0}, \tau_{\geq 0})\).

(iii) For any open and closed decomposition \(j : U \hookrightarrow X\) and \(i : Z \hookrightarrow X\) into \(S\)-schemes, the relative perverse \(t\)-structure on \(D^b(X)\) is obtained by gluing (recollement) from the relative perverse \(t\)-structures on \(D^b(U)\) and \(D^b(Z)\).

We note that (ii) and (iii) are formal consequences of (i), once the perverse \(t\)-structure properties have been established. We also note that if \(S = \text{Spec}(k)\) is a field and \(X\) is a finite-type \(k\)-scheme, the relative perverse \(t\)-structure is identified with the usual (middle) perverse \(t\)-structure on \(D^b(X)\), while if \(X = S\) then the relative perverse \(t\)-structure coincides with the standard \(t\)-structure on \(D^b(S)\). Therefore, in view of (ii), relative perverse sheaves on \(X\) pull back to absolute perverse sheaves on \(X_\eta\) and \(X_s\).
We apply the notion of relative perversity to the setting of Chevalley algebras \( \mathfrak{g} \) over \( S = \text{Spec}(\mathcal{O}_K) \) and their associated bundles \( \tilde{\mathfrak{g}} \). Recall the setting of Sections 3.4 and 3.5 so that in particular \( \mathfrak{g} \) is a simple simply-laced Chevalley \( \mathcal{O}_K \)-algebra and \( p = \text{char}(k) \) is very good for \( \mathfrak{g} \). The Grothendieck alteration \( \pi: \tilde{\mathfrak{g}} \to \mathfrak{g} \) restricts to a Galois \( W \)-torsor \( \pi^{rs}: \tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs} \) over the regular semisimple locus (Proposition 3.17).

For Lie algebras \( \mathfrak{g} \) over algebraically closed field \( k \), the above fact is classically known and furthermore it implies that the absolute perverse sheaf

\[
\mathcal{F} = R\pi_*\mathcal{Q}_\ell[\dim \mathfrak{g}]
\]

is equipped with a \( W \)-action, constructed by Borho–MacPherson (who attribute it originally to Lusztig) in the following way: \( \pi \) is a small morphism so in particular \( \mathcal{F} \) is an IC sheaf, i.e. the intermediate extension of local system \( \mathcal{G} = \pi^*\mathcal{Q}_\ell \) ([BM83], §1.8). Sheaf \( \mathcal{G} \) is \( W \)-equivariant and \( W \) acts by “deck transformations”, so by the functoriality of the intermediate extension functor, the \( W \)-action uniquely extends to \( \mathcal{F} \) (loc. cit., §2.6 Proposition 1).

We next show that this result of Borho–MacPherson extends to the relative Grothendieck alteration \( \pi \) in the setting of Chevalley \( \mathcal{O}_K \)-algebras, where now \( \mathcal{F} \) is relatively perverse and in particular an “intermediate extension” object in \( \text{D}^b_{\lambda}(\mathfrak{g}) \). By an IC sheaf in this setting we mean that, if \( j: \mathfrak{g}^{rs} \hookrightarrow \mathfrak{g} \) is the inclusion of the open dense subscheme of regular semisimple elements, then \( \mathcal{F} \simeq j_*\mathcal{G} \).

**Theorem 5.13.** Let \( \pi: \tilde{\mathfrak{g}} \to \mathfrak{g} \) denote the Grothendieck alteration of Chevalley \( \mathcal{O}_K \)-algebra \( \mathfrak{g} \). Then:

(i) Complex \( \mathcal{F} = R\pi_*\mathcal{Q}_\ell[\dim \mathfrak{g}] \in \text{D}^b_{\lambda}(\mathfrak{g}) \) is relatively perverse.

(ii) Complex \( \mathcal{F} \) is an IC sheaf.

**Proof.** Denote by \( j : \mathfrak{g}_\eta \hookrightarrow \mathfrak{g} \) and \( i : \mathfrak{g}_s \hookrightarrow \mathfrak{g} \) the respective open and closed immersions of the geometric generic and special fibers, with the understanding that \( \mathfrak{g}_\eta \) is Lie algebra \( \mathfrak{g}_\tau = \mathfrak{g}_\pi \) so that the constructions of Sections 3.4 and 3.5 make sense for it too. Part (i) is immediate from Theorem 5.12 since proper base-change gives

\[
\mathcal{F}|_{\mathfrak{g}_s} = j^*R\pi_*\mathcal{Q}_\ell[\dim \mathfrak{g}] \simeq R(\pi_\eta)_*\mathcal{Q}_\ell[\dim \mathfrak{g}]
\]

and \( \mathcal{F}|_{\mathfrak{g}_s} \simeq R(\pi_\eta)_*\mathcal{Q}_\ell[\dim \mathfrak{g}] \) as the formation of \( \tilde{\mathfrak{g}} \) commutes with basechange with \( S \) in good characteristics. Thus, both restrictions are perverse sheaves on \( \mathfrak{g}_\eta \) and \( \mathfrak{g}_s \), respectively; in fact they are IC sheaves by the same reasoning of ([BM83], §1.8).

For (ii), let \( j': \mathfrak{g}^{rs} \hookrightarrow \mathfrak{g} \) and \( i': \mathfrak{g}' = \mathfrak{g} \setminus \mathfrak{g}^{rs} \hookrightarrow \mathfrak{g} \) denote the respective open and closed immersions of the regular semisimple elements and their complement. Denote by \( \mathcal{H}^0 \) the composition \( \tau^{<0} \circ \tau^{>0} \) of the relative perverse truncation functors, which is itself a cohomological functor, and set \( \mathcal{G} = j'^*\mathcal{F} \) for brevity. We wish to show

\[
\mathcal{F} \simeq j'_*\mathcal{G} = \text{im}(\mathcal{H}^0(j'_*\mathcal{G}) \to \mathcal{H}^0(j'_*\mathcal{G}))
\]

Note that \( \mathcal{G} \) is a lisse sheaf on \( \mathfrak{g}^{rs} \) via proper base-change along cartesian diagram:

---

\(^3\)Note that Borho–MacPherson state a more general version here, in terms of a parabolic subgroup \( P \subset G \); we need only take \( P \) to be a Borel for our purposes.
That is, \( j^* \mathcal{F} \simeq \pi^r_\mathfrak{g}[\dim \mathfrak{g}] \) and \( \pi^r_\mathfrak{g} \) is a Galois \( W \)-torsor (Proposition 3.17), so that \( \pi^r_\mathfrak{g}[\dim \mathfrak{g}] \) is lisse by algebraic Ehresmann ([Mil], Thm. 20.2). Moreover \( \mathcal{F} = p^\mathcal{H}_0(\mathcal{F}) \) (via the perversity established in (i)) sits in long exact sequences

\[
\cdots \rightarrow p^\mathcal{H}_0(j_!^* \mathcal{G}) \rightarrow \mathcal{F} \rightarrow p^\mathcal{H}_0(i_*^* \mathcal{F}) \xrightarrow{[1]} \cdots
\]

(5.11)

\[
\cdots \rightarrow p^\mathcal{H}_0(i_*^* \mathcal{F}) \rightarrow \mathcal{F} \rightarrow p^\mathcal{H}_0(j'_! \mathcal{G}) \xrightarrow{[1]} \cdots
\]

(5.12)

which are respectively coming from the standard triangles

\[
j_!^* j^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow i_*^* i^* \mathcal{F} \xrightarrow{[1]} \quad i_*^* i^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow j'_! j^* \mathcal{F} \xrightarrow{[1]}
\]

(5.13)

Set \( C_1 = i_*^* i^* \mathcal{F} = \text{Cone}(j_!^* \mathcal{G} \rightarrow \mathcal{F}) \) and \( C_2 = i_*^* i^* \mathcal{F}[1] = \text{Cone}(\mathcal{F} \rightarrow j'_! \mathcal{G}) \). In order to show \( \mathcal{F} \simeq j'_! \mathcal{G} \) it suffices to show \( \mathcal{F} \rightarrow p^\mathcal{H}_0(j'_! \mathcal{G}) \) and \( p^\mathcal{H}_0(j'_! \mathcal{G}) \rightarrow \mathcal{F} \) i.e. the respective kernel and cokernel vanish. So it suffices to have \( p^\mathcal{H}_0(C_1) = 0 \) and \( p^\mathcal{H}_0^{-1}(C_2) = 0 \).

A brief note on why checking the above two conditions is enough: suppose \( p^\mathcal{H}_0(j'_! \mathcal{G}) \rightarrow \mathcal{F} \), then via right-exactness of \( i^* \) we have \( p^\mathcal{H} \circ (i')^* \) right-exact ([Ach21], Lemma A.7.14) so that \( p^\mathcal{H}_0(i^* \circ p^\mathcal{H}_0(j'_! \mathcal{G})) \rightarrow p^\mathcal{H}_0(i^* \mathcal{F}) \). \( p^\mathcal{H}_0 \circ i^* \circ p^\mathcal{H}_0 \circ j'_! \) is left-adjoint to

\[
p^\mathcal{H}_0 \circ j^* \circ p^\mathcal{H}_0 \circ i'_* \simeq p^\mathcal{H}_0 \circ j^* \circ i'_* = 0
\]

where the penultimate equivalence is due to the perverse \( t \)-exactness of \( j^* \) and \( i'_* \). Hence \( p^\mathcal{H}_0(i^* \mathcal{F}) = \tau_{\geq 0}(i^* \mathcal{F}) = 0 \), meaning \( i^* \mathcal{F} \in \text{p}/\text{sD}(\mathfrak{g})^{\geq 1} \). The dual argument for \( \mathcal{F} \hookrightarrow p^\mathcal{H}_0(j'_! \mathcal{G}) \) gives \( i^* \mathcal{F} \in \text{p}/\text{sD}(\mathfrak{g})_{\leq -1} \), altogether giving the familiar IC sheaf conditions for \( \mathcal{F} \) as a result of recollement ([BBD82], 1.4.24).

Consider \( p^\mathcal{H}_0(C_1) \) first; it is enough to show \( j^* p^\mathcal{H}_0(C_1) = i^* p^\mathcal{H}_0(C_1) = 0 \) since if \( p^\mathcal{H}_0(C_1) \) had nonempty support, it would have to intersect the supports of either of these complexes. Now \( j^* \) is perverse \( t \)-exact by Theorem 5.12 (ii), so applying \( j^* \) to the left exact triangle in (5.13) and then taking \( p^\mathcal{H}_0 \), which commutes with \( j^* \), we obtain

\[
\cdots \rightarrow p^\mathcal{H}_0(j^* j'_! \mathcal{G}) \rightarrow j^* \mathcal{F} \rightarrow p^\mathcal{H}_0(j^* C_1) \xrightarrow{[1]} \cdots
\]

(5.14)

Since \( \mathfrak{g}^\mathfrak{rs} \) represents the open subfunctor in \( \mathfrak{g} \) of regular semisimple elements, the following diagram is cartesian

\[
\begin{array}{ccc}
\mathfrak{g}^{\mathfrak{rs}} & \xrightarrow{j^{\mathfrak{rs}}} & \mathfrak{g}^{\mathfrak{rs}} \\
\downarrow^\pi^{\mathfrak{rs}} & & \downarrow^\pi \\
\mathfrak{g}^\mathfrak{rs} & \xrightarrow{j^{\mathfrak{rs}}} & \mathfrak{g}
\end{array}
\]

(5.15)
where \( j^{rs} : g^{rs}_\eta \to g^{rs} \) is induced from \( \eta \to S \) and \( j'_s : g^{rs}_\eta \to g_s \) is the open immersion induced from \( j' \). Hence base-change yields \( j^{rs*} j'^* \simeq j'_s (j^* \mathcal{F}) = j'_s (R(\pi_\eta)_* Q_\ell [\dim g]) \) so that the long exact sequence \((5.14)\) is

\[
\ldots \longrightarrow p\mathcal{H}^0(j'_!(R(\pi_\eta)_* Q_\ell [\dim g])) \big|_{g^{rs}_\eta} \longrightarrow (R(\pi_\eta)_* Q_\ell [\dim g]) \longrightarrow p\mathcal{H}^0(\mathcal{C}_1) \longrightarrow \ldots \tag{5.16}
\]

Now the second-left arrow in \((5.16)\) is surjective since \( R(\pi_\eta)_* Q_\ell [\dim g] \) is an IC sheaf, and \( j^{rs*} \mathcal{H}(j'_! j^{rs*} \mathcal{F}) = 0 \) since \( j'_s j^{rs*} \) is right \( t \)-exact, again by recollement \([BBD82, \text{Prop. } 1.4.12]\). We therefore get \( j^{rs*} \mathcal{H}(\mathcal{C}_1) \simeq p\mathcal{H}^0(\mathcal{C}_1) = 0 \). A similar argument yields \( i^{rs*} \mathcal{H}(\mathcal{C}_1) = 0 \), via long exact sequence

\[
\ldots \longrightarrow p\mathcal{H}^0(i^* j'_! \mathcal{G}) \longrightarrow i^* \mathcal{F} \longrightarrow p\mathcal{H}^0(i^* \mathcal{C}_1) \longrightarrow \ldots \tag{5.17}
\]

and \( i^* j'_! \mathcal{G} \simeq j'_s i^{rs*} \mathcal{G} \simeq j'_s (R(\pi_s)_* Q_\ell [\dim g]) \big|_{g^{rs}_s} \) coming from base-change along diagram

\[
\begin{array}{ccc}
g^{rs}_s & \xrightarrow{j^s} & g^{rs} \\
\downarrow & & \downarrow \\
g_s & \xrightarrow{i} & g
\end{array}
\]  

\[\tag{5.18}\]

where \( i^{rs} \) is induced from \( s \to S \) and \( j'_s \) is the base-change of \( j' \). We therefore yield \( p\mathcal{H}^0(\mathcal{C}_1) = 0 \). For \( p\mathcal{H}^{-1}(\mathcal{C}_2) \) we argue in an analogous manner, namely we apply \( t \)-exact functors \( j^*, i^* \) to the right triangle in \((5.13)\) and show \( j^{rs*} \mathcal{H}^{-1}(\mathcal{C}_2) \simeq i^{rs*} \mathcal{H}^{-1}(\mathcal{C}_2) = 0 \). For \( j^{rs*} \mathcal{H}^{-1}(\mathcal{C}_2) = 0 \) the reasoning is parallel to showing \( j^{rs*} \mathcal{H}^0(\mathcal{C}_1) = 0 \), since \( j^* j'_s \simeq j'_s j^{rs*} \) via flat base-change along diagram \((5.15)\); once again we reduce to \( j^* \mathcal{F} \) being an IC sheaf and \( p\mathcal{H}^{-1}(j'_s j^{rs*} \mathcal{F}) = 0 \) as \( j'_s j^{rs*} \) is left \( t \)-exact \([BBD82, \text{Prop. } 1.4.12]\). Now \( t \)-exactness of \( i^* \) yields exact sequence

\[
\ldots \longrightarrow i^{rs*} \mathcal{H}^{-1}(\mathcal{C}_2) \longrightarrow i^* \mathcal{F} \longrightarrow p\mathcal{H}^0(i^* j'_! \mathcal{G}) \longrightarrow \ldots \tag{5.19}
\]

induced from the right triangle in \((5.13)\), but we cannot immediately conclude by a standard base-change theorem. Consider instead the open/closed decompositions

\[
\begin{array}{ccc}
g^{rs}_s & \xrightarrow{j^s} & g^{rs} \\
\downarrow & & \downarrow \\
g_s & \xrightarrow{i} & g
\end{array}
\]

\[\tag{5.18}\]

induced from the respective decompositions on \( g, g_s \) and their Grothendieck alterations \( \pi, \pi_s \). Together with \( i : g_s \to g, i^{rs} : g^{rs}_s \to g^{rs} \), these maps fit into commutative diagrams

\[
\begin{array}{ccc}
g^{rs}_s & \xrightarrow{j^s} & g^{rs} \\
\downarrow & & \downarrow \\
g_s & \xrightarrow{i} & g
\end{array} \quad \begin{array}{ccc}
g^{rs}_s & \xrightarrow{j^{rs}} & g^{rs} \\
\downarrow & & \downarrow \\
g_s & \xrightarrow{i^{rs}} & g
\end{array}
\]

\[\tag{5.20}\]
by the same token as before. We aim to show \( i^* j'_* \mathcal{G} \simeq j'_s i^{rs*} \mathcal{G} \). Via the Grothendieck alteration \( \pi \), diagram (5.18) and the left diagram in (5.20) fit into the following commutative cube diagram (5.21), all of whose faces are cartesian:

\[
\begin{array}{c}
\tilde{g}_{rs}^i & \to & \tilde{g}_{rs}^s \\
\downarrow j & \downarrow & \downarrow j' \\
\tilde{g}_s & \to & \tilde{g} \\
\end{array}
\]

Here \( \pi_{rs}^s \), \( \pi_{rs}^s \) denote the obvious restrictions of \( \pi \). Now, suppressing that \( \pi_s \) is derived for notation purposes, \( i^* j'_* \mathcal{G} \) becomes

\[
i^* j'_* j^s \circ i^* \circ \pi_* Q_\ell[\dim g] \simeq i^* j'_* \pi_{rs}^s \circ j^s Q_\ell[\dim g]
\]

via proper base-change along the right-face and front-face diagrams, and similarly \( j'_s i^{rs*} \mathcal{G} \) becomes

\[
j'_s i^{rs*} j^s \circ i^* \circ \pi_* Q_\ell[\dim g] \simeq j'_s i^{rs*} \pi_{rs}^s \circ j^s Q_\ell[\dim g]
\]

visually base-change along the right-face and back-face diagrams. Then \( i^* j'_* \mathcal{G} \simeq j'_s i^{rs*} \mathcal{G} \) precisely when \( \tilde{i}^* j'_* Q_\ell \simeq \tilde{j}'_s Q_\ell \), ignoring the dimension shifts. To show this, take the standard exact triangle

\[
\tilde{i}_{s}^! Q_\ell \longrightarrow Q_{\ell, \tilde{g}} \longrightarrow \tilde{j}'_{s} Q_\ell \longrightarrow [1]
\]

We have \( \tilde{i}^! Q_\ell \simeq Q_\ell[-2](-1) \) by Lemma 5.14, which we prove right after this theorem. Applying the \( t \)-exact functor \( \tilde{i}^* : D^b_c(\tilde{g}) \to D^b_c(\tilde{g}_s) \) induces the following diagram from triangle (5.22) and the two diagrams in (5.20), where the vertical arrows are base-change morphisms:

\[
\begin{array}{c}
\tilde{i}_{s}^* Q_\ell[-2](1) \longrightarrow Q_{\ell, \tilde{g}_s} \longrightarrow \tilde{i}^* j'_s Q_\ell \longrightarrow +1 \\
\downarrow \simeq \downarrow \simeq \downarrow h \\
\tilde{j}'_{s} i^* Q_\ell[-2](1) \longrightarrow Q_{\ell, \tilde{g}_s} \longrightarrow \tilde{j}'_{s} Q_\ell \longrightarrow +1
\end{array}
\]

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The left vertical arrow is an isomorphism by proper base-change, and so is the middle vertical arrow, hence $h$ is also an isomorphism (eg. by the five lemma). We conclude that $i^*j^*_s \mathcal{G} \simeq j^*_s i^*s \mathcal{G}$ and so the long exact sequence (5.19) is

$$\cdots \longrightarrow i^*p \mathcal{H}^{-1}(\mathcal{C}_2) \longrightarrow R(\pi_s)_* \mathcal{Q}_\ell[\dim \mathfrak{g}] \longrightarrow p^* \mathcal{H}^0(j^*_s R(\pi_s)_* \mathcal{Q}_\ell[\dim \mathfrak{g}])|_{\mathfrak{g}^{rs}} \longrightarrow \cdots$$  \hspace{1cm} (5.23)

so that $i^*p \mathcal{H}^{-1}(\mathcal{C}_2) = 0$ as $R(\pi_s)_* \mathcal{Q}_\ell[\dim \mathfrak{g}]$ is an IC sheaf. This yields $\mathcal{F} \simeq j^*_s \mathcal{G} = j^*_s j^* \mathcal{F}$ so $\mathcal{F}$ is indeed an IC sheaf on $\mathfrak{g}$. \hfill \Box

**Lemma 5.14.** Let $\mathfrak{g}$ be a Chevalley $\mathcal{O}_K$–algebra so that $\mathfrak{g}_\pi$ and $\mathfrak{g}_s$ are simple Lie algebras of the same Dynkin type. Let $\widetilde{\chi} : \widetilde{\mathfrak{g}} \to \mathfrak{h}$ be the associated adjoint bundle and write $\widetilde{\chi}^* : \mathfrak{h}^{rs} \to \mathfrak{g}^{rs}$ for the complement of the preimage of $\mathfrak{g}^{rs}$ under the Grothendieck alteration. Then $i^\mathfrak{g} \mathcal{Q}_\ell \simeq \mathcal{Q}_\ell[-2](-1)$.

**Proof.** Call $\widetilde{\mathfrak{g}}' = \mathfrak{g}^{rs}$, then we aim to show $\widetilde{\mathfrak{g}}'$ is smooth over $S$. We first show it is flat. As a consequence of Proposition 3.17, restricting $\widetilde{\chi}$ to $\widetilde{\chi}_s : \widetilde{\mathfrak{g}}_s' \to \mathfrak{h}_s \simeq \mathbb{A}^e_K$ we have that every fiber is smooth over $k$, so we get flatness of $\widetilde{\chi}_s$ as follows: if $y = \widetilde{\chi}_s(x)$, the associated local homomorphism $\mathcal{O}_{\mathfrak{h}',y} \to \mathcal{O}_{\mathfrak{g}'_s,x}$ has regular source and a regular fiber ring $F = \mathcal{O}_{\mathfrak{g}'_s,x}/m_y \mathcal{O}_{\mathfrak{g}'_s,x}$, so we may pick a regular system of parameters $(x_1, \ldots, x_r)$ for $\mathcal{O}_{\mathfrak{h}',y}$ and $(y_1, \ldots, y_s)$ in $\mathcal{O}_{\mathfrak{g}'_s,x}$ so that their images in $F$ form a regular system of parameters. We have

$$\dim(\mathcal{O}_{\widetilde{\mathfrak{g}}_s,x}) = \dim(\mathcal{O}_{\mathfrak{h}',y}) + \dim(F)$$

so $(x_1, \ldots, x_r, y_1, \ldots, y_s)$ generates the maximal ideal of $\mathcal{O}_{\mathfrak{g}'_s,x}$ and so $\mathcal{O}_{\mathfrak{g}'_s,x}$ is regular. Then by miracle flatness ([Sta13 Tag 00R4]) we get $\widetilde{\chi}_s$ flat. Since every fiber of $\widetilde{\chi}_s$ is smooth and reduced, we get in fact that $\widetilde{\chi}_s$ is smooth, hence $\widetilde{\mathfrak{g}}'$ is smooth over $k$ and a similar analysis yields $\mathfrak{g}_\pi^{rs}$ smooth over $\overline{K}$.

The same local algebra argument in the previous paragraph yields $\mathfrak{g}$ flat over $S$; the only extra step required here is directly showing flatness of the associated local ring map $\mathcal{O}_K \to \mathcal{O}_{\mathfrak{g}',x}$ (for $x$ an $\mathcal{O}_K$–point of $\mathfrak{g}'$). As $\mathcal{O}_K/(\pi) = k$ and Koszul complex $K_*(\pi, \mathcal{O}_K)$ is a free resolution of $\mathcal{O}_K$–module $k$, we have

$$\text{Tor}_1^{\mathcal{O}_K}(k, \mathcal{O}_{\mathfrak{g}',x}) = H^1(K_*(\pi, \mathcal{O}_K) \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathfrak{g}',x}) = H^1(K_*(\pi, \mathcal{O}_{\mathfrak{g}',x}))$$

and the right-hand side is zero, since $\pi$ is a non-zerodivisor in $\mathcal{O}_{\mathfrak{g}',x}$, so $\mathcal{O}_{\mathfrak{g}',x}$ is flat over $S$. Now as the generic and special fiber are smooth we get that $\mathfrak{g}'$ is also smooth over $S$. Then the proof of ([Ach21], Thm. 2.2.13) shows pair $(\mathfrak{g}', \mathfrak{g})$ is smooth of relative codimension 1, so that $i^\mathfrak{g} \mathcal{Q}_\ell \simeq \mathcal{Q}_\ell[-2](-1)$, completing the claim. \hfill \Box

**Corollary 5.15.** Let $S \subset \mathfrak{g}$ be a relative Slodowy slice at a fiberwise subregular nilpotent $\mathcal{O}_K$–section $x \in N_\mathfrak{g}(\mathcal{O}_K)$. Let $\pi_S : \tilde{S} := \pi^{-1}(S) \longrightarrow S$ be the restriction of the Grothendieck alteration to $\tilde{S}$. Then $\mathcal{F}_S = R\pi_{S*} \mathcal{Q}_\ell[\dim S]$ is a relative IC sheaf.

**Proof.** The arguments of Theorem 5.13 carry over to this setting as soon as we have that the version of $\pi_S$ over a geometric point of $\text{Spec}(\mathcal{O}_K)$ is a small morphism. So we reduce to the case of $k$ an algebraically closed field, $x \in N_\mathfrak{g}$ a subregular nilpotent element in Lie
algebra $\mathfrak{g}$ over $k$, and $S$ the canonical Slodowy slice at $x$, transverse to its orbit (‘canonical’ here means that, in light of Lemma 4.2, any such Slodowy slice has locally the form of the slice described in ([Slo80b], §7.4)).

In this setting we know $S$ is transverse to every adjoint $G$-orbit ([Slo80b], §7.4 Corollary). Take a stratification of $\mathfrak{g}$ into locally closed subsets

$$\mathfrak{g} = X_{-1} \amalg (X_0 \setminus X_{-1}) \amalg \coprod_{n \geq 1} X_n, \quad X_{-1} = \mathfrak{g}^{rs}, X_n = \{x \mid \text{dim}(\pi^{-1}(x)) = n\}$$

which, after a possible refinement, induces a stratification on $S$. By transversality, $S \cap X_n = \emptyset$ since $S$ meets only the regular and subregular orbit in $\mathcal{N}_g$, so $S = S_1 \amalg S_2 \amalg S_3$ where $S_i = S \cap X_{n-2}$, and for any $y \in S_i$ we have

$$\text{dim}(\pi_S^{-1}(y)) \leq \frac{1}{2}(\text{dim}(S) - \text{dim}(S_i))$$

with equality if and only if $i = 0$ (where we get the dense open stratum $S \cap \mathfrak{g}^{rs}$. The only nontrivial case is $y \in S_3$ as $\text{dim}(\pi_S^{-1}(y)) = 1$; since by Theorem 4.21 we have that $S \to \mathfrak{h}/W$ realizes a miniversal deformation of surface singularity $x \in \mathcal{N}_g \cap S$, locus $S_3$ consists of the nearby singularities lying over discriminant divisor $\Delta \subset \mathfrak{h}/W$, hence by flatness $\text{dim}(S_3) = r - 1$. The above inequality therefore is just $1 < \frac{3}{2}$. $\square$

### 5.5 Weyl–Springer actions

We retain the definitions and assumptions of Section 3.1. In the setting of Proposition 3.17 and Theorem 5.13 we have seen that the Grothendieck alteration $\pi$ restricted to $\pi^{rs} : \tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$ is a finite étale $W$-torsor so that $\mathcal{G} = R\pi_*\mathcal{Q}_\ell|_{\mathfrak{g}^{rs}} \simeq \pi^{rs}_*\mathcal{Q}_\ell$ is a lisse sheaf on $\mathfrak{g}^{rs}$. To describe the $W$-action on it, we define an auxiliary geometric vector bundle related to $\tilde{\mathfrak{g}}$ as follows. Let

$$\tilde{\mathfrak{g}}_T := \{(gT, x) \in G/T \times_S \mathfrak{g} \mid x \in \text{Ad}_g(\mathfrak{h})\} \to G/T \times_S \mathfrak{g} \quad (5.24)$$

Here quotient $G/T$ is represented by a smooth quasi-affine $S$-scheme via ([Con14], Thm. 2.3.1). Projection to the second factor in (5.24) yields a morphism $\rho : \tilde{\mathfrak{g}}_T \to \mathfrak{g}$; note $\rho$ is not necessarily proper like $\pi$, since $G/T$ is in general only fiberwise quasi-affine. Projection to the first factor yields a morphism $\tilde{\mathfrak{g}}_T \to G/T$, and the arguments of Lemma 3.15 (except Zariski-local triviality) carry over to show:

**Lemma 5.16.** $\tilde{\mathfrak{g}}_T$ is a smooth étale-locally trivial $G$-torsor over $G/T$ with fiber $\mathfrak{h}$.

Define a right $W$-action on $\tilde{\mathfrak{g}}_T$ via $(gT, x) \cdot w = (gn_wT, x)$ where $n_w \in N_G(T)$ is a lift of $w$. The map

$$G/T \times_S \mathfrak{h} \to \tilde{\mathfrak{g}}_T, \quad (gT, h) \mapsto (gT, \text{Ad}_g(h)) \quad (5.25)$$

is a $W$-equivariant isomorphism, where the left-hand side is equipped with the $W$-action described in Definition 3.2(ii). We also define a morphism $\phi : G/T \times_S \mathfrak{g}^{rs} \to G/B \times_S \mathfrak{g}^{rs}$ via $(gT, x) \mapsto (gB, x)$ and set $\tilde{\mathfrak{g}}^{rs}_T = \{(gT, x) \in \tilde{\mathfrak{g}}_T \mid x \in \mathfrak{g}^{rs}\}$.

**Lemma 5.17.** Restricting $\phi$ to $G/T \times_S \mathfrak{h}^{rs}$ induces an isomorphism $\tilde{\mathfrak{g}}^{rs}_T \simeq \tilde{\mathfrak{g}}^{rs}$. [46]
Applying finite map $\rho$ is the content of ([JNJ04], geometric points). As before, by the fibral isomorphism criterion ([EGA IV], 17.9.5) it suffices to pass to geometric points $\overline{n}$ and $s$ of base $S$ and show we have an isomorphism over $\overline{K}$ and $k$. This is the content of ([JNJ04], §13.4 Lemma); we only explain the bijection input over field $k$.

Proof. It is clear that $G/T \times_S h^{rs} \simeq \tilde{g}^{rs}_T$ via the isomorphism in ([5.25]), since $g^{rs} = \text{Ad}_G(h^{rs})$. If $(gT, h) \in gT(k)$ then $h \in g\text{Lie}(T) g^{-1}$ i.e. $gTg^{-1} \subseteq C_G(h) = T$, so $g \in N_G(T)$ and we may write $\rho^{-1}(h) = \{(n_wT, h) \mid w \in W\}$. Similarly if $(GB, h) \in g^{rs}(k)$ we have $h \in \text{Lie}(gBg^{-1})$ so $h \in \text{Lie}(T')$ for some maximal torus $T' \subseteq gBg^{-1}$. At the same time $T' \subseteq C_G(h) = T$ so $T' = T$ and $T, g^{-1} T g$ are conjugate in $B$, say via $b \in B(k)$. It follows that $gb = n_w \in N_G(T)$ so $gB = n_wB$ and $\pi^{-1}(h) = \{(n_wB, h) \mid w \in W\}$. Hence $\phi$ maps $\rho^{-1}(h)$ bijectively to $\pi^{-1}(h)$.

Corollary 5.18. $\rho^{rs} : \tilde{g}^{rs}_T \rightarrow g^{rs}$ is a finite étale $W$-torsor and $G = \pi^{rs}_* Q_\ell \simeq \rho^{rs}_* Q_\ell$.

Proof. This is a direct consequence of Proposition 3.17 since $\rho^{rs} = \pi^{rs} \circ \phi$ and $\pi^{rs}$ is a finite étale $W$-torsor. By Lemma 5.17 we have a $W$-equivariant isomorphism $\tilde{g}^{rs}_T \simeq \tilde{g}^{rs}$ so $G \simeq \pi^{rs}_* \phi_* Q_\ell = \rho^{rs}_* Q_\ell$.

The upshot of this construction is that we may define the $W$-action on $G$ explicitly via its description as $\rho^{rs}_* Q_\ell$. The stalk of constant sheaf $Q_\ell$ at any geometric point $x \rightarrow \tilde{g}^{rs}_T$ is generated by a distinguished basis element $v_x$, and for $w \in W$ we have an isomorphism $i_w : w_* Q_\ell \rightarrow Q_\ell$ so that the map on stalks is $v_{x,w^{-1}} \mapsto v_x$ via the right $W$-action on $\tilde{g}^{rs}_T$. Applying finite map $\rho^{rs}_*$ to all $i_w$ and using $\rho^{rs} \circ w = \rho^{rs}$ we have

$$\rho^{rs}_* Q_\ell = (\rho^{rs} \circ w)_* Q_\ell \simeq \rho^{rs}_* w_* Q_\ell \xrightarrow{i_w} \rho^{rs}_* Q_\ell$$

(5.26)

where the rightmost map is denoted $i_w$ again by abuse of notation. By functoriality this is an automorphism of $\rho^{rs}_* Q_\ell$ and on each stalk at $\rho^{rs}(x)$ we have

$$(\rho^{rs}_* Q_\ell)_{\rho^{rs}(x)} = (v_{x,u^{-1}} \mid u \in W)$$

so that (5.26) maps basis vectors $v_{x,u^{-1}} \xrightarrow{i_w} v_{x,(u^{-1}w)}$. One checks $i_{w_1 w_2} = i_{w_1} \circ i_{w_2}$ and so we obtain an algebra homomorphism

$$Q_\ell[W] \rightarrow \text{End}(\rho^{rs}_* Q_\ell) \simeq \text{End}(\pi^{rs}_* Q_\ell)$$

Via the intermediate extension functor we get an induced action on $F = R\pi_* Q_\ell$ as follows; recall here that $j' : g^{rs} \hookrightarrow g$. $i' : g \setminus g^{rs} \rightarrow g$ are the open/closed decompositions of $g$ coming from the gluing construction.

Lemma 5.19. Sheaf $F \simeq j'_* G$ is $W$-equivariant via canonical isomorphism

$$\text{End}_{\text{Perv}(g^{rs})}(G) \simeq \text{End}_{\text{Perv}(g)}(j'_* G)$$

Proof. Both perverse sheaf categories here are assumed to come from the relative perverse $t$-structures defined in Section 5.4. Via adjunction

$$\text{Hom}(G, G) = \text{Hom}(j'^* j'_* G, G) \simeq \text{Hom}(j'_* G, j'_* G) \simeq \text{Hom}(p\mathcal{H}^0(j'_* G), p\mathcal{H}^0(j'_* G))$$

(5.27)
where the last (canonical) isomorphism follows from $j'_s \mathcal{G} \in \mathcal{P}/S \mathcal{D}^{\leq 0}$, $j'_s \mathcal{G} \in \mathcal{P}/S \mathcal{D}^{\geq 0}$. Since $i'_s$ is $t$-exact, for any $\mathcal{E}$ in $\operatorname{Perv}(\mathcal{g} \setminus \mathcal{g}^{rs})$ we have

$$\operatorname{Hom}(\mathcal{P}^0(j'_s \mathcal{G}), i'_s \mathcal{E}) = \operatorname{Hom}(j'_s \mathcal{G}, i'_s \mathcal{E}) = \operatorname{Hom}(i''_s \circ j'_s \mathcal{G}, \mathcal{E}) = 0$$

so $\mathcal{P}^0(j'_s \mathcal{G})$ has no nontrivial quotient objects from $i'_s \operatorname{Perv}(\mathcal{g} \setminus \mathcal{g}^{rs})$. Similarly, by the gluing construction we have $i'_s \circ j'_s = 0$ hence $\mathcal{P}^0(j'_s \mathcal{G})$ has no nontrivial subobjects from $i'_s \operatorname{Perv}(\mathcal{g} \setminus \mathcal{g}^{rs})$. So by the definition of $j'_s \mathcal{G}$, the right-hand side in (5.27) is precisely $\operatorname{End}(j'_s \mathcal{G})$ and the $W$-equivariance follows formally (see e.g. [KW13, §III.15.3]).

**Definition 5.20.**

(i) For a general $\mathcal{O}_K$-section $x$ of the nilpotent scheme $\mathcal{N}_g$, we denote by

$$B_x := \pi^{-1}(x) = \tilde{\mathcal{g}} \times_{\mathcal{g}} \lnmx$$

its projective *Springer fiber* with the reduced subscheme structure coming from $\tilde{\mathcal{g}}$. Here $\lnmx$ is the closure of the image of section $x : \operatorname{Spec}(\mathcal{O}_K) \to \mathcal{N}_g$.

(ii) The *Springer representations* are $\mathcal{H}^i(B_x, \mathbb{Q}_\ell)$ equipped with a $W$-action inherited from the $W$-action on $\mathcal{F} = R\pi_* \mathbb{Q}_\ell$ constructed above by taking stalks and proper base-change.

If we consider the special fiber $x_s \in \mathcal{N}_g(k)$ of $x$, then by Proposition 3.17 we get that the special fiber of $B_x$ is $B_{x_s}$, the usual Springer fiber corresponding to a nilpotent element of $\mathcal{g}_s$. Since $B_x$ is projective over $\mathcal{O}$, and the proper base-change isomorphism is canonical ([Full1], Thm. 7.3.1), we obtain an isomorphism of $W$-modules

$$\mathcal{H}^i(B_x, \mathbb{Q}_\ell) \simeq \mathcal{H}^i(B_{x_s}, \mathbb{Q}_\ell)$$

where the right-hand side has the classical Springer action of $W$ defined via the method of Borho–MacPherson ([BM83], §2.6). In fact, in very good characteristic more is true:

**Lemma 5.21.** We have $\mathcal{H}^i(B_{x_s}, \mathbb{Q}_\ell) \simeq \mathcal{H}^i(B_{x_\eta}, \mathbb{Q}_\ell)$ canonically as Galois modules.

**Proof.** Let $f : B_x \to \operatorname{Spec}(\mathcal{O}_K)$ be the projective structure map and let $\nu : \tilde{B}_x \to B_x$ be the normalization; concretely we have

$$\bigsqcup_{i=1}^r (\mathcal{P}^1_S)_i \xrightarrow{\nu} B_x \xrightarrow{f} \mathcal{F} = \operatorname{Spec}(\mathcal{O}_K)$$

where structure map $\tilde{f}$ is proper and smooth. In particular $R^i \tilde{f}_* \mathbb{Q}_\ell$ is a local system on $S$ and so on stalks we have

$$(R^i \tilde{f}_* \mathbb{Q}_\ell)_s \xrightarrow{\sim} (R^i \tilde{f}_* \mathbb{Q}_\ell)_\eta$$

(5.28)
Here the first isomorphism is the cospecialization map given by the data of an étale sheaf on $S$. Now by proper base–change (as $f$ and the special and generic fiber counterparts $\tilde{f}_s, f_s, \tilde{f}_\eta, f_\eta$ are all proper) we have

$$H^i(B_{xs}, Q_\ell) \simeq R^i(f_s)_*\mathbb{Q}_\ell \sim R^i(\tilde{f}_s)_*\mathbb{Q}_\ell \simeq (R^i\tilde{f}_s\mathbb{Q}_\ell)_s$$

$$H^i(B_{x\eta}, Q_\ell) \simeq R^i(f_\eta)_*\mathbb{Q}_\ell \sim R^i(\tilde{f}_\eta)_*\mathbb{Q}_\ell \simeq (R^i\tilde{f}_\eta\mathbb{Q}_\ell)_\eta$$

Hence, together with isomorphism (5.28) we have $H^i(B_{xs}, Q_\ell) \simeq H^i(B_{x\eta}, Q_\ell)$ canonically. The case of interest here is $i = 2$, whence

$$H^2(B_{xs}, Q_\ell) \simeq \bigoplus_{i=1}^r \mathbb{Q}_\ell(-1) \simeq H^2(B_{x\eta}, Q_\ell)$$

where the $I$–action factors through the mod $\ell^n$ cyclotomic characters on the Tate twists, hence is trivial as $K = K^{unr}$. The essential ingredient here is that $x$ is fiberwise subregular and characteristic $p$ is very good, so that both the generic and special subregular Springer fibers are the same arrangement of $r$ projective lines (Yun16, §1.3.8).

The above lemma may be interpreted as a ‘shadow’ of the more general principle that, even though a priori our base scheme is Spec($\mathcal{O}_K$), the adjoint quotient $\chi$ and its “resolved” version $\widetilde{\chi}$ have the same properties over $K$ and over $k$ of very good characteristic.

We now make a few remarks on the nature of Springer representations. The surprising fact about them is that the $W$–action on $H^i(B_x, Q_\ell)$ is not induced by a “physical” $W$–action on the Springer fiber. Nevertheless, Springer ([Spr76], Thm. 6.10) defined a natural correspondence between $H^i(B_x, Q_\ell)$ and representations of Weyl group (by a manner different from Borho–MacPherson).

**Theorem 5.22 (Springer correspondence).** Assume $G$ is a simple, simply–connected, simply–laced Lie group $G$ over algebraically closed field $k$ of good characteristic, and let $x \in \mathcal{N}(k)$ be a nilpotent element and $C = C_G(x)/C_G(x)^0$ be the connected component group of its centralizer. Set $n = \dim(B_x)$ and let $\chi$ be the character of an irreducible representation of $C$. There is a natural graded $W$–action on $H^i(B_x, Q_\ell)$ commuting with the $C$–action, and in particular

$$H^{2n}(B_x, Q_\ell) \simeq \bigoplus_{\chi \in \hat{C}} \chi \otimes V_{x,\chi}$$

where each non-zero $\chi$–isotypic component $V_{x,\chi}$ is an irreducible $W$–representation. Furthermore each irreducible representation of $W$ appears as a $V_{x,\chi}$ for a unique (up to conjugacy) pair $(x, \chi)$.

The convention we use here is that the trivial $W$–representation corresponds to $x$ regular nilpotent and trivial $\chi$, and the sign $W$–representation corresponds to $x = 0$, where one investigates the cohomology of the full flag variety $B$. This is the opposite convention of
but coincides with [BM83]; in fact the Springer representations we consider differ from [Spr76] by a twist of the sign character.

In our setting, $x$ is subregular (so $n = 1$) and $C$ is trivial for the $ADE$-type Lie algebras since the centralizer is connected ([Slo80b], §7.5 Lemma 4). In this case it is known (see e.g. [Yun16], §1.5.17) that the corresponding Springer representation on $H^2(B_x, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell^r$ is isomorphic to the irreducible reflection representation of $W$ on $\mathfrak{h}^\vee$, which in the $A_n$-case (where $W = S_{n+1}$) is just the standard $S_{n+1}$-representation.

5.6 Monodromy Weyl actions and the proof of the main theorem. We now go back to the setting of Sections 5.1 and 5.2, assuming in particular $S = \text{Spec}(\mathcal{O}_K)$ is a complete trait and $\text{char}(k)$ is very good. Since we require notation from previous sections as well, we gather here the relevant objects.

Notation 5.23. Let $\mathcal{X}/S$ be an integral proper flat surface with smooth generic fiber $\mathcal{X}_\eta$ and singular special fiber $\mathcal{X}_s$ containing a unique RDP $x_s \in \mathcal{X}_s(k)$. Let $r$ be the rank of the Dynkin diagram associated to the RDP (Theorem 2.2 (ii)). If $\mathfrak{g}$ is the rank–$r$ simple Chevalley $\mathcal{O}_K$-algebra associated to RDP $x_s$, we may identify $x_s$ with a generic point of the unique subregular orbit inside the nilpotent cone of the Lie $k$-algebra $\mathfrak{g}_s$ ([Slo80b], §6.4 Thm.). Extend $x_s$ to a fiberwise subregular element $x \in \mathfrak{g}(\mathcal{O}_K)$ (Definition 3.7) and let $S$ be the affine relative Slodowy slice which is transverse at $x$ in the sense of Section 4.1. Unless noted otherwise, $S_\eta$, resp. $\tilde{S}_h$ denote the geometric fibers of $\chi$, resp. $\tilde{\chi}$ over geometric points (i.e. field-valued points) $\overline{h} \to h/W$ and $h \to \mathfrak{h}$. The respective generic and special fibers will be denoted $(S)_\eta$ and $(S)_s$, and similarly for $\tilde{S}$ so that we can distinguish them from the geometric fibers of $\chi, \tilde{\chi}$.

By Corollary 5.6 the monodromy action of $I = \text{Gal}(\overline{\eta}/\eta)$ on $H^2(\mathcal{X}_s, \mathbb{Q}_\ell)$ depends only on $\text{Spec}(\mathcal{O}_{\mathcal{X}_{x_s}})$, which by Remark 2.26 we can view as an algebraic deformation of the affine singularity $\text{Spec}(\mathcal{O}_{\mathcal{X}_{x_s}})$ over base $S$. In other words, by Theorem 4.21 we have two cartesian diagrams

\[
\begin{array}{ccc}
\mathcal{X}_{(x_s)} = \text{Spec}(\hat{\mathcal{O}}_{\mathcal{X}_{x_s}}) & \longrightarrow & S \\
\downarrow \phi & & \downarrow \chi \\
S & \longrightarrow & \mathfrak{h}/W
\end{array}
\quad \quad
\begin{array}{ccc}
\mathcal{Z} = S \times_{\mathfrak{h}/W} \mathfrak{h} & \longrightarrow & \mathfrak{h} \\
\downarrow \phi & & \downarrow \psi \\
S & \longrightarrow & \mathfrak{h}/W
\end{array}
\] (5.29)

where $\phi$ is induced by miniversality, and the right diagram is the pullback of $\phi$ along finite cover $\psi$. Here $\chi$ is a map of henselianized schemes, $\mathfrak{h}$ and $W$ are part of the Lie algebraic data associated to RDP $x_s$ (see Section 4.6) and by Remark 4.23 we may identify the miniversal base as

$\mathfrak{h}/W \simeq \text{Spec}(\mathcal{O}_K[t_1, \ldots, t_r])$

Lemma 5.24. Let $\mathcal{X}/S$, $x_s \in \mathcal{X}_s(k)$ and $\psi : \mathfrak{h} \to \mathfrak{h}/W$ be as above.
(i) There exists a finite ramified extension $L/K$ and its associated trait $S_L = \{\eta_L, \mathfrak{s}\}$, which is minimal with respect to the following property: the base-change $X_L/S_L$ admits a local affine model of the RDP singularity $x_s \in X_s(k)$ which has a simultaneous resolution.

(ii) If $B_{x_s}$ is the exceptional divisor of the minimal resolution of $X_s$ induced by the simultaneous resolution of $X_L$, then the stalks of the nearby cycles are

$$(R\Psi_X \mathbb{Q}_\ell)_{x_s} \sim R\Gamma(B_{x_s}, \mathbb{Q}_\ell)$$

and the isomorphism is $\text{Gal}(\mathfrak{p}/\eta)$-equivariant.

Proof. For part (i), note that $S \to \mathfrak{h}/W$ maps $s \mapsto 0 \in \mathfrak{h}/W(k)$ and $\eta \mapsto \mathfrak{h}^{rs}/W(K)$ since the generic fiber $X_\eta$ is smooth. Thus, $Z$ is a finite flat, generically étale $S$-scheme and so in particular it is $S$-affine, say $Z = \text{Spec}(R)$ for some $\mathcal{O}_K$-algebra $R$. By assumption $R \otimes \mathcal{O}_K K$ is a finite étale $K$-algebra, whence

$$R \otimes \mathcal{O}_K K \simeq \prod_i L_i$$

is a finite product of finite separable extensions $L_i/K$, totally ramified since $K = \mathcal{K}$. From the right diagram in (5.29) we obtain a cartesian diagram

$$\begin{array}{ccc}
\prod_i \text{Spec}(L_i) & \longrightarrow & \mathfrak{h}^{rs} \\
\downarrow & & \downarrow \psi^{rs} \\
\eta & \longrightarrow & \mathfrak{h}^{rs}/W
\end{array}$$

where $\psi^{rs}$ is a Galois $W$-cover, hence we get a transitive $W$-action on $\prod_i \text{Spec}(L_i)$. So all extensions $L_i/K$ are isomorphic and we may fix one $L = L_i$ with Galois group $\text{Gal}(L/K) = W_1 \subseteq W$, the stabilizer of $\text{Spec}(L)$ inside $\prod \text{Spec}(L_i)$.

Since $R$ is excellent, we can replace it with its normalization $R \to \tilde{R} \simeq \prod_i \mathcal{O}_{L_i}$, which is finite over $R$. In particular we get a map

$$S_L = \text{Spec}(\mathcal{O}_L) \to \tilde{Z} = \text{Spec}(\tilde{R}) \to \mathfrak{h}$$

and via Theorem [4.21] we may produce a fiber product from this map and the ‘versal’ simultaneous resolution $\tilde{\chi} : \tilde{S} \to \mathfrak{h}$. In other words, define $\mathcal{Y}$ via the cartesian square

$$\begin{array}{ccc}
\mathcal{Y} = S_L \times_{\mathfrak{h}} \tilde{S} & \longrightarrow & \tilde{S} \\
\downarrow & & \downarrow \tilde{\chi} \\
S_L & \longrightarrow & \mathfrak{h}
\end{array}$$

(5.31)

By the universal property of pullbacks we get a unique map

$$\pi_L : \mathcal{Y} \to X_{(x_s)L} = X_{(x_s)} \times_S S_L \simeq \text{Spec}(\mathcal{O}_X \mathcal{X}_{x_s} \otimes \mathcal{O}_K \mathcal{O}_L)$$
We show $\pi_L$ is the desired simultaneous resolution, i.e. that $\mathcal{Y}$ simultaneously resolves the RDP singularity of the local affine neighborhood $\mathcal{X}_{(x_s),L}$ in $\mathcal{X}_L$. Scheme $\mathcal{Y}$ is smooth over $S_L$ via pullback in (5.31) and $\pi_L$ is proper as the base-change of proper morphism $\pi : \tilde{S} \to S \times_{h/W} h$ along $S_L \to h$. Lastly we need only pass to the special fiber and note that, via proper base-change, the induced map $(\pi_L)_s : \mathcal{Y}_s \to (\mathcal{X}_{(x_s),L})_s = (\mathcal{X}_{(x_s)})_s$ is just the minimal resolution of surfaces $\tilde{S}_0 \to S_0$, where the subscript denotes the fiber over zero, so we are done.

To show that $L/K$ has minimal degree, let $M/K$ be any finite extension so that $\mathcal{X}_M$ has a local model of the singularity over $S_M = \text{Spec}(\mathcal{O}_M)$ admitting a simultaneous resolution. Via diagram (5.29) we obtain a non-zero morphism $S_M \to e_Z \simeq \text{Spec}(L_i) \to S_L$ exhibiting $M$ as an extension of $L$ as well.

Finally, we show (ii). Part (i) together with ([Art74], Thms. 1 and 2) implies that over base $S_L$ there exists a simultaneous resolution $\tilde{\pi}_L : \tilde{X} \to X_L$, where $\tilde{X}$ is an algebraic space; note $\tilde{\pi}_L$ is obtained from the ‘local model’ resolution $\pi_L$ of (i). Since the fibers of $\tilde{X}$ are smooth 2–dimensional, they are schemes themselves, thus in particular $R\Psi_{\tilde{X}}\mathbb{Q}_\ell$ makes sense as a sheaf on $\tilde{X}_s$. Since the map of generic fibers $(\tilde{\pi}_L)_{\eta L} : \tilde{X}_{\eta L} \to (X_L)_{\eta}$ is an isomorphism by construction, proper base-change gives

$$R\Psi_{X}\mathbb{Q}_\ell = R\Psi_{X_L}((\tilde{\pi}_L)_{\eta L}, \mathbb{Q}_\ell) \simeq R(\tilde{\pi}_L)_{\eta ss} R\Psi_{\tilde{X}}\mathbb{Q}_\ell \simeq R(\tilde{\pi}_L)_{\eta ss} \mathbb{Q}_\ell$$

as $\tilde{X}$ is smooth. Passing to the stalk at singularity $x_s \in X_s(k)$ yields (5.30) since the exceptional divisor is $\tilde{\pi}_L^{-1}(x_s)$ by definition.

We now advance towards defining a $W$-action on the nontrivial nearby cycles stalk $(R\Psi_{X}\mathbb{Q}_\ell)_{x_s} \simeq R\Gamma(B_{x_s}, \mathbb{Q}_\ell)$ mimicking the monodromy Weyl action defined in ([Slo80a], §4.2 and §4.3); we will eventually identify this new action with the Springer $W$-action as it is given in Theorem 5.22.

By Lemma 5.17 we have that

$$\chi^{rs} : \tilde{g}^{rs} \simeq G/T \times_S h^{rs} \to h^{rs}$$

is just the projection to the second factor, whence we can restrict the simultaneous resolution diagram (3.3) to obtain a cartesian square

$$\begin{array}{ccc}
\tilde{g}^{rs} & \xrightarrow{\chi^{rs}} & g^{rs} \\
\downarrow{\chi^{rs}} & & \downarrow{\chi^{rs}} \\
h^{rs} & \xrightarrow{\psi^{rs}} & h^{rs}/W
\end{array}$$

where the horizontal maps are Galois $W$-torsors. The content of the following lemma ensures the square stays cartesian when we restrict the Grothendieck alteration to the
Slodowy slice (or, equivalently, when we restrict diagram (4.5) to the regular semisimple locus). Below we set $S^{rs} = S \times g^{rs}$ and $\tilde{S}^{rs} = \tilde{S} \times g^{rs}$.

Lemma 5.25 ([Spr83], Lem. 3). The following diagram is cartesian.

\[ \begin{array}{ccc} \tilde{S}^{rs} & \xrightarrow{\pi^{rs}} & S^{rs} \\ \downarrow \chi^{rs} & & \downarrow \chi^{rs} \\ h^{rs} & \xrightarrow{\cdot} & h^{rs} \end{array} \] (5.32)

The desired $W$–action will in a sense follow from the “monodromy” of diagram (5.32). We first note a ‘homotopical’ lemma for $\ell$-adic cohomology. It is originally due to Springer ([Spr83], Prop. 1) but Laumon has generalized it to the relative scheme setting ([Lau03], Lemme 5.5), and this is the version we use.

Lemma 5.26. Let $f : X \rightarrow S$ be an $S$-scheme endowed with a $\mathbb{G}_{m,S}$-action contracting $X$ to a section $p : Y \simeq S$ and let $\mathcal{F}$ be a $\mathbb{G}_{m}$-equivariant sheaf on $X$. Then $Rf_{\ast}\mathcal{F} \simeq p_{\ast}\mathcal{F}|_{Y}$.

Lemma 5.27. If $\tilde{S}_{0}$ is the fiber of $\tilde{\chi}$ over the zero $\mathcal{O}_{K}$-section of $h$, then we have canonical isomorphisms

\[ H^i((\tilde{S})_{\eta}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i((\tilde{S}_{0})_{\eta}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i(B_{x_{\eta}}, \mathbb{Q}_\ell) \] (5.33)

\[ H^i((\tilde{S})_{s}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i((\tilde{S}_{0})_{s}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i(B_{x_{s}}, \mathbb{Q}_\ell) \] (5.34)

Proof. Recall that, by Definition 4.15, $S, \tilde{S}$ and $h$ are equipped with appropriate $\mathbb{G}_{m,S}$-actions so that both $R\pi_{\ast}\mathbb{Q}_\ell$ and $R\chi_{\ast}\mathbb{Q}_\ell$ are $\mathbb{G}_{m,S}$-equivariant sheaves; the first and last actions contract $S$ and $h$ to $x$ and the origin, respectively. Let $f : \tilde{S} \rightarrow S$ and $g : S \rightarrow S$ be the structure morphisms over $S = \text{Spec}(\mathcal{O}_{K})$, then by the Leray spectral sequence

\[ E_{pq}^{2} = R^{p}g_{\ast}(R^{q}\pi_{\ast}\mathbb{Q}_\ell) \Rightarrow R^{p+q}f_{\ast}\mathbb{Q}_\ell \]

By Lemma 5.26 since the $\mathbb{G}_{m,S}$-action on $S$ contracts it to the fiberwise nilpotent subregular section $x \simeq S$ (with image $\text{im}(x) \subset S$), we have $E_{pq}^{2} = 0$ for $p > 0$ and

\[ R^{q}f_{\ast}\mathbb{Q}_\ell \simeq E_{2q}^{q} = R^{q}\pi_{\ast}\mathbb{Q}_\ell|_{\text{im}(x)} \] (5.35)

Let $j : \eta \hookrightarrow S$ and $i : s \hookrightarrow S$ be the inclusions of the geometric generic and special points of the trait. By properness of $\pi$ and base-change, (5.35) yields

\[ j^{\ast}R^{q}f_{\ast}\mathbb{Q}_\ell \simeq j_{x_{\pi}}^{\ast}R^{q}\pi_{\ast}\mathbb{Q}_\ell \simeq H^{q}(B_{x_{\pi}}, \mathbb{Q}_\ell) \] (5.36)

\[ i^{\ast}R^{q}f_{\ast}\mathbb{Q}_\ell \simeq i_{x_{s}}^{\ast}R^{q}\pi_{\ast}\mathbb{Q}_\ell \simeq H^{q}(B_{x_{s}}, \mathbb{Q}_\ell) \] (5.37)

One needs to ensure these fiber products are nonempty. To this end, one observes that $S_{\eta}$ and $S_{s}$ intersect only regular and subregular orbits, which are nonempty ([Slo80b], §5.5).
where \( j_{x_\eta} : x_\eta \hookrightarrow S \) and \( i_{x_s} : x_s \hookrightarrow S \) are the inclusions of the generic and special fibers of section \( x \) into \( S \). Now the absolute version of the aforementioned Leray spectral sequence 
\[
E_2^{pq} = H^p(S, R^q\pi_*\mathbb{Q}_\ell) \implies H^{p+q}(\mathcal{S}, \mathbb{Q}_\ell)
\]

yields isomorphisms
\[
H^i((\mathcal{S}_\eta), \mathbb{Q}_\ell) \xrightarrow{\sim} H^i(B_{x_\eta}, \mathbb{Q}_\ell), \quad H^i((\mathcal{S}_s), \mathbb{Q}_\ell) \xrightarrow{\sim} H^i(B_{x_s}, \mathbb{Q}_\ell)
\]

by ([Spr83, Lemma 2]), since \((S)_\eta\) and \((S)_s\) are Slodowy slices over \( K \) resp. \( k \) in the usual sense ([Slo80b, §5.1]). Proper base-change maps are canonical, hence all isomorphisms in (5.36), (5.37) and (5.38) are canonical. Applying the same Leray spectral sequence argument to the map \( \pi_0 : \mathcal{S}_0 \to S_0 \), which is still \( \mathbb{G}_{m,S} \)-equivariant (loc. cit., Lemma 2), we obtain after base-change again that \( H^i((\mathcal{S}_0)_\eta, \mathbb{Q}_\ell) \simeq H^i(B_{x_\eta}, \mathbb{Q}_\ell) \) and \( H^i((\mathcal{S}_0)_s, \mathbb{Q}_\ell) \simeq H^i(B_{x_s}, \mathbb{Q}_\ell) \), so that altogether we get the canonical isomorphisms (5.33), induced from inclusions \( B_x \hookrightarrow \mathcal{S}_0 \hookrightarrow \mathcal{S} \).

**Proposition 5.28.** For a geometric point \( \overline{h} \to \mathfrak{h}/\!\!/W \) we have \( H^i(S_{\overline{h}}, \mathbb{Q}_\ell) \simeq H^i(B_{x_\overline{h}}, \mathbb{Q}_\ell) \).

**Proof.** Note that \( \overline{h} \) necessarily specializes to \( 0 \in \mathfrak{h}/\!\!/W(k) \). We first investigate the case where the residue field of \( \overline{h} \) is \( k \), following [Spr83]: let \( h \) be a lift of \( \overline{h} \) along \( W \)-covering \( \psi : \mathfrak{h} \to \mathfrak{h}/\!\!/W \) and let \( L \) be the line connecting \( h \) and \( 0 \) in \( \mathfrak{h} \), \( \tilde{\chi}_L : \tilde{S}_L \to L \) the \( \mathbb{G}_m \)-equivariant restriction of \( \tilde{\chi} \) to \( L \). Note \( \tilde{S}_L \) is \( \mathbb{G}_m \)-stable by the action \( \tilde{\mu} \) defined in Section 4.5 (factoring through \( \mathbb{G}_{m,S} \to \mathbb{G}_{m,k} \)).

We claim \( \mathcal{E} := R^i\tilde{\chi}_L, \mathbb{Q}_\ell \) is a lisse sheaf. \( \mathcal{E}|_{L \setminus \{0\}} \) is locally constant since the map \( \mathbb{G}_m \times \tilde{S}_h \to \tilde{S}_L \setminus \tilde{S}_0 \) given by \((t,s) \mapsto \tilde{\mu}(t,s)\) is an étale covering. Now by ([Sta18, Tag 03Q7]) the stalk at zero is
\[
\mathcal{E}_0 = H^i(\tilde{S}_L \times_L L(0), \mathbb{Q}_\ell), \quad L(0) = \text{Spec}(\mathcal{O}_{L,0}^{sh})
\]

Applying the argument of Lemma 5.27 to \( \mathbb{G}_m \)-equivariant map \( \tilde{\chi}_L \) (i.e. looking at its associated Leray spectral sequence) we observe \((R^\bullet\tilde{\chi}_L, \mathbb{Q}_\ell)_0 \simeq H^i((\tilde{S}_h), \mathbb{Q}_\ell) \) and hence \( \mathcal{E}_0 \simeq H^i((\tilde{S}_0)_s, \mathbb{Q}_\ell) \) by Lemma 5.27 Equation (5.34).

As \( L(0) \) is a DVR there is a unique generic (geometric) point \( \eta_L \) of \( L(0) \) with specialization \( \eta_L \to 0 \). We then have
\[
\mathcal{E}_{\eta_L} \simeq H^i(\tilde{S}_{\eta_L}, F) \simeq H^i(\tilde{S}_L \times_L L(0), \mathbb{Q}_\ell)
\]

with the first isomorphism due to \( \mathcal{E}|_{L(0)} \) being locally constant, and the second isomorphism due to \( \tilde{\chi}_L \) being smooth and ([SGA 4\frac{1}{2}, p. 56]). Altogether we get that specialization \( \mathcal{E}_0 \to \mathcal{E}_\eta \) is bijective, so by the criterion for local constancy of sheaves ([Sta18, Tag 0GJ7]) we conclude that \( \mathcal{E} \) is lisse. In particular \( H^i(\tilde{S}_h, \mathbb{Q}_\ell) \simeq H^i((\tilde{S}_0)_s, \mathbb{Q}_\ell) \simeq H^i(B_{x_\overline{h}}, \mathbb{Q}_\ell) \). We have \( \tilde{S}_h \simeq S_\overline{h} \) via Lemma 5.32 since the Grothendieck alteration induces an isomorphism over the regular semisimple locus, so altogether we get the desired isomorphism.

The case of \( \overline{h} \) having zero-characteristic residue field works similarly by using Lemma 5.27 which yields \( H^i(S_{\overline{h}}, \mathbb{Q}_\ell) \simeq H^i(B_{x_\overline{h}}, \mathbb{Q}_\ell) \). By Lemma 5.21 we are therefore done.  \[\square\]
We isolate from the proof a Weyl action of interest. Let $\alpha_h : H^i(S_h, \mathbb{Q}_\ell) \to H^i(B_{x_s}, \mathbb{Q}_\ell)$ denote the isomorphism of Proposition 5.28. By Lemma 5.25 $h \to \mathfrak{h}^{rs}$ is a lift of $\mathfrak{h}$ and so $S_h \simeq S_h$. In the proof of Proposition 5.28 we constructed an isomorphism $H^i(S_h, \mathbb{Q}_\ell) \simeq H^i(B_{x_s}, \mathbb{Q}_\ell)$ and so in particular $H^i(S_h, \mathbb{Q}_\ell) \simeq H^i(S_{w-h}, \mathbb{Q}_\ell)$, where $w : h \to \mathfrak{h}^{rs}$ is a $W$–translate of $h$ under the reflection $W$–action on $\mathfrak{h}^{rs}$. Thus $\mathfrak{h} = \psi(h) = \psi(w \cdot h)$ and $H^i(S_{w-h}, \mathbb{Q}_\ell) \simeq H^i(S_h, \mathbb{Q}_\ell)$, altogether giving the composite automorphism

$$H^i(B_{x_s}, \mathbb{Q}_\ell)(\alpha_h)^{-1} \to H^i(S_{\psi(h)}, \mathbb{Q}_\ell) \simeq H^i(S_{\psi(w-h)}, \mathbb{Q}_\ell) \cong \alpha_{w-h}(H^i(B_{x_s}, \mathbb{Q}_\ell)) \quad (5.39)$$

**Corollary 5.29.** The Gal($L/K$)-action on $H^2(B_{x_s}, \mathbb{Q}_\ell)$ is the restriction of the “monodromy” $W$–action coming from the automorphisms $\alpha_{w-h} \circ \alpha_h^{-1}$ in (5.39).

**Proof.** By Lemma 5.24 there exists a finite extension $L/K$ so that the singularity of $X/S$ admits a simultaneous resolution. We therefore have that the monodromy action of Gal($\eta/\eta$) factors through finite quotient Gal($L/K$), and we get a map Spec($\mathcal{O}_L$) $\to \mathfrak{h}$ with closed point $s \mapsto 0$ and generic point $\eta \mapsto h \in \mathfrak{h}^{rs}$; let $\mathfrak{h}$ be the geometric point over $h$ (NB. this is different than $\mathfrak{h}$, defined in Proposition 5.28: here $\mathfrak{h} \to \mathfrak{h}^{rs}$).

We also set $\chi' : S' \to \mathfrak{h}$ to be the base–change of $\chi$ along $\psi : \mathfrak{h} \to \mathfrak{h}/W$. The singularity $x_s \in X_s$ viewed as the generic point in the subregular orbit of $S_h$, is fixed by the $W$–cover, so that $\chi'$ is still a map of henselianized schemes ($S'$ completed at $x_s$ and $\mathfrak{h}$ completed at 0).

Let $R\Psi^0_h \mathbb{Q}_\ell \in D^b(S'_0)$ be the sliced nearby cycles corresponding to data $(x_s, \mathfrak{h}, \text{sp})$ with sp : $\mathfrak{h} \to 0$ (Definition 5.8). If $i : S'_0 \hookrightarrow S'$ and $j : S' \times_{\mathfrak{h}} (\mathfrak{h}/\mathfrak{h}) \to S'$ are the respective immersions, with $j$ induced by the specialization map sp, we have

$$R\Psi^0_h \mathbb{Q}_\ell = i^*Rj_* \mathbb{Q}_\ell$$

Now the Grothendieck alteration $\pi : \tilde{S} \to S$ has Stein factorization $\tilde{S} \to S' \to S$; we denote $\tilde{S} \to S'$ also by $\pi$. Then $\pi$ is proper, and so the formation of

$$R\pi_\mathfrak{h}_{\mathfrak{h}} : D^b(\tilde{S} \times_{\mathfrak{h}} \mathfrak{h}) \longrightarrow D^b(S' \times_{\mathfrak{h}} \mathfrak{h})$$

commutes with base–change on $S'$ and $\mathfrak{h}$ (see Section 5.3). Note that $\bar{\chi} : \tilde{S} \to \mathfrak{h}$ is smooth, so $R\Psi^0_{\mathfrak{h}} \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell$ (III.17, Ex. 1.7(b)) and in particular base–changing to $S'_0 \to S'$ and $0 \to \mathfrak{h}$ yields (cf. Equation (5.2))

$$R\Psi^0_{\mathfrak{h}} \mathbb{Q}_\ell \simeq R(\pi_0)_* \mathbb{Q}_\ell \quad (5.40)$$

Here $\pi_0 : \tilde{S}_0 \to S'_0$ is the induced minimal resolution of surface $S'_0$, and the left–hand side is $R\Psi^0_{\mathfrak{h}}(R(\pi_0)_* \mathbb{Q}_\ell)$ since $\pi_0 : \tilde{S}_0 \to S'_0$ factors through $\mathfrak{h}^{rs}$, and $\mathfrak{h}^{rs}$ is smooth (hence locally acyclic) outside the finite set of isolated singularities of each fiber. Therefore by Theorem 5.10 the sliced
nearby cycles $R\Psi^0 Q_\ell$ are $\Psi$–good, i.e. they commute with any base–change on $\mathfrak{h}$. Base–changing to $\text{Spec}(O_k) \hookrightarrow \mathfrak{h}$ yields the following computation of Milnor tube cohomology (see Remark 5.11 and Equation (5.8))

$$(R\Psi^0 Q_\ell)_{x_s} \simeq R\Gamma(S' \times_{\mathfrak{h}} h(\mathfrak{p}_1), Q_\ell) \isom R\Gamma(S' \times_{\mathfrak{h}} h, Q_\ell) \simeq (R\Psi x Q_\ell)_{x_s} \quad (5.41)$$

where the middle restriction isomorphism is a consequence of $\Psi$–goodness and the last isomorphism follows upon identifying $S' \times_{\mathfrak{h}} \text{Spec}(O_k) \simeq \chi(x_s)$. The right–hand side of (5.41) is naturally equipped with the monodromy $\text{Gal}(L/K)$-action (Section 5.1), so via this restriction we see that the $\text{Gal}(L/K)$-action on $H^2(B_{x_s}, Q_\ell)$ comes from restricting the monodromy $W$-action on $H^2(B_{x_s}, Q_\ell)$.

\[\square\]

**Remark 5.30.** Implicit in Corollary 5.29 is the fact that the monodromy $W$-action on $H^2(B_{x_s}, Q_\ell)$ comes from the action of $\pi^e_1(h^{rs}/W)$ on the stalks $H^2(\mathfrak{S}_\mathfrak{p}, Q_\ell)$ of local system $R^2\chi_\mathfrak{p} Q_\ell$, as explained in ([Spr83], §5). This retrieves the cohomological version of the monodromy results in ([Slo80a], §4.2-4.3). In the complex situation, Slodowy proves a stronger result, i.e. that $B_{x_s}$ is homotopy-equivalent to a general fiber of $\mathfrak{S}_\mathfrak{p}$ and so we get a $W$-action on the homotopy type of $B_{x_s}$. By comparison, we obtain a weaker result here because we cannot pass to the differentiable category the way Slodowy does over $\mathbb{C}$.

So far we have defined two $W$-actions on $H^2(B_{x_s}, Q_\ell)$, the Springer $W$-action and the monodromy $W$-action. We next show that they coincide, by adapting the argument of ([Hot88], §1.9) to our $\ell$-adic setting. Since $B_{x_s}$ is defined over $k$, in the following proposition we may work over $k$ instead of $S$.

**Proposition 5.31.** The Springer and monodromy $W$-actions on $H^2(B_{x_s}, Q_\ell)$ coincide.

**Proof.** Let $h \to 0$ be a specialization from a point in $h^{rs}/W(k)$ and consider its associated cospecialization map $H^2((\mathfrak{S}_0)_s, R\pi_\ast Q_\ell) \xrightarrow{\text{cospec}} H^2(\mathfrak{S}_h, R\pi_\ast Q_\ell)$. Since $R\pi_\ast Q_\ell|_{\mathfrak{S}_h} \simeq \pi^e_1 Q_\ell$ and $\pi^{rs}$ is a finite étale $W$-cover, we have

$$H^2(\mathfrak{S}_h, R\pi_\ast Q_\ell) = H^2(\mathfrak{S}_h, \pi^{rs}_\ast Q_\ell) \simeq H^2\left(\bigcoprod_{w \in W} \mathfrak{S}_{w \cdot h}, Q_\ell\right) \simeq H^2((\bar{\mathfrak{S}}_0)_s, Q_\ell) \otimes Q_\ell[W]$$

where $h$ is a chosen lift of $\bar{h}$ in $h^{rs}$ yielding all other lifts $w \cdot h$, and the last isomorphism follows from Proposition 5.28. The right-hand side is further identified with $H^1(B_{x_s}, Q_\ell) \otimes Q_\ell[W]$ via Lemma 5.27 equipped with the monodromy $W$-action on the left. On the other hand by proper base-change we have

$$H^2((\mathfrak{S}_0)_s, R\pi_\ast Q_\ell) \simeq H^2(B_{x_s}, Q_\ell) \simeq H^2((\bar{\mathfrak{S}}_0)_s, Q_\ell)$$

with the latter isomorphism again due to Lemma 5.27 and the right-hand side is equipped with the Springer $W$-action coming from the left-hand side and the Springer $W$-action on $R\pi_\ast Q_\ell$. These $W$-modules fit together into diagram

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\[
\begin{array}{c}
\text{H}^2(\mathcal{S}_h, \text{R} \pi_s \mathbb{Q}_\ell) \xrightarrow{\sim} \bigoplus_{w \in W} \text{H}^2(\mathcal{S}_{w,h}, \mathbb{Q}_\ell) \xrightarrow{\sim} \text{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell[W] \\
\text{H}^2((\mathcal{S}_0)_s, \text{R} \pi_s \mathbb{Q}_\ell) \xrightarrow{\sim} \text{H}^2((\mathcal{S}_0)_s, \mathbb{Q}_\ell) \xrightarrow{i} \text{H}^2((\mathcal{S}_0)_s \times W \cdot h, \mathbb{Q}_\ell) \cong \text{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell[W]
\end{array}
\]

where the bottom-right isomorphism is the Künneth map, the bottom row has \( \text{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \) equipped with the Springer \( W \)-action and the top row has \( \text{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \) equipped with the monodromy \( W \)-action. The image of \( i \) is by definition \( \text{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell[W] \cong \text{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \), so the right vertical isomorphism identifies the monodromy and Springer \( W \)-actions on \( \text{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \).

We conclude this section with the proof of parts (i) and (ii) of the Main Theorem \[1.1\].

**Proof of Theorem \[1.1\] (i), (ii).** By assumption \( p = \text{char}(k) \) is sufficiently good, so \( p \nmid |\text{Gal}(L/K)| \) and hence the monodromy action of \( \text{Gal}(\pi/\eta) \) is tame, hence generated by the Kummer character \( t_\ell \). We may therefore identify \( \text{Gal}(L/K) = \langle t_\ell \rangle \) with a cyclic subgroup \( \langle w \rangle \) of \( W \), and so by Proposition \[3.31\], Theorem \[5.22\] and Corollary \[3.6\] we have that the monodromy action on \( \text{H}^2(X, \mathbb{Q}_\ell) \) is the restriction to \( \langle w \rangle \) of the reflection action of \( W \) on \( \mathfrak{h}^\vee \cong \text{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \). This retrieves (i), and by construction we achieve good reduction after base-change \( L/K \), yielding (ii) as well.

**Corollary 5.32.** Suppose the base \( S = \text{Spec}(\mathcal{O}_K) \) of model \( \mathcal{X} \) intersects the discriminant divisor \( \Delta \subset \mathfrak{h}/W \) transversely, then \( \text{Gal}(L/K) \) is a multiple of the Coxeter number.

**Proof.** As a consequence of Theorem \[1.1\] we have an equivalent description of the action of monodromy in the complex case and the tame mixed-characteristic case. We may thus identify the fundamental homogeneous generators of \( \mathcal{O}_K[\mathfrak{h}]^W \) with coordinates on \( \mathfrak{h}/W \cong \text{Spec}(\mathcal{O}_K[t_1, \ldots, t_n]) \) so that the highest degree of a fundamental generator is \( \text{Cox}(\mathfrak{g}) \). Then the line \( \text{Spec}(\mathcal{O}_K) \) is transverse to the discriminant iff all \( W \)-invariant polynomials of degree \( < \text{Cox}(\mathfrak{g}) \) vanish on \( \text{Spec}(\mathcal{O}_K) \) (Slo80a, p. 38). Conversely any Coxeter element \( w \in W \) acting as a reflection on \( \mathfrak{h} \cong \mathfrak{h}^\vee \) has a 1-dimensional eigenspace for eigenvalue \( \zeta_N, N = \text{Cox}(\mathfrak{g}) \), which is mapped to a transversal line in \( \mathfrak{h}/W \) under a degree \( \text{Cox}(\mathfrak{g}) \) cover.

Since \( L \) is a field extension on which \( \mathcal{X}/S \) admits a simultaneous resolution, we have that \( \text{Cox}(\mathfrak{g}) = \text{ord}(w) | \text{Gal}(L/K) \).

### 5.7 Explicit monodromy actions on degenerations.

This section concerns part (iii) of Theorem \[1.1\]. For certain types of singularities we may directly compute (up to conjugacy) the elements acting as monodromy operators on \( \text{H}^2(\mathcal{X}_\eta, \mathbb{Q}_\ell) \). We assume the situation in Notation \[5.23\] once more, i.e. we work over a complete trait \( S = \text{Spec}(\mathcal{O}_K) \) and consider flat proper surfaces \( \mathcal{X}/S \) so that \( \mathcal{X}_s \) has a singularity of type \( A_n \) and \( p = \text{char}(k) \) is sufficiently good for the singularity \( (p > n + 1) \).
After coordinate change we therefore have a formal affine neighborhood (see Example 2.24)
\[
\text{Spec}\left( \frac{\mathcal{O}_K[x, y, z]}{x^2 + z^2 + y^{n+1} + \pi^{a_n} y^{n-1} + \cdots + \pi^{a_2} y + \pi^{a_1}} \right) \rightarrow \text{Spec}(\mathcal{O}_K) \quad (5.42)
\]
In this case \( W = S_{n+1} \) and the \( W \)-cover \( \mathfrak{h} \rightarrow \overline{\mathfrak{h}}/W \) is given in coordinates by
\[
\text{Spec}\left( \frac{\mathcal{O}_K[s_1, \ldots, s_{n+1}]}{s_1 + \cdots + s_{n+1}} \right) \rightarrow \text{Spec}(\mathcal{O}_K[t_1, \ldots, t_n]),
\]
\[
(s_1, \ldots, s_{n+1}) \mapsto \sigma_i(s_1, \ldots, s_{n+1}), \quad i = 2, \ldots, n + 1 \quad (5.43)
\]
where \( \sigma_i \) are the symmetric polynomials (see [Tyu70], §3 or [Bri66], §2.7). The full monodromy group \( W \) acts on \( H^2(\mathcal{B}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell^n \) via the standard \( S_{n+1} \)-representation. We also know in this case (loc. cit.) that the base-changed miniversal family
\[
V = \{ x^2 + z^2 + (y - s_1) \cdots (y - s_{n+1}) = 0 \}
\]
over \( \mathfrak{h} \) may be simultaneously resolved by means of the graph of the map
\[
V \rightarrow \mathbb{P}^1 \times \cdots \times \mathbb{P}^1, \quad (x, y, z) \mapsto \left[ x : \prod_{i=1}^{n+1} (y - s_i) \right]^{n+1} \quad (5.44)
\]
We describe here a general principle that will allow us to pass between the mixed-characteristic setting of \( \text{Spec}(K) \) and the equal-characteristic setting of \( \text{Spec}(k((t))) \). The following is a direct application of Abhyankar’s lemma ([SGA 1], §XIII.5).

**Lemma 5.33.** Let \( S = \text{Spec}\left( \frac{\mathcal{O}_K[[t, u]]}{tu - \pi} \right) \) and \( \text{Spec}(\mathcal{O}_K), \text{Spec}(k[[t]]) \rightarrow S \) be the fiber maps induced respectively from \( u \mapsto 1, u \mapsto 0 \). Restricting to the open dense scheme \( S_{(t)} \simeq \mathbb{G}_m, \mathcal{O}_K \) and the associated maps \( \text{Spec}(K) \rightarrow \mathbb{G}_m, \mathcal{O}_K \leftarrow \text{Spec}(k((t))) \) yields an isomorphism of (tame) étale fundamental groups
\[
\pi_1^{\text{tame}}(\text{Spec}(K), \overline{\eta}) \sim \pi_1^{\text{ét}}(\mathbb{G}_m, \mathcal{O}_K, \overline{\eta}) \sim \pi_1^{\text{tame}}(\text{Spec}(k((t))), s)
\]
Under our restrictions on the characteristic, the ramified covers of \( K \) that we consider are automatically tame, so in what follows we may view family \( (5.42) \) as a family over \( \text{Spec}(k[[t]]) \) with parameter \( t \) instead of \( \pi \) for the purposes of ramification theory. We use this fact implicitly in the statements below.

**Proposition 5.34.** Suppose we have a factorization \( x^2 + z^2 + (y^{r_1} - \pi^{b_1}) \cdots (y^{r_k} - \pi^{b_k}) \) of the family in (5.42) with \( \sum r_i = n + 1, \sum b_i = a_1 \) and \( (r_i, b_i) \) coprime. Then the associated model achieves good reduction after a totally ramified base-change of order \( \text{lcm}(r_1, \ldots, r_k) \) and the element \( w \in S_{n+1} \) acting as the monodromy operator has cycle-type \( (r_1, \ldots, r_k) \). As a result, every conjugacy class in \( S_{n+1} \) may act as monodromy on a degenerating family.
Proof. Let \( f(y) = (y^{r_1} - \pi^{b_1}) \cdots (y^{r_k} - \pi^{b_k}) \in \mathcal{O}_K[y] \). By assumption \( \overline{f}(y) = y^{n+1} \) for the mod \( \pi \) polynomial so we may assume \( b_i \geq 1 \). By the Newton polygon of \( f \) we get slopes \( \mu_i = \frac{b_i}{r_i} \), which may appear with multiplicity (equal to the number of factors of form \( y^{r_i} - \pi^{b_i} \)). Since \( \mathcal{O}_K \) is henselian and the degrees \( r_i \) are coprime to \( b_i \), we get that each factor is irreducible by Gauss's lemma, hence the degree of its splitting field is divisible by \( r_i \). On the other hand \( r_i < p \) and \( \mathcal{O} \) contains all roots of unity prime to \( p \) so the splitting field degree is exactly \( r_i \), yielding an \( r_i \)-cycle in \( S_n+1 \).

Since any element of \( S_n+1 \) can be written as a product of disjoint cycles and disjoint cycles commute, we observe that \( f(y) \) splits after an extension of degree \( \text{lcm}(r_1, \cdots, r_k) \), corresponding to an element of cycle-type \( (r_1, \cdots, r_k) \) as we may reorder the factors in decreasing order of \( r_i \). The base-changed model then admits a (small) resolution of the form (5.44) where the \( s_i \) correspond to the roots of \( f(y) \); in other words we get a smooth model obtained as a base-change of \( \overline{S} \to \mathfrak{h} \) to \( \text{Spec}(\mathcal{O}_L) \), where \( \mathcal{O}_L/\mathcal{O}_K \) is the unique totally ramified extension of \( \mathcal{O}_K \) of degree \( \text{lcm}(r_1, \cdots, r_k) \). Conjugacy classes in \( S_n+1 \) bijectively correspond to cycle-types of this form, which proves the last statement.

For \( A_n \)-singularities, we derive another proof of Corollary 5.32 using the explicit equation of the miniversal deformation.

Corollary 5.35. Assume \( \text{Spec}(\mathcal{O}_K) \) intersects the discriminant divisor \( \Delta \) transversely, then for the resulting model with an \( A_n \)-type singularity at the special fiber we have that monodromy acts a Coxeter element.

Proof. We assume again that there is an affine étale neighborhood of the singular point in \( X \) of the form \( \{ x^2 + z^2 + f(y) = 0 \} \) for \( f(y) = y^{n+1} + \pi^{a_n}y^{n-1} + \cdots + \pi^{a_2}y + \pi^{a_1} \) with \( a_i \geq 1 \) (otherwise the proof of Proposition 5.34 shows that we get a singularity of \( A_m \)-type for \( m < n \) when we reduce mod \( \pi \)). Viewing \( \text{Spec}(\mathcal{O}_K) \) as a line inside \( \mathfrak{h}/W \), it intersects \( \Delta \) transversely exactly when it is not tangent to the tangent cone of \( \Delta \), which is \( V(\sigma_{n+1}) \) and has multiplicity \( n \). Here \( \sigma_{n+1} \) is the \( (n+1) \)-th fundamental generator of the symmetric group \( S_{n+1} \). When the affine étale neighborhood of the singular point in \( X \) is smooth, we satisfy the non-tangency condition.

By the Jacobian criterion we see that the point corresponding to ideal \( (x, y, z, \pi) \) is singular if and only if \( a_1 \geq 2 \) so we have \( a_1 = 1 \). In this case \( f(y) \) is Eisenstein, hence irreducible in \( \mathcal{O}_K \) by Gauss's lemma. The Galois group of its splitting field therefore contains an element of order \( n+1 \) i.e. an \( (n+1) \)-cycle, and it is cyclic, hence generated by an \( (n+1) \)-cycle. We thus get the conjugacy class of Coxeter elements in \( S_{n+1} \).

We can compute a few low-degree examples to see all possible monodromy operators.

Example 5.36.

(i) In the \( A_1 \) case, the possible affine neighborhoods are given by \( \{ x^2 + z^2 + f(y) = 0 \} \) with \( f(y) = y^2 - \pi^N \) for \( N \geq 1 \). If \( N = 2n+1 \) is odd, \( f(y) \) is irreducible and splits after a base-change of order \( \mathbb{Z}/2 \); the resulting model \( \{ x^2 + z^2 + (y - \pi^{2n+1})(y - \pi^{2n+1}) = 0 \} \) admits a small resolution (see Example 2.11) and via the standard
representation we get that the Coxeter element $\sigma \in S_2$ acts as $(-1)$. If $N = 2n$ is even, we have a factorization $\{x^2 + z^2 + (y - \pi^n)(y + \pi^n) = 0\}$ and hence obtain good reduction without base-change, corresponding to the trivial conjugacy class in $S_2$. This retrieves the main theorem of [Kim20] for the case of one ordinary double point on the special fiber.

(ii) In the $A_2$ case, we obtain an affine neighborhood as before with $f(y) = y^3 + \pi^a y + \pi^b$, $a, b \geq 1$ (including $\infty$ for the case of vanishing coefficients). The particular case $f(y) = y^3 - \pi$ gives a map $\text{Spec}(O_K) \to \text{h}/W$ transverse to the discriminant, hence we obtain good reduction after a base-change of order $\mathbb{Z}/3$; indeed, passing to $L = \mathcal{Q}_p(\sqrt[p]{p})$ factors $f(y)$ completely, and we can resolve each fiber akin to (5.44). The corresponding Coxeter element acts on a 2-dimensional subspace of $H^2(X_\eta, \mathbb{Q}_\ell)$ (up to conjugation) as $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.

Case $f(y) = y^3 - \pi^3$ factors without base-change, hence we directly obtain good reduction and trivial monodromy. Case $f(y) = y^3 - \pi y = y(y^2 - \pi)$ factors after a quadratic base-change again, yielding monodromy acting as a transposition on a 2-dimensional subspace of $H^2(X_\eta, \mathbb{Q}_\ell)$; up to conjugation, the associated matrix is $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$.

We conclude with a conjectural statement, which is work in progress.

**Conjecture 5.37.** For singularities of type $D_n$ or $E_n$ one may construct an integral model of some proper smooth surface $X/K$ so that the monodromy operator may lie in any conjugacy class of $W$.

The $D_n$ case should work similar to the $A_n$ case, since we can describe conjugacy classes of the Weyl group $W$ as certain partition classes. The more interesting cases are the exceptional ones ($E_6, E_7, E_8$) for which there is no clear description (at least to the author) of the conjugacy classes of $W$.

**References**


[Slo80a] P. Slodowy. Four lectures on simple groups and singularities. *(No Title)*, 1980.


