The Beauville-Laszlo Descent Theorem

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1 The Beauville-Laszlo Theorem: Gluing Quasicoherent Sheaves from the formal disk and its Zariski Open Complement

This short note is to give a short exposition of a short theorem of Beauville-Laszlo. No results or arguments are original; for the original reference, see [1].

Namely, we consider an affine setting, where A is a ring with $f \in A$ a nonzerodivisor, and $\widehat{A}$ the f-adic completion of A. Recall that an A-module M is f-regular if f is a nonzerodivisor on M. We will prove the following theorem

Theorem 1. *There is a functorial bijection:*

$$\left\{(M,\alpha,\beta)\colon \begin{smallmatrix} M \text{ is an } A\text{-module and} \\ \alpha\colon M_f \xrightarrow{\sim} F \text{ and } \beta\colon M\otimes_A \widehat{A} \xrightarrow{\sim} G \end{smallmatrix}\right\} \longleftrightarrow \left\{(F,G,\varphi)\colon \begin{smallmatrix} F \text{ is an } A_f\text{-module, } G \text{ an } \widehat{A}\text{-module} \\ \text{and } \varphi\colon F\otimes_A \widehat{A} \xrightarrow{\sim} G_f \end{smallmatrix}\right\}$$

Moreover, restricting the lefthand side to finitely generated, flat, or projective and finitely generated modules and the righthand side to the corresponding properties for F, G preserves the above bijection.

Remark 2. *One direction is clear: Given (M,α,β), we recover (F,G,φ) by $F = M_f$, $G = \widehat{A}\otimes_A M$, and*

$$\varphi\colon \widehat{A}\otimes_A M_f \to (\widehat{A}\otimes_A M)_f$$

is the natural identification.

Remark 3. *Restricting to the case where F, G are free of rank r, we see that an A module M with trivializations over A_f and $\hat{A}$ is equivalent to an element of $\mathrm{GL}_r(\widehat{A}_f)$*

Remark 4. *This correspondence gives a way of building vector bundles on the variety $\mathrm{Spec}(A)$ from bundles on the open set $\mathrm{Spec}(A_f)$ and the infinitesmal disk $D = \mathrm{Spec}(\widehat{A})$ with gluing data on the (infinitesmal) intersection. For a curve X over a field k, a k-algebra R, and a point $p \in X(R)$ with local coordiate z for example, this gives the result: Isomorphism classes of rank r vector bundles E on X with trivializations τ, σ over $(X-p)_R$ and D_R are in (functorial) bijection with $\mathrm{GL}_r\left(R(\!(z)\!)\right)$.*

Remark 5. *Compactly stated, this result says that, if $M_f(R)$ denotes the category of f-regular R-modules, then we have a Cartesian diagram of categories*

$$\begin{array}{ccc} M_f(A) & \longrightarrow & M_f(\widehat{A}) \\ \downarrow & & \downarrow \\ M(A_f) & \longrightarrow & M(\widehat{A}_f) \end{array}$$

2 An Argument in a Special Case: Faithfully Flat Descent

In this section, we will aim to prove Theorem 1 in the case that A is also noetherian. Indeed, in this case, the maps $A \to \widehat{A}$ and $A \to A_f$ are both flat maps, and the associated maps on Spec form a cover

$$\mathrm{Spec}(A_f) \sqcup \mathrm{Spec}(\widehat{A}) \to \mathrm{Spec}(A)$$

of $\mathrm{Spec}(A)$ in the fpqc topology. We recall some key points of Grothendieck's finite, flat descent theory:

Let X be a scheme. Given a fpqc cover $\{h_i \colon U_i \to X\}$ of X, a ***descent datum*** is the data of qasicoherent sheaves $\mathscr{F}_i$ on U_i and (gluing) isomorphisms $\alpha_{ij} \colon \mathscr{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathscr{F}_j|_{U_{ij}}$ which satisfy the cocycle condition $\alpha_{ij} \circ \alpha_{jk} \circ \alpha_{ki} = 1$ on U_{ijk}. We say that a descent datum ***descends to*** X if there exists a sheaf $\mathscr{F}$ on X and isomorphisms $\varphi_i \colon \mathscr{F}|_{U_i} \xrightarrow{\sim} \mathscr{F}_i$ on U_i such that $\alpha_{ij} = \varphi_j \circ \varphi_i^{-1}$ on U_{ij}. The main result is now the following:

Theorem 6. *(See [3, Proposition 023T]) Every fpqc descent datum descends to X.*

Consider now our problem: The data of a triple (F, G, φ) as in Theorem 1 gives a descent datum for a cover of $X = \mathrm{Spec}(A)$ in the fpqc topology, and so descends to a quasicoherent sheaf $\mathscr{F} = \widetilde{M}$ on X, as desired.

3 Proof in Full Generality

We now give an elementary proof of the main result.

Proof of Theorem 1. The main observation will appear already in the proof of uniqueness. Namely, suppose that (M, α, β) is an A-module corresponding to a triple (F, G, φ). We demand the compatibility that φ is the composition

$$\widehat{A} \otimes_A F \xrightarrow{1 \otimes \alpha^{-1}} \widehat{A} \otimes_A M_f \xrightarrow{\beta_f} G_f$$

We claim that (M, α, β) is then unique with this property. For this, we make the following crucial observation:

Claim 1. *With the assumptions on M as above, there is a short exact sequence*

$$0 \to M \to F \xrightarrow{\overline{\varphi}} G_f/G \to 0$$

Proof of Claim 1. We start by observing that $A_f/A \simeq \varinjlim_n A/f^n A$ and $\mathrm{Tor}_1^A(A/f^n, M) = 0$ (since f is a nonzerodivisor on A and M, Tor is computed by tensoring the resolution of $A \xrightarrow{\cdot f^n} A$ by M to get the injection $M \xrightarrow{\cdot f^n} M$), we have $\mathrm{Tor}_1^A(A_f/A, M) = 0$. Therefore, the short exact sequence

$$0 \to A \to A_f \to A_f/A \to 0$$

remains exact after tensoring with M.

But A_f/A is already f-adically complete as an A-module, so the canonical map $A_f/A \to \widehat{A} \otimes_A A_f/A \simeq \widehat{A}_f/\widehat{A}$ is an isomorphism. Hence, we get a short exact sequence

$$0 \to M \to M_f \xrightarrow{mf^{-n} \mapsto (f^{-n} \otimes m)} (\widehat{A}_f \otimes_A M)/(\widehat{A} \otimes_A M) \to 0$$

The isomorphisms α, β give a an isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & M_f & \longrightarrow & \frac{(\widehat{A}_f \otimes_A M)}{(\widehat{A} \otimes_A M)} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & M & \longrightarrow & F & \xrightarrow{\overline{\varphi}} & G_f/G & \longrightarrow & 0 \end{array}$$

as claimed. □

The Claim above immediately implies uniqueness; the triple (M, α, β) is the unique triple determining the isomorphism of short exact sequences above.

Now, consider the problem of existence. We reverse the argument above. Consider (F, G, φ) in the righthand side of the correspondence. We seek to reconstruct the short exact sequence of the claim. Composing φ with the projection $G_f \to G_f/G$ gives a map $\overline{\varphi} \colon F \to G_f/G$.

Claim 2. *$\overline{\varphi}$ is surjective.*

Proof of Claim 2. The map $\rho \colon A \to A_f \times \widehat{A} =: B$ is faithful, i.e. a map $M \to N$ of A modules is surjective if and only if the base change $M_B \to N_B$ is surjective. (This follows since, for any maximal ideal $\mathfrak{m}$, $\rho_{\mathfrak{m}} \colon A_{\mathfrak{m}} \to B_{\mathfrak{m}} \simeq A_{\mathfrak{m}}$ is the identity map.) Hence, it suffices to check that $\overline{\varphi} \otimes_A B$ is surjective. Already,

$$A_f \otimes_A (G_f/G) = (A_f \otimes_A G_f)/(A_f \otimes_A G) = G_f/G_f = 0,$$

so we are reduced to studying the base change to $\widehat{A}$. But this returns the map $\widehat{A} \otimes_A F \to G_f/G$ by $1_{\widehat{A}} \otimes \overline{\varphi}$ since G_f/G is already f-adically complete. Hence, $\overline{\varphi}$ is surjective. □

Defining $M = \ker(\overline{\varphi})$ now gives the desired short exact sequence

$$0 \to M \xrightarrow{i} F \xrightarrow{\overline{\varphi}} G_f/G \to 0$$

Again using $A_f \otimes_A (G_f/G) = 0$, we get the induced map $\alpha := i_f \colon M_f \to F$ is an isomorphism. Finally, to recover β, we first claim that $\mathrm{Tor}_1^A(\widehat{A}, G_f/G) = 0$. Indeed, we can write $G_f/G = \varinjlim_n G/f^n G$, so it suffices to show $\mathrm{Tor}_1^A(\widehat{A}, G/f^n G) = 0$. For this, we first compute $\mathrm{Tor}_1^A(\widehat{A}, A/f^n A) = 0$ using the long exact sequence on Tor from the exact sequence

$$0 \to A \xrightarrow{\cdot f^n} A \to A/f^n A \to 0$$

Then, we choose a presentation

$$0 \to N \xrightarrow{j} (A/f^n A)^I \xrightarrow{p} G/f^n G \to 0$$

for G/f^nG and use this to compute Tor; namely, the sequence remains exact on tensoring with $\widehat{A}$ since f^n kills all terms and so we have a surjection

$$\mathrm{Tor}_1^A(\widehat{A}, (A/f^nA)^I) \to \mathrm{Tor}_1^A(\widehat{A}, G/f^nG) \to 0$$

where we have already shown the former term is 0.

Now, we get a short exact sequence

$$0 \to \widehat{A} \otimes_A M \xrightarrow{1 \otimes i} \widehat{A} \otimes_A F \xrightarrow{\pi \circ \varphi} G_f/G \to 0$$

β can now be recovered as the composition $\varphi \circ (1 \circ i)$, which from the short exact sequence gives an isomorphism $\widehat{A} \otimes_A M \xrightarrow{\sim} G$. □

4 A Related Construction: Principal G-Bundles as Double Coset Spaces

Here, we recall the adelic formulation of the moduli of principal G-bundles. The construction somehow feels analogous to the above with "glue at a (fixed) point" replaced by "glue at finitely many (nonfixed) points."

Let X be a smooth, projective curve over a field k, $F = k(X)$ its function field, and $\mathrm{Bun}_G = \mathrm{Bun}_G(X)$ the moduli of G-bundles on X. Moreover, we let $\eta = \mathrm{Spec}(F)$ be the generic point of X and $D_X = \mathrm{Spec}\, F_x$ the closed (infinitesimal) disk at x. We will give an adelic reformulation of the k-points of X.

Let P be a principal G-bundle on X. First we claim:

Proposition 7. *There exists a Zariski open set $U \subset X$ such that $P|_U$ is trivial. That is, $P|_\eta$ is a trivial G-torsor.*

Proof. This follows from the theorem of Steinburg and Bott-Springer:

$$H^1(k(C), G) = 0 \text{ for semisimple, connected } G \text{ ovver } C.$$

This implies that $\eta = \mathrm{Spec}(F)$ only admits trivial torsors. □

Now, any torsor P on C is trivial on some Zariski open $U \subset C$. On each of the finitely many points in the complement of U, P is trivial over the infinitismal disks in the complement by smoothness of $G \times C$. Hence, the above proposition implies that a principal G-bundle P is obtained by gluing the trivial torsor over η with trivial torsors over D_x for finitely many closed points $x \in |X|$. The gluing data for such a G-bundle is given by elements of $G(D_x^\circ) = G(\mathrm{Spec}(F_x))$, and so a convenient way to encode this gluing data is as an element of the adelic group $G(\mathbb{A}_F)$. Such gluing data is uniquely determined up to sections of D_x and η, and so we get a 1-1 correspondence

$$\mathrm{Bun}_G(k) = G(F)\backslash G(\mathbb{A}_F)/G(\mathbb{O}_F)$$

where $\mathbb{O}_F := \prod_{v \in |F|} \mathcal{O}_x$ consists of the integral adeles.

5 Some Remarks towards Heinloth's More Modern Approach

The previous section discussed the adelic picture for the moduli space of principal G-bundles. We can significantly generalize this phenomenon. Indeed, if $\mathscr{G}$ is a "parahoric" group scheme over a projective curve C (over k), then Heinloth proved the following uniformization theorem:

Theorem 8 (Conjectured by Pappas-Rapaport, Proved by Heinloth, [2]). *Let $\mathscr{P} \in \mathrm{Bun}_{\mathscr{G}}(S)$ for a noetherian scheme S over a field (or an excellent Dedekind domain). Then on the complement of any closed point x in C, there is a faithfully flat neighborhood $S' \to S$ of S such that $\mathscr{P}|_{(C-x)\times S'}$ is trivial.*

In particular, this wider class of parahoric group schemes can handle the classification of vector bundles with bilinear forms and Prym conditions.

This uniformization theorem gives some remarkable structure to the moduli space $\mathrm{Bun}_{\mathscr{G}}$ (which was well known in the classical case much earlier). In particular, Heinloth proves that

$$\pi_0(\mathrm{Bun}_{\mathscr{G}}) \simeq \pi_1(\mathscr{G}_{\overline{\eta}})_{G_{k(C)}}$$

and that we have a short exact sequence

$$0 \to \prod_{x \in \mathrm{Ram}(\mathscr{G})} X^*(\mathscr{G}_x) \to \mathrm{Pic}(\mathrm{Bun}_{\mathscr{G}}) \to \mathbb{Z} \to 0$$

References

[1] Arnaud Beauville and Yves Laszlo. "Un lemme de descente". In: *Comptes Rendus de l'Academie des Sciences-Serie I-Mathematique* 320.3 (1995), pp. 335–340.

[2] Jochen Heinloth. "Uniformization of $\mathscr{G}$-bundles". In: *Mathematische Annalen* 347.3 (2010), pp. 499–528.

[3] The Stacks project authors. *The Stacks project*. `https://stacks.math.columbia.edu`. 2020.