Research Statement

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My research focuses on the geometry of moduli spaces of Higgs bundles and Hitchin type fibrations. My central project is joint work with Benedict Morrissey studying the geometry of Hitchin fibrations associated to symmetric spaces. While the majority of this document deals with this project, I also include a brief report on some joint work with Ngô Bảo Châu studying spectral covers and invariant tensors for classical groups, including the case of G_2.

1 The Hitchin Fibration for Symmetric Spaces and Relative Fundamental Lemmas

In this section, I discuss a research program around my joint work with Benedict Morrissey on the geometry of the Hitchin fibration for symmetric spaces. This geometry lies at the intersection of numerous areas of mathematics, including differential geometry, algebraic geometry, mathematical physics and number theory. My research takes an algebro-geometric perspective, with potential applications to arithmetic and the theory of automorphic forms.

1.1 Background and Introduction

For a smooth, projective, complex curve C and reductive group G, Corlette and Simpson introduced the notion of Higgs bundles to classify reductive representations of the fundamental group $\pi_1(C)$ into $G(\mathbb{C})$ [3, 22]. The moduli space of G Higgs bundles $\mathcal{M}_G$ is a hyperkähler variety and exhibits interesting and deep duality phenomena stemming from physics and mirror symmetry.

In [6], Hitchin introduced a beautiful fibration

$$h_G \colon \mathcal{M}_G \to \mathcal{A}_G$$

over an affine space $\mathcal{A}_G$ of half the dimension of $\mathcal{M}_G$, which is a global analogue of the characteristic polynomial of a matrix. Hitchin used spectral covers of C to argue that connected components of the fibers of h_G are generically abelian varieties and that $\mathcal{M}_G$ is a completely integrable system.

In [17] and [18], Ngô develops the geometry of h_G from a more invariant theoretic point of view, working over fields k of more general characteristic. His ultimate goal was to relate point counts over finite fields of fibers of h_G with certain expressions of orbital integrals in order to establish the Fundamental Lemma of Langlands, Shelstad, and Waldspurger. The essential ingredients for doing so included a product formula relating Hitchin fibers to local analogues and a support theorem inspired by Goresky and McPherson.

Let us now work again over $\mathbb{C}$ and consider a real form $G_{\mathbb{R}} \subset G(\mathbb{C})$. It is a natural question to ask which Higgs bundles correspond to representations of $\pi_1(C)$ whose image lies in $G_{\mathbb{R}}$. This was answered by García-Prada, Gothen, and Iriera in [4]; namely, in a construction dating back to Cartan, any real form of G determines a unique holomorphic involution θ on G, and hence an associated symmetric space $X = G/(G^{\theta})^{\circ}$. One can associate to X a moduli space $\mathcal{M}_X$ of Higgs bundles for the symmetric space, and it is $\mathcal{M}_X$ that classifies representations of $\pi_1(C)$ in $G_{\mathbb{R}}$ under the Corlette-Simpson correspondence.

Similar to the classical case, there is a fibration

$$h_X \colon \mathcal{M}_X \to \mathcal{A}_X$$

over an affine space $\mathcal{A}_X$ whose geometry is governed roughly by the representation of G acting on the cotangent bundle T^*X. However, the fibration h_X exhibits two novel behaviors:

- There may be exceptional components. For example, in Schapostnik's thesis [21] for the symmetric space $X = \mathrm{GL}_{2n} / \mathrm{GL}_n \times \mathrm{GL}_n$, fibers are generically identified with a disjoint union of 2^ℓ copies of the Picard stack classifying line bundles on a spectral curve for an explicit ℓ.

  In addition, there are special "Hitchin components" in the case of a split involution that have been studied from the perspective of character varieties.
- The connected components may still fail to be abelian. For example, Hitchin and Schapostnik give a symmetric pair for which the fibers of h_X over a generic a are identified with the space of rank two vector bundles on a spectral curve [7].

My joint work with Benedict Morrissey in [15] seeks to understand the geometry of the Hitchin fibration for symmetric spaces by explicitly computing an invariant theoretic structure first observed in [5] and by introducing a new method of studying regular centralizers. Our work explains the unfamiliar behaviors above from their shadows in invariant theory.

The Hitchin fibration for X is an important example in the Relative Langlands Program of Sakellaridis, Venkatesh and Ben-Zvi. In particular, I expect it to be the geometric structure underlying a Relative Fundamental Lemma, which has been proven in certain cases and conjectured more generally in work of Spencer Leslie [9, 11, 12]. The geometry for symmetric spaces is significantly simpler than geometry for spherical varieties in general, and therefore, it presents a wonderful testing ground for conjectures in larger generality. For example, certain geometric structures for symmetric spaces can be described in high codimension, and cases for which arithmetic is difficult (like those with type N roots or those for which regular centralizers are nonabelian) may still be approached through the invariant theoretic formalism we study.

1.2 New Results

In this section, I will summarize my joint work with Benedict Morrissey in [15]. Throughout this section, we work over an algebraically closed field k with certain mild restrictions on its characteristic.

Let $\mathfrak{g}^{reg}$ denote the set of elements of the Lie algebra $\mathfrak{g}$ whose centralizer in G is of minimal dimension. For $G = \mathrm{GL}_n$, this is the set of matrices which have at most one Jordan block of any given eigenvalue. One important perspective taken in [18] is that the geometry of the Hitchin fibration for G Higgs bundles can be abstracted to properties of the Chevalley map

$$\chi_G \colon \mathfrak{g}^{reg}/G \to \mathfrak{g}/\!\!/G$$

from the stack quotient $\mathfrak{g}^{reg}/G$ to the GIT quotient $\mathfrak{g}/\!\!/G = \operatorname{Spec} k[\mathfrak{g}]^G$. For instance, this map is a gerbe banded by a smooth commutative group scheme over $\mathfrak{g}/\!\!/G$, and as a consequence, the generic fibers of the Hitchin fibration are abelian varieties.

Now, fix an involution θ on G. Let $K = (G^\theta)^\circ$ be the connected component of the fixed points of θ acting on G, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ into $+1$ and -1 eigenspaces of θ, respectively. Let H be a closed subgroup of G such that

$$K \subset H \subset N_G(K)$$

For technical reasons, it is important that we work in the slightly more general setting of (G, θ, H) Higgs bundles, which are associated to the representation of H acting on $\mathfrak{p}$. As in the case of G

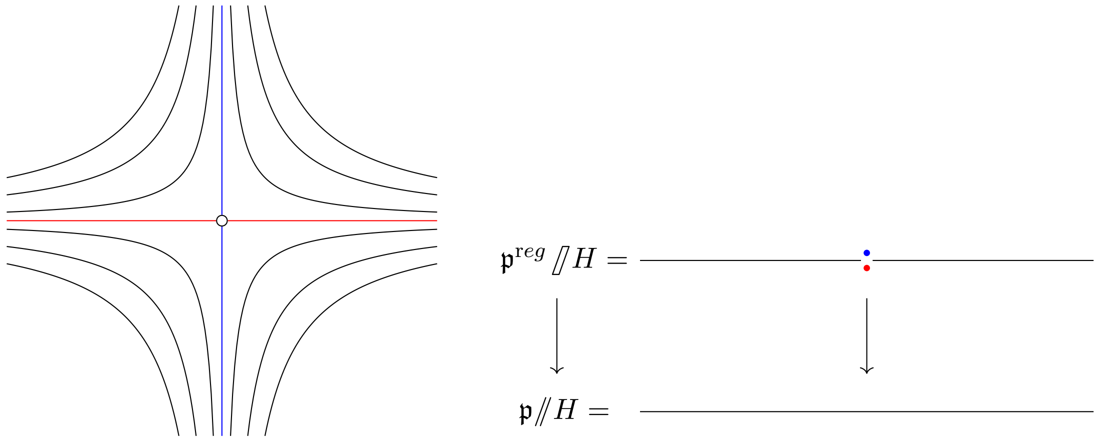

Figure 1: (Left) The orbits of $H = S(\mathbb{G}_m \times \mathbb{G}_m)$ acting on $\mathfrak{p}^{reg} \simeq \mathbf{A}^2 \setminus \{0\}$ for the symmetric space $X = \mathrm{SL}_2 / S(\mathbb{G}_m \times \mathbb{G}_m)$. Note the two orbits, drawn in blue and red, whose closure includes the (non-regular) closed orbit $\{0\}$. The regular quotient for this symmetric pair (pictured right) is the affine line with doubled origin.

Higgs bundles, we define the regular locus $\mathfrak{p}^{reg}$ in $\mathfrak{p}$ to be the set of elements of $\mathfrak{p}$ whose centralizer in H is of minimal dimension. This may not agree with the notion of regularity in $\mathfrak{g}$. Then, the geometry of the (G, θ, H) Hitchin fibration is determined by the geometry of the Chevalley type map

$$\chi \colon \mathfrak{p}^{reg}/H \to \mathfrak{p} /\!\!/ H$$

from the stack quotient $\mathfrak{p}^{reg}/H$ to the GIT quotient $\mathfrak{p} /\!\!/ H = \operatorname{Spec} k[\mathfrak{p}]^H$.

It was observed by [5] that χ is no longer a gerbe. Indeed, the action of $H = K$ on $\mathfrak{p}$ for the symmetric space $X = \mathrm{SL}_2 / S(\mathbb{G}_m \times \mathbb{G}_m)$ is the hyperbolic action of $\mathbb{G}_m$ on $\mathbf{A}^2$, for which there are two regular orbits whose closure includes the origin; see Figure 1. In [5], García-Prada and Peón-Nieto prove the existence of an alternative quotient, which we will call the regular quotient of $\mathfrak{p}^{reg}$ by H and will denote by $\mathfrak{p}^{reg} /\!\!/ H$, by rigidifying the stack $\mathfrak{p}^{reg}/H$. The existence of a regular quotient has since been proven in far greater generality in forthcoming work of Morrissey and Ngô, where it plays an important role in the study of generalized Hitchin systems [16].

Our first, and central, result is a complete description of $\mathfrak{p}^{reg} /\!\!/ H$.

Theorem 1. *(H.-Morrissey)*

1. *Any symmetric pair is isogenous to a product of symmetric pairs (G_i, θ_i, H_i) where G_i is either simple or a product $G_{i,1} \times G_{i,1}$ with involution swapping factors. The regular quotient for (G, θ, H) is the product of the regular quotients for the (G_i, θ_i, H_i).*
2. *For any (G, θ, H) with G simple except for the split form on SO_{4n}, there is a Zariski open $U \subset \mathfrak{p}^{reg} /\!\!/ H$ such that the regular quotient is given by gluing two copies of $\mathfrak{p} /\!\!/ H$ along U.*
3. *There is an inductive process to determine the gluing locus U explicitly.*

The key steps in the proof of part (ii) are as follows. Levy proves the existence of sections of the map

$$\mathfrak{p}^{reg} \to \mathfrak{p} /\!\!/ H$$

through any nilpotent element in $\mathfrak{p}^{reg}$ [13]. Choose sections through each of the finitely many regular nilpotent H orbits. One can use the $\mathbb{G}_m$ action on $\mathfrak{p} /\!\!/ H$ to show that the regular quotient is isomorphic to a collection of copies of $\mathfrak{p} /\!\!/ H$, indexed by the set of regular nilpotent H orbits in

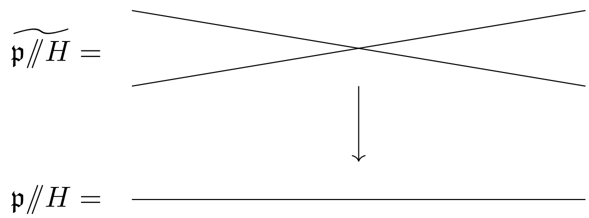

Figure 2: The cameral cover for the symmetric space $X = \mathrm{SL}_3 / S(\mathbb{G}_m \times \mathrm{GL}_2)$. In this case, $\widetilde{\mathfrak{p}}^{reg}$ classifies Borels of H.

$\mathfrak{p}^{reg}$, glued wherever the sections become conjugate. When G is simple and (G, θ, H) is not the split form on SO_{4n}, the number of such orbits is either 1 or 2 (see [13]). Finally, an Levi induction argument reduces to computing the number of regular nilpotent H orbits for symmetric pairs on associated Levi subgroups.

We also determine the regular quotient for the split form on SO_{4n} above, but its form is slightly more complicated due to the presence of 4 nilpotent H orbits in $\mathfrak{p}^{reg}$.

As an immediate corollary of Theorem 1, we generalize results of Schapostnik [21] for symmetric pairs (G, θ, H) for which there exist spectral covers. Moreover, this explains conceptually a source of special components appearing in [21], coming from lifting along the nonseparated cover

$$\mathfrak{p}^{\mathrm{reg}} \,\widetilde{/\!\!/}\, H \to \mathfrak{p}^{\mathrm{reg}} /\!\!/ H.$$

As another corollary of our results, Morrissey has proven a Fourier-Mukai duality predicted by mathematical physics for the quasisplit form on GL_{2n} [14].

Our work also describes the regular centralizer group scheme $I_H^{\mathrm{reg}} \subset H \times \mathfrak{p}^{\mathrm{reg}}$. Past work in [10] and [5] has focused on studying regular centralizers in the quasisplit case, i.e. when I_H^{reg} is abelian, by using a version of the Grothendieck-Springer map classifying Borels of G satisfying a compatibility condition with respect to θ. Our method replaces this with a cover $\widetilde{\mathfrak{p}}^{reg}$ of $\mathfrak{p}^{\mathrm{reg}}$ which is the incidence variety classifying parabolic subgroups of H of a certain type. While $\widetilde{\mathfrak{p}}^{reg}$ does classify Borels of H in certain familiar quasisplit examples, for example for the symmetric spaces $X = \mathrm{GL}_{2n} / \mathrm{GL}_n \times \mathrm{GL}_n$ and $X = \mathrm{GL}_{2n+1} / \mathrm{GL}_n \times \mathrm{GL}_{n+1}$, the use of parabolics is necessary even for quasisplit forms, for example for $X = \mathrm{SO}_{2n+2} / \mathrm{SO}_{n+2} \times \mathrm{SO}_n$.

In a case-by-case argument over all quasisplit forms except the quasisplit form on E_6, this produces a cover

$$\widetilde{\mathfrak{p}/\!\!/H} \to \mathfrak{p}/\!\!/H$$

from which $\widetilde{\mathfrak{p}}^{reg}$ is obtained as a base change. The key technical difficulty is that the space $\widetilde{\mathfrak{p}}^{reg}$, and hence $\widetilde{\mathfrak{p}/\!\!/H}$, is not irreducible. In particular, the number of irreducible components is measured by the quotient of two Weyl groups: one associated to a torus obtained as a centralizer of a regular, semisimple element and one associated to the center of the corresponding Levi subgroup. For example, in the case of the symmetric space $X = \mathrm{SL}_3 / S(\mathbb{G}_m \times \mathrm{GL}_2)$, the components are measured by the difference between the Weyl group $W_H = S_2$ of $H = S(\mathbb{G}_m \times \mathrm{GL}_2)$ and the trivial group $W_C = \{1\}$ associated to the subtorus

$$C = \{A = \mathrm{diag}(x, x, y) \colon x, y \in k,\ \det(A) = 1\}.$$

In the case of a quasisplit form not of type E_6, this means an explicit description of the regular quotient can be given as it relates to the Galois descent of a constant torus on $\widetilde{\mathfrak{p}/\!\!/H}$. It is expected that this should hold for the form E_6 and that an analogous statement should hold for forms with nonabelian centralizers with minimal modification of our current argument.

1.3 Future Work

While I expect to take on additional questions during the course of a postdoc, I outline here two initial lines of inquiry that I expect to be attainable in the time frame of a postdoc: Generalizing results to the case of nonabelian regular centralizers and initiating work on a support theorem for the Hitchin system of a symmetric pair.

The former goal should result in generalizations of the work of Schapostnik and Hitchin [7], in which fibers of the map

$$h_X : \mathcal{M}_X \to \mathcal{A}_X$$

for the symmetric space $X = \mathrm{GL}_{2n} / \mathrm{Sp}_{2n}$ are described as moduli spaces of rank 2 bundles on a spectral curve over a generic locus of the base $\mathcal{A}_X$. This is a particularly surprising phenomena as it provides the first examples of generalized Hitchin systems with nonabelian regular centralizers that can be described explicitly. Much remains unknown about Higgs bundles for these pairs, though Branco's thesis [2] suggests that there may still be nontrivial duality statements to be made. It is also the author's hope that this helps extend the work of [16], which is developing a framework in which to study generalized Hitchin systems and which does not at present apply to this class of examples.

As we have noted, the generalized Hitchin system for a symmetric space is the key geometric object of interest in a Relative Fundamental Lemma. Using the work of [18] as a guide, there is a natural program of study in proving such a result. Namely, in his proof of the Fundamental Lemma, Ngô studied the perverse cohomology groups of the pushforward along the Hitchin map h_G of the constant sheaf $\mathbb{Q}_\ell$ on a suitable open subset of $\mathcal{M}_G$[1]. This object decomposes into simple pieces, each of which is the IC sheaf of a local system on some closed subset of the Hitchin base. The key result describes the supports of these simple constituents. We propose to study a relative version of this support theorem.

Question. *For quasisplit X, describe the supports of the IC sheaves appearing in the decomposition of the pushforward of the constant sheaf $\mathbb{Q}_\ell$ along the relative Hitchin fibration h_X.*

Finally, symmetric spaces are a first example of spherical varieties, and therefore have a natural place inside the framework of the Relative Langlands program pioneered by Sakellaridis, Venkatesh, and Ben-Zvi [1]. A Relative Fundamental Lemma would be a powerful tool in establishing arithmetic results predicted by this theory. Moreover, it is expected that further geometric results can be gleaned from the geometric picture of Hitchin systems for symmetric spaces, with applications to relative duality. For example, it is expected that the Fourier-Mukai dual of $\mathcal{M}_X$ with respect to the Poincaré bundle on $\mathcal{M}_G$ is related to the dual group and associated representation used in [20, 1].

2 Companion Matrix Constructions and Invariant Tensors for Classical Groups

In this section, I will discuss a research program around joint work with Ngô Bảo Châu [19] which is not my main research focus, but remains of interest. This work originated in the study of sections of the Hitchin fibration coming from companion matrix constructions. In particular, we were motivated in finding canonical invariant tensors for the groups $G = \mathrm{Sp}_{2n}, \mathrm{SO}_n$, and G_2, and giving an algebro-geometric explanation for their existence. In the future work section below, I discuss some exciting directions growing out of these relatively simple ideas.

[1] The "anisotropic" locus, over which the map h_G is proper

2.1 Introduction

Let G be a reductive group over k, and denote by $\mathfrak{g}$ its Lie algebra. The Chevalley map

$$\chi\colon \mathfrak{g} \to \mathfrak{g}/\!\!/G,$$

where $\mathfrak{g}/\!\!/G := \mathrm{Spec}(k[\mathfrak{g}]^G)$ denotes the invariant theoretic quotient of $\mathfrak{g}$ by the adjoint action of G, is of fundamental importance in the construction of the Hitchin fibration. In particular, for $\mathfrak{g} = \mathfrak{gl}_n$, χ sends a matrix to its characteristic polynomial.

In [8], Kostant exhibited a section of the Chevalley map for a general reductive group G under the assumption that the characteristic of k does not divide the order of the Weyl group. As explained in [18], this section can be used to construct sections of the Hitchin fibration and can be used to define affine Springer fibers. However, Kostant's construction has two important detractions: on the one hand, it requires a bound on the characteristic of the field that depends on the group G, which is usually not optimal for classical groups, and on the other hand, it can be counter-intuitive for computations. To illustrate this latter point, consider the case $G = \mathrm{GL}_3$, in which case $\mathfrak{g}/\!\!/G = \mathbf{A}^3$ is the 3-dimensional affine space. The Kostant section is the map sending

$$(a_1, a_2, a_3) \in \mathfrak{g}/\!\!/G \quad \mapsto \quad \begin{pmatrix} \frac{a_1}{3} & \frac{a_1^2}{6} + \frac{a_2}{2} & -\frac{4a_1^3}{27} - \frac{a_1 a_2}{3} - a_3 \\ 1 & \frac{a_1}{3} & \frac{a_1^2}{6} + \frac{a_2}{2} \\ 0 & 1 & \frac{a_1}{3} \end{pmatrix} \in \mathfrak{g}$$

If you introduced this problem to an undergraduate student of linear algebra, of course, they would not give you the answer above; they might instead suggest the map:

$$(a_1, a_2, a_3) \in \mathfrak{g}/\!\!/G \quad \mapsto \quad \begin{pmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{pmatrix} \in \mathfrak{g}$$

sending a characteristic polynomial to its *companion matrix*. The section to the Hitchin map that Hitchin constructed in [6] is not strictly the same as the one of [18] in the sense that he does not rely on the Kostant section but another section that feels more like a generalization of the companion matrix. Instead of the companion matrix, a map $\mathfrak{g}/\!\!/G \to \mathfrak{g}$, we construct a map $\mathfrak{g}/\!\!/G \to [\mathfrak{g}/G]$, where $[\mathfrak{g}/G]$ is the stack quotient of $\mathfrak{g}$ by the adjoint action of G. This section is completely canonical and works in characteristic not 2 for $G = \mathrm{Sp}_{2n}$ and SO_n and in characteristic not 2 or 3 for $G = G_2$.

Let $G = \mathrm{GL}_n$, and let $A_n = k[\mathfrak{g}]^G$ be the coordinate ring of the GIT quotient. In characteristic not 2, we have

$$A_n \simeq k[\mathfrak{t}]^W = k[a_1, \ldots, a_n]$$

where a_i is the degree i elementary symmetric polynomial. We consider

$$B_n = k[\mathfrak{t}]^{S_{n-1}} = A_n[x]/(f(x))$$

where $f(x) = x^n - a_1 x^{n-1} + \cdots + (-1)^n a_n$ is a generic characteristic polynomial. For $\mathfrak{s}_n = \mathrm{Spec}(B_n)$, the corresponding morphism

$$\mathfrak{s}_n \to \mathfrak{c}_n$$

is the universal spectral cover. Then, B_n is a free, rank n module over A_n equipped with the multiplication-by-x endomorphism whose characteristic polynomial is $f(x)$. This gives the data of an A_n point of the stack $[\mathfrak{g}/G]$ corresponding to a section of χ.

For G symplectic, special orthogonal, and G_2, G is defined as a subgroup of GL_n fixing certain tensors. We call the inclusion $G \to \mathrm{GL}_n$ the standard representation of G. We also have the induced inclusion of Lie algebras $\mathfrak{g} \to \mathfrak{gl}_n$ compatible with the adjoint actions of G and GL_n. The main result of our work can be formulated as follows:

Theorem 2. *Let G be a symplectic group, odd special orthogonal group, or G_2 group and $G \to \mathrm{GL}_n$ its standard representation. Let $\mathfrak{c} \to \mathfrak{c}_n$ be the induced map of Chevalley quotients which is a closed embedding in these cases. Then the restriction $\mathcal{O}_{\mathfrak{s}_n}$ to $\mathfrak{c}$*

$$\mathcal{V} = \mathcal{O}_{\mathfrak{c}} \otimes_{\mathcal{O}_{\mathfrak{c}_n}} \mathcal{O}_{\mathfrak{s}_n}$$

as locally free $\mathcal{O}_{\mathfrak{c}}$-module affords a canonical tensor defining a G-reduction and the companion matrix for GL_n defines a canonical map $\mathfrak{g}/\!\!/G \to [\mathfrak{g}/G]$ which is a section of the natural map $[\mathfrak{g}/G] \to \mathfrak{g}/\!\!/G$. This statement remains valid for even orthogonal groups after replacing $\mathfrak{c} \times_{\mathfrak{c}_n} \mathfrak{s}_n$ by its normalization.

While these forms can be constructed case-by-case, they afford an algebro-geometric description. Let us take the case of $G = \mathrm{Sp}_{2n}$ as an example. The spectral cover $\mathfrak{s} = \mathfrak{c} \times_{\mathfrak{c}_n} \mathfrak{s}_n$ comes equipped with an involution τ over $\mathfrak{c}$, and hence factors through a subcover

$$\mathfrak{s} \xrightarrow{2} \mathfrak{s}/\tau \xrightarrow{n} \mathfrak{c}$$

where the degrees are shown above the arrows. This intermediate cover determines an irreducible component

$$\mathfrak{s}/\tau \xrightarrow{\sim} \Delta_\tau \subset \mathrm{Sym}^2_{\mathfrak{c}}(\mathfrak{s})$$

of the symmetric power of $\mathfrak{s}$ relative to $\mathfrak{c}$. The canonical tensor on $\mathcal{V}$ is obtained by considering alternating 2-forms which are supported on the distinguished component Δ_τ: The collection of such alternating forms is a free, rank 1 sheaf on Δ_τ, and one obtains the canonical tensor on $\mathcal{V}$ as a generator of this sheaf as a $\mathcal{O}_{\mathcal{V}}$ module. The explicit form of this alternating 2-form is given by Euler's form

$$\langle h_1, h_2 \rangle = \mathrm{Tr}\left(\frac{h_1(x)h_2(-x)}{f'(x)}\right)$$

where Tr denotes the trace map from the function field of $\mathfrak{s}$ to the function field of $\mathfrak{c}$.

In the case of $G = G_2$, the spectral cover is a degree 6 cover which admits two factorizations. The first is similar to the symplectic case, whereas the second is a factorization through a degree 3 cover of $\mathfrak{c}$. The latter determines a distinguished component of the symmetric power $\mathrm{Sym}^3_{\mathfrak{c}}(\mathfrak{s})$, from which the canonical 3-form can be derived in a similar fashion to the symplectic case. The explicit tensor, recorded in our paper [19], has not yet appeared in the literature.

2.2 Future Work

For any semisimple Lie algebra $\mathfrak{g}$, the bracket on $\mathfrak{g}$ is recovered by the data of the Killing form b together with the Cartan 3-form t defined by

$$t(x, y, z) = b(x, [y, z]).$$

We expect that the methods outlined above can be applied in this more general setting to recover both b and t from the root data in a process involving little choice. In place of the distinguished components arising from spectral covers, we will have components determined by the vanishing of sums of roots; for example, in the case of 3-forms, t is supported on a component of a certain

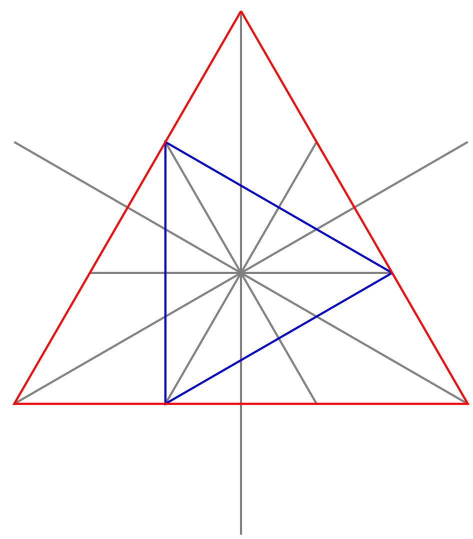

Figure 3: Some triangles of roots summing to zero in the root diagram for G_2.

symmetric power given by triples of roots (α, β, γ) such that $\alpha+\beta+\gamma = 0$. (See Figure 3.) Such constructions are also expected to produce a canonical section to the Chevalley map $\chi\colon [\mathfrak{g}/G] \to \mathfrak{g}/\!\!/G$, although our methods will still require that the characteristic of k be larger than the order of the Weyl group. We expect that completing the relevant arguments may involve some derived algebraic geometry to correctly define the module of forms with support on a distinguished component.

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