THE HITCHIN FIBRATION FOR SYMMETRIC PAIRS

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(Todo: This Draft is work in progress.)

Contents

1. Introduction

The Hitchin fibration is a geometric space appearing in a wide range of mathematical works including automorphic forms, mathematical physics, and non abelian Hodge theory. The Hitchin fibration for symmetric pairs (also referred to in the literature as the $G(\mathbb{R})$-Hitchin fibration, or the Hitchin fibration for real Higgs bundles) is an analogue of the Hitchin fibration that also appears in many of these fields. Essentially it corresponds to replacing the adjoint action of G on $\mathfrak{g}$ by the action of H on $\mathfrak{p}$, where $(\mathfrak{p}, H)$ is a symmetric pair.

In particular we note that it appears in non abelian Hodge theory when the moduli space of reductive representations from $\pi_1(X)$ (X a Riemann surface) to $G(\mathbb{C})$ is replaced by $G(\mathbb{R})$ (a real form of some reductive algebraic group over $\mathbb{C}$) [10]. In mathematical physics it is associated to boundary conditions in theories of class S, and to $N = 4$, $d = 4$ supersymmetric Yang–Mills theory considered on a 4 manifold which is of the form $X \times [0, 1] \times \mathbb{R}$. This leads to the image of this version of the Hitchin moduli space, in the Hitchin moduli space for G being the support of a BAA brane. Finally it is related to infinitesimal forms of relative trace formulas for symmetric spaces G/H.

As such it is an important open problem to describe the geometry of the Hitchin fibration for symmetric pairs as precisely as possible. For technical reasons at some points we have to restrict to quasisplit symmetric pairs in many parts of this paper, in particular with respect ot describing the analogue of the group scheme of regular centralizers. We will remove this assumption in later work. In the case of the usual Hitchin fibration the geometry of the fibration is a deeply researched topic, with the most general statements for arbitrary reductive algebraic groups contained in [8] and [23]. In the symmetric pair case there is a large literature including [31, 32, 14, 33, 34, 5] studying these via spectral covers. We use cameral cover methods as in [8, 23] to describe the geometry of the Hitchin fibration for symmetric pairs extending the earlier work of [25, 26, 18, 11].

1.1. **Related Work.** There are many papers describing fibers of the Hitchin fibration for symmetric pairs via spectral covers including [31, 32, 14, 33, 34, 5, 6, 2, 26]. Note that the papers [14, 5, 6, 2] treat cases where the symmetric pair is not quasisplit. In the case of the symmetric pair associated to the real form $U(n, n)$ or equivalently $H = K = GL(n) \times GL(n) \hookrightarrow GL(2n) = G$ we provide a direct description of the equivalence between the description of [33] and our description via cameral covers in section **??**.

When it comes to the use of cameral covers this is considered in general in [11], and in the case of $SU(n, n+1)$ in [26]. Finally the cameral cover approach to Hitchin type fibrations relies fundamentally on the Gröthendieck–Springer resolution, an analogue of this for quasisplit symmetric pairs with $H = G^\theta$ is considered in [18, 11]. We want to outline how the results of our paper relate to this previous work. The paper [11] of O. Garcia–Prada and A. Peón–Nieto includes results describing some parts of the Hitchin fibration for quasisplit symmetric pairs of the form $(\mathfrak{p}, K)$, $(\mathfrak{p}, G^\theta)$, and $(\mathfrak{p}, N_G(K))$. S. Leslie [18] develops a quasisplit Gröthendieck–Springer resolution for symmetric pairs where certain Borels of G are used. We take a different approach to the Gröthendieck–Springer resolution in this paper using instead parabolics of the group K. Leslie uses this to describe regular centralizers for quasisplit symmetric pair $(\mathfrak{p}, G^\theta)$, and we expect that this could be generalized to other symmetric pairs $(\mathfrak{p}, H)$.

For quasisplit $(\mathfrak{p}, K)$ [11] show that there is a space $\mathfrak{p}^{reg} /\!\!/_I G$ (to our knowledge this is the first use we know of what is called the regular quotient in this paper and [21]), with respect to which the stabilizer group I descends to a group scheme J, this is also the first place where we are aware of the use of a regular quotient. The fact that this can be a non-separated cover of the GIT quotient is seen in examples. New to this paper is the explicit description of the geometry of $\mathfrak{p}^{reg} /\!\!/_I G$.

For quasisplit $(\mathfrak{p}, G^\theta)$ the regular centralizer scheme I is shown in [11] to descend to a group scheme $J \to \mathfrak{p}/\!\!/G^\theta \cong \mathfrak{a}/\!\!/W_a$. Furthermore an explicit description of the restriction of this to regular semisimple locus is given. Leslie [18] gives a description of J via Weil restriction of a finite extension of a torus over the entire space $\mathfrak{a}/\!\!/W_a$.

Finally in [11] the case of quasisplit $(\mathfrak{p}, N_G(K))$, J is described in terms of fixed points of an involution on the regular centralizers of the adjoint action of G on $\mathfrak{g}$. Furthermore the gerbe $\mathfrak{p}^{reg}/N_G(K) \to \mathfrak{p}/\!\!/N_G(K) \cong \mathfrak{a}/\!\!/W_a$ is described as fixed points for an involution on the Donagi–Gaitsgory description [8] of the gerbe of regular Higgs bundles (in the usual sense).

We avoid the two above methods because for the following reasons. The second of these can not be easily generalized to $H \neq N_G(K)$. The first we avoid because (i) we want to use this paper as an example of the more general approach of [21], and (ii) we want as many as possible of our methods here to generalize to the case of tempered spherical varieties which we will consider in forthcoming work together with Z. Luo.

We use throughout foundational work on symmetric pairs and their invariant theory in particular [17, 19, 35, 28].

We note also that some constructions are analogous to those used in the study of the multiplicative Hitchin fibration via the Vinberg monoid in [4, 3, 7, 38], where the phenomenon of there being multiple regular orbits corresponding to a single point of the GIT quotient also occurs.

More generally the paper of Ngô B.-C. and the second author [21] considers regular quotients in higher generality, and we use this both explicitly and implicitly. Finally in joint work of the authors with Z. Luo, we hope to extend the results of this paper to the setting $\mathfrak{p}/K$ is replaced with T^*X/G for X a spherical variety, and replacing the quasisplit condition by tempered when necessary. Note that this recovers most of the cases considered in this paper upon considering $X = G/H$ a symmetric space.

1.2. **Future Work.** There are some aspects of the geometry of the Hitchin fibration for symmetric pairs where our understanding is behind that of for the usual Hitchin fibration. Some points that will be addressed in future work follow.

1.3. Notation and Conventions.

1.3.1. *Reductive Groups.* For a scheme M acted on by an algebraic group G we denote the stack quotient by M/G, and the GIT quotient by $M/\!\!/G$.

- G- a reductive algebraic group.
- $\mathfrak{g}$ - the Lie algebra of G.
- θ: An involution $G \to G$.
- G^θ – scheme of fixed points of the involution theta.
- $K := (G^\theta)^0$
- H a group satisfying $K \subset H \subset N_G(K)$.
- $\mathfrak{p}$, The -1 eigenspace of the action of θ on $\mathfrak{g}$.
- Φ is the root system of G.
- W - the Weyl group of G.
- $\mathfrak{a}$ - A Cartan subspace in $\mathfrak{p}$, see definition 2.4
- Φ_r is the set of restricted roots, see definition 2.6.
- $W_\mathfrak{a}$ - the little Weyl group, see definition 2.8
- $\mathcal{N} \subset \mathfrak{g}$ is the nilpotent cone.
- $\mathcal{N}_\mathfrak{p} := \mathcal{N} \cap \mathfrak{p}$.
- $\mathfrak{p}^{reg}$ – the subscheme of regular points of $\mathfrak{p}$, see 2.23
- $I_E \to \mathfrak{p}$ is a group scheme defined in Definition 3.8.
- $\kappa_i : \mathfrak{a}/\!\!/W_a \to \mathfrak{p}^{reg}$ is a Kostant–Rallis section.
- $\mathcal{S}_i$ is the image of the Kostant–Rallis section.
- $\mathfrak{p}^{\kappa_i,H}$ is the subset of $\mathfrak{p}^{reg}$ consisting of the H-orbit of the Kostant–Rallis section κ_i, see section 3.2.
- F^* - a subgroup of A controlling the quotient $N_G(K)/K$, defined in Proposition 2.13.
- C_- - the connected component $(F^*)^0$.
- r - the restriction map $\Phi \to \Phi_r \cup \{0\}$, see definition 2.6.
- L - a Levi subgroup of G, often of the form in definition 3.36
- φ_L - the comparison map $\mathfrak{a}/\!\!/W_{\mathfrak{a},L} \to \mathfrak{a}/\!\!/W_\mathfrak{a}$, see Proposition 3.38.
- χ_L - is the morphism $\chi_L : \mathfrak{p}_L^{reg}/\!\!/H_L \times_{\mathfrak{a}/\!\!/W_{a,L}} U_L \to \mathfrak{p}^{reg}/\!\!/H$ of Lemma 3.40.
- i_L - the embedding $\mathfrak{p}_L \to \mathfrak{p}$.
- $\mathfrak{sym}_n$ - the space of symmetric $n \times n$ matrices.

1.3.2. *Geometry.*

- $\widetilde{\mathfrak{p}/\!\!/H}$ is defined in section 3.2.
- (Todo: The regular quotient!)

1.3.3. *Hitchin Fibration.*

- $r : \mathcal{A}^{reg} \to \mathcal{A}$ is the map from the regular Hitchin base to the Hitchin base.

2. Background on Symmetric Pairs

Let G be a reductive group over an algebraically closed field k, and $\mathfrak{g}$ its Lie algebra. Throughout, we make the following assumption on the characteristic of k:

$\ell = char(k)$ is good for G and the character and cocharacter lattices $X_*(A)/\mathbb{Z}\check{\Phi}_r$ and $X^*(A)/\mathbb{Z}\Phi_r$ associated with the restricted root system (defined in 2.6) have no ℓ-torsion

Let θ be an algebraic involution on G, i.e. an algebraic map $\theta\colon G \to G$ such that $\theta^2 = 1$.

Let $K = (G^\theta)^\circ$ denote the neutral component of the invariant subgroup. The involution θ induces a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ denote the $(+1)$ and (-1) eigenspaces of θ, respectively. The adjoint action of G on $\mathfrak{g}$ restricts to an action of G^θ on $\mathfrak{p}$; in particular, any subgroup of G^θ acts on $\mathfrak{p}$.

Definition 2.1. A *symmetric pair* with respect to the involution θ is the data of a closed subgroup $H \subset G$ such that $K \subset H \subset N_G(K)$.

Remark 2.2. In characteristic p we will often need to add the condition that H is smooth. This is not automatic, as can be seen by the example of considering $H = \mu_p \subset \mathbb{G}_m^p$ (with the involution being inversion) in characteristic p.

2.1. Maximal θ-split Tori, the Little Weyl Group, and the Restricted Root System.

In this section, we define the restricted root system associated to a symmetric pair. This root system is the crucial ingredient in the classification of simple symmetric pairs. See [37], chapter 26 for a more detailed exposition.

We start by considering root systems of G for θ-stable tori. A complicating fact in the theory of symmetric pairs is that, although all maximal tori of G are conjugate, conjugation will not respect the involution θ restricted to a torus. In this paper, we will be primarily focused on the following class of tori of G.

Definition 2.3. A θ-split torus A of G is a torus of G such that $\theta(a) = a^{-1}$ for all $a \in A$.

Let A be a fixed θ-split torus of G which is maximal among such tori. It is not true in general that A is a maximal torus of G; it is the case that $\mathfrak{a} = Lie(A)$ is a maximal abelian subalgebra of $\mathfrak{p}$. These subalgebras are important enough to warrant their own name.

Definition 2.4. A *Cartan* of $\mathfrak{p}$ is a maximal, abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$.

Cartans of $\mathfrak{p}$ and maximal θ-split tori A of G are in bijection via the exponential map. We will always fix $\mathfrak{a}$ and A in the sequel; this choice is justified by the following Proposition.

Proposition 2.5. *([37], Lem 26.15) All maximal θ-split tori A are conjugate.*

We will abuse notation slightly and call any extension of A to a maximal torus of G a *maximally θ-split torus.*

Fix $T \supset A$ a maximally θ-split torus as above. Let Φ be the set of roots of G with respect to T.

Definition 2.6. The set of *restricted roots* is

$$\Phi_r := \{\alpha|_{\mathfrak{a}} \in \mathfrak{a}^* \colon \alpha \in \Phi,\ \alpha|_{\mathfrak{a}} \neq 0\}.$$

We denote by $r\colon \Phi \to \Phi_r \cup \{0\}$ the restriction map taking $\alpha \mapsto \alpha|_{\mathfrak{a}}$.

Theorem 2.7. *([37], Lem 26.16) Φ_r forms a (possibly nonreduced) root system.*

Definition 2.8. The *little Weyl group*, denoted $W_{\mathfrak{a}}$, is the Weyl group associated to the root system Φ_r.

Proposition 2.9. *([37], Prop. 26.19) There is an isomorphism $W_{\mathfrak{a}} \simeq N_G(\mathfrak{a})/C_G(\mathfrak{a})$. In particular, the latter acts as a reflection group on $\mathfrak{a}$.*

As the root system Φ_r may be nonreduced, we recall the associated reduced root system.

Definition 2.10. To a nonreduced root system Φ, we define the (reduced) root system

$$\Phi^{\mathrm{red}} := \left\{\alpha \in \Phi \colon \frac{1}{2}\alpha \notin \Phi\right\}$$

which we call the reduced root system associated to Φ.

We note that the Weyl group $W_{\mathfrak{a}}$ associated to Φ_r is the same as the Weyl group of the reduced root system Φ_r^{red}.

It is useful to note that that the root system Φ_r^{red} on $\mathfrak{a}$ can be seen as the root system associated to a reductive algebraic group. In fact, such a group is given by the Gaitsgory-Nadler group, introduced for symmetric varieties in [22] and generalized in [9] and [16]. This dual group plays an important role in Langlands duality phenomena, for example as in [30].

Proposition 2.11. *([30], Theorem 3.3.1) To any spherical variety, there exists a subgroup $G_X^\vee \subset G^\vee$ with maximal torus the canonical torus A_X^*, canonical up to conjugation by the canonical torus $T^\vee \subset G^\vee$.*

Fix a Killing form identifying $\mathfrak{g} \simeq \mathfrak{g}^*$. In the special case of a symmetric variety $X = H\backslash G$, we can take A_X^* such that the killing form identifies $Lie(A_X^*) \subset \mathfrak{g}^*$ with $\mathfrak{a} \subset \mathfrak{g}$

Proof. The identification of $\mathfrak{a}$ with $\mathrm{L}ie(A_X^*)$ and the root system of the dual group $G_X^\vee$ is done in Theorem 6.7 of [15]. ☐

Definition 2.12. The *dual group* $G_X^\vee$ of the symmetric space $X = H\backslash G$ is the subgroup $G_X^\vee \subset G^\vee$ corresponding to the dual root data to $(\Delta_r^{\mathrm{red}}, \Phi_r^{\mathrm{red}})$.

For our purposes, it will be useful to understand the discrepancy between K, G^θ, and $N_G(K)$. In the following Proposition, by I^2 for I a group, we mean the collection $I^2 = \{a^2 \colon a \in I\}$, this is a group when I is abelian.

Proposition 2.13. *(a) The normalizer is given explicitly by*

$$N_G(K) = \{g \in G \colon g\theta(g^{-1}) \in Z(G)\}$$

(b) We have $N_G(K) = F^*K$ where $F^* = \{a \in A \colon a^2 \in Z(G)\}$. Note that F^* depends on the choice of A. Furthermore, $(F^*)^\circ = C_-$ is the connected subgroup of $Z(G)$ whose Lie algebra is the (-1) eigenspace of $Z(G)$.

(c) There is a short exact sequence

$$1 \to G^\theta \to N_G(K) \xrightarrow{\tau} (F^*)^2 \to 1$$

where $\tau(g) = g\theta(g^{-1})$.

(d) The group $G^\theta/K = \pi_0(G^\theta)$ is a discrete group.

(e) Any closed subgroup H in $N_G(K)$ containing K has H° reductive.

Proof. Part (a) follows from the proof of Lemma 1.1 of [35]. Part (b) is Lemma 8.1 in [28]. Part (c) follows immediately from (b). Part (d) is clear as G^θ is finite type. Part (e) is directly from Lemma 8.1 of [28]. ☐

We have a Chevalley-style result on the GIT quotient $\mathfrak{p}/\!\!/K := \operatorname{Spec} k[\mathfrak{p}]^K$.

Theorem 2.14. *([19], Theorem 4.9 and Corollary 4.10) The natural inclusion map $\mathfrak{a} \to \mathfrak{p}$ induces a isomorphisms $\mathfrak{a}/\!\!/W_{\mathfrak{a}} \simeq \mathfrak{p}/\!\!/K \simeq \mathfrak{p}/\!\!/N_G(K)$. In particular, for any closed $K \subset H \subset N_G(K)$, we have $\mathfrak{a}/\!\!/W_{\mathfrak{a}} \simeq \mathfrak{p}/\!\!/H$. Note in particular that this map is $\mathbb{G}_m$-equivariant under the homothety actions on both $\mathfrak{a}$ and $\mathfrak{p}$.*

Corollary 2.15. *There is a natural identification of the GIT quotients $\mathfrak{p}/\!\!/K \simeq \mathfrak{g}_X^\vee/\!\!/G_X^\vee$.*

Proof. The lefthand side is isomorphic to $\mathfrak{a}/\!\!/W_{\mathfrak{a}}$ by Theorem 2.14 while the righthand side is isomorphic to $\mathfrak{a}_X^*/\!\!/W_{G_X}$ where $\mathfrak{a}_X^* \subset \mathfrak{g}_X^\vee$ is a Cartan of G_X. By Proposition 2.11, we can choose a Killing form on $\mathfrak{g}$ that identifies $\mathfrak{a}_X^* \subset \mathfrak{g}_X^\vee \subset \mathfrak{g}^\vee$ and $\mathfrak{a} \subset \mathfrak{g}$. Since G_X by definition has its root system the dual of Φ_r^{red}, there is a canonical isomorphism $W_{G_X} \simeq W_{\mathfrak{a}}$. The identification above now follows. ☐

The invariant theory of this GIT quotient is well studied. We will make use of the following fact.

Lemma 2.16. *([19], Lemma 4.11) We can write $k[\mathfrak{a}]^{W_{\mathfrak{a}}} = k[f_1, \dots, f_r]$ for $r = \dim(\mathfrak{a})$ algebraically independent homogeneous polynomials $f_1, \dots, f_r$ or degrees $m_1, \dots, m_r$, respectively, which we will call the* exponents of the root system Φ_r. *Moreover, the sum of these exponents can be computed as*

$$\sum_i m_i = r + \frac{\#\Phi_r^{red}}{2}.$$

Proof. Follows from taking degree of the left hand and right hand side of the equality in [19], Lemma 4.11, noting that the length of the longest element in $W_{\mathfrak{a}}$ is given by the number of positive roots in the reduced root system. ☐

2.2. **Examples.** In the following, we label examples in the notation $H \subset G$ with the involution being implied.

Example 2.17. **The Diagonal Case.** Let G_1 be a reductive group, and consider $G = G_1 \times G_1$ with involution

$$\theta(g, h) = (h, g)$$

swapping the two factors. Then, $K = G_1$ is the diagonal copy of G_1 in $G_1 \times G_1$, and we have the Cartan decomposition

$$\mathfrak{g} = \{(X, X) \colon X \in \mathfrak{g}_1\} \oplus \{(X, -X) \colon X \in \mathfrak{g}_1\}.$$

A Cartan in $\mathfrak{p}$ is a Cartan $\mathfrak{t} \subset \mathfrak{g}_1$ of G_1 embedded in $\mathfrak{p}$ by

$$\mathfrak{a} = \{(X, -X) \colon X \in \mathfrak{t}\}.$$

while a maximal θ-split Cartan is given by

$$\mathfrak{t} = \{(X, Y) \colon X, Y \in \mathfrak{t}\}.$$

Note that in this example, this maximal θ-split Cartan is also maximal θ-fixed. The restricted root system of $(G_1 \times G_1, G_1)$ is given by the root system for G_1.

Example 2.18. **The Case** $\mathrm{GL}_n \times \mathrm{GL}_n \subset \mathrm{GL}_{2n}$**.** Let $G = \mathrm{GL}_{2n}$ and consider the involution

$$\theta(X) = I_{n,n} X I_{n,n} \quad \text{where } I_{n,n} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$$

We have $K = G^\theta = \mathrm{GL}_n \times \mathrm{GL}_n \subset \mathrm{GL}_{2n}$ embedded block diagonally, and the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \right\}.$$

A Cartan in $\mathfrak{p}$ is given by

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} : \delta \text{ is diagonal} \right\}$$

A maximally θ-split torus is

$$\mathfrak{t} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) = \left\{ \begin{pmatrix} \eta & \delta \\ \delta & \eta \end{pmatrix} : \delta, \eta \text{ are diagonal} \right\}$$

The restricted root system is computed as

$$\Phi_r = \{\pm(\delta_j^* \pm \delta_k^*) | j \neq k \text{ and } 1 \leq j, k \leq n\} \cup \{\pm 2\delta_j^* | 1 \leq j \leq n\}$$

where δ_j^* denotes the dual basis element to the j-th coordinate of δ in $\mathfrak{a}$.

Example 2.19. **The Case** $\mathrm{SO}_n \times \mathrm{SO}_n \subset \mathrm{SO}_{2n}$**.** Let $G = \mathrm{SO}_{2n}$ with the involution

$$\theta(X) = I_{n,n} X I_{n,n} \quad \text{where } I_{n,n} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$$

(compare with Example 2.18). Then, $K = \mathrm{SO}_n \times \mathrm{SO}_n \subset \mathrm{SO}_{2n}$ is embedded block diagonally. Note that this is an index 2 subgroup in $G^\theta = S(O_n \times O_n)$. The Cartan decomposition is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A, B \in \mathfrak{so}_n \right\} \oplus \left\{ \begin{pmatrix} 0 & C \\ -C^t & 0 \end{pmatrix} \right\}$$

We fix a Cartan in $\mathfrak{p}$

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} : \delta \text{ is diagonal} \right\}$$

The restricted root system is

$$\Phi_r = \{ i(\pm\delta_j^* \pm \delta_k^*) \colon 1 \le j < k \le n \}$$

where δ_j^* denotes the dual basis element to the j-th coordinate of δ in $\mathfrak{a}$ and i is the imaginary unit.

Example 2.20. **The Case** $\mathrm{SO}_m \times \mathrm{SO}_{2n-m} \subset \mathrm{SO}_{2n}$**,** $m < n$**.** Fix $m < n$, and consider the case of $G = \mathrm{SO}_{2n}$ with the involution

$$\theta(X) = I_{m,2n-m} X I_{m,2n-m} \quad \text{where } I_{m,2n-m} = \begin{pmatrix} I_m & 0 \\ 0 & -I_{2n-m} \end{pmatrix}$$

Then, $K = \mathrm{SO}_m \times \mathrm{SO}_{2n-m} \subset \mathrm{SO}_{2n}$ is embedded block diagonally, and the Cartan decomposition is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in \mathfrak{so}_m,\ B \in \mathfrak{so}_{2n-m} \right\} \oplus \left\{ \begin{pmatrix} 0 & C \\ -C^t & 0 \end{pmatrix} : C \text{ is a } 2m \times 2n-m \text{ matrix} \right\}$$

We choose

$$\mathfrak{a} = \left\{ \left(\begin{array}{c|c} & \mathbf{0}_{m\times n-m} \quad \delta \quad \mathbf{0}_{m\times n-m} \\ \hline \begin{matrix} \mathbf{0}_{n-m\times m} \\ -\delta \\ \mathbf{0}_{n-m\times m} \end{matrix} & \end{array} \right) : \delta \text{ is diagonal } m \times m \right\}$$

Note that this extends to a Cartan of SO_{2n} given by

$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} : \delta \text{ is diagonal } n \times n \right\}$$

and that the root system for SO_{2n} with respect to $\mathfrak{t}$ is

$$\Phi = \{ i(\pm\delta_j^* \pm \delta_k^*) \colon 1 \le j < k \le n \}$$

As $m < n$ the restricted root system is given by

$$\Phi_r = \{ i(\pm\delta_j^* \pm \delta_k^*) \colon 1 \le j < k \le m \} \cup \{ \pm i\delta_j^* \colon 1 \le j \le m \}.$$

Example 2.21. **The Case** $\mathrm{SO}_m \times \mathrm{SO}_{2n-m+1} \subset \mathrm{SO}_{2n+1}$**,** $m \le n$**.** Fix $m \le n$ and consider the case of $G = \mathrm{SO}_{2n+1}$ with the involution

$$\theta(X) = I_{m,2n-m+1} X I_{m,2n-m+1} \quad \text{where } I_{m,2n-m+1} = \begin{pmatrix} I_m & 0 \\ 0 & -I_{2n-m+1} \end{pmatrix}$$

Then, $K = \mathrm{SO}_m \times \mathrm{SO}_{2n-m+1} \subset \mathrm{SO}_{2n+1}$ is embedded block diagonally, and the Cartan decomposition is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in \mathfrak{so}_m,\ B \in \mathfrak{so}_{2n-m+1} \right\} \oplus \left\{ \begin{pmatrix} 0 & C \\ -C^t & 0 \end{pmatrix} : C \text{ is a } 2m \times 2n-m+1 \text{ matrix} \right\}$$

We choose

$$\mathfrak{a} = \left\{ \left(\begin{array}{c|ccc} & \mathbf{0}_{m\times(n-m)} & \delta & \mathbf{0}_{m\times(n-m+1)} \\ \hline \mathbf{0}_{(n-m)\times m} & & & \\ -\delta & & & \\ \mathbf{0}_{(n-m+1)\times m} & & & \end{array} \right) : \delta \text{ is diagonal } m\times m \right\}$$

which sits inside the Cartan of SO_{2n+1}

$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & \delta & \mathbf{0}_{m\times 1} \\ -\delta & 0 & \\ \mathbf{0}_{1\times m} & & \end{pmatrix} : \delta \text{ is diagonal } n\times n \right\}$$

The root system with respect to the Cartan $\mathfrak{t}$ is

$$\Phi = \{i(\pm\delta_j^* \pm \delta_k^*) \colon 1 \le j < k \le n\} \cup \{\pm i\delta_j^* \colon 1 \le j \le n\}$$

In particular, we conclude that the restricted root system is the type B_m root system

$$\Phi_r = \{i(\pm\delta_j^* \pm \delta_k^*) \colon 1 \le j < k \le m\} \cup \{\pm i\delta_j^* \colon 1 \le j \le m\}.$$

Example 2.22. **The Case $\mathrm{GL}_n \subset \mathrm{Sp}_{2n}$.** Let $G = \mathrm{Sp}_{2n}$ with the involution

$$\theta(X) = I_{n,n} X I_{n,n} \quad \text{where } I_{n,n} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$$

We have

$$K = \left\{ \begin{pmatrix} g & \\ & g^{-t} \end{pmatrix} : g \in \mathrm{GL}_n \right\} \subset \mathrm{Sp}_{2n}$$

We fix a Cartan in $\mathfrak{p}$

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} : \delta \text{ is diagonal} \right\}$$

Note that $\mathfrak{a}$ is also a Cartan of Sp_{2n}, making this a split symmetric pair. The restricted root system is thus equal to the root system of Sp_{2n}:

$$\Phi_r = \{\pm\delta_j^* \pm \delta_k^* \colon 1 \le j < k \le n\} \cup \{\pm 2\delta_j^* \colon 1 \le j \le n\}$$

where δ_j^* denotes the dual basis element to the j-th coordinate of δ in $\mathfrak{a}$.

2.3. **Regularity and the Quasi-Split Condition.**

Definition 2.23. We denote by $I \subset \mathfrak{p} \times H$ the group scheme of centralizers over $\mathfrak{p}$, i.e.

$$I = \{(X, h) \colon h \cdot X = X\}.$$

An element $X \in \mathfrak{p}$ is called *regular* if $\dim(I_X)$ is minimal[1]. Let $\mathfrak{p}^{reg} \subset \mathfrak{p}$ denote the open subscheme of regular elements in $\mathfrak{p}$.

Remark 2.24. Note that since H is a finite extension of K, the notion of regularity does not depend on the choice of subgroup H–only on the involution θ.

Proposition 2.25. *(*[19]*, Lemma 4.3) For any $x \in \mathfrak{p}$, x is regular if and only if* $\dim Z_G(x) = \dim \mathfrak{a} + \dim Z_K(A)$.

We call an element $x \in \mathfrak{p}$ *semisimple* if x is semisimple in $\mathfrak{g}$. That this definition is correct is motivated in part by the following compatibility.

Lemma 2.26. *Any $x \in \mathfrak{p}$ admits a decomposition $x = s + n$ for $s, n \in \mathfrak{p}$, s semisimple, n nilpotent, and $n \in Z_G(s)$.*

[1]We note that this is not quite the same as the definition of [21]. However under the assumptions we make on the charactersitic of our field this is equivalent to the definition in *loc cit.*

Proof. See Lemma 2.1 in [19]. ☐

Proposition 2.27. *Let $x \in \mathfrak{p}$ have decomposition $x = s + n$ as in Lemma 2.26, and put $L = Z_G(s)^0$, $\mathfrak{l} = \mathrm{Lie}(L)$, and $\mathfrak{p}_L = \mathfrak{l} \cap \mathfrak{p}$. Then L is θ-stable and x is regular in $\mathfrak{p}$ if and only if n is regular as an element of $\mathfrak{p}_L$.*

Proof. We follow an identical argument to the proof of Proposition 9.12 in [28]. We assume without loss of generality that $s \in \mathfrak{a}$ so that $A \subset L$. By Proposition 2.25, x is regular if and only if $\dim Z_K(x) = \dim Z_K(A)$ and n is regular in L if and only if $\dim Z_{K\cap L}(n) = \dim Z_{K\cap L}(A)$. We have $Z_K(x)^0 = (Z_K(s) \cap Z_K(n))^0 = Z_{K\cap L}(n)^0$, so that $\dim Z_K(x) = \dim Z_{K\cap L}(n)$. Since $x \in A$, $Z_K(A) = Z_{K\cap L}(x)$, so the result follows. ☐

Lemma 2.28. *([19], Lemma 4.2) We have $\mathrm{Lie}(Z_G(A)^\circ) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$.*

As in the Lie algebra $\mathfrak{g}$, the regular, semisimple locus is dense and easy to understand.

Lemma 2.29. *Let $\mathfrak{p}^{rs} \subset \mathfrak{p}$ denote the subscheme of regular, semisimple elements in $\mathfrak{p}$.*

(a) Let $x \in \mathfrak{p}$. Then, x is semisimple if and only if x is contained in a Cartan of $\mathfrak{p}$.

(b) The regular, semisimple locus $\mathfrak{p}^{rs}$ is dense in $\mathfrak{p}$.

(c) If $x \in \mathfrak{a}$ is regular, semisimple, then $\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$.

Proof. (a) and (b) follow immediately from [19], Corollary 2.10 and Theorem 2.11. (c) follows from [19], Lemma 4.3. ☐

The action of H on $\mathfrak{p}$ does not have abelian centralizers in general. In the sequel, we will restrict ourselves to the special subclass of forms with this property, namely those which are *quasi-split*.

Definition 2.30. We say a symmetric pair $(\mathfrak{p}, H)$ is *quasi-split* if $\mathfrak{p}^{reg} \subset \mathfrak{g}^{reg}$; that is, the notion of regularity in $\mathfrak{p}$ under the action of H and $\mathfrak{g}$ under the action of G coincide.

Remark 2.31. The quasi-split condition does not depend on the choice of subgroup H, only on the involution θ.

Proposition 2.32. *The following are equivalent:*

(1) $(\mathfrak{p}, H)$ is quasi-split;
(2) $Z_G(A) = T$ is a maximal (maximally θ-split) torus;
(3) $I|_{\mathfrak{p}^{reg}}$ is a commutative group scheme;

Proof. The centralizer $Z_G(A)$ includes a maximal θ split torus T; it is abelian if and only if $Z_G(A) = T$. The pair $(\mathfrak{p}, H)$ is quasi-split if and only if for all $x \in \mathfrak{a}^{reg}$, we have

$$\dim Z_G(x) = \mathrm{rank}(G)$$

so that all inclusions in $T \subset Z_G(A) \subset Z_G(x)$ are equalities. Hence, (1) and (2) are equivalent.

Assume (1) and (2) hold now. Then if $x \in \mathfrak{p}^{reg}$, $I|_{\mathfrak{p}^{reg}}$ is contained in the regular centralizer group scheme in G, which is abelian. Hence, (3) holds.

Conversely, if (3) holds, then by Lemma 2.29, there exists $x \in \mathfrak{p}^{reg}$ such that $\mathfrak{z}_{\mathfrak{g}}(A) = \mathfrak{z}_{\mathfrak{g}}(x)$. Then $\dim \mathfrak{z}_{\mathfrak{g}}(A) = \dim \mathfrak{z}_{\mathfrak{g}}(x) = r$ is the rank of G and by Lemma 2.28 $Z_G(A)$, we conclude that $Z_G(A)$ is a maximal torus. ☐

Proposition 2.33. *([18], Lemma 1.6) For θ quasi-split, the little Weyl group $W_{\mathfrak{a}}$ is naturally a subgroup $W_{\mathfrak{a}} \subset W$.*

Let T be a maximal θ-split torus, Φ the root system of G with respect to T, and Φ_r the restricted root system with restriction map

$$r \colon \Phi \to \Phi_r \cup \{0\}$$

As noted in Section 2.1, roots of G may, a priori, restrict to zero in $\mathfrak{a} \subset \mathfrak{t}$. For quasi-split forms, this does not happen as shown in the following lemma.

Lemma 2.34. *For (G,θ) quasi-split, the set $r^{-1}(0)$ is empty; that is, no root in Φ restricts to zero on $\mathfrak{a}$.*

Proof. Fix θ-stable Cartan $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{a}$ where $\mathfrak{t}_0 \subset \mathfrak{t}_K$ is the $(+1)$ eigenspace of θ on $\mathfrak{t}$. Suppose that $\alpha \in \Phi$ restricts to zero on $\mathfrak{a}$. Let $(X,Y) \in \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$ be an eigenvector with eigencharacter α. Then, using the compatibility of the bracket on $\mathfrak{g}$ with the Cartan decomposition, we have, for all $t \in \mathfrak{t}_0$ and $a \in \mathfrak{a}$,

$$\alpha(t)(X,Y) = \alpha(t+a)(X,Y) = \big(\mathrm{a}d(t)(X) + \mathrm{a}d(a)(Y), \mathrm{a}d(a)(X) + \mathrm{a}d(t)(Y)\big). \tag{2.1}$$

In particular, $\mathrm{a}d(a)(Y)$ is independent of a, so $Y \in \mathfrak{c}_{\mathfrak{p}}(\mathfrak{a})$. But since the form is quasi-split, $\mathfrak{c}_{\mathfrak{p}}(\mathfrak{a}) = \mathfrak{a} \subset \mathfrak{c}_{\mathfrak{p}}(\mathfrak{t}_0)$. [c.f. Levy, Lemma 2.3] Hence, $\mathrm{a}d(t)(Y) = 0$ and equation 2.1 implies that

$$\mathrm{a}d(a)(X) = \alpha(t)Y$$

for all $t \in \mathfrak{t}_0$ and $a \in \mathfrak{a}$. This can only be true if both sides of the expression are uniformly zero, so $\alpha = 0$ is not in Φ, a contradiction. ☐

Fix a Cartan $\mathfrak{t} \supseteq \mathfrak{a}$ of $\mathfrak{g}$ extending $\mathfrak{a}$. In general for an involution θ with symmetric pair $(\mathfrak{p}, K)$, there is a finite, ramified cover $\mathfrak{a}/\!\!/W_{\mathfrak{a}} \to \mathfrak{t}/\!\!/W$ produced by the map of GIT quotients $\mathfrak{p}/\!\!/K \to \mathfrak{g}/\!\!/G$. This map is unramified precisely in the quasi-split case:

Lemma 2.35. *([24], Theorem 3.6) If the involution θ is quasi-split, then the map $\mathfrak{a}/\!\!/W_{\mathfrak{a}} \to \mathfrak{t}/\!\!/W$ is unramified.*

We record here an identity that will be important for dimension counts later.

Lemma 2.36. *([28], Lemmas 3.1 and 3.2) We have the identity*

$$\dim \mathfrak{k} - \dim \mathfrak{p} = \dim Z_K(\mathfrak{a}) - \dim \mathfrak{a}.$$

In particular, if the form is quasi-split, then

$$\dim \mathfrak{k} - \dim \mathfrak{p} = r - 2r_\theta$$

where r is the rank of the group G and $r_\theta = \dim \mathfrak{a}$ is the rank of the involution.

2.4. **Nilpotent Orbits.** In this section, we review results of [17], [35], and [19] on nilpotent K-orbits.

Recall that K acts on the nilpotent cone $\mathcal{N}_{\mathfrak{p}} = \mathcal{N} \cap \mathfrak{p}$ of $\mathfrak{p}$. The nilpotent cone is not necessarily irreducible, reflecting the fact that there is not necessarily a single K-orbit of regular nipotents in $\mathfrak{p}$.

Theorem 2.37. *([19], Theorem 5.1) Each irreducible component of $\mathcal{N}_{\mathfrak{p}}$ contains a unique regular K-orbit as an open, dense subset. In particular, irreducible components of $\mathcal{N}_{\mathfrak{p}}$ are in 1-1 correspondence with connected components of $\mathcal{N}_{\mathfrak{p}}^{reg}$, i.e. $\mathrm{I}rr(\mathcal{N}_{\mathfrak{p}}) = \pi_0(\mathcal{N}_{\mathfrak{p}}^{reg})$.*

Corollary 2.38. *The space $\mathcal{N}_{\mathfrak{p}} \backslash \mathcal{N}_{\mathfrak{p}}^{reg}$ is of codimension ≥ 1 in $\mathcal{N}_{\mathfrak{p}}$.*

The number of K-conjugacy classes of regular nilpotents was studied by [35] over $\mathbb{C}$ and by [19] in good characteristic.

Proposition 2.39. *([19], Proposition 6.21) The number of regular nilpotent orbits (and hence the number of irreducible components of the nilpotent cone) is exactly two for each of the cases (listed as pairs $(G, H = K)$):*

- $(\mathrm{GL}_{2n}, \mathrm{SO}_{2n})$;
- $(\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n)$;
- $(\mathrm{SO}_{2n+1}, \mathrm{SO}_{2m} \times \mathrm{SO}_{2(n-m)+1})$, $2m < 2(n-m)+1$;
- $(\mathrm{Sp}_{2n}, \mathrm{GL}_n)$;
- $(\mathrm{SO}_{2n}, \mathrm{SO}_{2m} \times \mathrm{SO}_{2(n-m)})$, $m \neq n/2$;
- $(\mathrm{SO}_{4n}, \mathrm{GL}_{2n})$;
- $(\mathrm{SO}_{4n+2}, \mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n+1})$;
- (G, SL_8), *for G simple of type E_7;*

- *$(G, G' \times \mathbb{G}_a)$, for G simple of type E_7 and G' simple of type E_6;*

In addition, the form $(\mathrm{SO}_{4n}, \mathrm{SO}_{2n} \times \mathrm{SO}_{2n})$ has exactly 4 regular nilpotent orbits. All other symmetric pairs with G simple and $H = K$ have irreducible nilpotent cone in $\mathfrak{p}$, and hence a single regular nilpotent orbit.

Remark 2.40. Among the above involutions, only the following are quasi-split:

- $(\mathrm{GL}_{2n}, \mathrm{SO}_{2n})$;
- $(\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n)$;
- $(\mathrm{SO}_{2n+1}, \mathrm{SO}_n \times \mathrm{SO}_{n+1})$;
- (G, SL_8), for G simple of type E_7;
- $(\mathrm{SO}_{4n}, \mathrm{SO}_{2n} \times \mathrm{SO}_{2n})$ (which has 4, not 2, nilpotent orbits)

Remark 2.41. Note that $(F^*)^2$ acts trivially on $\mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}$. For any symmetric pair (G, H), by Proposition 2.13, we have $H = C \cdot H'$ where $C \subset (F^*)^2$ and $K \subset H' \subset G^\theta$. The H-orbits on $\mathcal{N}_{\mathfrak{p}}^{reg}$ are therefore given by $(\mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}/K)/\pi_0(H') = (\mathcal{N}_{\mathfrak{p}}^{reg}/K)/\pi_0(H)$.

If one sets $H = N_G(K)$, then the classification of regular, nilpotent orbits becomes trivial.

Theorem 2.42. *([19], Theorem 5.16) The normalizer group $N_G(K)$ acts transitively on the set of regular nilpotents.*

In particular, for $H = N_G(K)$ as in Remark 2.41, it is necessarily true that $\pi_0(H)$ acts transitively on $\mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}/K$.

For our purposes later, we will also need the classification of nilpotent orbits for the diagonal case.

Lemma 2.43. *The diagonal case $G_1 \subset G_1 \times G_1$ of Example 2.17 has a single nilpotent K orbit on $\mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}$.*

Proof. There is an isomorphism of stacks $\mathfrak{p}/K \simeq \mathfrak{g}_1/G_1$ given by projecting onto the first variable. In particular, this map preserves regularity and induces an isomorphism $\mathcal{N}_{\mathfrak{p}}^{reg}/K \simeq \mathcal{N}^{\mathrm{reg}}/G_1$ where $\mathcal{N}$ is the nilpotent cone in G_1. Since any complex group G_1 has a unique regular G_1 orbit, the lemma follows. □

We will study the map $\mathfrak{p} \to \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ produced by Theorem 2.14 in some detail; it will provide the underlying structure of the Hitchin fibration for symmetric pairs. In this spirit, we now prove this map's flatness.

Lemma 2.44. *The map $\mathfrak{p} \to \mathfrak{p}/\!\!/K \simeq \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ is flat, as is the map $\mathfrak{p}^{reg} \to \mathfrak{p}/\!\!/K$.*

Proof. This is a morphism between two smooth schemes. Hence, by miracle flatness, it suffices to show that the fibers are equidimensional. Let $U \subset \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ be the subset of $x \in \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ whose fiber in $\mathfrak{p}$ is $\dim(K) - r_\theta$ dimensional, with $r_\theta = \dim(\mathfrak{a})$ the rank of the group. Then U contains $0 \in \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ as Theorem 2.37 guarantees $\dim(\mathcal{N}_{\mathfrak{p}}) = \dim(K) - r_\theta$, and U is stable under the action of $\mathbb{G}_m$ as the map is $\mathbb{G}_m$ equivariant. Moreover, U contains the open complement of the image of the root hyperplanes $\cup_\alpha H_\alpha \subset \mathfrak{a}$ of $\mathfrak{a}$. Hence, $U = \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ and the map is flat. □

2.5. **Generalities on Kostant-Rallis Sections.** In this subsection, we review the theory of Kostant Rallis sections, as introduced in [17] and generalized in [19]. We work in the generality of [19]; in particular, in this section, it is important that $char(k) = p$ is good for G, namely if we let Δ be a basis for the root system Φ of G and if we express the longest element of Φ relative to Δ as $\check{\alpha} = \sum_{\beta \in \Delta} m_\beta \beta$, then p is good for G if and only if $p > m_\beta$ for all $\beta \in \Delta$.

In positive characteristic, associated characters replace the $\mathfrak{sl}_2$ triples used in [17]. As this paper will only rely on the existence of sections, we leave such the theory of associated characters and their relationship to the more explicit $\mathfrak{sl}_2$ triples to Appendix 4.

Lemma 2.45. *([19], Corollary 6.29) Let $e \in \mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}$ be a regular nilpotent. Then there exists a slice $e + \mathfrak{v} \subset \mathfrak{p}^{reg}$ contained in the regular locus of $\mathfrak{p}$ such that the map*

$$e + \mathfrak{v} \to \mathfrak{p}/\!\!/K$$

is an isomorphism whose fiber over $0 \in \mathfrak{p}/\!\!/K$ is e.

Remark 2.46. In general, the space $\mathfrak{v} \subset \mathfrak{p}$ is constructed by taking a normal associated character λ to e (see Definition 4.6) and then constructing a certain Lie subalgebra $\mathfrak{g}^* \subset \mathfrak{g}$ with corresponding Cartan decomposition $\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*$. The slice is given by taking $\mathfrak{v}$ to be an $\mathrm{A}d(\lambda)$ graded complement to $[\mathfrak{k}^*, e]$ inside of $\mathfrak{p}^*$.

If we suppose that the characteristic of k is either zero or greater than the Coxeter number of G, then the results of Appendix 4 give a bijection between H-conjugacy classes of assocaited characters and H-conjugacy clsses of $\mathfrak{sl}_2$ triples. In this case, we can complete $e \in \mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}$ to a normal $\mathfrak{sl}_2$ triple (e, h, f) (see Definition 4.1) uniquely up to $C_K(e)^\circ$ conjugacy, and we can take $e + \mathfrak{v} = e + \mathfrak{c}_{\mathfrak{p}}(f)$ as in [17], Theorem 11.

We will need a bit more on the differential of the action map

$$H \times \mathcal{S} \to \mathfrak{p}^{reg}.$$

To do so, we will need the following auxiliary construction, introduced by [17] in characteristic zero and by [19] in characteristic p; namely, the following asserts the existence of a subgroup $G^* \subset G$ which is θ stable, for which θ acts as a split form, and which is sufficiently large to contain the image of Kostant-Rallis sections of G. This construction will only be used to prove Lemma 2.48 and will not be used elsewhere in the paper.

Lemma 2.47. *([19], Thm 6.18, Cor 6.26, Lem 6.27 and Lem 6.29) Fix a maximal abelian $\mathfrak{a} \subset \mathfrak{p}$ for the symmetric pair (G, H) and fix $e \in \mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}$. Then there exists a reductive sub-Lie algebra $\mathfrak{g}^* \subset \mathfrak{g}$ and reductive group G^* with $\mathrm{Lie}(G^*) = \mathfrak{g}^*$ with the following properties:*

(1) $\mathfrak{g}^* \subset \mathfrak{g}$ is θ stable, and the involution $\theta^* = \theta|_{\mathfrak{g}^*}$ lifts to an involution θ^* on G^*.
(2) Let $\mathfrak{p}^*, K^*$ denote the analogous constructions for θ^* acting on $\mathfrak{g}^*$. Then, $\mathfrak{a} \subset \mathfrak{p}^*$ and the involution θ^* on G^* is split, i.e. $\mathfrak{a} \subset \mathfrak{g}^*$ is a Cartan of $\mathfrak{g}^*$.
(3) The restriction map produces an isomorphism of GIT quotients $\mathfrak{p}^*/\!\!/K^* \simeq \mathfrak{p}/\!\!/K$.
(4) The regular locus in $\mathfrak{p}$ and the regular locus in $\mathfrak{p}^*$ agree, i.e. $\mathfrak{p}^{\mathrm{reg}} \cap \mathfrak{p}^* = (\mathfrak{p}^*)^{\mathrm{reg}}$
(5) $e \in \mathfrak{p}^*$ and in Lemma 2.45, one can chose $\mathfrak{v} \subset \mathfrak{p}^*$. In particular, the Kostant-Rallis section to

$$\mathfrak{p}^{reg} \to \mathfrak{p}/\!\!/K$$

of Lemma 2.45 has image in $(\mathfrak{p}^*)^{reg}$ and so also gives a section to

$$(\mathfrak{p}^*)^{\mathrm{reg}} \to \mathfrak{p}^*/\!\!/K \simeq \mathfrak{g}^*/\!\!/G^*.$$

(6) The group G^* satisfies the "standard hypotheses": i.e. p is good for G^*; the derived subgroup of G^* is simply connected; and there exists a G^*-equivariant, nondegenerate, symmetric bilinear form $\kappa \colon \mathfrak{g}^* \times \mathfrak{g}^* \to k$.

Lemma 2.48. *Fix $e \in \mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}$ and consider the section $\mathcal{S} := e + \mathfrak{v}$ as in Lemma 2.45. We assume further that the cocharacter lattice $\mathbf{X}_*(A)/\mathbb{Z}\Phi_r^*$ of the restricted root system has no p torsion for p the characteristic of k. Let*

$$H \times \mathcal{S} \to \mathfrak{p}^{reg}$$

be the action map for the adjoint action of H on $\mathcal{S} \subset \mathfrak{p}^{reg}$. Then the differential of this map at $(1, e)$ is surjective.

Proof. The differential of the above map at $(1, e)$ is identified with the map

$$\mathfrak{k} \oplus \mathfrak{v} \to \mathfrak{p}, \quad (x, v) \mapsto [x, e] + v.$$

Replacing (G,θ) with (G^*,θ^*) and using Lemma 2.47, we may assume that the symmetric pair is split (and in particular, quasisplit) and that G satisfies the standard hypotheses as well as the condition that $\mathbf{X}_*(A)/\mathbb{Z}\Phi_r^*$ has no p torsion. Therefore, the result follows by [29], Lemma 3.1.3, by intersecting with $\mathfrak{p}$. □

3. The regular quotient

3.1. **Generalities on the regular quotient.** One motivation for the introduction of the regular quotient is introduced by Ngô B.-C. and the second author in [21], is to allow generalization of the invariant theory of the adjoint action of G on $\mathfrak{g}$ which is used in the analysis of the Hitchin fibration in [8, 23]. The key point is that the regular quotient $M^{reg} /\!\!/_{I'} G$ is a space such that the quotient stack M^{reg}/G is a gerbe over $M^{reg} /\!\!/_{I'} G$.

We first recall the definition of the regular quotient from [21]. We also introduce some of the basic properties before giving an elementary description in our case. We stress that the majority of this paper can be read merely knowing that the regular quotient of the action of H on $\mathfrak{p}$ is a space parameterizing the H-orbits of maximal possible dimension, and the characterization given by Proposition 3.3. Theorem 3.19 shows that under our assumptions on p we have that $\mathfrak{p}^{reg}$ is precisely the open subscheme such that the stabilizers are of minimal dimension.

Let M be an affine variety acted on by a reductive algebraic group H. Let $I_M \subset M \times H$ be the group scheme over M of stabilizers of the H-action. Following [21] the regular locus is the subset maximal open subscheme $M^{reg} \to M$ such that $Lie(I_{M^{reg}}) \to M^{reg}$ is a vector bundle. We note that this is an H-scheme. Secondly we note that in characteristic zero the k-points of M^{reg} are precisely those such that the stabilizer of m in H is of the minimal possible dimension. We will be using $(M,H) = (\mathfrak{p},H)$ a symmetric pair.

Assume that $I_M^0|_{M^{reg}} \subset I' \hookrightarrow I_M|_{M^{reg}}$ is a smooth subgroup scheme over M^{reg}, that is G-equivariant.

Definition 3.1 (Regular Quotient [21])**.** Let M, I', and H be as above. We define the *regular quotient* $M /\!\!/_{I'} H$ to be the stack quotient of the groupoid in algebraic spaces

$$(M^{reg} \times H)/I' \rightrightarrows M^{reg}.$$

Note that this does depend on the precise choice of I' used. It is describing a rigidification by I' of the stack quotient M^{reg}/H.

By definition $M /\!\!/_{I'} H$ satisfies a 2-coequalizer property for the diagram $(M^{reg} \times H)/I' \rightrightarrows M^{reg}$.

The regular quotient has the following properties [21]:

Proposition 3.2 (Properties of Regular Quotient from [21])**.** *The regular quotient has the following properties:*

- *If I' is abelian then it descends to a group scheme $J \to M^{reg} /\!\!/_{I'} H$.*
- *I' descends to a band in the sense of Giraud* [13] *$J_{band} \to M^{reg} /\!\!/_{I'} H$.*
- *The map $M^{reg}/H \to M^{reg} /\!\!/_{I'} H$ is a gerbe banded by J_{band}; when I' is abelian it is a J-gerbe.*

The second property is in fact a defining property of the regular quotient:

Proposition 3.3 ([21])**.** *Let $I' = I_M|_{M^{reg}}$, and V be a scheme such that each fiber of $M^{reg} \to V$ consists of a single G-orbit. Then $V = M^{reg} /\!\!/_{I'} G$.*

We will use this property to describe $\mathfrak{p}^{reg} /\!\!/_{I'} H$, for $I' = I_{\mathfrak{p}}|_{\mathfrak{p}^{reg}}$. For the other choices of I' we consider we will also need to use the following theorem.

Proposition 3.4 ([21])**.** *Let $H \subset H'$, such that H is a normal subgroup. Suppose that the H action on M and on M^{reg} extends to a H' action. Suppose that I' is also H'-equivariant, and a normal subgroup of $N \times H'$.*

Then the map $M^{reg} /\!\!/_{I'} H \to M^{reg} /\!\!/_{I'} H'$ is a principal H'/H-bundle.

3.1.1. *Equivariant Case.* Suppose there is also a action of $\mathbb{G}_m$ [2] on M commuting with the action of H. In addition to the assumptions of the previous section ($I' \to M^{reg}$ is smooth over M^{reg} and G-equivariant), we also assume that I' is $\mathbb{G}_m$-equivariant.

Then [21] show the following:

Proposition 3.5 ([21], $\mathbb{G}_m$-equivariant version of 3.2)**.** *For M, H, I' and $\mathbb{G}_m$ as above we have that:*

- *If I' is abelian then it descends to a group scheme $\mathbb{J} \to (M^{reg} /\!\!/_{I'} H)/\mathbb{G}_m$.*
- *I' descends to a band $\mathbb{J}_{band} \to (M^{reg} /\!\!/_{I'} H)/\mathbb{G}_m$.*
- *The map $M^{reg}/(H \times \mathbb{G}_m) \to (M^{reg} /\!\!/_{I'} H)/\mathbb{G}_m$ is a gerbe banded by $\mathbb{J}_{band}$, when I' is abelian it is a $\mathbb{J}$-gerbe.*

Proposition 3.6 ([21], $\mathbb{G}_m$-equivariant version of Proposition3.4)**.** *Let M, H, I' and $\mathbb{G}_m$ be as above. Let $H \subset H'$, such that H is a normal subgroup. Suppose that the H action on M and on M^{reg} extends to a H' action. Suppose that I' is also H'-equivariant. Finally suppose that the actions of H' and $\mathbb{G}_m$ commute.*

Then the map $(M^{reg} /\!\!/_{I'} H)/\mathbb{G}_m \to (M^{reg} /\!\!/_{I'} H')/\mathbb{G}_m$ is a principal H'/H-bundle.

In other words the map $(M^{reg} /\!\!/_{I'} H) \to (M^{reg} /\!\!/_{I'} H')$ is $\mathbb{G}_m$-equivariant.

In the setting of Proposition 3.3 there is at most one action of $\mathbb{G}_m$ on V such that the map $M^{reg} \to V$ is equivariant. This shows:

Proposition 3.7 ([21], Equivariant version of Proposition 3.3)**.** *Let $I' = I_M|_{M^{reg}}$, and V be a scheme such that each fiber of $M^{reg} \to V$ consists of a single G-orbit, and the morphism $M^{reg} \to V$ is $\mathbb{G}_m$-equivariant. Then $V \cong M^{reg} /\!\!/_{I'} G$, and the morphism is $\mathbb{G}_m$-equivariant.*

3.1.2. *The Case of Symmetric Pairs.* We now give a brief overview of the quotients we will consider. We will of course be using $(M, H) = (\mathfrak{p}, H)$ where $(\mathfrak{p}, H)$ is a symmetric pair as in section 2. This is equipped with a commuting action of the group $\mathbb{G}_m$ which acts on the vector space $\mathfrak{p}$ by scaling.

There are multiple possible choices of I' in this setting, and we will use several different choices. Let C be the set of irreducible components of the nilpotent cone $\mathcal{N}_{\mathfrak{p}}$. There is an action of $\pi_0(H) = H/H^0$ on C. Let E be a normal subgroup of $\pi_0(H)$. Let $H^0 \subset H_E \subset H$ be the normal subgroup of H such that $\pi_0(H_E) = E$.

Definition 3.8. For E, H_E as above let I_E denote the stabilizer scheme of the action of H_E on $\mathfrak{p}$, and I_E^{reg} denote the restriction of this to $\mathfrak{p}^{reg}$. We also sometimes denote this as $I_{H_E}^{reg}$.

These are the group schemes we will use to define the regular quotient for symmetric pairs. We show in the next section that the subgroup scheme $I_E^{reg} \subset \mathfrak{p} \times H_E$ of stabilizers is a smooth group scheme over $\mathfrak{p}^{reg}$.

Lemma 3.9. *The subgroup scheme $I_E^{reg} \subset \mathfrak{p}^{reg} \times H_E$ of stabilizers is H-equivariant with respect to the action on $\mathfrak{p}^{reg} \times H$ given by the adjoint action and conjugation respectively.*

Proof. This is immediate from the definition. ☐

Lemma 3.10. *The subgroup scheme I_E^{reg} is $\mathbb{G}_m$-equivariant.*

Proof. This is immediate because I_E is the stabilizer group scheme of the action of H_E on $\mathfrak{p}$, and the action of $\mathbb{G}_m$ commutes with the action of H_E. Hence I_E is $\mathbb{G}_m$-equivariant, so I_E^{reg} is $\mathbb{G}_m$-equivariant. ☐

Proposition 3.11. *If $(\mathfrak{p}, H)$ is quasi-split then the subgroup scheme I_E^{reg} is abelian.*

Proof. Since $(\mathfrak{p}, H)$ is quasi-split, we have $\mathfrak{p}^{reg} \subset \mathfrak{g}^{reg}$. Consequently, I_E^{reg} is a subgroup scheme of $I|_{\mathfrak{p}^{reg}}$ where I is the group scheme of centralizers for G. Since $I|_{\mathfrak{g}^{reg}}$ is abelian, the result follows. ☐

[2]The paper [21] considers the case of a commuting action of an arbitrary reductive group, but we will here state the results for the case of $\mathbb{G}_m$

Hence the general results on regular quotient give us that:

Proposition 3.12. *Let $(\mathfrak{p}, H)$ be a symmetric pair, and E as above.*
Then I_E^{reg} descends to bands $J_{band} \to \mathfrak{p}^{reg} /\!\!/_{I_E^{reg}} H$, and $\mathbb{J}_{band} \to (\mathfrak{p}^{reg} /\!\!/_{I_E^{reg}} H)/\mathbb{G}_m$.
Then

$$\mathfrak{p}^{reg}/H \to \mathfrak{p}^{reg} /\!\!/_{I_E^{reg}} H$$

is a gerbe banded by J_{band}, and

$$\mathfrak{p}^{reg}/(H \times \mathbb{G}_m) \to \mathfrak{p}^{reg} /\!\!/_{I_E^{reg}} (H \times \mathbb{G}_m)$$

is a gerbe banded by $\mathbb{J}_{band}$.

If $(\mathfrak{p}, H)$ is quasisplit then I_E^{reg} descends to a smooth commutative group schemes $J_E \to \mathfrak{p}^{reg} /\!\!/_{I_E^{reg}} H$, and $\mathbb{J}_E \to (\mathfrak{p}^{reg} /\!\!/_{I_E}^{reg} H)/\mathbb{G}_m$.

The gerbes $\mathfrak{p}^{reg}/H \to \mathfrak{p}^{reg} /\!\!/_{I_E^{reg}} H$ and $\mathfrak{p}^{reg}/(H \times \mathbb{G}_m) \to \mathfrak{p}^{reg} /\!\!/_{I_E^{reg}} (H \times \mathbb{G}_m)$ are a J_E-gerbe and $\mathbb{J}_E$-gerbe respectively.

Proof. Immediate from Proposition 3.2 and Proposition 3.5. □

We can now give a brief overview of how we will describe the quotient $\mathfrak{p}^{reg} /\!\!/_{I_E^{reg}} H$. We reduce to the case where $E = \pi_0(H)$, in which case we will simply denote $I_{\pi_0(H)}$ by I (when H is clear from the context). More specifically we have:

Corollary 3.13 (Corollary of Proposition 3.4)**.**

$$\mathfrak{p}^{reg} /\!\!/_{I_E^{reg}} H = (\mathfrak{p} /\!\!/_{I_E^{reg}} H_E)/(H/H_E).$$

Note that $H/H_E = \pi_0(H)/E$.

Proof. Immediate from Proposition 3.4. □

3.2. **Regular quotient and smoothness of stabilizer group schemes via Kostant–Rallis Sections.** In this section we describe the regular locus $\mathfrak{p}^{reg}$ as the union of the H-orbits of potentially multiple Kostant–Rallis sections. We use this to deduce smoothness of several of the group schemes considered in the previous section in a way completely analogous to the case of the adjoint action of G on $\mathfrak{g}$ as considered in [12, 29]. We then have that the regular quotient in the case where $I' = I_E^{reg}$ and $E = \pi_0(H)$ can be obtained by gluing together multiple copies of the GIT quotient together. An explicit description of the gluing will be described in the subsequent sections. This is a modification of an argument for the case of the Vinberg monoid found in Proposition 2.12 of [3] and Equation 2.7 and Lemma 2.2.8 of [7]. We will then give a direct argument that I_E^{reg} descends to the regular quotient.

Lemma 3.14 (Analogue of Lemma 2.2.8 of [7], see also Proposition 2.12 of [3])**.** *Let $U \subset \mathfrak{p}^{reg}$ be stable under the $H \times \mathbb{G}_m$-action. If $U \cap \mathcal{N} = \mathcal{N}^{reg}$ then $U = \mathfrak{p}^{reg}$.*

The following proof is identical to that of [7], we provide it here for completeness.

Proof. We let $F := \mathfrak{p}^{reg} \backslash U$. By assumption this is a $\mathbb{G}_m \times H$ subscheme of $\mathfrak{p}$. Let $\chi|_F$ denote the restriction of $\chi : \mathfrak{p} \to \mathfrak{a} /\!\!/ W_a$ to F.

We let $V \subset F$ be the inverse image under $\chi|_F$ of the subset $\{x \in \mathfrak{a}/\!\!/W_a | dim(\chi_F^{-1}(x)) < dim(H) - dim(\mathfrak{a})\}$. This is an open subscheme of F by upper semicontinuity. Furthermore it includes $0 \in \mathfrak{p}$ by Lemma 2.38. As V is preserved by $\mathbb{G}_m$ and 0 is in the closure of every $\mathbb{G}_m$-orbit of F we have that $V = F$.

By Lemma 2.44 we have that each fiber of χ is of dimension $dim(\mathfrak{p}) - dim(\mathfrak{a})$. Suppose that $x \in \mathfrak{p}^{reg} \cap F$, then as F is stable under H, the dimension of the orbit of x (which is inside F) is $dim(\mathfrak{p}) - dim(\mathfrak{a})$. This contradicts F being of codimension ≥ 1 in each fiber. □

We now recall that, by Theorem 2.37, for each irreducible component $S \in Irr(\mathcal{N}_{\mathfrak{p}})$ there is a unique regular K-orbit of $\mathfrak{p}$ in S. Furthermore, by Lemma 2.45, there is a (not unique) Kostant–Rallis section $\kappa_S : \mathfrak{a}/\!\!/W_a \to \mathfrak{p}^{reg}$ such that $\kappa_S(0) \in S$.

Pick a set of representations $\{S_i | i \in I\}$ for the $\pi_0(H)$-orbits in $Irr(\mathcal{N}_{\mathfrak{p}})$. For each representative pick a Kostant–Rallis section κ_i. Let $\mathcal{S}_i$ be the image as used to define the Kostant–Rallis section in section 2.5. We then have a morphism

$$(H \times \mathcal{S}_i)/I_H \to \mathfrak{p}^{reg},$$

which is quasifinite and an isomorphism over the regular semisimple locus $\mathfrak{p}^{reg,ss}$. Hence it is birational. As $\mathfrak{p}^{reg}$ is normal, Zariski's main theorem implies that this morphism is an open embedding. Hence we can define $\mathfrak{p}^{\kappa_i,H}$ to be the open subscheme of $\mathfrak{p}^{reg}$ which is the image of this morphism.

Proposition 3.15.

$$\mathfrak{p}^{reg} = \cup_{i \in I} \mathfrak{p}^{\kappa_i,H}.$$

Proof. This is an immediate consequence of Lemma 3.14 applied to $U = \cup_{i \in I} \mathfrak{p}^{\kappa_i,H}$. □

3.2.1. *Application to smoothness of I_E^{reg}, I'_K and I_K^0.* We start by proving a Lemma.

Lemma 3.16. *If H is smooth then the morphism $H \times \mathcal{S}_i \to \mathfrak{p}^{\kappa_i,H}$ is smooth and surjective.*

Proof. This morphism is surjective by Lemma 2.48, so it is sufficient to prove that it is smooth.

The morphism is H-equivariant and $\mathbb{G}_m$-equivariant. Hence the morphism is smooth in a $H \times \mathbb{G}_m$-invariant open neighbourhood of $(1, \kappa_i(0))$. The only such neighbourhood is $H \times \mathcal{S}_i$, hence the morphism is smooth. □

Proposition 3.17. *The composition $\chi|_{\mathfrak{p}^{\kappa_i,H}} : \mathfrak{p}^{\kappa_i,H} \hookrightarrow \mathfrak{p} \to \mathfrak{p}/\!\!/H$ is smooth and surjective.*

Proof. This is identical to the proof of Proposition 3.3.3 of [29], namely the composition $H \times \mathcal{S}_i \to \mathfrak{p}^{\kappa_i,H} \to \mathfrak{p}/\!\!/H \cong \mathcal{S}_i$ is identified with the projection to $\mathcal{S}_i$. Hence [1] Tag 02K5 imples $\chi|_{\mathfrak{p}^{\kappa_i,H}}$ is smooth and surjective. □

We denote by $I_{\mathcal{S}_i} := I_H^{reg} \times_{\mathfrak{p}^{reg}} \mathcal{S}_i$ the restriction of I_H^{reg} to $\mathcal{S}_i$.

Proposition 3.18. *The map $I_{\mathcal{S}_i} \to \mathcal{S}_i$ is smooth.*

Proof. This proof of Proposition 3.3.5 of [29] carries over to this setting. For completeness we sumarize: As schemes over $\mathcal{S}_i$ we have isomorphisms

$$I_{\mathcal{S}_i} \cong \mathcal{S}_i \times_{\mathfrak{p} \times \mathcal{S}_i} (H \times \mathcal{S}_i) \cong \mathcal{S}_i \times_{\mathfrak{p} \times_{\mathfrak{p}/\!\!/H} \mathcal{S}_i} (H \times \mathcal{S}_i) \cong \mathcal{S}_i \times_{\mathfrak{p}} (H \times \mathcal{S}_i).$$

Hence as $H \times \mathcal{S}_i \to \mathfrak{p}^{reg} \hookrightarrow \mathfrak{p}$ is smooth we have that $I_{\mathcal{S}_i} \to \mathcal{S}_i$ is smooth. □

Theorem 3.19. *If H is smooth then the group schemes $I_E^{reg} \to \mathfrak{p}^{reg}$, I'_K and I_K^0 are smooth.*

Note that for (G, θ) quasisplit in characteristic 0 this is proved for I_{G^θ} in [11]. In Theorem 4.7 of [18] this is generalized to the case where $p > 2$ and p is such that $I^{reg} \to \mathfrak{g}^{reg}$ (that is to say the regular centralizers for the adjoint action of G on $\mathfrak{g}$) is smooth (see condition $C.3$ of [29] for such conditions).

Proof. By Proposition 3.15 it is sufficient to show that for each $i \in I$ (recall this denoted the represented the representatives of the $\pi_0(H)$-orbits in $Irr(\mathcal{N}_{\mathfrak{p}})$) the morphism $I_W^{reg}|_{\mathfrak{p}^{\kappa_i,H}} \to \mathfrak{p}^{\kappa_i,H}$ is smooth.

As these are open subschemes of I_H^{reg} it is sufficient to show that for each $i \in I$ we have $I_H^{reg}|_{\mathfrak{p}^{\kappa_i,H}} \to \mathfrak{p}^{\kappa_i,H}$ is smooth.

The proof is now identical to Corollary 3.6 in [29], we provide it only for completeness. The diagram

$$\begin{tikzcd} H \times I_{\mathcal{S}_i} \arrow[r] \arrow[d] & I_H^{reg}|_{\mathfrak{p}^{\kappa_i,H}} \arrow[d] \\ H \times \mathcal{S}_i \arrow[r] & \mathfrak{p}^{\kappa_i,H} \end{tikzcd}$$

is Cartesian. Hence Lemma 3.16 implies that $H \times I_{\mathbb{S}_i} \to I_H^{reg}$ is smooth and surjective. Furthermore the Lemma 3.16 and Proposition 3.18 tell us that the composition $H \times I_{\mathbb{S}_i} \to \mathfrak{p}^{\kappa_i, H}$ is smooth. Hence $I_H^{reg}|_{\mathfrak{p}^{\kappa_i,H}} \to \mathfrak{p}^{\kappa_i,H}$ is smooth by Tag 0K25 of [1]. $\square$

3.2.2. *Application to the regular quotient.* Let $\widetilde{\mathfrak{p}/\!\!/H}$ be the union of I copies of $\mathfrak{a}/\!\!/W_a$ where we glue the copies labelled by i and j on the subscheme $U \subset \mathfrak{a}/\!\!/W_a$ where the sections κ_i and κ_j are conjugate. Note that *a priori.* it is not clear that the gluing is done in a fashion compatible with the $\mathbb{G}_m$-action, and thus it is not clear that $\widetilde{\mathfrak{p}/\!\!/H}$ has a $\mathbb{G}_m$-action coming from the $\mathbb{G}_m$-actions on $\mathfrak{a}/\!\!/W_a$.

Theorem 3.20. *Assuming that H is smooth we have a $\mathbb{G}_m$-equivariant isomorphism of schemes*

$$\widetilde{\mathfrak{p}/\!\!/H} \xrightarrow{\cong} \mathfrak{p}^{reg} /\!\!/_{I_H^{reg}} H,$$

where the $\mathbb{G}_m$-action comes on $\widetilde{\mathfrak{p}/\!\!/H}$ comes from the $\mathbb{G}_m$-action on each copy of $\mathfrak{a}/\!\!/W$. Furthermore this isomorphism commutes with the ($\mathbb{G}_m$)-equivariant morphisms to $\mathfrak{p}/\!\!/H$.

Remark 3.21. Note that the left hand side is definable regardless of whether or not H is smooth while the right hand side is only known to be definable when H is smooth.

Proof. This argument is essentially identical to a similar argument in the Vinberg monoid case found in [21]. The non-equivariant version isomorphism follows immediately from Proposition 3.3. and the fact that the map $\mathfrak{p} \to \mathfrak{p}/\!\!/K \cong \mathfrak{a}/\!\!/W_a$ is $\mathbb{G}_m$-equivariant.

These identifications commute with the morphism to $\mathfrak{p}/\!\!/H$, because these are the unique morphisms to $\mathfrak{p}/\!\!/H$ such that the diagrams

$$\begin{tikzcd} \mathfrak{p}^{reg} \arrow[r] \arrow[d] & \mathfrak{p} \arrow[d] \\ \mathfrak{p}^{reg} /\!\!/_{I'} H \arrow[r] & \mathfrak{p}/\!\!/H \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{p}^{reg} \arrow[r] \arrow[d] & \mathfrak{p} \arrow[d] \\ \widetilde{\mathfrak{p}/\!\!/H} \arrow[r] & \mathfrak{p}/\!\!/H \end{tikzcd}$$

commute.

We hence have a $\mathbb{G}_m$-action on $\widetilde{\mathfrak{p}/\!\!/H}$ via the identification with the regular quotient. Because the morphism $\widetilde{\mathfrak{p}/\!\!/H} \to \mathfrak{p}/\!\!/H$ is $\mathbb{G}_m$-equivariant we hence must have that this $\mathbb{G}_m$ action comes from the $\mathbb{G}_m$-action on each copy of $\mathfrak{a}/\!\!/W_a$.

The equivariant isomorphism result now follows immediately, or via Proposition 3.7. $\square$

3.2.3. *Direct Proof of Gerbe Structure.* We now prove some of the results of proposition 3.12 without the use of results from [21]. We also note that the proofs of these results is identical to those of both the Lie algebra case (e.g. [23]) and to those for regular quotients in [21], as such we only include these for completeness.

Proposition 3.22. *The maps*

$$\mathfrak{p}^{reg}/H \to \widetilde{\mathfrak{p}/\!\!/H} \cong \mathfrak{p}^{reg} /\!\!/_{I_H^{reg}} H$$

and

$$(\mathfrak{p}^{reg}/H)/\mathbb{G}_m \to (\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m \cong (\mathfrak{p}^{reg} /\!\!/_{I_H^{reg}} H)/\mathbb{G}_m$$

are smooth gerbes.

Proof. It is enough to show this for the case of $(\mathfrak{p}^{reg}/H)/\mathbb{G}_m \to (\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m$, the remaining case will follow by pullback to $\widetilde{\mathfrak{p}^{reg}/\!\!/H}$. Consider the pullback off this map along $\mathfrak{p}^{reg} \to (\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m$. We have a section of the pullback given by the diagonal section

$$\mathfrak{p}^{reg} \to \mathfrak{p}^{reg} \times_{(\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m} \mathfrak{p}^{reg} \to \mathfrak{p}^{reg} \times_{\widetilde{\mathfrak{p}/\!\!/H}/\mathbb{G}_m} (\mathfrak{p}^{reg}/H).$$

This gives an identification of $\mathfrak{p}^{reg} \times_{\widetilde{\mathfrak{p}/\!\!/H}/\mathbb{G}_m} (\mathfrak{p}^{reg}/H)$ with $BI_{\pi_0(H)}$, concluding the proof. $\square$

Proposition 3.23. *If $(\mathfrak{p}, H)$ is quasisplit then $I^{reg}_{\pi_0(H)}$ descends to a smooth group scheme*

$$J_{\pi_0(H)} \to \widetilde{\mathfrak{p}/\!\!/H} \cong \mathfrak{p}^{reg} /\!\!/_{I_H^{reg}} H,$$

and descends further to a smooth group scheme

$$\mathbb{J}_{\pi_0(H)} \to (\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m \cong (\mathfrak{p}^{reg} /\!\!/_{I_H^{reg}} H)/\mathbb{G}_m.$$

The proof used to define the group scheme of regular centralizers (this can be found in e.g. [23]) for a Lie algebra generalizes immediately to this setting, and indeed to the general setting of regular quotients with I' abelian. We include it for completeness.

Proof. Consider the coequalizer diagram

$$\mathfrak{p}^{reg} \times_{\widetilde{\mathfrak{p}^{reg}}/H} \mathfrak{p}^{reg} \rightrightarrows \mathfrak{p}^{reg}.$$

et π_1, π_2 be the two projection maps. Let $act : H \times \mathfrak{p}^{reg} \to \mathfrak{p}^{reg} \times_{\widetilde{\mathfrak{p}^{reg}}/H} \mathfrak{p}^{reg} \rightrightarrows \mathfrak{p}^{reg}$ be the map $(h, x) \mapsto (x, ad_h(x))$.

We can identify $act^*\pi_1^* I_{\pi_0(H)}$ and $act^*\pi_2^* I_{\pi_0(H)}$ as follows. Given $(x, g) \in \mathfrak{p}^{reg} \times_{\widetilde{\mathfrak{p}^{reg}}/H} \mathfrak{p}^{reg}$ we can pick $g \in G$ conjugates $act^*\pi_1^* I_{\pi_0(H)}$ to $act^*\pi_2^* I_{\pi_0(H)}$ We then have that because $I_{\pi_0(H)}$ is abelian, and the map act is an $I_{\pi_0(H)}$-bundle, the identification of $act^*\pi_1^* I_{\pi_0(H)}$ and $act^*\pi_2^* I_{\pi_0(H)}$ descends to an identification of We can identify $\pi_1^* I_{\pi_0(H)}$ and $\pi_2^* I_{\pi_0(H)}$ on $\mathfrak{p}^{reg} \times_{\widetilde{\mathfrak{p}^{reg}}/H} \mathfrak{p}^{reg}$. Hence $I_{\pi_0(H)}$ descends to a smooth group scheme on $\mathfrak{p}^{reg}$. ☐

Proposition 3.24. *If $(\mathfrak{p}, H)$ is and H is smooth then the map*

$$\mathfrak{p}^{reg}/H \to \widetilde{\mathfrak{p}/\!\!/H} \cong \mathfrak{p} /\!\!/_{I_H^{reg}} H$$

is a $J_{\pi_0(H)}$-gerbe. Similarly the map

$$(\mathfrak{p}^{reg}/H)/\mathbb{G}_m \to (\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m \cong (\mathfrak{p} /\!\!/_{I^{reg}_{\pi_0(H)}} H)/\mathbb{G}_m$$

is a $\mathbb{J}_{\pi_0(H)}$-gerbe.

Proof. It has already been shown that these spaces are gerbes. Hence as we have identifications $\chi^* J_E \cong I_E^{reg}$ and $\overline{\chi}^* J_E \cong I_E^{reg}$ (for the maps $\chi : \mathfrak{p}^{reg} \to \widetilde{\mathfrak{p}/\!\!/H}$, and $\overline{\chi} : \mathfrak{p}^{reg} \to (\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m$) we have that these are J_E and $\mathbb{J}_E$ gerbes respectively. ☐

Remark 3.25. It is important to note that generalizations and refinements of several of the above results are expected. We point out a few of these now.

If $\widetilde{\mathfrak{p}/\!\!/H} \cong \mathfrak{p}/\!\!/H$ (or equivalently there is only one regular nilpotent H-orbit), then we can use a Kostant–Rallis section κ to pull back I_H^{reg} to get a group scheme $\kappa^* I_H^{reg}$ on $\widetilde{\mathfrak{p}/\!\!/H}$. If $(\mathfrak{p}, H)$ is quasisplit, but we do not assume that H is smooth, we can still get that $p^*(\kappa^* I_H^{reg}) \cong I_H^{reg}$ (for $p : \mathfrak{p}^{reg} \to \widetilde{\mathfrak{p}/\!\!/H}$).

Secondly under the same assumption that there is one regular nilpotent H-orbit, we can consider $\kappa^* I_H^{reg}$ when H is smooth, but $(\mathfrak{p}^{reg}, H)$ is not quasisplit. Note that we can also use $\mathbb{G}_m^{[2]}$-equivariance (that is to say we consider the usual $\mathbb{G}_m$-action, but we precompose by the squaring map $\mathbb{G}_m \to \mathbb{G}_m$) to get a group scheme on $[(\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m^{[2]}]$.

In this case $p^*(\kappa^* I_H^{reg})$ is I_H^{reg}. This in particular provides a *group* that $[\mathfrak{p}^{reg}/H \times \mathbb{G}_m^{[2]}] \to [(\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m^{[2]}]$ is a gerbe for.

As such it is an important question to see whether there are sections of $\mathfrak{p}^{reg} \to \widetilde{\mathfrak{p}/\!\!/H}$ and $[\mathfrak{p}^{reg}/\mathbb{G}_m^{[2]}] \to [\widetilde{\mathfrak{p}/\!\!/H}/\mathbb{G}_m^{[2]}]$ which would allow generalization of these considerations to arbitrary symmetric pairs (with H smooth).

Finally one could ask whether there are then descriptions of the group scheme $\kappa^* I_H^{reg}$ via Weil restriction. We note that the work of Hitchin–Schaposnik [14] and Branco [6] strongly suggests that in certain examples, one can describe at least a connected component of the Weil restriction of SL_2 from a spectral cover.

3.3. **Overview of Explicit Description of the Regular Quotient.** Suppose that we are trying to describe the geometry of the regular quotients $\mathfrak{p}^{reg} /\!\!/_{I_E} H$, $\mathfrak{p}^{reg} /\!\!/_{I'_K} K$, or $\mathfrak{p}^{reg} /\!\!/_{I^0_K} K$. We ultimately provide two different descriptions.

The first appears in Theorem 3.29, where we describe the regular quotient in terms of certain quotients of component groups.

The second is done by the following multistep procedure:

- Firstly for $\mathfrak{p}^{reg} /\!\!/_{I_E} H$ reduce describing this to describing$\mathfrak{p}^{reg} /\!\!/_{I^{reg}_{H_E}} H_E$ where H_E for the E involved in the definition of I_E. This is done by Corollary 3.13. For $\mathfrak{p}^{reg} /\!\!/_{I'_K} K$, or $\mathfrak{p}^{reg} /\!\!/_{I^0_K} K$ we have that by Proposition **??** we can reduce to the case of describing $\mathfrak{p}^{reg} /\!\!/_{I^{reg}_K} K$, which is a special case of describing $\mathfrak{p}^{reg} /\!\!/_{I^{reg}_{H_E}} H_E$.
- Secondly we reduce to the case of simply connected simple groups and the case $H = G_1 \overset{\Delta}{\hookrightarrow} G_1 \times G_1$ of Example 2.17, using Theorem 3.35.
- Thirdly we reduce understanding the orbits above a point in $\mathfrak{a}/\!\!/W_a$ to the case of nilpotent cones of certain Levi's, that we call distinguished Levi's, in Theorem 3.42 and Proposition 3.45.
- We use the immediately preceding point to describe the structure for simple, simply connected groups and the case of Example 2.17. Except for the case of $SO(n) \times SO(n) \hookrightarrow SO(2n)$ (considered in Example 3.56) this is not complicated due to the fact that there are at most 2 regular H-orbits in the nilpotent cone. The resulting explicit description of the regular quotient is included as Theorem 3.46. This description is as $\mathfrak{p} /\!\!/_{I^{reg}_H} H \cong \mathfrak{a}/\!\!/W_a \coprod_U \mathfrak{a}/\!\!/W_a$ for an explicitly described open $U \subset \mathfrak{a}/\!\!/W_a$.

In section 3.7 we explicitly compute the regular quotient in some cases. In the case of Example 2.17 we show that the regular quotient is simply $\mathfrak{a}/\!\!/W_a$ in Example 3.50.

We explicitly compute U for $H = K = \mathrm{GL}_n \times \mathrm{GL}_n \subset \mathrm{GL}_{2n} = G$ (Example 2.18) in Example 3.51.

We explicitly compute U for the split form $K = SO_n \subset SL_n$ in Example 3.53.

Finally we consider the case of $K = SO(n) \times SO(n) \hookrightarrow SO(2n)$ in Example 3.56, again using the results of section 3.4.

3.4. **The Action of $N_G(K)/Z_- \cdot K$ on the Regular Quotient.**

Lemma 3.26. *The regular centralizer group scheme I^{reg}_H is affine for any $K \subset H \subset N_G(K)$.*

Proof. The regular centralizer scheme I^{reg}_H is a subgroup of the constant group scheme $H \times \mathfrak{p}^{\mathrm{reg}} \to \mathfrak{p}^{\mathrm{reg}}$ is affine defined by vanishing of commutators. Hence, I^{reg}_H is a closed subgroup of $H \times \mathfrak{p}^{\mathrm{reg}}$ over $\mathfrak{p}^{\mathrm{reg}}$, and so is affine. □

Let $\mathscr{A} \subset G \times \mathfrak{p}^{\mathrm{rss}}$ be the family over $\mathfrak{p}^{rss}$ whose fiber over $X \in \mathfrak{p}^{rss}$ is the maximal θ-split torus A_X such that $\mathrm{L}ie(A_X) = \mathfrak{z}_{\mathfrak{g}}(X) \cap \mathfrak{p}$. (Note, the existence and uniqueness of such a torus $A \subset G$ is given in [19], Lemma 0.1.)

Moreover, we let $\mathscr{F}^* \subset \mathscr{A}$ denote the family over $\mathfrak{p}^{rss}$ whose fiber over $X \in \mathfrak{p}^{rss}$ is the subgroup

$$\{a \in A_X : a^2 \in Z(G)\} \subset A_X$$

Recall from Proposition 2.13, Part (a) that, for a given choice of A, we have $N_G(K) = F^* \cdot K$ where F^* is chosen with respect to A. We use this to determine the structure of $I^{\mathrm{reg}}_{N_G(K)}/I^{\mathrm{reg}}_K$.

Lemma 3.27. *Fix a choice of $X \in \mathfrak{p}^{\mathrm{rss}}$ determining A and $F^* \subset A$. Moreover, let Z_- denote the subgroup of the center $Z = Z(G)$ on which θ acts by inversion.*

(1) We have an isomorphism over $\mathfrak{p}^{\mathrm{rss}}$

$$\left(I^{\mathrm{reg}}_{N_G(K)}/Z_- \cdot I^{\mathrm{reg}}_K\right)\Big|_{\mathfrak{p}^{rss}} \simeq \coprod_{a \in F^*/Z_- \cdot (F^* \cap K)} \mathfrak{p}^{\mathrm{rss}} \tag{3.1}$$

(2) The isomorphism (3.1) *extends to an isomorphism*

$$I^{\mathrm{reg}}_{N_G(K)}/Z_- \cdot I^{\mathrm{reg}}_K \simeq \coprod_{a \in F^*/Z_- \cdot (F^* \cap K)} U_a \tag{3.2}$$

where $U_a \to \mathfrak{p}^{\mathrm{reg}}$ is the inclusion map for an open set $\mathfrak{p}^{\mathrm{rss}} \subset U_a \subset \mathfrak{p}^{\mathrm{reg}}$.

Proof. The inclusion $I^{\mathrm{reg}}_{N_G(K)} \subset N_G(K) \times \mathfrak{p}^{\mathrm{reg}}$ defines a map

$$I^{\mathrm{reg}}_{N_G(K)}/Z_- \cdot I^{\mathrm{reg}}_K \to \big(N_G(K)/Z_- \cdot K\big) \times \mathfrak{p}^{\mathrm{reg}} \tag{3.3}$$

with target a constant group scheme with discrete fiber. For any fixed $Y \in \mathfrak{p}^{rss}$, let A_Y be the fiber of $\mathscr{A}$ at Y and $F^*_Y \subset A_Y$ be the fiber of the group scheme $\mathscr{F}^*$ over Y. Then, it is clear from Proposition 2.13 that $I^{\mathrm{reg}}_{N_G(K),Y} = F^*_Y \cdot Z_- \cdot I^{\mathrm{reg}}_{K,Y}$. Therefore, the fiber of the map (3.3) at Y is identified with the identity map

$$F^*_Y/Z_- \cdot (F^*_Y \cap K) \to F^*_Y/Z_- \cdot (F^*_Y \cap K)$$

In particular, (3.3) is an isomorphism over the regular, semisimple locus, proving part (1).

For (2), we claim that the map (3.3) remains an injection over $\mathfrak{p}^{\mathrm{reg}}$. In particular, this amounts to the following claim:

Claim: Let $Y \in \mathfrak{p}^{\mathrm{reg}}$. For any $g_1, g_2 \in I^{\mathrm{reg}}_{N_G(K),Y}$, if $g_1 = hg_2$ for $h \in K$, then in fact $h \in I^{\mathrm{reg}}_{K,Y}$.

Proof of Claim. Since g_1 and g_2 centralize Y, we have

$$Y = \mathrm{ad}(hg_2) \cdot Y = \mathrm{ad}(h)Y$$

Hence, $h \in I_{G,Y} \cap K = I^{\mathrm{reg}}_{K,Y}$. □

It follows that the map (3.3) describes the quotient $I^{\mathrm{reg}}_{N_G(K)}/Z_- \cdot I^{\mathrm{reg}}_K$ as the disjoint union of open subsets of $\mathfrak{p}^{reg}$ extending the sheets $\big(N_G(K)/Z_- \cdot K\big) \times \mathfrak{p}^{rss}$. □

Lemma 3.28. *The group $N_G(K)$ acts transitively on the fibers of the map $\mathfrak{p}^{\mathrm{reg}} \to \mathfrak{p}/\!\!/K$.*

Proof. Theorem 3.20 reduces this to the $N_G(K)$ action on the zero fiber $\mathcal{N}^{\mathrm{reg}}_{\mathfrak{p}}$. Theorem 2.42 proves this case. □

Theorem 3.29. *Consider the natural action of the constant group scheme $\underline{N_G(K)} := N_G(K) \times \mathfrak{p}/\!\!/K$ on $\mathfrak{p}^{\mathrm{reg}}$ over $\mathfrak{p}/\!\!/K$.*

(1) A choice of Kostant-Rallis section $\kappa\colon \mathfrak{p}/\!\!/K \to \mathfrak{p}^{\mathrm{reg}}$ gives an identification

$$\mathfrak{p}^{\mathrm{reg}} \simeq \underline{N_G(K)}/\kappa^* I^{\mathrm{reg}}_{N_G(K)},$$

as schemes over $\mathfrak{p}/\!\!/K$.

(2) The regular quotient $\mathfrak{p}^{\mathrm{reg}} /\!\!/_{I^{reg}_H} H$ is identified with the quotient

$$\mathfrak{p}^{\mathrm{reg}} /\!\!/_{I^{\mathrm{reg}}_H} H \simeq \frac{\underline{N_G(K)}/\underline{H}}{\kappa^*(I^{\mathrm{reg}}_{N_G(K)}/I^{\mathrm{reg}}_H)} = \frac{\underline{N_G(K)}/\underline{Z_- \cdot H}}{\kappa^*(I^{\mathrm{reg}}_{N_G(K)}/Z_- \cdot I^{\mathrm{reg}}_H)}$$

Proof. By acting on the image of the Kostant–Rallis section κ we gain a surjective morphism $\underline{N_G(K)} \to \mathfrak{p}^{reg}$. This clearly factors through an isomorphism $\underline{N_G(K)}/\kappa^* I^{\mathrm{reg}}_{N_G(K)} \to \mathfrak{p}^{reg}$.

Part (2) then follows by considering the transitive $\underline{N_G(K)}$ action on the right hand side of the description of $\mathfrak{p}^{reg} /\!\!/ H$ given in Theorem 3.20. □

3.5. **Reduction to the Simple, Simply Connected Case.** We begin by reducing to the case of G simple, simply connected. We begin by lifting along covering maps.

Lemma 3.30. *([19], Lemma 1.3) Let $\hat{G} \to G$ be the simply connected cover of a simple group G, and let θ be an involution on G. Then, θ lifts uniquely to an involution on $\hat{G}$.*

Recall that any reductive group G admits a central isogeny

$$\xi\colon Z(G)^0 \times G^{der} \to G$$

where G^{der} is semisimple and $Z(G)^0$ is a torus. Then G^{der} can be written as a product $G^{der} = \prod_j G_j$ where each G_j is simple. We consider the cover

$$q\colon \hat{G} = Z(G)^0 \times \prod_j \hat{G}_j \to Z(G)^0 \times \prod_j G_j$$

where $\hat{G}_j$ is the universal cover of G_j and q is the product of the universal covering maps. Denote by $\hat{\xi}$ the composition $q \circ \xi\colon \hat{G} \to G$.

Lemma 3.31. *The involution θ on G lifts along $\hat{\xi}$ to an involution $\hat{\theta}$ on $\hat{G}$ with respect to which $\hat{G}$ admits a decomposition*

$$\hat{G} = Z(G)^0 \times \prod_j \hat{G}_j \tag{3.4}$$

where each $\hat{G}_j$ is preserved by the involution $\hat{\theta}$ and is either simple or a product of two isomorphic simples with $\hat{\theta}$ acting by permuting the factors.

Proof. Denote the composition

$$\xi_j\colon G_j \hookrightarrow Z(G)^0 \times G^{\mathrm{der}} \to G$$

The map ξ_j is an inclusion. By simplicity of G_j, we have that $\xi_j(G_j) \cap \theta(\xi_j(G_j))$ is either $\xi_j(G_j)$ or the identity. In the former case, θ restricts to an involution on $\xi_j(G_j)$, and there is a unique lifting along the embedding ξ_j to an involution on G_j. By Lemma 3.30, this lifts uniquely along the cover $q|_{\hat{G}_j}$ to an involution on $\hat{G}_j \subset \hat{G}$.

In the case that $\xi_j(G_j)$ intersects trivially with G_j, $\xi_j(G_j)$ is carried to another simple normal subgroup of G. Such subgroups are enumerated by the images $\xi_i(G_i)$ for $i \neq j$; we fix $i \neq j$ so that $\theta(\xi_j(G_j)) = \xi_i(G_i)$. Then, θ acts on $\xi_i(G_i) \times \xi_j(G_j)$ by automorphisms $\xi_i(G_i) \to \xi_j(G_j)$ and $\xi_j(G_j) \times \xi_i(G_i)$ which compose to the identity. These maps lift to maps $\hat{G}_i \to \hat{G}_j$ and $\hat{G}_j \to \hat{G}_i$, and so for such pairs (i,j) for $i \neq j$, θ lifts to an involution on $\hat{G}_i \times \hat{G}_j \subset \hat{G}$. □

We now use this to reduce the description of the regular quotient to the case where either G is simple, simply connected, or $G = G_1 \times G_1$ where G_1 is simple, simply connected and θ swaps the two copies of G_1.

Lemma 3.32. *For the isogeny $\xi\colon \hat{G} \to G$ as in Lemma 3.31, we have $\xi^{-1}(Z(G)) = Z(\hat{G})$.*

Proof. Suppose that $\hat{z} \in Z(\hat{G})$ and let $g \in G$. Since ξ is surjective, there exists $\hat{g} \in \hat{G}$ such that $\xi(\hat{g}) = g$. Then,

$$\xi(\hat{z})g = \xi(\hat{z}\hat{g}) = \xi(\hat{g}\hat{z}) = g\xi(\hat{z})$$

Hence, $Z(\hat{G}) \subset \xi^{-1}(Z(G))$.

Conversely, if $z \in Z(G)$ and $\hat{z} \in \xi^{-1}(z)$, then for any $\hat{g} \in \hat{G}$, we have

$$\xi(\hat{z}\hat{g}) = z\xi(\hat{g}) = \xi(\hat{g})z = \xi(\hat{g}\hat{z})$$

Hence, the data of $\hat{z}$ determines a continuous map

$$\hat{G} \to \ker(\xi), \quad \text{by } \hat{g} \mapsto \hat{z}^{-1}\hat{g}^{-1}\hat{z}\hat{g}$$

Since $\hat{G}$ is continuous and $\ker(\xi)$ is finite, it follows that the image of this map is $\{1\}$, and so $\hat{z} \in Z(\hat{G})$. □

Proposition 3.33. *Let $\xi\colon \hat{G} \to G$ be the isogeny of Lemma 3.31, and let $\hat{K} = \big((\hat{G})^{\theta}\big)^{\circ}$. Then, we have $\xi^{-1}\big(N_G(K)\big) \subset N_{\hat{G}}(\hat{K})$.*

Proof. Let $\hat{\theta}$ be the lift of θ to $\hat{G}$ from Lemma 3.31. We have a commutative diagram

$$\begin{tikzcd} \hat{G} \arrow[r, "\hat{\theta}"] \arrow[d, "\xi"'] & \hat{G} \arrow[d, "\xi"] \\ G \arrow[r, "\theta"] & G \end{tikzcd}$$

Recall that by Proposition 2.13

$$N_G(K) = \{g \in G\colon g\theta(g^{-1}) \in Z(G)\}$$

Let $g \in N_G(K)$, and take $\hat{g} \in \xi^{-1}(g)$. Then,

$$\xi\big(\hat{g}\hat{\theta}(\hat{g}^{-1})\big) = g\xi(\hat{\theta}(\hat{g}^{-1})) = g\theta(g^{-1}) \in Z(G)$$

Therefore, by Lemma 3.32,

$$\hat{g}\hat{\theta}(\hat{g}^{-1}) \in \xi^{-1}(Z(G)) = Z(\hat{G}).$$

We conclude $\xi^{-1}\big(N_G(K)\big) \subset N_{\hat{G}}(\hat{K})$, as desired. □

Lemma 3.34. *For any isogeny $\xi\colon \hat{G} \to G$, we have $\hat{K} \subset \xi^{-1}(K)$.*

Proof. Note that the lifting of Lemma 3.31 satisfies $\xi^{-1}(G^{\theta}) \supset \hat{G}^{\hat{\theta}}$. Taking connected components, we have an inclusion $\xi^{-1}(G^{\theta})^{\circ} \supset \hat{K}$. Now, since the preimage of a connected component is a union of connected components, we have $\big(\xi^{-1}(G^{\theta})\big)^{\circ} \subset \xi^{-1}(K)$, so we conclude the result. □

Theorem 3.35. *Let $(\mathfrak{p}, H)$ be a symmetric pair corresponding to (G, θ). Let $\hat{G} = Z(G)^0 \times \prod_j \hat{G}_j$ be as in Equation 3.4 of Lemma 3.31. Let $(\mathfrak{p}_j, K_j)$ be the symmetric pair (with K_j) connected corresponding to (G_j, θ_j). Let $H_j = \xi^{-1}(H) \cap G_j$. Then $(\mathfrak{p}_j, H_j)$ is a symmetric pair, and there is a $\mathbb{G}_m$-equivariant isomorphism*

$$(\mathfrak{p}^{reg} /\!\!/_{I_H^{reg}} H) \cong ((Lie(Z(G)^0) \cap \mathfrak{p}) /\!\!/ (\xi^{-1}(H) \cap Z(G)^0) \times \prod_j \mathfrak{p}_j^{reg} /\!\!/_{I_{H_j}^{reg}} H_j$$

Proof. Firstly let $\hat{H} = \xi^{-1}(H)$. By Lemma 3.34 and Proposition 3.33, $(\mathfrak{p}, \hat{H})$ is a symmetric pair of $(\hat{G}, \hat{\theta})$. Furthermore we note that by the decomposition of Equation 3.4 we have that $\hat{H} = (Z(G)^0 \cap H) \times \prod_j H_j$. Since $\hat{\theta}$ respects the decomposition of Equation 3.4, $\hat{K}$ and $N_{\hat{G}}(\hat{K})$ decompose as products

$$\hat{K} = (\hat{K} \cap Z(G)^{\circ}) \times \prod_j \hat{K}_j \quad \text{and} \quad N_{\hat{G}}(\hat{K}) = (N_{\hat{G}}(\hat{K}) \cap Z(G)^{\circ}) \times \prod_j N_{\hat{G}_j}(\hat{K}_j).$$

Hence, $(\mathfrak{p}_j, H_j)$ is a symmetric pair.

Since the action of $\hat{H}$ on $\mathfrak{p}$ factors through H with finite quotient, the regular locus of these two actions agree. Furthermore, the group scheme of centralizers $\hat{I}$ in $\hat{H}$ decomposes as a product of centralizers, so $\mathfrak{p}^{reg} = \mathfrak{z}(\mathfrak{g}) \times \prod_j \mathfrak{p}_j^{reg}$, where $\mathfrak{p}_j^{reg}$ is the regular locus of $\mathfrak{p}_j$ under the action of H_j. This gives a decomposition of stacks

$$\mathfrak{p}^{\mathrm{reg}}/H \simeq \mathfrak{z}(\mathfrak{g})/(H \cap Z(G)^0) \times \prod_j \mathfrak{p}_j^{\mathrm{reg}}/H_j.$$

Note we also get an isomorphism of GIT quotients:

$$\mathfrak{p}^{reg} /\!\!/ H \cong \mathfrak{z}(\mathfrak{g}) /\!\!/ (H \cap Z(G)^0) \times \prod_j \mathfrak{p}_j^{\mathrm{reg}} /\!\!/ H_j.$$

Both of the above identifications are $\mathbb{G}_m$-equivariant.

Pick a section of $\mathfrak{z}(\mathfrak{g})^{reg} \to ((Lie(Z(G)^0) \cap \mathfrak{p}) /\!\!/ (\xi^{-1}(H) \cap Z(G)^0)$, and for each j pick a set of Kostant–Rallis sections for $(\mathfrak{p}_j, H_j)$ as in Proposition 3.15. For any choice of one section for each $\mathfrak{p}_j$, and the chosen section of $\mathfrak{z}(\mathfrak{g})^{reg}$, adding these sections together gives a section of $\mathfrak{p}^{reg}$ over $\mathfrak{p}^{reg} /\!\!/ H$. Suppose we get l sections in total. We note that these sections are H conjugate if and only if they are $\xi^{-1}(H)$ conjugate. Exactly the same arguments as used in Theorem 3.20 show that $\mathfrak{p}^{reg} /\!\!/_{I_H^{reg}} H$ is given by gluing together l copies of $\mathfrak{a}/\!\!/W_a \cong \mathfrak{p}/\!\!/H$ on the open sets where these are conjugate. This immediately gives the identification of the Proposition. $\square$

3.6. **Reduction to Levi Subgroups.** From now on, we assume that G is simple, simply connected. We keep notation as in Section 2.1. In particular, we fix a maximal θ-split torus T of G. Let $S \subset \Phi_r$ be a subset of restricted roots with the following properties:

(1) (Primitivity) For any $\alpha \in S$, if $c\alpha \in \Phi_r$ for $c \in \mathbb{Q}$ then $c\alpha \in S$.
(2) (Sub-root System) S is a sub-root system of Φ_r.

Such $S \subset \Phi_r$ are exactly those coming from subsets of root hyperplanes in $\mathfrak{a}$ as follows: For any collection of root hyperplanes $H_1, \dots, H_k \subset \mathfrak{a}$ such that $\cap_{i=1}^k H_i$ is not contained in any other root hyperplane, take S to be the collection of all restricted roots $\alpha \in \Phi_r$ which vanish on one of the H_j.

Definition 3.36. Let L_S be the connected Levi subgroup of G whose Lie algebra is the sum

$$\mathfrak{l}_S = \mathfrak{t} + \sum_{\beta \in r^{-1}(S)} \mathfrak{g}_\beta$$

where r is the restriction map $r\colon \Phi \to \Phi_r \cup \{0\}$ as in definition 2.6. We refer to L_S as a *distinguished Levi.*

We denote $K_L, \mathfrak{p}_L, \mathfrak{a}_L, A_L$, etc. for the corresponding objects in the Levi $L = L_S$, and $H_L := H \cap L$. Note that $T \subset L$, so that $A_L = A$ and $\mathfrak{a}_L = \mathfrak{a}$. We will relate the structure of the stack $\mathfrak{p}/H$ to the stack $\mathfrak{p}_L/H_L$.

Remark 3.37. Note that if one takes $H = K$, it is *not* true in general that $H_L = K_L$. For example, consider the symmetric pair corresponding to the diagonally embedded $\mathrm{SL}_2 \times \mathrm{SL}_2 \subset \mathrm{SL}_4$ (see example 2.18). Then, one choice of Levi L of the above form corresponds to the Lie algebra

$$\mathfrak{l} = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} : \mathrm{T}r(A) = 0 \right\}.$$

One computes

$$K \cap L = \left\{ \begin{pmatrix} g & \\ & g \end{pmatrix} : \det(g) = \pm 1 \right\}.$$

In particular, $K \cap L$ is disconnected. By definition, $K_L = (L^\theta)^\circ = (K \cap L)^\circ$ is the digaonally embedded copy of SL_2.

We relate the Weyl groups and GIT quotients as follows.

Proposition 3.38. *The little Weyl group $W_{\mathfrak{a},L}$ of the Levi L is a subgroup of $W_{\mathfrak{a}}$. Let $D_L \subset \mathfrak{a}$ be the union of hyperplanes $\mathfrak{h}_\alpha$ in $\mathfrak{a}$ such that $\alpha \notin \Phi_{r,L}$ and let $\pi\colon \mathfrak{a} \to \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ be the projection map. The map of GIT quotients*

$$\varphi_L \colon \mathfrak{a}/\!\!/W_{\mathfrak{a},L} \to \mathfrak{a}/\!\!/W_{\mathfrak{a}}$$

is étale away from $\pi(D_L) \subset \mathfrak{a}/\!\!/W_{\mathfrak{a}}$.

Proof. Recall that $W_{\mathfrak{a}}$ is generated by reflections given by roots in the restricted root system Φ_r, and similarly for $W_{\mathfrak{a},L}$ with the restricted roots system for the Levi, $\Phi_{r,L}$. [See Richardson, Lemma 4.5.] We claim that $\Phi_{r,L}$ is a subroot system of Φ_r corresponding to roots in L. Indeed, by construction $\mathfrak{a}_L = \mathfrak{a}$, and the root system of L with respect to a maximally θ-split torus T is a subroot system of G with respect to T. Therefore, restricting to $\mathfrak{a}$ gives a subroot system $\Phi_{\mathfrak{a},L}$ of $\Phi_{\mathfrak{a}}$. It follow that $W_{\mathfrak{a},L} \subset W_{\mathfrak{a}}$ is a subgroup.

Now, consider the map φ as above. The projection π factors through φ, giving covers π and π' as below.

$$\begin{tikzcd}
\mathfrak{a} \arrow[d, "\pi'"'] \arrow[dr, "\pi"] & \\
\mathfrak{a}/\!\!/W_{\mathfrak{a},L} \arrow[r, "\varphi_L"'] & \mathfrak{a}/\!\!/W_{\mathfrak{a}}
\end{tikzcd}$$

As both sides are quotients of $\mathfrak{a}$, the ramification locus of φ is exactly those images $\pi(a) \in \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ such that $a \in \mathfrak{a}^w$ for some $w \in W_{\mathfrak{a}} \setminus W_{\mathfrak{a},L}$. But for any $w \in W_{\mathfrak{a}}$ with minimal presentation $w = s_1 \dots s_n$ for simple reflections s_j, the fixed locus is $\mathfrak{a}^w = \bigcap \mathfrak{h}_j$ where $\mathfrak{h}_j$ is the hyperplane fixed by s_j. In particular, from our earlier description of $W_{\mathfrak{a},L}$, it follows that φ is ramified exactly on those $\pi(a)$ such that $a \in \mathfrak{h}_\alpha$ for some $\alpha \in \Phi_r \setminus \Phi_{r,L}$. $\square$

Definition 3.39. Let φ_L, π, π', and D_L be as above. We let U be the complement of $\pi(D_L)$ in $\mathfrak{a}/\!\!/W_{\mathfrak{a}}$ and U_L the complement of $\pi'(D_L)$ in $\mathfrak{a}/\!\!/W_{\mathfrak{a},L}$.

Similarly, let U^{enh} and U_L^{enh} denote the corresponding preimages in $\mathfrak{p}^{\mathrm{reg}} /\!\!/ H$ and $\mathfrak{p}_L^{\mathrm{reg}} /\!\!/ H_L$, respectively.

Let $i_L : \mathfrak{p}_L \to \mathfrak{p}$ be the inclusion map and let

$$p\colon \mathfrak{p} \to \mathfrak{p}/\!\!/H \quad \text{and} \quad p_L\colon \mathfrak{p}_L \to \mathfrak{p}_L/\!\!/H_L$$

be the projection maps.

Lemma 3.40. *There is a morphism*

$$\chi_L : \mathfrak{p}_L^{reg} /\!\!/_{I_{H_L}^{reg}} H_L \times_{\mathfrak{a}/\!\!/W_{a,L}} U_L \to \mathfrak{p}^{reg} /\!\!/_{I_H^{reg}} H.$$

Proof. For any $x \in p_L^{-1}(U_L)$, $(I_G)_x = (I_L)_x$ and hence $(I_{H_L})_x = (I_H)_x \cap (I_L)_x$. Hence the map φ sends regular elements of $p_L^{-1}(U_L) \subset \mathfrak{p}_L$ to regular elements of $\mathfrak{p}$. The result follows. $\square$

Lemma 3.41. *We have a canonical isomorphism $\chi_L^* J|_{U_L} \simeq J_L|_{U_L}$.*

Proof. Now, there is a map $J_L|_{U_L} \to \chi_L^* J$ from the inclusion $H_L \subset H$. To show this is an isomorphism on U_L, it suffices to check on fibers. Let $y \in U_L$ have preimage $x \in \mathfrak{p}_L^{\mathrm{reg}}$. Then, we can identify $(J_L)_y = (I_{H_L})_x$ and $(\varphi^* J)_y = (I_H)_x$, where the result now follows. $\square$

Theorem 3.42. *The morphism χ_L induces an isomorphism*

$$\mathfrak{p}_L^{\mathrm{reg}} /\!\!/ H_L\big|_{U_L^{enh}} \to \mathfrak{p}^{\mathrm{reg}} /\!\!/ H\big|_{U^{enh}} \times_{\mathfrak{a}/\!\!/W_{\mathfrak{a}}} (\mathfrak{a}/\!\!/W_{\mathfrak{a},L})$$

Proof. Suppose that $p_1, p_2 \in U_L^{\mathrm{enh}}$ are H_L conjugate. Then, there is a T torsor $C \subset L$ conjugating p_1 to p_2 and since $T \subset L$ is also a maximal torus of G, this must also be the set of elements of G conjugating p_1 to p_2. Hence the map is injective.

For surjectivity, we must show that for any $p \in U^{enh}$, $(H \cdot p) \cap \mathfrak{p}_L^{\mathrm{reg}} \neq \emptyset$. Let $x \in U_{enh}$. Then by Lemma 2.26, we express $x = s + n$.

We first claim that s conjugate to some $s' \in \mathfrak{p}_L$. Indeed, we will prove the stronger statement that any two semisimple s, s' in the same fiber of $\mathfrak{p} \to \mathfrak{p}^{\mathrm{reg}}/\!\!/K$ are K-conjugate. By Lemma 2.29, part (a) and Proposition 2.5, we are reduced to the case where $s, s' \in \mathfrak{a}$ are in the same Cartan of $\mathfrak{p}$. In this case, the restriction of the map $\mathfrak{p} \to \mathfrak{p}^{\mathrm{reg}}/\!\!/K$ can be identified with $\mathfrak{a} \to \mathfrak{a}/\!\!/W_{\mathfrak{a}}$, and s, s' lying in the same fiber is equivalent to them differing by the action of $W_{\mathfrak{a}}$. Since any $w \in W_{\mathfrak{a}}$ has representatives in K, this is true.

Now, let $h \in K$ be such that $s' = hsh^{-1} \in \mathfrak{p}_L$. Then, we have

$$hnh^{-1} \in \mathrm{Lie}(hZ_G(s)h^{-1}) = \mathrm{Lie}(Z_G(hsh^{-1})) = \mathfrak{z}_{\mathfrak{g}}(hsh^{-1})$$

where the final equality follows from Lemma 2.28. Then, since s lies away from the root hyperplanes not associated to L, $\mathfrak{z}_{\mathfrak{g}}(hsh^{-1}) \subset \mathfrak{l}$ is a subset of the Lie algebra of L. We conclude that h conjugates x to $\mathfrak{p}_L$.

The claim now follows as $\mathfrak{p}_L^{\mathrm{reg}} = \mathfrak{p}^{\mathrm{reg}} \cap \mathfrak{p}_L$. $\square$

Theorem 3.43. *The map $\psi\colon \mathfrak{p}_L^{\mathrm{reg}}/H_L \to \varphi^*(\mathfrak{p}^{reg}/H)$ is an isomorphism when restricted to U_L^{enh}.*

Proof. Recall that $\mathfrak{p}^{reg}/H$ is a gerbe over $\mathfrak{p}^{reg} /\!\!/ H$ banded by J, and similarly, $\mathfrak{p}_L^{reg}/H_L$ is a gerbe over $\mathfrak{p}_L^{\mathrm{reg}} /\!\!/ H_L$ banded by J_L. Hence, to conclude, it suffices to note that the map $\psi|_U$ is a map of $J_L \simeq \varphi^* J$ gerbes. $\square$

To conclude, we will reduce to computations of regular nilpotent orbits. To do this we will use Proposition 3.45.

Lemma 3.44. *Let (G, θ) be a semisimple group with involution θ. Then the intersection of all root hyperplanes of the restricted root system is*

$$\cap_{\alpha\in\Phi_r} H_\alpha = 0 \in \mathfrak{a}.$$

Proof. For every root hyperplane H_α in $\mathfrak{a}$, let S_α denote the set of all hyperplanes of $\mathfrak{t}$ which restrict to H_α. Let S denote the set of all hyperplanes in $\mathfrak{t}$ which contain $\mathfrak{a}$. Note that for every root hyperplane $H \subset \mathfrak{t}$, $H \cap \mathfrak{a}$ is either a root hyperplane in $\mathfrak{a}$ or is all of $\mathfrak{a}$. Hence, S and S_α as α varies gives a partition of all root hyperplanes of $\mathfrak{t}$. We conclude that

$$\bigcap_{\alpha\in\Phi_r} H_\alpha = \mathfrak{a} \cap \bigcap_{\alpha\in\Phi_r} H_\alpha \subset \left(\bigcap_{H\in S} H\right) \cap \bigcap_{\alpha}\left(\bigcap_{H\in S_\alpha} H\right) = \bigcap_{H\subset\mathfrak{t}:\ \text{root hyperplane}} = \{0\} \qquad \square$$

Proposition 3.45. *Let $(\mathfrak{p}, H)$ be a symmetric pair associated with (G, θ). Let $p : \mathfrak{a} \to \mathfrak{a}/\!\!/W_a$. Let $Y = \cap_\alpha H_\alpha \subset \mathfrak{a}$ be the intersection of all root hyperplanes in $\mathfrak{a}$. There is then an isomorphism of stacks over $p(Y)$:*

$$\mathfrak{p}/H \times_{\mathfrak{a}/\!\!/W_a} p(Y) \cong \mathcal{N}_{\mathfrak{p}}/H \times p(Y)$$

Restricting to regular elements gives:

$$\mathfrak{p}^{reg}/H \times_{\mathfrak{a}/\!\!/W_a} p(Y) \cong \mathcal{N}_{\mathfrak{p}}^{reg}/H \times p(Y)$$

Proof. As in Lemma 3.31 we denote $\hat{G} \xrightarrow{\xi} G$ the product of $Z(G)^0$ and the simply connected covers of G_j. We then have by Lemma 3.31 that θ lifts to an involution $\hat{\theta}$ on $\hat{G}$. Furthermore this restricts to involutions $\theta_{Z(G)^0}$ and $\theta_{G^{der}}$ on $Z(G)^0$ and G^{der} respectively. Let $\hat{H} = \xi^{-1}(H)$. By the proof of Lemma 3.31 we have that $(\mathfrak{p}, \hat{H})$ is a symmetric pair.

We then have that $\mathfrak{a}$ is the same for G and for $\hat{G}$. Furthermore we also have that the restricted root hyperplanes and hence Z is the same. We have that $\mathfrak{a} = (Lie(Z(G)^0) \cap \mathfrak{p}) \times \mathfrak{a}_{G^{der}}$ As such we have that $Y_G = (Lie(Z(G)^0) \cap \mathfrak{p}) \times Y_{G^{der}}$.

Let $H_{G^{der}} = \hat{H} \cap G^{der}$. We then have that $\hat{H} = (H \cap Z(G)^0) \times H_{G^{der}}$. We note that $\mathfrak{p}/\hat{H} = (((Lie(Z(G)^0) \cap \mathfrak{p})/(H \cap Z(G)^0)) \times (\mathfrak{p}_{G^{der}}/H)$. Furthermore with respect to this decomposition, the morphism to $\mathfrak{a}/\!\!/W_a \cong ((Lie(Z(G)^0) \cap (\mathfrak{p})) \times \mathfrak{a}_{G^{der}}/\!\!/W_{G^{der},a}$ corresponds to the morphisms $\mathfrak{p}_{G^{der}}/\hat{H} \to \mathfrak{a}_{G^{der}}/\!\!/W_{G^{der},a}$ and $((Lie(Z(G)^0) \cap \mathfrak{p})/(H \cap Z(G)^0) \to (Lie(Z(G)^0) \cap \mathfrak{p})$. Hence we have that

$$\mathfrak{p}/\hat{H} \times_{\mathfrak{a}/\!\!/W_a} p(Y) \cong (((Lie(Z(G)^0) \cap \mathfrak{p})/(H \cap Z(G)^0) \times \mathcal{N}_{\mathfrak{p}_{G^{der}}}/H_{G^{der}},$$

where we are using Lemma 3.44.

We note that $\mathcal{N}_{\mathfrak{p}_{G^{der}}} = \mathcal{N}_{\mathfrak{p}}$ (the latter does not depend on whether G or $\hat{G}$ is considered). As $H \cap Z(G)^0$ is acting trivially we have that

$$\mathfrak{p}/\hat{H} \times_{\mathfrak{a}/\!\!/W_a} p(Y) \cong p(Y) \times \mathcal{N}_{\mathfrak{p}}/\hat{H}.$$

To describe $\mathfrak{p}/H$ rather than $\mathfrak{p}/\hat{H}$. We can then recover the wanted identification by rigidification. We have that $\mathfrak{p}/\hat{H} \times_{\mathfrak{a}/\!\!/W_a} p(Y) \cong (\mathfrak{p} \cap q^{-1}(p(Y)))/\hat{H}$ where $q : \mathfrak{p} \to \mathfrak{a}/\!\!/W_a$. Let $Z_1 = Ker(\hat{H} \to H)$ considered as an $\hat{H}$-equivariant subgroup scheme of the the stabilizer scheme of both the $\hat{H}$ action on $(\mathfrak{p} \cap q^{-1}(p(Y)))$, and of that on $p(Y) \times \mathcal{N}_{\mathfrak{p}}$, where the action on the first factor is trivial. Then we have that

$$(\mathfrak{p} \cap q^{-1}(p(Y)))/H \cong (\mathfrak{p} \cap q^{-1}(p(Y))) /\!\!/_{Z_1} \hat{H} \cong (p(Y) \times \mathcal{N}_{\mathfrak{p}}) /\!\!/_{Z_1} \hat{H} \cong (p(Y) \times \mathcal{N}_{\mathfrak{p}})/H,$$

giving the first isomorphism.

Restricting to regular elements gives us

$$\mathfrak{p}^{reg}/H \times_{\mathfrak{a}/\!/W_a} p(Y) \cong \mathcal{N}_{\mathfrak{p}}^{reg}/H \times p(Y)$$

because all steps of the proof were compatible with restricting to regular elements. □

We conclude the description of the enhanced quotient $\mathfrak{p}/\!/_{I_H} H$ for G simple. Recall from Theorem 3.20 that it suffices to describe the gluing on $\widetilde{\mathfrak{p}/\!/H}$ explicitly. To describe the gluing on the intersection of some hyperplanes $\cap_i H_i$, we take the Levi L associated to the H_i. Then, by Theorem 3.42 and Proposition 3.45 the number of sheets of $\widetilde{\mathfrak{p}/\!/H}$ over $\cap_i H_i$ is determined by the regular K-orbits of the nilpotent cone for L, $\mathcal{N}_{\mathfrak{p}_L}^{\mathrm{reg}}$. This is determined by the list in Proposition 2.39. For all simple groups except $\mathrm{SO}_n \times \mathrm{SO}_n \subset \mathrm{SO}_{2n}$, there are at most 2 regular nilpotent orbits, so it suffices to describe only the number of sheets in fibers of the map $\mathfrak{p}^{\mathrm{reg}} /\!/_{I_H^{reg}} H \to \mathfrak{p}^{\mathrm{reg}}/\!/H$. For the $\mathrm{SO}_n \times \mathrm{SO}_n \subset \mathrm{SO}_{2n}$ case, one also needs to compute the gluing pattern of the 4 sheets at the origin as it degenerates. Some results on this case are given in Example 3.56.

More formally:

Theorem 3.46. *Let $(\mathfrak{p}, H)$ be a symmetric pair corresponding to a simple group G,such that $(K,G) \neq (SO(n) \times SO(n) \subset SO(2n))$. Let $U = \cup_L U_L$, where U_L as in Definition 3.39 and L ranges over the subgroups L of the form in Definition 3.36, such that $(\mathcal{N}_{\mathfrak{p}_L})$ has a single regular H_L-orbit*[3]*. We then have that $\mathfrak{p}/\!/_{I_H} H \cong \mathfrak{a}/\!/W_a \coprod_U \mathfrak{a}/\!/W_a$, and this identification is $\mathbb{G}_m$-equivariant.*

Remark 3.47. We note that if $\mathcal{N}_{\mathfrak{p}}$ has one irreducible component by Proposition 2.39 we can just directly state that $\mathfrak{p}/\!/_{I_H} H \cong \mathfrak{a}/\!/W_a$.

Proof of Theorem 3.46. This follows immediately from Theorem 3.20. □

We note that the remaining case of $(K,G) = (G_1, G_1 \times G_1)$ there is no nonseparrated structure as shown in Example 3.50. Finally the following proposition show that the regular semisimple locus is always in the open set U of Theorem 3.46

Proposition 3.48. *The map $\mathfrak{p}^{\mathrm{reg}} /\!/_{I_H} H \to \mathfrak{p}^{reg}/\!/H \simeq \mathfrak{a}/\!/W_{\mathfrak{a}}$ is an isomorphism on the complement of the image of all root hyperplanes in $\mathfrak{a}$.*

Proof. The complement of hyperplanes in $\mathfrak{a}/\!/W_{\mathfrak{a}}$ is the space of semisimple, regular elements $\mathfrak{a}^{rs} \subset \mathfrak{a}$. Let $x \in \mathfrak{p}^{reg}$ lie over the image of $s \in \mathfrak{a}^{rs}$ in $\mathfrak{a}^{rs}/\!/W_{\mathfrak{a}}$. Then by Lemma 2.26 we have the Jordan decomposition $x = s + n$. Since s is regular and n is regular nilpotent in its centralizer, by Proposition 2.27 we must have $n = 0$. Then, the result follows as there is a unique (closed) orbit of semisimple elements in each fiber of the map $\mathfrak{p}^{reg} \to \mathfrak{a}/\!/W_{\mathfrak{a}}$. □

Theorem 3.49. *Let $(\mathfrak{p}, H)$ be a symmetric pair corresponding to a simple group G,such that $(K,G) \neq (SO(n) \times SO(n) \subset SO(2n))$.*

Then $\mathfrak{p}/\!/_{I_H^{reg}} H \cong \mathfrak{a}/\!/W_a \coprod_U \mathfrak{a}/\!/W_a$ where U is the complement of a closed subvariety which is the union of intersections of root hyperplanes.

Proof. This follows immediately from Theorem 3.46 and Proposition 3.48. □

3.7. **Examples.**

Example 3.50. Consider the diagonal case $G_1 \overset{\Delta}{\subset} G_1 \times G_1$ from Example 2.17. In this case, we have an isomorphism of stacks

$$\mathfrak{p}/G_1 \to \mathfrak{g}_1/G_1$$

[3]We note that this can be worked out using Proposition 2.39, together with, if necessary, computing $\pi_0(H_L)$ and its action on irreducible components of $\mathcal{N}_{\mathfrak{p}_L}$.

by projecting onto the first factor. The latter quotient is well studied over the regular locus. In particular, it is shown in [8, 23] that the map

$$\mathfrak{g}_1^{\mathrm{reg}}/G_1 \to \mathfrak{g}_1^{\mathrm{reg}}/\!\!/G_1$$

is a gerbe for the descent of $I_{G_1}^{reg}$ to $\mathfrak{g}_1^{\mathrm{reg}}/\!\!/G_1$. The regular quotient $\mathfrak{g}_1^{\mathrm{reg}}/\!\!/_{I_{G_1}^{reg}} G_1$ is therefore just the GIT quotient $\mathfrak{g}_1^{reg}/\!\!/G_1$. We verify below that this agrees with our inductive construction.

Fix a maximal torus T in G_1 and recall that a Cartan in $\mathfrak{p}$ is given by

$$\mathfrak{a} = \{(X, -X) \colon X \in \mathfrak{t}\},$$

while the restricted root system agrees with the root system on G_1. The distinguished Levi subgroup associated to the sub root system $(\alpha_i, -\alpha_i)$ is of the form $L \simeq L_1 \times L_1$ for L_1 the connected Levi of G_1 with

$$\mathfrak{l}_1 = \mathfrak{t} \oplus \sum_i (\mathfrak{g}_1)_{\alpha_i}$$

The involution θ acts on $L_1 \times L_1$ by swapping factors. But by Lemma 2.43, there is a single regular nilpotent H_L orbit for this form. Hence, there is no non-separated structure anywhere on $\mathfrak{p}^{\mathrm{reg}}/\!\!/_{I_{G_1}^{reg}} G_1$.

Example 3.51. We revisit Example 2.18. Recall $H = K = \mathrm{GL}_n \times \mathrm{GL}_n \subset \mathrm{GL}_{2n} = G$,

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} : \delta \text{ is diagonal} \right\} \subset \left\{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \right\} = \mathfrak{p}$$

and

$$\Phi_r = \{\pm(\delta_j^* \pm \delta_k^*) | j \neq k \text{ and } 1 \leq j, k \leq n\} \cup \{\pm 2\delta_j^* | 1 \leq j \leq n\}$$

is a simple root system of type C_n. The little Weyl group has the form $W_{\mathfrak{a}} = \{\pm 1\}^n \ltimes S_n$, with S_n permuting the dual basis δ_j^* and $\{\pm 1\}^n$ acting by changing the sign of the coordinates δ_j^*.

Note also that if $2\delta_j^* = 2\delta_k^* = 0$, then also $\pm\delta_j^* \pm \delta_k^* = 0$. Hence, we need only deal with distinguished Levis associated to subroot systems $S \subset \Phi_r$ satisfying:

$(*)$ For every distinct $1 \leq j < k \leq n$, either $\{\pm 2\delta_j^*, \pm 2\delta_k^*\} \not\subset S$ or else $\{\pm 2\delta_j^*, \pm 2\delta_k^*, \pm\delta_j^* \pm \delta_k^*\} \subset S$.

Now we see that any *simple* subroot system of Φ_r satisfying condition $(*)$ is $W_{\mathfrak{a}}$ conjugate to one of the following:

(1) The subroot system $\Phi_r = \{\pm\delta_j^* \pm \delta_k^* \colon 1 \leq j, k \leq n_1\}$ for some $n_1 \leq n$. In this case,

$$\mathfrak{l} = \left\{ \left(\begin{array}{cc|cc} * & & * & \\ & \eta' & & \delta' \\ \hline * & & * & \\ & \delta' & & \eta' \end{array} \right) : \delta' \text{ and } \eta' \text{ and diagonal } (n - n_1) \times (n - n_1) \text{ matrices} \right\}$$

so that $L \simeq \mathrm{GL}_{n_1} \times T'$ for a torus T' of rank $2(n - n_1)$.

(2) The subroot system $\Phi_r = \{\pm(\delta_j^* - \delta_k^*) \colon 1 \leq j < k \leq n_1\}$ for some $n_1 \leq n$. In this case,

$$\mathfrak{l} = \left\{ \left(\begin{array}{cc|cc} A & & B & \\ & \eta'' & & \delta'' \\ \hline B & & A & \\ & \delta'' & & \eta'' \end{array} \right) : \delta'' \text{ and } \eta'' \text{ are diagonal } (n - n_1) \times (n - n_1) \text{ matrices} \right\}$$

so that $L \simeq (\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_1}) \times T''$ for T'' a torus and the involution acting on $\mathrm{GL}_2 \times \mathrm{GL}_2$ by swapping factors. There is a unique nilpotent orbit in this case.

For an arbitrary subroot system $S \subset \Phi_r$ satisfying condition $(*)$, S is a product $S_1 \times \cdots \times S_a \times S_1' \times \cdots \times S_b'$ where S_j is $W_{\mathfrak{a}}$ conjugate to a root system of type (1) and S_j' is $W_{\mathfrak{a}}$ conjugate to a root system of type (2). However, we note that condition $(*)$ immediately implies that $a = 1$. Hence, we reduce to distinguished Levis associated with $S_1 \times S_1' \times \cdots \times S_b'$

For such a root system, we have

$$L \simeq \mathrm{GL}_{2n_1} \times (\mathrm{GL}_{m_1} \times \mathrm{GL}_{m_1}) \times \cdots \times (\mathrm{GL}_{m_b} \times \mathrm{GL}_{m_b}) \times T'$$

for T' a torus of rank $2(n - \sum_j n_j - \sum_k m_k)$. We can see easily that

$$H_L = (\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_1}) \times \prod_{k=1}^{b} \mathrm{GL}_{m_k} \times (T')^\theta$$

where $(T')^\theta$ is a (connected) torus of rank $(n - \sum_j n_j - \sum_k m_k)$. In particular, we see that $|\mathcal{N}^{\mathrm{reg}}_{\mathfrak{p}_L}/H_L| = 2$ by comparing with the list in Proposition 2.39 for the first factor of $(\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_1}) \subset \mathrm{GL}_{2n_1}$.

We hence conclude with a description of the regular quotient $\mathfrak{p}^{reg} /\!\!/_{I_K^{reg}} K$ in this case.

Proposition 3.52. *For H, $\mathfrak{p}$, and K as in this example, let $V \subset \mathfrak{a}$ be the subscheme that is the complement of all root hyperplanes for roots of the form $\pm 2\delta_i^*$. Then, we have $\mathfrak{p}^{reg} /\!\!/_{I_K^{reg}} K \cong \mathfrak{a}/\!\!/W_a \coprod_U \mathfrak{a}/\!\!/W_a$, where $U := V/\!\!/W_a \subset \mathfrak{a}/\!\!/W_a$.*

Proof. Follows immediately from the above computations and Theorem 3.49. □

Example 3.53. Consider the split form $\mathrm{SO}_n \subset \mathrm{SL}_n$. If n is odd, there is only one nilpotent orbit and the regular quotient and GIT quotient agree. We will assume therefore that n is even. We have $\mathfrak{a} = \mathfrak{t}$ is the diagonal Cartan inside $\mathfrak{p} = \mathfrak{sym}_n$ (symmetric $n \times n$ matrices). The restricted root system agrees with the root system for SL_n and so is type A_{n-1}. Any distinguished Levi is $W = W_{\mathfrak{a}}$ conjugate to a block diagonal Levi

$$L = S(\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_l}) \subset \mathrm{SL}_n$$

where $n_j \geq 1$ and $\sum_j n_j = n$. It is easy to see that

$$H_L = S(O_{n_1} \times \cdots \times O_{n_l})$$

Recall that the symmetric pair $\mathrm{SO}_{n_j} \subset \mathrm{GL}_{n_j}$ has 2 regular nilpotent orbits in the case where n_j is even and 1 when n_j is odd (see Proposition 2.39). Without loss of generality, suppose that $n_1, \dots, n_a$ are even while $n_{a+1}, \dots, n_l$ are odd.

If $l > a$ (i.e. some n_j is odd), then the quotient map

$$L = S(\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_l}) \to \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_a} =: L_1$$

is surjective and carries H_L to the subgroup

$$H_1 := O_{n_1} \times \cdots \times O_{n_a}$$

(L_1, H_1) is a symmetric pair, and one checks that $\pi_0(H_1)$ acts freely on components of $\mathcal{N}^{\mathrm{reg}}_{\mathfrak{p}_{L_1}}$. In particular, comparing with list of Proposition 2.39, we see that

$$\#(\mathcal{N}^{\mathrm{reg}}_{\mathfrak{p}_L}/H_L) = \#(\mathcal{N}^{\mathrm{reg}}_{\mathfrak{p}_{L_1}}/H_1) = 2^a/2^a = 1.$$

If $l = a$ (i.e. all indices are even), then $\pi_0(H_L)$ acts freely on the components of $\mathcal{N}^{reg}_{\mathfrak{p}_L}$ and comparing with the list in Proposition 2.39 gives

$$\#(\mathcal{N}^{\mathrm{reg}}_{\mathfrak{p}_L}/H_L) = 2^a/2^{a-1} = 2.$$

We conclude:

Proposition 3.54. *Let n be even, and let $\epsilon_1, \dots, \epsilon_n$ be coordinates for diagonal matrices, so that $\mathfrak{a} = \mathfrak{t}$ is the locus $\sum_j \epsilon_j = 0$. Let $\mathrm{Even}Par_n$ denote the set of even partitions of $\{1, \dots, n\}$, i.e. the set of decompositions $\{1, \dots, n\} = S_1 \sqcup \cdots \sqcup S_k$ with each $|S_i|$ even. Put*

$$Z_{j_1, \dots, j_l} = \{\epsilon_{j_1} = \epsilon_{j_2} = \cdots = \epsilon_{j_l}\} \subset \mathfrak{a}$$

and $Z_S = \cap_m Z_{S_m}$ for $S \in \mathrm{Even}Par_n$.

Define V to be the complement of

$$\cup_{S \in \mathrm{Even}Par_n} Z_S \subset \mathfrak{a}$$

and let $U = V/\!\!/W_{\mathfrak{a}} \subset \mathfrak{a}/\!\!/W_{\mathfrak{a}}$. Then in this example, we have $\mathfrak{p}^{\mathrm{reg}} /\!\!/_{I_K^{\mathrm{reg}}} K \simeq \mathfrak{a}/\!\!/W_{\mathfrak{a}} \coprod_U \mathfrak{a}/\!\!/W_{\mathfrak{a}}$.

Proof. This follows by Theorem 3.49 and the computations above. ☐

Remark 3.55. In the statement of Proposition 3.54, one can replace $EvenPar_n$ with only those partitions $\{1,\dots,n\} = S_1 \sqcup \dots \sqcup S_l$ for which $\#S_j = 2$ for all j. The open set U in the Proposition has complement of codimension $n - (n/2)$. Hence this gives an example of a symmetric pair for which the gluing does not occur along the complement of a divisor.

Example 3.56. Consider the split form $\mathrm{SO}_{2n} \times \mathrm{SO}_{2n} \subset \mathrm{SO}_{4n}$ of Example 2.19. Recall that restricted roots are of the form

$$\Phi_r = \{i(\pm\delta_j^* \pm \delta_k^*) \colon j \neq k, 1 \leq j,k \leq 2n\}$$

This gives the root system of type D_{2n}, which is simple and simply-laced when $n \geq 2$ and is the product $D_2 = A_1 \times A_1$ when $n = 1$. Recall that for $n \geq 2$ this is the unique family of simple symmetric pairs up to isogeny for which there are 4 regular nilpotent orbits.

For inductive purposes, we will need to describe the case when $n = 1$: When $n = 1$, the root system D_2 is not simple, and the isogeny of Theorem 3.35 is the map

$$\xi \colon \mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}_4$$

with θ lifting to the involution $\theta(g) = g^{-t}$ on each copy of SL_2, and

$$\xi^{-1}(\mathrm{SO}_2 \times \mathrm{SO}_2) = \mathrm{SO}_2 \times \mathrm{SO}_2 \subset \mathrm{SL}_2 \times \mathrm{SL}_2\,.$$

In particular, the regular quotient can be described as a product:

Proposition 3.57. *The form* $\mathrm{SO}_2 \times \mathrm{SO}_2 \subset \mathrm{SO}_4$ *has regular quotient given by the product of regular quotients*

$$\mathfrak{p} /\!\!/_{I_K^{reg}} K = \mathfrak{p}_1 /\!\!/_{I_{K_1}^{reg}} K_1 \times \mathfrak{p}_1 /\!\!/_{I_{K_1}^{reg}} K_1$$

where $(G_1, \theta_1, H_1) = (\mathrm{SL}_2, g \mapsto g^{-t}, \mathrm{SO}_2)$.

Now, consider the case $\mathrm{SO}_{2n} \times \mathrm{SO}_{2n} \subset \mathrm{SO}_{4n}$ for $n \geq 2$. In this case, the root system is type D_{2n}, is simple, and is simply laced. The little Weyl group is $W_{\mathfrak{a}} = \{\pm 1\}^{n-1} \ltimes S_n$, where S_n acts on the coordinates δ_j^* by permuting the indices j and $\{\pm 1\}^{n-1}$ acts by changing an even number of signs on the δ_j^*. Any *simple* root subsystem of Φ_r is $W_{\mathfrak{a}}$ conjugate to one of the following:

(1) $S = \{i(\pm\delta_j^* \pm \delta_k^*) \colon 1 \leq j < k \leq m_1\}$ for some $m_1 \leq 2n$. The Levi associated to this root subsystem is

$$\mathfrak{l} = \left\{ \left(\begin{array}{cc|cc} A & & B & \\ & \mathbf{0}_{2n-m_1} & & \delta' \\ \hline -B^t & & A & \\ & -\delta' & & \mathbf{0}_{2n-m_1} \end{array} \right) : A \in \mathfrak{so}_{m_1},\ \text{and } \delta' \text{ is a diagonal } (2n-m_1)\times(2n-m_1) \text{ matrix} \right\}$$

giving $L = \mathrm{SO}_{2m_1} \times T'$ for T' a split torus and θ acting on SO_{2m_1} by conjugation by $diag(I_{m_1}, -I_{m_1})$.

(2) $S = \{\pm i(\delta_j^* - \delta_k^*) \colon 1 \leq j < k \leq m_1\}$ for some $m_1 \leq 2n$. The Levi associated to this S is

$$\mathfrak{l} = \left\{ \left(\begin{array}{cc|cc} A & & B & \\ & \mathbf{0}_{2n-m_1} & & \delta' \\ \hline -B & & A & \\ & -\delta' & & \mathbf{0}_{2n-m_1} \end{array} \right) : A \in \mathfrak{so}_{m_1},\ B \in \mathfrak{sym}_{m_1},\ \text{and } \delta' \text{ is a diagonal } (2n-m_1)\times(2n-m_1) \text{ matrix} \right\}$$

giving $L \simeq \mathrm{GL}_{m_1} \times T'$ for T' a split torus and θ acting on GL_{m_1} by $g \mapsto g^{-t}$.

(3) $S = \{\pm i(\delta_j^* + \delta_k^*) \colon 1 \leq j < k \leq m_1\}$ for some *odd* $m_1 < 2n$. The Levi associated to this root subsystem is

$$\mathfrak{l} = \left\{ \left(\begin{array}{cc|cc} A & & B & \\ & \mathbf{0}_{2n-m_1} & & \delta' \\ \hline -B & & A & \\ & -\delta' & & \mathbf{0}_{2n-m_1} \end{array} \right) : A \in \mathfrak{so}_{m_1},\ B = B^{\ddagger},\ \text{and } \delta' \text{ is a diagonal } (2n-m_1)\times(2n-m_1) \text{ matrix} \right\}$$

where $B^\ddagger = \left((-1)^{ij} b_{ji}\right)_{1\leq i,j\leq m_1}$. This again gives $L \simeq \mathrm{GL}_{m_1} \times T'$.

Any arbitrary subroot system of Φ_r is a product $S = S_1 \times \cdots S_l$ where each S_j is conjugate to one of the three above root subsystems. Of the above, types (2) [when m_1 is even] and (1) can contribute nontrivial regular nilpotent orbits. Suppose that

$$S = \prod_{j=1}^{a} S_j^{(1)} \times \prod_{j=1}^{b} S_j^{(2)} \times \prod_{j=1}^{c} S_j^{(3)}$$

where $S_j^{(k)}$ has rank $m_j^{(k)}$.

Let $H_L^{(2)} = \prod_{j=1}^{b} O_{m_j^{(2)}}$ and $H_L^{(3)} = \prod_{j=1}^{b} O_{m_j^{(3)}}$. Then, we compute

$$H_L = \left(\prod_{j=1}^{a} O_{m_j^{(1)}} \times S\left(\prod_{k=1}^{a} O_{m_k^{(1)}} \times H_L^{(2)} \times H_L^{(3)}\right)\right) \cap \left(S\left(\prod_{j=1}^{a} O_{m_j^{(1)}} \times H_L^{(2)} \times H_L^{(3)}\right) \times \prod_{k=1}^{a} O_{m_k^{(1)}}\right)$$

with the intersection being taken inside

$$\prod_{j=1}^{a} (O_{m_j^{(1)}} \times O_{m_j^{(1)}}) \times H_L^{(2)} \times H_L^{(3)}$$

We recall that $\mathrm{SO}_{m_j} \subset \mathrm{GL}_{m_j}$ has 1 regular, nilpotent K orbit when m_j is odd and 2 when m_j is even, and that $\mathrm{SO}_{m_j} \times \mathrm{SO}_{m_j} \subset \mathrm{SO}_{2m_j}$ has 2 regular nilpotent orbits when m_j is odd and 4 when m_j is even. Define the following invariants:

Let N_1^e be the number of Levis of type (1) with m_j even and N_1^o the number of Levis of type (1) with m_j odd. Let N_2^e and N_2^o be defined similarly for type (2) Levis, and N_3 the number of type (3) Levis. Let ϵ_1 be zero if both $N_1^o = 0$ and $N_1^e \neq 0$ and 1 otherwise, and let ϵ_2 be zero if $\sum_j m_j = 2n$ and $N_2^o = N_3 = 0$ and 1 otherwise. Then, we can count the nilpotent orbits by studying the components of H_L. We find:

$$\#(\mathcal{N}_{\mathfrak{p}_L}^{reg}/H_L) = 2^{2N_1^e + N_1^o + N_2^e} / (2^{N_1^e + \epsilon_1 - 1} \cdot 2^{N_1^e + N_1^o + N_2^e + \epsilon_2 - 1}) = 2^{2-\epsilon_1-\epsilon_2}$$

For example, for $\mathrm{SO}_4 \times \mathrm{SO}_4 \subset \mathrm{SO}_8$, we have:

(1) There are 4 sheets over the loci:
   (i) Fix an ordering i_j of $\{1,2,3,4\}$. $\{\delta_{i_1}^* = \delta_{i_2}^* = 0,\ \delta_{i_3}^* = \pm\delta_{i_4}^*\}$, i.e. strata corresponding to distinguished Levis $H_L \subset L$ that are W conjugate to $(\mathrm{SO}_2 \times \mathrm{SO}_2) \times \mathrm{SO}_2 \subset \mathrm{SO}_4 \times \mathrm{GL}_2$.
   (ii) The origin $\{\delta_i^* = 0 \text{ for all } i\}$, i.e. the strata corresponding to the distinguished Levi SO_8.
(2) There are 2 sheets over the loci:
   (i) Fix an ordering i_j of $\{1,2,3,4\}$, and fix signs $\epsilon_j \in \{\pm 1\}$. $\{\delta_{i_j}^* = \epsilon_j \delta_{i_{j+1}}^* : j = 1,2,3\}$, i.e. strata corresponding to distinguished Levis $H_L \subset L$ which are W-conjugate to $\mathrm{SO}_4 \subset \mathrm{GL}_4$.
   (ii) Fix i_j an ordering of $\{1,2,3,4\}$ and signs $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. $\{\delta_{i_1}^* = \epsilon_1 \delta_{i_2}^*, \delta_{i_3}^* = \epsilon_2 \delta_{i_4}^*\}$, i.e. strata corresponding to distinguished Levis $H_L \subset L$ which are W conjugate to $S(O_2 \times O_2) \subset \mathrm{GL}_2 \times \mathrm{GL}_2$.
(3) 1 sheet over all other strata.

Note that the above list allows us to reduce to *only* Levis conjugate to $(\mathrm{SO}_2 \times \mathrm{SO}_2) \times \mathrm{SO}_2 \subset \mathrm{SO}_4 \times \mathrm{GL}_2$. Note that the regular quotient of this symmetric pair is given by the regular quotient of $\mathrm{SO}_2 \times \mathrm{SO}_2 \subset \mathrm{SO}_4$, whose gluing pattern was studied above.

Example 3.58. Consider the more general case of $\mathrm{SO}_m \times \mathrm{SO}_{2n-m} \subset \mathrm{SO}_{2n}$ from Example 2.20. Note that this is split for $m = n$ (see previous example for this case with m even) and quasi-split for $m = n-1$. Note furthermore that if m were odd, then there would be a single regular, nilpotent orbit for this pair, and the regular quotient would be equal to the GIT quotient. We will therefore consider only the case where m is even. The restricted root system for this pair is given by

$$\Phi_r = \{i(\pm\delta_j^* \pm \delta_k^*) \colon 1 \leq j < k \leq m\} \cup \{\pm i\delta_j^* \colon 1 \leq j \leq m\}.$$

This is a simple root system of type B_m, and the little Weyl group $W_{\mathfrak{a}} = \{\pm 1\}^m \ltimes S_m$ acts by permutations and sign changes on the δ_j^*, $1 \le j \le m$. Note that we again have a condition on subroot systems of Φ_r that can arise as the set of roots vanishing a collection of hyperplanes in $\mathfrak{a}$. Namely,

($*$) For every distinct $1 \le j < k \le m$, either $\{\pm i\delta_j^*, \pm i\delta_k^*\} \not\subset S$ or else $\{\pm i\delta_j^*, \pm i\delta_k^*, i(\pm\delta_j^* \pm \delta_k^*)\} \subset S$.

A *simple* root subsystem $S \subset \Phi_r$ satisfying condition ($*$) is $W_{\mathfrak{a}}$ conjugate to one of the following:

(1) $\{i(\pm\delta_j^* \pm \delta_k^*)\colon 1 \le j < k \le m_1\} \cup \{i(\pm\delta_j^*)\colon 1 \le j \le m_1\}$ for some $m_1 \le m$. This has distinguished Levi given by $L \simeq \mathrm{SO}_{2(m_1+n-m)} \times T'$ where T' is a (not split) torus, and θ acts on $\mathrm{SO}_{2(m_1+n-m)}$ by conjugation by $\mathrm{diag}(I_{m_1, 2n-2m-m_1})$.

(2) $\{\pm i(\delta_j^* - \delta_k^*)\colon 1 \le j < k \le m_1\}$ for some $m_1 \le m$. This has associated Levi given by $L = \mathrm{GL}_{m_1} \times T'$ where T' is a (nonsplit) torus and θ acts on GL_{m_1} by the split involution $\theta(g) = g^{-t}$.

An arbitrary subroot system satisfying ($*$) is a product of at most one Levi of type (1) and an arbitrary number of Levis of type (2). There are two sheets over the strata:

- Fix $m_1, \dots, m_r$ such that each m_k is even and $\sum_l m_l = 2n$. The $W_{\mathfrak{a}}$ orbit of $\{\delta_j^* = 0\colon 1 \le j \le m_1\} \cap \bigcap_{l=2}^{r-1} \{\delta_j^* = \delta_k^*\colon m_l + 1 \le j < k \le m_{l+1}\}$.
- Fix $m_1, \dots, m_r$ such that each m_k is even and $\sum_l m_l = 2n$. The $W_{\mathfrak{a}}$ orbit of $\bigcap_{l=1}^{r-1} \{\delta_j^* = \delta_k^*\colon m_l + 1 \le j < k \le m_{l+1}\}$.

and map $\mathfrak{p} /\!\!/_{I_K^{reg}} K \to \mathfrak{p} /\!\!/ K$ is an isomorphism elsewhere.

Example 3.59. Consider the case of $\mathrm{SO}_m \times \mathrm{SO}_{2n+1-m} \subset \mathrm{SO}_{2n+1}$, $m \le n$, of Example 2.21. Recall that the root system is the simple type B_m root system

$$\Phi_r = \{i(\pm\delta_j^* \pm \delta_k^*)\colon 1 \le j < k \le m\} \cup \{\pm i\delta_j^*\colon 1 \le j \le m\}.$$

The little Weyl group $W_{\mathfrak{a}} = \{\pm 1\}^m \ltimes S_m$ acts on Φ_r by permutation and sign change on the δ_j^*. Note that by Proposition 2.39, there is a unique regular, nilpotent K orbit in $\mathfrak{p}$ when m is odd and two when m is even. We therefore restrict to the case when m is even.

Note also that if $i\delta_j^* = i\delta_k^* = 0$, then also $i(\pm\delta_j^* \pm \delta_k^*) = 0$. Hence, we need only deal with distinguished Levis associated to subroot systems $S \subset \Phi_r$ satisfying:

($*$) For every distinct $1 \le j < k \le n$, either $\{\pm i\delta_j^*, \pm i\delta_k^*\} \not\subset S$ or else $\{\pm i\delta_j^*, \pm i\delta_k^*, i(\pm\delta_j^* \pm \delta_k^*)\} \subset S$.

Any *simple* subroot system of Φ_r is $W_{\mathfrak{a}}$ conjugate to one of the following:

(1) $\{i(\pm\delta_j^* \pm \delta_k^*), \pm i\delta_j^*\colon 1 \le j < k \le m_1\}$ for some $m_1 \le m$. The associated Levi for this subroot system is $L \simeq \mathrm{SO}_{2(m_1+n-m)+1} \times T'$ for T' a torus and θ acting on $\mathrm{SO}_{2(m_1+n-m)+1}$ by conjugation by the matrix $I_{m_1, m_1+2(n-m)+1} = \mathrm{diag}(I_{m_1}, -I_{m_1+2(n-m)+1})$.

(2) $\{i(\pm\delta_j^* - \delta_k^*)\colon 1 \le j < k \le m_1\}$ for some $m_1 \le m$. The associated Levi for this subroot system is $L \simeq \mathrm{GL}_{m_1} \times T'$ for T' a torus, and θ acting on GL_{m_1} by $\theta(g) = g^{-t}$.

An arbitrary subroot system of Φ_r is a product of Levis of type (1) and (2) with at most one type (1) Levi appearing. There are 2 sheets precisely over the following strata in $\mathfrak{a} /\!\!/ W_{\mathfrak{a}}$:

- Fix $m_1, \dots, m_r$ such that each m_k is even and $\sum_l m_l = 2n$. The $W_{\mathfrak{a}}$ orbit of $\{\delta_j^* = 0\colon 1 \le j \le m_1\} \cap \bigcap_{l=2}^{r-1} \{\delta_j^* = \delta_k^*\colon m_l + 1 \le j < k \le m_{l+1}\}$.
- Fix $m_1, \dots, m_r$ such that each m_k is even and $\sum_l m_l = 2n$. The $W_{\mathfrak{a}}$ orbit of $\bigcap_{l=1}^{r-1} \{\delta_j^* = \delta_k^*\colon m_l + 1 \le j < k \le m_{l+1}\}$.

and the map $\mathfrak{p} /\!\!/_{I_K^{reg}} K \to \mathfrak{p} /\!\!/ K$ is an isomorphism elsewhere.

Example 3.60. Consider the split form $\mathrm{GL}_n \subset \mathrm{Sp}_{2n}$ of example 2.22. The restricted root system agrees with the usual root system, which is type C_n. We use the presentation

$$\Phi_r = \{\pm\delta_j^* \pm \delta_k^*\colon 1 \le j < k \le n\} \cup \{\pm 2\delta_j^*\colon 1 \le j \le n\}$$

for the root system, where the dual basis δ_j^* is chosen with respect to the coordinate vectors for the Cartan $\mathfrak{a}$ in example 2.22. The little Weyl group $W_{\mathfrak{a}} = W = \{\pm 1\}^n \ltimes S_n$ acts on Φ_r by letting S_n act by permuting the δ_j^* and $\{\pm 1\}^n$ act by sign changes on the δ_j^*.

Note also that if $2\delta_j^* = 2\delta_k^* = 0$, then also $\pm\delta_j^* \pm \delta_k^* = 0$. Hence, we need only deal with distinguished Levis associated to subroot systems $S \subset \Phi_r$ satisfying:

($*$) For every distinct $1 \leq j < k \leq n$, either $\{\pm 2\delta_j^*, \pm 2\delta_k^*\} \not\subset S$ or else $\{\pm 2\delta_j^*, \pm 2\delta_k^*, \pm\delta_j^* \pm \delta_k^*\} \subset S$.

Now we see that any *simple* subroot system S of Φ_r satisfying condition ($*$) is $W_{\mathfrak{a}}$ conjugate to one of the following:

(1) $\{\pm\delta_j^* \pm \delta_k^* \colon 1 \leq j < k \leq n_1\} \cup \{\pm 2\delta_j \colon 1 \leq j \leq n_1\}$ for some $n_1 \leq n$. The associated Levi has

$$\mathfrak{l} = \left\{ \left(\begin{array}{cc|cc} * & & * & \\ & 0 & & \delta' \\ \hline * & & * & \\ & \delta' & & 0 \end{array} \right) : \delta' \text{ is a diagonal } (n-n_1) \times (n-n_1) \text{ matrix} \right\}$$

giving $L \simeq \mathrm{Sp}_{2n_1} \times T'$ for T' a split torus and θ acting on Sp_{2n_1} by conjugation by $diag(I_{n_1}, -I_{n_1})$.

(2) $\{\pm(\delta_j^* - \delta_k^*) \colon 1 \leq j < k \leq n_1\}$ for some $n_1 \leq n$. The associated Levi has

$$\mathfrak{l} = \left\{ \left(\begin{array}{cc|cc} A & & B & \\ & \mathbf{0}_{n-n_1} & & \delta' \\ \hline B & & -A^t & \\ & \delta' & & \mathbf{0}_{n-n_1} \end{array} \right) : A \in \mathfrak{so}_{n_1},\ B \in \mathfrak{sym}_{n_1},\ \text{and } \delta' \text{ is a diagonal } (n-n_1) \times (n-n_1) \text{ matrix} \right\}$$

giving $L \simeq \mathrm{GL}_{n_1} \times T'$ where T' is a split torus and θ acts on GL_{n_1} by the split form $g \mapsto g^{-t}$.

An arbitrary root subsystem of Φ_r satisfying ($*$) is a product of the above types, with at most one factor of type (1) appearing. In particular, the strata of $\mathfrak{a}/\!\!/W_{\mathfrak{a}}$ with two sheets are the following:

(1) For any subset $T \subset \{1, \ldots, n\}$, $\{\delta_j^* = 0 \colon j \in T\}$. These are the strata corresponding to Levis of type (1).

(2) Fix the following data: Even integers n_k such that $\sum_k n_k = n$; an ordering i_j of the numbers $\{1, \ldots, n\}$; signs $\epsilon_j \in \{\pm 1\}$. Then, consider the strata

$$\bigcap_k \left\{ \delta_{i_j}^* = \epsilon_j \delta_{i_{j+1}}^* \colon \left(\sum_{l=1}^k n_l\right) + 1 \leq j \leq \sum_{l=1}^{k+1} n_l \right\}.$$

This corresponds to Levis which are products of type (2) Levis $L \simeq \prod_k \mathrm{GL}_{n_k}$ where all the n_k are even.

3.8. **Galois Description of J via the Parabolic Cover for Quasi-split Symmetric Pairs.** We continue to assume all symmetric pairs in this section are quasi-split. We will follow the general construction outlined in [21]; namely, we will study the parabolic cover associated to the action of H on $\mathfrak{p}$. We begin by a lemma relating parabolics of H to parabolics of G.

Lemma 3.61. *Any Borel B of H is the intersection $B_G \cap H$ for some θ-stable Borel B_G of G. Moreover, suppose we have a Levi $M \subset H$ of H and suppose that there exists a Levi $M_G \subset G$ such that $M_G \cap H = M$. Then, there exists parabolics $Q \subset H$ with Levi M and $Q_G \subset G$ with Levi M_G such that for any parabolic P of H conjugate to Q, there exists a parabolic P_G of G conjugate to Q_G such that $P = P_G \cap H$.*

Proof. We begin with the statement on Borels, first showing that the intersection any θ-stable Borel of G with H is a Borel of H. Let B_G be a θ-stable Borel of G and put $B = B_G \cap H$. It is clear that B is solvable; we must show that H/B is projective, or equivalently that $H/B \simeq H \cdot B_G \subset G/B_G$ is a closed embedding. We claim that, in fact, $H \cdot B_G = (G/B_G)^\theta$. Let B'_G be another θ-stable Borel of G, and let $g \in G$ be such that $g \cdot B_G = B'_G$. Applying θ gives $\theta(g) \cdot B_G = B'_G$, so that $g^{-1}\theta(g) \in N_G(B_G) = B_G$ and further

$$g^{-1}\theta(g) \in \{b \in B_G \colon \theta(b) = b^{-1}\}.$$

Recall that the image of the morphism $G \to G$ sending $g \mapsto g^{-1}\theta(g)$ is precisely those g in G such that $\theta(g) = g^{-1}$. As B_G is θ-stable, the image of the morphism $B_G \to B_G$ is therefore the intersection

$$\{g \in G \colon \theta(g) = g^{-1}\} \cap B_G = \{b \in B_G \colon \theta(b) = b^{-1}\},$$

and so there exists $b \in B_G$ so that $g^{-1}\theta(g) = b^{-1}\theta(b)$. We have $gb^{-1} \cdot B_G = B'_G$ while also

$$(gb^{-1})^{-1}\theta(gb^{-1}) = b(g^{-1}\theta(g))\theta(b)^{-1} = b(b^{-1}\theta(b))\theta(b)^{-1} = 1.$$

We conclude that B_G and B'_G are G^θ conjugate. Hence, $G^\theta \cdot B_G = (G/B_G)^\theta$ and since the flag variety is invariant under central isogeny, also $H \cdot B_G = (G/B_G)^\theta$. Now, since the fixed locus of an algebraic involution is closed, we conclude that $B_G \cap H$ is Borel.

Since all Borels of H are conjugate under K, and since K conjugacy preserves θ-stability, we conclude that every Borel of H is the intersection of a Borel of G with H.

Now, suppose we have a Levi $M \subset H$ of H and suppose that there exists a Levi $M_G \subset G$ such that $M_G \cap H = M$. Choose any parabolic Q_G of G with Levi factor M_G, and put $Q = Q_G \cap H$. We must show first that Q is a parabolic with Levi M. Let $B_G \subset Q_G$ be a Borel in Q_G and $B = B_G \cap H$ the corresponding Borel of H. Since we have a commutative diagram

$$\begin{array}{ccc} H/B & \longrightarrow & H/Q \\ \downarrow & & \downarrow \\ G/B_G & \longrightarrow & G/Q_G \end{array}$$

with the horizontal arrows being surjective and the left vertical arrow being a closed immersion, it follows that H/Q is a closed subvariety of G/Q_G. In particular, H/Q is projective, and Q is a parabolic in H. That the Levi factor of Q is M follows from intersecting the decomposition $Q_G = M_G \cdot U_G$ where U_G is the unipotent radical of Q_G.

Now, for any parabolic P of H conjugate to Q by $h \in H$, we may take the corresponding conjugate parabolic $h \cdot Q_G$ of G. The result follows. ☐

We now introduce the central object of study; the parabolic cameral cover. To make sense of the definition, we need the following.

Lemma 3.62. *For any $X \in \mathfrak{p}^{rss}$, $Z_K(X)^\circ$ is a (not necessarily maximal) torus of H, and $Z_{\mathfrak{k}}(X) = \mathrm{L}ie(Z_K(X)^\circ)$.*

Proof. Assume without loss of generality that $X \in A$. By Proposition 2.32, $Z_K(X)^\circ \subset Z_K(A)$ is a connected abelian subgroup of a maximal torus, so is a torus in H.

The statement on Lie algebras is Lemma 4.2 of [19]. ☐

Definition 3.63. Fix a regular, semisimple element $X \in \mathfrak{a}$. We define Levi subgroups $M := Z_H(X)^\circ$ of H and $M_G := Z_G(X)^\circ$ of G.

Let Q and Q_G be as in the statement of Lemma 3.61 for the Levis $M \subset M_G$.

Remark 3.64. Note that M, M_G are well defined up to conjugation by H. The parabolics Q and Q_G may involve further choice as parabolics with fixed Levi type are not necessarily conjugate.

Definition 3.65. Fix the data of Definition 3.63. Then, let

$$\widetilde{\mathfrak{p}}^{\mathrm{rss}} = \{(X,P) \colon X \in \mathfrak{p}^{reg},\ \mathrm{L}ie(P) \supset C_{\mathfrak{k}}(C_{\mathfrak{k}}(X)) \text{ and } P \text{ is a parabolic of } H \text{ which is } H \text{ conjugate to } Q\}$$

and denote by $\pi_{\mathfrak{p}} \colon \widetilde{\mathfrak{p}}^{\mathrm{reg}} \to \mathfrak{p}^{reg}$ the projection map. We denote by $\tilde{\mathfrak{p}}^{rss} = \tilde{\mathfrak{p}}^{reg}|_{\mathfrak{p}^{rss}}$.

Proposition 3.66. *Let $W_{H,M} = N_H(M)/M$ denote the relative Weyl group of M. There is a canonical map $\widetilde{\mathfrak{p}}^{\mathrm{rss}} \to \mathfrak{a}/\!\!/W_{\mathrm{ker}} \times_{W_{\mathfrak{a}}/W_{\mathrm{ker}}} W_{H,M} = \mathfrak{a} \times_{W_{\mathfrak{a}}} W_{H,M}$ fitting into a Cartesian diagram*

$$\begin{tikzcd}
\tilde{\mathfrak{p}}^{rss} \arrow[rr] \arrow[d] & & \mathfrak{a} \times_{W_{\mathfrak{a}}} W_{H,M} \arrow[d] \\
\mathfrak{p}^{rss} \arrow[r] & \mathfrak{p}/\!\!/H \arrow[r, "\cong"] & \mathfrak{a}/\!\!/W_a
\end{tikzcd}$$

Proof. Let $\widetilde{\mathfrak{a}}^{reg} = \pi_{\mathfrak{p}}^{-1}(\mathfrak{a}^{reg})$ be the closed subscheme of $\widetilde{\mathfrak{p}}^{reg}$ over the subscheme $\mathfrak{a}^{reg} \subset \mathfrak{p}^{rss}$. Then, we have a projection map $\widetilde{\mathfrak{a}}^{reg} \to \mathfrak{a}^{reg}$.

The relative Weyl group $W_{H,M}$ acts on the set of all parabolics of H with Levi M, and the set of such parabolics conjugate to Q is a $W_{H,M}$ torsor under this action.

The map $\mathfrak{a}^{reg} \times W_{H,M} \to \widetilde{\mathfrak{a}}^{reg}$ sending $(X, w) \mapsto (X, w \cdot Q)$ is an isomorphism, and the resulting diagram

$$\begin{tikzcd}
\tilde{\mathfrak{a}}^{reg} \arrow[r, "\cong"] \arrow[d] & \mathfrak{a}^{reg} \times W_{H,M} \arrow[r] & \mathfrak{a} \times_{W_{\mathfrak{a}}} W_{H,M} \arrow[d] \\
\mathfrak{a}^{reg} \arrow[r] & \mathfrak{p}/\!\!/H \arrow[r, "\cong"] & \mathfrak{a}/\!\!/W_a
\end{tikzcd}$$

with the top right arrow given by quotienting by the diagonal action of $W_{\mathfrak{a}}$, is commutative and Cartesian.

Recall from Lemma 2.29 that $H \cdot \mathfrak{a}^{reg} = \mathfrak{p}^{rss}$. The H-orbit $H \cdot (\widetilde{\mathfrak{a}}^{reg}) \to \mathfrak{p}^{rss}$ is therefore a surjective $W_{H,M}$ cover. Since there is an inclusion of $W_{H,M}$ covers

$$H \cdot (\widetilde{\mathfrak{a}}^{reg}) \hookrightarrow \widetilde{\mathfrak{p}}^{\mathrm{rss}}$$

we conclude that this map is an isomorphism. In particular, since the map $\widetilde{\mathfrak{a}}^{reg} \to \mathfrak{a} \times_{W_{\mathfrak{a}}} W_{H,M}$ is $Z_H(\mathfrak{a}^{reg})$ equivariant, it extends to a map

$$p^{\mathrm{rss}} \colon \widetilde{\mathfrak{p}}^{\mathrm{rss}} \to \mathfrak{a} \times_{W_{\mathfrak{a}}} W_{H,M}$$

by taking

$$p^{\mathrm{rss}}(h \cdot \gamma) = p^{\mathrm{rss}}(\gamma).$$

This map is H invariant, and hence the diagram

$$\begin{tikzcd}
\tilde{\mathfrak{p}}^{rss} \arrow[rr] \arrow[d] & & \mathfrak{a} \times_{W_{\mathfrak{a}}} W_{H,M} \arrow[d] \\
\mathfrak{p}^{rss} \arrow[r] & \mathfrak{p}/\!\!/H \arrow[r, "\cong"] & \mathfrak{a}/\!\!/W_a
\end{tikzcd}$$

is a Cartesian square. □

We now seek to extend Proposition 3.66 over the regular locus. For this, we will need Lemma **??**, which demonstrates the structure of the parabolic cover over the regular nilpotent locus. We begin with some preliminary lemmas.

First, let us set some notation. Let B_G be the unique Borel in G such that $e \in Lie(B_G)$, and let $T_G \subset B_G$ be the unique maximal torus of B_G. Note that B_G is necessarily θ stable, and hence T_G contains a maximal torus T_H of H. Choose $\mathfrak{a}$ so that $C = C_H(\mathfrak{a}) \subset T_H$, and let S_H denote the set of simple roots of H with respect to T_H and the Borel $B_G \cap H$ (which is Borel by the proof of Lemma 3.61). Denote by $V \subset S_H$ the set of simple roots of H which are trivial on C. Then, consider the nilpotent element

$$e' := \sum_{\alpha \in S_H \setminus V} e_\alpha^{\mathfrak{k}} \in \mathfrak{k}$$

Lemma 3.67. *For all simple quasisplit forms except possibly the quasisplit form on E_6, we have:*

(1) The nilpotent

$$\widetilde{e} = \sum_{\alpha \in S_G} e_\alpha^{\mathfrak{g}}$$

lies in $\mathfrak{p}$.

(2) The nilpotents $\widetilde{e} \in \mathfrak{p}$ and $e' \in \mathfrak{k}$ defined above commute.

Proof. We proceed case-by-case through the classification of quasisplit simple symmetric pairs. As the definitions of e, e' do not depend on isogeny class or center, we further assume that all pairs are of the form (G, θ, K) for G simple semisimple.

In the case of any split pair (G, θ, K), we have that $C = Z(G) \cap K$, and so $V = S_K$. Hence, $e' = 0$ and the result is trivial.

In the case of $(G, \theta, K) = (\mathrm{SL}_{2n}, \theta, S(\mathrm{GL}_n \times \mathrm{GL}_n))$ from example 2.18, we have T_K the set of diagonal matrices, and

$$\widetilde{e} = \begin{pmatrix} 0 & I_n \\ N_n & 0 \end{pmatrix} \quad \text{where } N_n = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \text{ is } n \times n$$

On the other hand, $V = \emptyset$ and

$$e' = \begin{pmatrix} N_n & \\ & N_n \end{pmatrix}$$

It is easy to check $[\widetilde{e}, e'] = 0$.

In the case of $(G, \theta, K) = (\mathrm{SL}_{2n+1}, \theta, S(\mathrm{GL}_n \times \mathrm{GL}_{n+1}))$, we have T_K is again the diagonal matrices, and

$$\widetilde{e} = \begin{pmatrix} & \vec{0} & I_n \\ I_n & & \\ \vec{0}^t & & \end{pmatrix} \quad \text{where } \vec{0} \text{ is the } n \times 1 \text{ zero vector.}$$

Then, again $V = \emptyset$ and

$$e' = \begin{pmatrix} N_n & \\ & N_{n+1} \end{pmatrix}$$

It is again an easy check that $[\widetilde{e}, e'] = 0$.

Now consider the case $(G, \theta, K) = (\mathrm{SO}_{2n+2}, \theta, SO_n \times \mathrm{SO}_{n+2})$ of example **??**, with θ given by

$$\theta \begin{pmatrix} A & B \\ -B^t & D \end{pmatrix} = \begin{pmatrix} A & -B \\ B^t & D \end{pmatrix}$$

In this case, we take

$$T_G = \left\{ \begin{pmatrix} & a_1 & & & & & & \\ -a_1 & & & & & & & \\ & & & a_2 & & & & \\ & & -a_2 & & & & & \\ & & & & & \ddots & & \\ & & & & \ddots & & & \\ & & & & & & & a_{n+1} \\ & & & & & & -a_{n+1} & \end{pmatrix} : a_j \in k^\times \right\}$$

When n is even, this gives $T_K = T_G$ while when n is odd, T_K is the $n-1$ dimensional subtorus given by $a_{(n+1)/2} = 0$. We divide into cases based on the parity of n: If n is even, we get:

$$\widetilde{e} = \begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix} \quad \text{where } B = \begin{pmatrix} i & 1 & i & -1 & & & & & & \\ -1 & i & 1 & i & & & & & & \\ & & i & 1 & i & -1 & & & & \\ & & -1 & i & 1 & i & & & & \\ & & & \ddots & & \ddots & & & & \\ & & & & i & 1 & i & -1 & & \\ & & & & -1 & i & 1 & i & & \\ & & & & & & i & 1 & 2i & 0 \\ & & & & & & -1 & i & 2 & 0 \end{pmatrix} \text{ is } n \times (n+1)$$

Moreover, V consists of all but two roots of SO_{n+1}, i.e.

$$e' = \begin{pmatrix} \mathbf{0}_n & & & & & \\ & \mathbf{0}_{n-2} & & & & \\ & & & & 0 & 2 \\ & & & & 0 & 2i \\ & & 0 & 0 & & \\ & & -2 & -2i & & \end{pmatrix}$$

So that $[\widetilde{e}, e'] = 0$.

In the case of n odd, we instead get

$$\widetilde{e} = \begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix} \quad \text{where } B = \begin{pmatrix} 0 & i & 1 & i & -1 & & & & \\ 0 & -1 & i & 1 & i & & & & \\ & & & i & 1 & i & -1 & & \\ & & & -1 & i & 1 & i & & \\ & & & & \ddots & & \ddots & & \\ & & & & & i & 1 & i & -1 \\ & & & & & -1 & i & 1 & i \\ & & & & & & & 2i & 2 \end{pmatrix} \text{ is } n \times (n+1)$$

Again, V consists of all but two roots of SO_{n+1}, and we have

$$e' = \begin{pmatrix} \mathbf{0}_n & & & & & & \\ & \mathbf{0}_{n-2} & & & & & \\ & & & & 0 & 2 & 1 \\ & & & & 0 & 2i & i \\ & & 0 & 0 & & & \\ & & -2 & -2i & & & \\ & & -1 & -i & & & \end{pmatrix}$$

We can again verify the commuting property.

The only remaining simple, semisimple quasisplit involution is the quasisplit form on E_6. $\square$

Lemma 3.68. *We keep notation as in Definition 3.63. Fix a regular, nilpotent element $e \in \mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}$. There exists a parabolic P of H, resp. P_G of G, so that P is conjugate to Q and $P \supset C_K(C_K(e)^\circ)$, resp. P_G is conjugate to Q_G and $P_G \supset C_G(C_K(e)^\circ)$. For any such P, there exists a corresponding P_G so that $P = P_G \cap H$.*

Proof. Let e' be as in the previous Lemma. We have $C_K(e') \supset C_K(C_K(e)^\circ)$ and $C_G(e') \supset C_G(C_K(e)^\circ)$. Moreover, we can complete e' to an $\mathfrak{sl}_2$ triple (e', h', f') in $\mathfrak{k}$ with h' regular in $\mathrm{Lie}(C)$. In particular, a minimal parabolic containing $C_K(e')$, resp. $C_G(e')$ will have Levi type $C_K(h') = C_K(C)$, resp. $C_G(h') = C_G(C)$, as required.

For the final statement, choose first a parabolic P_G with Levi factor $M_G = C_G(C)$. Note that $M = C_K(C) \subset M_G = C_G(C)$ are Levis of H and G respectively such that $M_G \cap H = M$. Hence, by the proof of Lemma 3.61, $P_G \cap H$ is a parabolic of H, and hence, all parabolics conjugate to P, and hence Q, can be written as intersections of such parabolics P_G. $\square$

Corollary 3.69. *The map $\widetilde{\mathfrak{p}}^{reg} \to \mathfrak{p}^{reg}$ is quasifinite, and there is an action of H on $\widetilde{\mathfrak{p}}^{reg}$ such that the GIT quotient*

$$\widetilde{\mathfrak{a}/\!\!/W_{\mathfrak{a}}} := \widetilde{\mathfrak{p}}^{reg}/\!\!/H$$

fits into a Cartesian diagram

$$\begin{tikzcd} \widetilde{\mathfrak{p}}^{reg} \arrow[r] \arrow[d] & \widetilde{\mathfrak{a}/\!\!/W_{\mathfrak{a}}} \arrow[d] \\ \mathfrak{p}^{reg} \arrow[r] & \mathfrak{a}/\!\!/W_{\mathfrak{a}} \end{tikzcd}$$

for a finite map $\widetilde{\mathfrak{a}/\!\!/W_{\mathfrak{a}}} \to \mathfrak{a}/\!\!/W_{\mathfrak{a}}$

(Todo: Describe gluing pattern explicitly to form $\widetilde{\mathfrak{a}/\!\!/W_{\mathfrak{a}}}$)

4. Sections from $\mathfrak{sl}_2$ Triples

In this appendix, we review the construction of [17] and note an extension of those results to positive characteristic when p is greater than the Coxeter number of G based on the results of [27], [19], [36], and [20]. In particular, we review the theory of normal $\mathfrak{sl}_2$ triples, and derive the Kostant-Rallis section from the construction of the Kostant section. We compare this with the results of [19], reviewed in Section 2.5.

Definition 4.1. We say an $\mathfrak{sl}_2$ triple (e, h, f) is *normal* if $e, f \in \mathfrak{p}$ and $h \in \mathfrak{k}$. We say that an $\mathfrak{sl}_2$-triple is *principal* if e is regular as an element of $\mathfrak{p}$.

Remark 4.2. Note that a principal, normal $\mathfrak{sl}_2$ triple in the sense of Definition 4.1 is a principal $\mathfrak{sl}_2$ triple of $\mathfrak{g}$ in the usual sense only in the case of a quasi-split involution.

In the characteristic p case, we will need to pass to associated characters.

Definition 4.3. Fix a nilpotent $e \in \mathcal{N}$. An *associated character* of e is a character $\lambda\colon \mathbb{G}_m \to G$ such that $e \in \mathfrak{g}(2; \lambda)$ (where $\mathfrak{g}(k; \lambda)$ is the k-th graded piece of $\mathfrak{g}$ under the grading induced by λ) and there is a Levi subgroup $L \subset G$ such that $\lambda(\mathbb{G}_m) \subset L^{\mathrm{der}}$ and e is distinguished in $\mathrm{Lie}(L)$, i.e. $Z_{L^{\mathrm{der}}}(e)^\circ$ is unipotent.

Lemma 4.4. *([27], Prop. 4) Given an associated character λ to a nilpotent e, one can extend e to a unique $\mathfrak{sl}_2$ triple (e, h, f) with $h \in \mathrm{Lie}(image(\lambda))$ and $f \in \mathfrak{g}(-2; \lambda)$.*

We recall the following facts about associated characters and $\mathfrak{sl}_2$ of G up to conjugation.

Lemma 4.5. *([20], Prop 18 and [36], Theorem 1.1) Consider the projection*

$$\{(e, \lambda)\colon e \in \mathcal{N} \text{ and } \lambda \text{ is an associated character for } e\}/G \to \mathcal{N}/G$$

where G acts by conjugation on each set.

(1) This map is a bijection in good characteristic.
(2) The bijection above factors through

$$\{(e, \lambda)\colon e \in \mathcal{N} \text{ and } \lambda \text{ is an associated character for } e\}/G \to \{\mathfrak{sl}_2\text{-triples}\}/G \to \mathcal{N}/G$$

where the first map comes from Lemma 4.4. The map from G-orbits of $\mathfrak{sl}_2$-triples to $\mathcal{N}/G$ is a bijection if and only if the characteristic of the field is greater than the Coxeter number.

Proof. Part (1) follows from Prop. 18, part 2, of [20]. Part 2 follows from Thoerem 1.1 of [36]. □

We will demand, in addition, that associated characters be compatible with the involution on G in the following sense.

Definition 4.6. We say that a character λ is a *normal* associated character with respect to a nilpotent $e \in \mathcal{N}_{\mathfrak{p}}$ of $\mathfrak{p}$ if it is an associated character or e and I*mage*$(\lambda) \subset K$.

Lemma 4.7. *(*[19]*, Cor. 5.4) For any $e \in \mathcal{N}_{\mathfrak{p}}$, there exists a normal associated character λ for e. Moreover, such a character is unique up to conjugation by the connected component of the centralizer $Z_K(e)^\circ$.*

We now deduce the results on $\mathfrak{sl}_2$ triples relevant to our paper.

Lemma 4.8. *The map*

$$\{H\text{-orbits of normal } \mathfrak{sl}_2 \text{ triples}\} \to \{H\text{-orbits of nilpotents in } \mathfrak{p}\}$$

is surjective, i.e. for any $e \in \mathcal{N}_{\mathfrak{p}}$, there exists a normal $\mathfrak{sl}_2$ triple (e, h, f) extending e. Assuming that the characteristic of the field is greater than the Coxeter number, this map is a bijection.

Proof. It suffices to prove this Lemma for $H = K$. In characteristic zero, this follows from [17], Proposition 4.

In characteristic $p > 0$, surjectivity follows from Lemma 4.4 and Lemma 4.7. Now assume the characteristic is greater than the Coxeter number. Then, we have a sequence of maps

$$\begin{tikzcd} \left\{\begin{smallmatrix} K\text{-orbits of pairs } (e,\lambda) \text{ for} \\ \lambda \text{ associated to } e, \text{ valued in } K \end{smallmatrix}\right\} \arrow[r, "\phi"] \arrow[rr, bend right, "\simeq"'] & \{K\text{-orbits of normal } \mathfrak{sl}_2\text{-triples}\} \arrow[r, two heads] & \{K\text{-orbits of nilpotents in } \mathfrak{p}\} \end{tikzcd}$$

Since the composite map is an isomorphism, the map ϕ is injective. We claim that it is also surjective. Suppose that a normal $\mathfrak{sl}_2$ triple (e, h, f) is not in the image of ϕ. Then, by Lemma 4.7, there is a character λ valued in K associated to e. Moreover, by Lemma 4.5, any two associated characters of e are conjugate by an element of $Z_G(e)^\circ$, and there is a unique character λ' associated to e for which (e, h, f) is the corresponding $\mathfrak{sl}_2$ triple. Let $g \in Z_G(e)^\circ$ conjugate λ' and λ, so that g also conjugates (e, h, f) to a normal $\mathfrak{sl}_2$ triple (e, h', f'). Since this g preserves normality of the $\mathfrak{sl}_2$ triple, L*ie*(*image* $g \cdot \lambda) \subset \mathfrak{k}$. Since $g \cdot \lambda$ is a one-parameter subgroup, it is connected and hence has image in K. We conclude that $g \cdot \lambda$ is an associated character to e valued in K whose associated $\mathfrak{sl}_2$ triple is (e, h, f). □

Now let $e \in \mathcal{N}_{\mathfrak{p}}^{reg}$ be a regular nilpotent. From a principal, normal $\mathfrak{sl}_2$ triple (e, h, f), one produces a Kostant-Rallis section by considering the slice $e + \mathfrak{c}_{\mathfrak{p}}(f)$.

Theorem 4.9. *The map $e + \mathfrak{c}_{\mathfrak{p}}(f) \to \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ is an isomorphism. We will call its inverse a* Kostant-Rallis section *associated to e.*

Moreover, for a given regular nilpotent e in $\mathfrak{p}$, this section is unique up to conjugation by $Z_K(e)^\circ$. In particular, this gives a bijection

$$\{K\text{-orbits of Kostant-Rallis sections}\} \to \{K\text{-orbits of regular nilpotents in } \mathfrak{p}\}.$$

Proof. In characteristic zero, this is the content of [17], Theorem 11.

In characteristic $p > 0$, by [19], Lemma 6.29, it suffices to check that $e + \mathfrak{c}_{\mathfrak{p}}(f)$ is an A$d(\lambda)$-graded complement of $[\mathfrak{k}, e]$, where $\lambda \colon k^\times \to K$ is an associated character to e. Certainly the slice is A$d(\lambda)$ graded as e and f are homogeneous with respect to the grading. To show that the slice gives a complement, it suffices to show

$$\mathfrak{p} = \big(e + \mathfrak{c}_{\mathfrak{p}}(f)\big) \oplus [\mathfrak{k}, e]$$

By the proof of Lemma 3.1.3 of [29], we have that

$$\mathfrak{g} = \big(e + \mathfrak{c}_{\mathfrak{g}}(f)\big) \oplus [e, \mathfrak{g}].$$

Intersecting this with $\mathfrak{p}$ and using the fact that $e \in \mathfrak{p}$ gives this result. ☐

Corollary 4.10. *Let $\mathfrak{s} = e + \mathfrak{c}_{\mathfrak{p}}(f)$ be the Kostant-Rallis slice in $\mathfrak{p}$. Then, $\mathfrak{p} = \mathfrak{s} + [e, \mathfrak{p}]$. In particular, if*

$$a \colon H \times \mathfrak{s} \to \mathfrak{p}^{reg}$$

is the action map. Then the differential of a at $0 \in \mathfrak{s}$ is surjective.

Proof. The first claim follows from the proof of Theorem 4.9. For the second, we note that the differential is identified with the map

$$\mathfrak{h} \times \mathfrak{s} \to \mathfrak{h}, \quad (x, s) \mapsto [x, e] + s.$$

By the first claim together with the observation that $\mathfrak{h} = \mathfrak{k}$, this is surjective. ☐

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