# Cobordism Theory

# Lecture notes of a course taught by Daniel Quillen

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# Contents

# Chapter 1

# Differentiable Manifolds

Let $X$ be a topological space. A *chart* on $X$ is a triple $c = (U, L, \phi)$, where $U$ is an open subset of $X$, $L$ is an $n$- dimensional vector space over $\mathbb{R}$ and $\phi : U \longrightarrow L$ is a continuous map such that $\phi :\longrightarrow \phi(U)$ is a homeomorphism and $\phi(U) \subset L$ is open in the natural topology of $L$.

Two charts, $c = (U, L, \phi), c' = (U', L', \phi')$ on $X$ are $\mathcal{C}^\infty$-compatible if $\phi' \circ \phi^{-1} : \phi'(U \cap U') \longrightarrow \phi(U \cap U')$ are $\mathcal{C}^\infty$-maps. A $\mathcal{C}^\infty$-atlas on $X$ is a family of mutually $\mathcal{C}^\infty$-compatible charts $A = \{c_\alpha = (U_\alpha, L_\alpha, \phi_\alpha)\}$ such that $\cup_\alpha U_\alpha = X$. A $\mathcal{C}^\infty$-atlas $A$ is full if given any chart $c$ compatible with all charts in $A$, $c$ is already in $A$. A $\mathcal{C}^\infty$ manifold is a Hausdorff space $X$ with a countable basis for its topology provided with a full $\mathcal{C}^\infty$-atlas. Observe that any $\mathcal{C}^\infty$-atlas extends uniquely to a full $\mathcal{C}^\infty$-atlas.

**Example:** The Grassmanian $Gr_q(V)$ of an $n$-dimensional real vector space $V$. By definition

$$Gr_q(V) = \{A \subset V : A \text{ is a } q\text{-dimensional subspace of } V\}.$$

$Gr_q(V)$ has a natural topology as a homogeneous space. It also inherits a $\mathcal{C}^\infty$-structure that way. We describe this $\mathcal{C}^\infty$-structure. If $B$ is an $(n-q)$-dimensional subspace of $V$, let $U_B = \{A \in Gr_q(V) : A \cap B = \{0\}\}$. If $A \in U_B$ is fixed, every $A' \in U_B$ is the graph of a linear map $u : A \longrightarrow B$, this defines a map $\phi_{A,B} : U_B \simeq \mathrm{Hom}_{\mathbb{R}}(A, B)$. A $\mathcal{C}^\infty$-atlas for $Gr_q(V)$ that gives its $\mathcal{C}^\infty$-structure is

$$\{(U_B, \mathrm{Hom}_{\mathbb{R}}(A, B)\phi_{A,B}) : A \subset U_B, B \in Gr_{n-q}(V)\}.$$

Let $X$ be a manifold and $c = (U, L, \phi)$, $c' = (U', L', \phi')$ be 2 charts centered at $x \in X$, (i.e. $\phi(x) = \phi'(x) = 0$). Let $\theta_{cc'} : L' \longrightarrow L$ be the Jacobian of $\phi \circ \phi' : \phi'(U \cap U') \longrightarrow \phi(U \cap U')$ at the origin. The we have de identities:

(i) $\theta_{cc'} \circ \theta_{c'c''} = \theta_{cc''}$;

(ii) $\theta_{c'c} = \theta_{cc'}^{-1}$;

(iii) $\theta_{cc} = id$.

Associate to each chart $c = (U, L, \phi)$ centered at $x$ the space $L$ and to each pair $c, c'$ of such charts the isomorphism $\theta_{cc'} : L' \longrightarrow L$. These data define a vector space at $x$ uniquely up to a canonical isomorfism which is called the *tangent space of* $X$ *at* $x$, $T_x(X)$.

**Example:** The Grassmanian again. One shows that canonically $T_A(Gr_q(V)) \simeq \mathrm{Hom}_{\mathbb{R}}(A, V/A)$.

Let $X, Y$ be $\mathcal{C}^\infty$-manifolds. Let $f : X \longrightarrow Y$ be a map. We call $f$ a $\mathcal{C}^\infty$-map if:

A $f$ is continuous

B $f$ is locally $\mathcal{C}^\infty$, i.e. if looked trough convenient charts it is $\mathcal{C}^\infty$.

Clear identities and composition of $\mathcal{C}^\infty$-maps are $\mathcal{C}^\infty$-maps, so we get the category $\mathcal{M}$ of $\mathcal{C}^\infty$ manifolds and maps. Further, if $f : X \longrightarrow Y$ is a $\mathcal{C}^\infty$-map and $x \in X$, one can define the differential

$$(df)_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

as a linear map of real vector spaces.

We want now to study properties of the category $\mathcal{M}$ as a category.

$\mathcal{M}$ is closed under finite products.

$\mathcal{M}$ is not however closed under arbitrary fibre products.

**Example:** In general if $f : X \longrightarrow Z$, $g : Y \longrightarrow Z$, are injective then $X \times_z Y = X \cap Y$ in $Z$ if the category has fibre products.

Now take $Z = \mathbb{R}^2$, $X = x$-axis, $Y =$ graph of a smooth function such that $f^{-1}(0)$ is compact and not locally euclidean. Then, as a set $X \times_Z Y$ is homoeomorphic to $f^{-1}(0)$ and is not, then a manifold.

Two maps $f : X \longrightarrow Z$, $g : Y \longrightarrow Z$ in $\mathcal{M}$ are *transversal* if for all $x \in X$, $y \in Y$ such that $f(x) = g(x) = z$, the map

$$T_x(X) \oplus T_y(Y) \stackrel{df_x \oplus dg_y}{\longrightarrow} T_z(Z)$$

is surjective.

**Examples:**

1. If $Z$ is a vector space and $f, g$ are inclusions of subspaces $X, Y$ transversality of $f, g$ means that $X + Y = Z$, i.e.,

   a. $\dim X + \dim Y \geq \dim Z$,

   b. $X \cap Y$ has minimal dimension ($= \dim X + \dim Y - \dim Z$).

2. Let $Y = e$, a point, $g(e) = z_0 \in Z$. Then $f$ is transversal to $g$, if and only if, for all $x \in f^{-1}(z_0)$, $df_x : T_x(X) \longrightarrow T_{z_0}(Z)$ is surjective. Such a point $z_0$ is called a *regular value* of f.

3. $f : X \longrightarrow Z$ is transversal to all maps $g$ with target $Z$ if and only if, for all $x \in X$, $df_x : T_x(X) \longrightarrow T_{f(x)}(Z)$ is surjective.

The following theorem is a consequence of the implicit function theorem.

**Theorem.** *If $f : X \longrightarrow Z$, $g : Y \longrightarrow Z$ are transversal then the fibre product $X \times_Z Y$ exists and $T_{(x,y)}(X \times_z Y) = T_x(X) \times_{T_z(Z)} T_y(Y)$ where $f(x) = z = g(y)$.*

A subset $A$ of a manifold $X$ has measure 0 if for any chart $c = (U, L, \phi)$, $\phi(A \cap U)$ has measure 0 in $L$ (Lebesgue measure).

**Theorem.** *Sard-Brown If $f : X \longrightarrow Y$ is a map in $\mathcal{M}$, then the set of regular values of $f$ is the complement of a set of measure 0.*

In fact one can show that the set of singular values of $f$ is a contable union of nowhere dense sets and has measure 0.

**Corollary.** *If $\dim_x X < \dim_y Y$ (observe a manifold need not have constant dimension) for all $x \in X$, $y \in Y$, then $f(X)$ has measure 0 in $Y$*

# Chapter 2

# Vector Bundles

A $\mathcal{C}^\infty$ vector bundle over a manifold $X$ is a map $\pi : E \longrightarrow X$ in $\mathcal{M}$ such that each fibre $E_x = \pi^{-1}(x)$, $x \in X$, has real vector space structure, and such that for every $x \in X$ there is an open neighborhood $U$ and an isomorphism (in $\mathcal{M}$)

$$\pi^{-1}(U) \xrightarrow{\simeq} U \times L$$

where $L$ is a finite dimensional vector space, compatible with the vector space structure on fibres. Let $\operatorname{rank}_x E = \dim E_x$ for $x \in E$.

A section $s$ of $E$ is a smooth map $s : X \longrightarrow E$ such that $\pi \circ s = id_x$. We denote by $\Gamma(X, E)$ the space of sections of $E$.

**Theorem.** *If $m \geq \dim_x X + \operatorname{rank}_x E$ for all $x \in X$, then there are $s_1, \ldots, s_m \in \Gamma(X, E)$ such that for every $x \in X$, $s_1(x), \ldots, s_m(x)$ genetate $E_x$.*

*Proof.* First we prove this for $X$ compact.

**Step 1:** *To produce a finite dimensional subspace $V \subset \Gamma(X, E)$ such that for all $x \in X$ the evaluation map $ev_x : V \longrightarrow E_x$ is surjective:*

We assume $\operatorname{rank}_x E$ is constantly equal to $r$. For each $x \in X$ choose a neighborhood $N_x$ of $x$ over which $E$ is trivial, i.e. $\pi^{-1}(N_x) \simeq N_x \times \mathbb{R}^r$; then, a section $s$ of $E$ over $N_x$ can be identified with a $\mathcal{C}^\infty$ function $N_x \longrightarrow \mathbb{R}^r$. By using $\mathcal{C}^\infty$ which are not 0 at $x$ and vanish outside a compact neighborhood of $x$ in $N_x$, we obtain $s_1^x, \ldots, s_r^x \in \Gamma(X, E)$ such that $\{s_i^x\}$ form a basis for $E_y$ when $y$ is in some neighborhood $V_x$ of $X$. By compactness of $X$, $X = \cup_{j=1}^N V_{x_j}$.

Let $V \subset \Gamma(X, E)$ to be the space spanned by $\{s_i^{x_j}\}_{i,j}$.

**Step 2:** *To show that for almost all $W \in Gr_m(V)$ the evaluation map $ev_x : W \longrightarrow E_x$ is surjective for all $x \in X$.*

Note that $W$ is *bad* if and only if there is $x \in X$ and a hyperplane $H \subset E_x$ such that $ev_x(W) \subset H$, i.e. $W \subseteq ev_x^{-1}(H)$. Let

$$Z = \{(x, H, W) : x \in X, H \text{ is a hyperplane in } E_x, W \in Gr_m(ev_x^{-1}(H))\}.$$

In general, if $\pi : E \longrightarrow X$ is a vector bundle, we can define the manifold $Gr_k(E) = \{(x, H) : x \in X, H \text{ is a } k\text{-plane in } E_x\}$ with a natural projection on $X$ and a natural locally trivial structure. Now $\dim Gr_k(E) = \dim X + k(\dim L - k)$ (where $L$ is the typical fibre of $E$). Also over $Gr_k(E)$ we have the vector bundle $F_k$ whose fibre at $(x, H)$ is the space $ev_x^{-1}(H)$. We see then that $Z = Gr_m(F_{r-1})$. Now $\dim Z = \dim Gr_{r-1}(E) + m(\dim V - 1 - m) = \dim X + (r - 1) + m(\dim V - 1 - m)$.

The set of bad $W$ in $Gr_m(V)$ is the image of the map $f : Z \longrightarrow Gr_m(V)$, $(x, H, W) \longrightarrow W$. By Sard's then it suffices to show that $\dim Z < \dim Gr_m(V) = m(\dim V - m)$. But: $\dim X + (r - 1) + m(\dim V - 1 - m) - m(\dim V - m) = \dim X + (r - 1) - m < 0$ since by hypothesis $\dim X + r \leq m$.

For $X$ non compact, recall that $X = \cup_{n=1}^{\infty} K_n$, $K_n$ compact and $K_n \subset$ interior$(K_{n+1})$. The theorem follows from the following lemma.

**Lemma.** *Let $s_1, \ldots, s_m \in \Gamma(X, E)$ span $E_x$ for all $x \in F$ where $F$ is closed in $X$ ($m \geq \dim_x X + \operatorname{rank}_x E$ for all $x \in X$); then $s_1, \ldots, s_m$ can be modified outside $F$ so as to span $E_x$ for all $x \in X$.*

*Proof.* To prove the lemma, its sufficient to show that for any compact $K \subset X$, we can modify $s_1, \ldots, s_m$ outside $F$ so as to spam $E_x$ for $x \in F \cup K$ (the process will converge since $K_n \subset \text{Int } K_{n+1}$).

**Step 1:** Choose a finite dimensional vector space $V' \subset \Gamma(X, E)$ such that all $s \in V'$ vanish on $F$ and $s_1, \ldots, s_m$ and $s \in V'$ span $E_x$ for all $x \in F \cup K$. This can be easily achieved. Let $V = \mathbb{R}^n \oplus V'$ and define $\omega_x : V \longrightarrow E_x$ by $(\lambda_1, \ldots, \lambda_m, s) \mapsto \sum_{i=1}^{m} \lambda_i s_i(x) + s(x)$.

**Step 2:** Almost all $W \in Gr_m(V)$ are such that

(i) for all $x \in F \cup K, \omega_x : W \longrightarrow E_x$ is surjective,

(ii) $W \cap V' = 0$.

This is seen in a similar way to that of step 2 of theorem.

Now choose such a $W$ and let $w_i$ be the unique element in $W$ such that $w_i + V' = (0, \ldots, 1, 0, \ldots, 0) + V'$. Let $s'_i$ be the image of $w_i$ in $\Gamma(X, E)$. Clearly $s'_1, \ldots, s'_m$ span $E_x$ for $x \in F \cap K$ and $s'_i \Big| F = s_i \Big| F$. □

□

In a similar way one can prove:

**Theorem** (Stability Theorem). *Let $\pi : E \longrightarrow X$ be a vector bundle with* $\dim E > \dim X$. *Then $\exists s \in \Gamma(X, E)$ such that $s(x) \neq 0$ for all $x \in X$.*

**Theorem** (Bertini's Theorem). *Let $X$ be a submanifold of $\mathbb{P}_n$ then almost all hyperplanes in $\mathbb{P}_n$ intersect $X$ transversally.*

# Chapter 3

# Imbeddings

A map $f : X \longrightarrow Y$ in $\mathcal{M}$ is called an *immersion* if $df_x : T_xX \longrightarrow T_{f(x)}Y$ is injective for all $x \in X$; it is called a *submersion* if $df_x : T_xX \longrightarrow T_{f(x)}Y$ is surjective for all $x \in X$; it is called an *imbedding* if it is a immersion and a homeomorfism of $X$ with the subspace $f(X) \subset Y$. Also $f$ is said ti be *étale* if it is both an inmmersion and a submersion. We say that $f : X \longrightarrow Y$ is *proper* if for any compact $K \subset Y$, $f^{-1}K$ is compact or equivalently: if $x_n$ is a sequence in $X$ such that $f(x_n) \to y \in Y$, then $\exists$ a convergent subsequence $x_{n_k} \to x \in X$.

*Remarks:* by the implicit function theorem, we have

1. $f$ is an immersion at $x$ if it is locally equivalent to the linear immersion $\mathbb{R}^m \subset \mathbb{R}^n$.

2. $f$ is a submersion at $x$ if it is locally equivalent to the linear projection $\mathbb{R}^n \longrightarrow \mathbb{R}^m$.

3. $f$ is étale at $x$ if it is locally equivalent to the identity isomorphism $\mathbb{R}^n \simeq \mathbb{R}^n$.

An imbedding is *open* if $f : X \hookrightarrow Y$ as an open subset. An imbedding $f : X \longrightarrow Y$ is *closed* if $f(X)$ is closed in $Y$.

Easily: any imbedding $f : X \longrightarrow Y$ factors into a closed imbedding $f_1$, followed by an open imbedding $i$

$$X \xrightarrow{f_1} U \xrightarrow{i} Y$$

(A locally compact subspace of a locally compact space is locally closed i.e, it is the intersection of an open and a closed set).

**Proposition.** *An imbedding is proper if and only if it is closed. A submersion is proper if and only if it is locally equivalent to $pr_1 : Y \times Z \longrightarrow Y$ with $Z$ a compact manifold.*

**Theorem.** *Let $f : X \longrightarrow Y$ be a proper map in $\mathcal{M}$ such that $2\dim_x X + 1 \leq \dim_{f(x)} Y$. Let $F \subset X$ closed and suppose that $f|_F$ is a 1-1 immersion. Then there is an imbedding $f' : X \longrightarrow Y$ such that $g|_F = f|_F$.*

*Proof.* We assume first that $X$ is compact. We express $X = \cup_{i=1}^{n} X_i$, where $\overline{U_i}$ is compact and $f(\overline{U_i}) \subset$ some chart in $Y$. By induction, it is enough to assume that $f$ is an imbedding on $F \cup (\cup_{i<m} \overline{U}_i)$ and to show that $f$ can be modified on $(\overline{U}_m)$ without changing it on $F \cup (\cup_{i<m} \overline{U}_i)$ so that it becomes an imbedding on $F \cup (\cup_{i \leq m} \overline{U}_i)$.

"Hence" we assume $Y = \mathbb{R}^b$.

**Step 1:** First we show that there is a function $g : X \longrightarrow V$ ($V$ a finite dimensional vector space) such that $g(F) = 0$ and

$$(f, g) : X \longrightarrow Y \times V$$

is an imbedding:

(1) Cover $X$ by coordinate open sets $U_i$ and use the coordinates functions to get a finite number of functions $u_\alpha : X \longrightarrow \mathbb{R}$ such that $u_\alpha(F) = 0$ and $du_\alpha$ span $T_x X$ for all $x$ not in the open set where $f$ is an immersion.

(2) This gives an immersion $(f, g) : X \longrightarrow Y \times V$ with $g(F) = 0$.

(3) Now consider the set

$$Z = \{(x, x') \in X \times X | (f, g)(x) = (f, g)(x') \text{ and } x \neq x'\}.$$

This $Z$ is closed in $X \times X$ and does not meet $F \times F$.

(4) So given $(x, x') \in Z$, there is a function $u : X \longrightarrow \mathbb{R}$ such that $u(x) \neq u(x')$ and $u(F) = 0$.

(5) For each $(x, x') \in Z$ pick such a $u$, call it $u_{(x,x')}$ and cover $Z$ by finitely many of the open sets

$$Z_{(x,x')} = \{(x_1, x_1') \in Z | u_{(x,x')}(x_1) \neq u_{(x,x')}(x_1')\}.$$

(6) Take the resulting family of $u_{(x,x')}$. Thus we obtain a finite family of functions $\{u_\alpha\}$ such that the map $X \longrightarrow Y \times V$,

$$x \mapsto (f(x), u_\alpha(x))$$

is an imbedding.

**Step 2:** Here we show that for almost all quotient spaces $Q$ of $Y \times X$ with $\dim Q = \dim Y$ we have the 2 properties:

(i) $X \longrightarrow Y \times V \longrightarrow Q$ is an imbedding.

(ii) $Y \longrightarrow Y \times V \longrightarrow Q$ is an isomorphism.

We show that the set

$$\{A \in Gr(Y \otimes V) |\dim A = \dim V, A \cap Y = 0 \text{ and}$$

$$X \longrightarrow Y \otimes V \longrightarrow (Y \otimes V)/A \text{ is an imbedding}\},$$

is everywhere dense. In particular there is one such $A$, so if $f' : X \longrightarrow Y$ is the map given by $f'(x) - f(x) - g(x) \in A$, the $f'$ is an imbedding and $f'|_F = f|_F$.

$$\begin{array}{ccc} X & \xrightarrow{(f,g)} & Y \otimes V \\ {\scriptstyle f'}\downarrow & & \downarrow \\ Y & \xrightarrow{\simeq} & (Y \otimes V)/A \end{array}$$

(1) First we determine the size of $A \in Gr(Y \otimes V)$ for which $X \longrightarrow Y \otimes V \longrightarrow (Y \otimes V)/A$ is not an immersion, i.e. if $h = (f, g)$ then there is $x$ and a line $l \subset T_xX$ such that $dh_x(l) \subset A$. Let

$$Z = \{(x, l, A)|x \in X, l \in \mathbb{P}T_x(X), dh_x(l) \subset A\},$$

$W = Y \otimes V$ and $\dim V = r$.

(2) Notice that $dh$ can be thought of as a map $TX \longrightarrow h^*TW = X \times_W TW$, and it is 1-1 since $h$ is an immersion.

(3) Also, $Z$ can be thought of as fibered over $\mathbb{P}TX$

$$F \longrightarrow Z \longrightarrow \mathbb{P}TX$$

$$\{A|dh(l) \subset A\} \longrightarrow \{x, l, A\} \longrightarrow \{x, l\}$$

so $Z$ is a Grassmannian bundle for a vector bundle over $\mathbb{P}TX$.

(4) Hence $Z$ is a manifold. Now $\dim Z = 2 \dim X - 1 + (r - 1)(\dim W - r)$.

(5) As far as immersion goes, bad $A$ is the same as being in the image of $Z \longrightarrow Gr(W)$, $(x, l, A) \mapsto A$.

(6) In calculating $\dim Z$ we use the fact that a subspace $dh(l) \subset A \subset W$ is the same as an $(r-1)$-subspace in $W/dh(l)$.

(7) Now $\dim Z < \dim Gr_r(W)$ for $2\dim X \leq \dim Y$. Hence, $\{\text{bad } A\}$ has measure 0.

**To get 1-1:** $A$ is bad if there are $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $h(x_1) - h(x_2) \in A$. Let

$$Z = \{(x_1, x_2, A) | h(x_1) - h(x_2) \in A \text{ and } x_1 \neq x_2\}.$$

Since $h$ is an imbedding, $Z$ will be a manifold, a Grassmannian bundle of the vector bundle

$$\{(x_1, x_2, v) | (x_1, x_2) \in X \times X - \Delta \text{ and } v \in W/\mathbb{R}(h(x_1) - h(x_2))\}.$$

Now $\dim Z = 2\dim X + (r-1)(\dim W - r)$. So $\dim Z < \dim Gr_r(W)$ if $2\dim X < \dim Y$. Hence $Z$ has measure 0.

The proof for the non compact case goes along similar lines. □

An intuitive way to explain why the proof works is this: consider $X$ imbedded in a large vector space $V$. We look for lines $L$ in $V$ such that projection along $l$ onto $V/L$ leaves $X$ imbedded in $V/L$. Bad lines $L$ are those such that

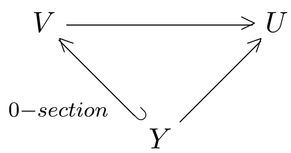
(a) $L$ passes through 2 points of $X$: there are $2\dim X$ of these.

(b) Tangent lines at $x \in X$, parallel to $L$: there are $\dim X + \dim X - 1$ (points + lines), such lines.

Hence $\dim V \geq 2\dim X + 1$ will work for the imbedding because if $\dim V > 2\dim X + 1$ we can project down.

**Corollary** (Whitney imbedding theorem)**.** *If* $\dim X \leq n$. *Then* $X$ *can be imbedded in* $\mathbb{R}^{2n+1}$ *as a closed submanifold.*

**Theorem** (Tubular neighbourhood theorem)**.** *Let* $Y$ *be a submanifold of* $X$ *and let* $V = T(X)|_Y/T(Y)$. *Then there is an open neighbourhood* $U$ *of* $Y$ *in* $X$ *and a diffeomorphism* $h : V \longrightarrow U$ *such that* $h$ *carries the* 0*-section of* $V$ *to* $Y$, *i.e. the diagram*




*commutes.*

*Proof.* Choose an imbedding of $X$ in $\mathbb{R}^n$, so $Y \subset \mathbb{R}^n$. Now $V_{_{Y\subset\mathbb{R}^n}}$ can be identified with $\{(y, v)|y \in Y, v \in \mathbb{R}^n,$ and $v \perp T_y(Y)\}$. So we can map $h : V_{_{Y\subset R^n}} \longrightarrow \mathbb{R}^n$ by $h(y, v) = y + v$. Now $h$ is smooth and étale along the 0-section. Hence there are open neighbourhoods $U \subset V_{_{Y\subset\mathbb{R}^n}}$ of the 0-section and $U' \subset \mathbb{R}^n$ of $Y$ such that $h : U' \longrightarrow U'$ is a diffeomorphism.

However we can always choose a function $\rho : Y \longrightarrow R^+$ such that $\{(y, v) : ||v|| < \rho(y)\} \subset U$ and always there si a diffeomorphism of $\{(y, v) : ||v|| < \rho(y)\}$ with $V_{_{Y\subset\mathbb{R}^n}}$ by a change of scale.

Now, within the tubular neighbourhood of $Y$ in $\mathbb{R}^n$ map $V_{Y\subset X,Y} \longrightarrow X$ by letting $v \in V_{Y\subset X,Y}$ go to the unique $x$ that goes into $v$ by orthogonal projection (one has to see this). One may have to shrink the neighbourhood U, may be. □

# Chapter 4

# Transversality

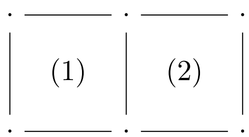
In a category, we say that a commutative square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

is *cartesian* (or a *pull back diagram*) if $X' = Y' \times_Y X$, i.e. if for al $T$

$$\mathrm{Hom}(T, X') \simeq \mathrm{Hom}(T, Y') \times_{\mathrm{Hom}(T,Y)} \mathrm{Hom}(T, X).$$

**Tautology.** *Suppose the diagram*




*is commutative. Then*

i. *(1) and (2) cartesian* $\Longrightarrow$ *the rectangle is cartesian.*

ii. *(2) and the rectangle cartesian* $\Longrightarrow$ *(2) cartesian.*

**Remarks:**

1. If $f : X \longrightarrow Y$ and $g : Y' \longrightarrow Y$ are transversal, then the fibre product $Y' \times_Y X$ exists and for all $z = (y', x) \in Y' \times_Y X$, $T_z(Y' \times_Y X) = T_{y'}(Y') \times_{T_yY} T_xX$.

2. Given the commutative square (1) of vector spaces:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow f & & \downarrow & & \\ 0 & \longrightarrow & K_2 & \longrightarrow & W' & \underset{g}{\longrightarrow} & W & \longrightarrow & C_2 & \longrightarrow & 0 \end{array}$$

we have the induced maps of kernels $K_1 \longrightarrow K_2$ and of cokernels $C_1 \longrightarrow C_2$.

i. Now (1) is cartesian if and only if $K_1 \xrightarrow{\simeq} K_2$.

ii. Also (1) is *bicartesian* if and only if $K_1 \xrightarrow{\simeq} K_2$. and $C_1 \xrightarrow{\simeq} C_2$.

3. In the Remark 1 the square

$$\begin{array}{ccc} T_z(Y' \times_Y X) & \longrightarrow & T_x X \\ \downarrow & & \downarrow df \\ T_{y'}Y' & \xrightarrow{dg} & T_y Y \end{array}$$

is bicartesian because $df + df : T_{y'}(Y') \oplus T_x(X) \longrightarrow T_y(Y)$ is onto, and because in the Remark 2 the square (1) is bicartesian if and only if (1) is cartesian and $g + f : W' \oplus V \longrightarrow W$ is surjectie.

**Definition.** A cartesian square in $\mathcal{M}$ is clean if the square of tangent spaces

$$\begin{array}{ccc} T_x X' & \longrightarrow & T_x X \\ \downarrow & & \downarrow \\ T_{y'}Y' & \longrightarrow & T_y Y \end{array}$$

is cartesian for every $x' \in X'$.

The usual for cartesian squares also work for clean cartesian squares.

**Lemma** (Transversality Lemma)**.** *Given maps in* $\mathcal{M}$

$$\begin{array}{ccc} & & Z \\ & & \downarrow g \\ Y \times S & \xrightarrow{h} & X \end{array}$$

*such that $h$ is transversal to $g$, then for almost all $s \in S$ the map $h(\cdot, s) = h_s : Y \longrightarrow X$ is transversal to $g$.*

*Proof.* Consider de diagram:

$$\begin{array}{ccccc} U_s & \longrightarrow & W & \longrightarrow & Z \\ \downarrow & (2) & \downarrow & (3) & \downarrow g \\ Y & \xrightarrow{y \mapsto (y,s)} & Y \times S & \xrightarrow{h} & X \\ \downarrow & (1) & \downarrow & & \\ pt & \xrightarrow{s} & S & & \end{array}$$

Here $W$ is the fibre product $W = (Y \times S) \times_X Z$ which exists because $h, g$ are transversal. So (3) is transversal cartesian. By Sard's Theorem,, for almost all $s \in S$, the maps $pt \longrightarrow^s S$ and $W \longrightarrow Y \times S \longrightarrow S$ are transversal, so $U_s$ exists for almost all $s \in S$ and the map $U_1 \longrightarrow pt$ factors $U_s \longrightarrow Y \longrightarrow pt$. Thus the rectangle (1)(2) is transversal cartesian and obviously (1) is transversal cartesian, then (2) is.

Hence the rectangle (2)(3) is transversal cartesian, so $h_s$ and $g$ are transversal for almost all $s \in S$.

□

**Theorem** (Transversality Theorem)**.** *Given maps in* $\mathcal{M}$

$$\begin{array}{ccc} & & Z \\ & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

*an a closed subset* $F \subset Y$ *such that* $f, g$ *are transversal over* $F$*, the there is* $f'$ *such that* $f \simeq f' \operatorname{rel} F$ *and* $f'$ *is transversal to* $g$*.*

*Proof.* (1) Factor $f$ as $Y \xrightarrow{\Gamma_f} Y \times X \xrightarrow{pr_2} X$ and consider de diagram

$$\begin{array}{ccccc} & & Y \times Z & \longrightarrow & Z \\ & & {\scriptstyle 1\times g}\downarrow & & \downarrow g \\ Y & \xrightarrow[\Gamma_f]{} & Y \times X & \xrightarrow[pr_2]{} & X \end{array}$$

(2) Since the square is transversal cartesian, $f$ is transversal to $g$ if and only if $\Gamma_f$ is transversal to $1 \times g$. Hence we can assume $f$ to be an imbedding (this argument also shows that given a factorization $Y \xrightarrow{i} U \xrightarrow{p} X$ of $f$, it suffices to $i$ transversally to $U \times_X Z \longrightarrow U$).

(3) Let $U$ be a tubular neighbourhood of the imbedding $f : Y \longrightarrow X$. By the tubular neighbourhood theorem, we may assume that $f$ is the 0-section of a vector bundle $X$ over $Y$.

(4) Choose a finite dimensional space $S \subset \Gamma(X \to Y)$ (sections of the "bundle" $X \to Y$) which spans the fibre everywhere.

(5) Also choose a $\mathcal{C}^\infty$ function $\rho : Y \longrightarrow \mathbb{R}$ such that $\rho^{-1}(0) = F$.

(6) Take $h : Y \times S \longrightarrow X$ given by $h(y, s) = \rho(y)s(y)$. Then $h$ is transversal to $g$: because, true on $F$ since $h(F \times S) = 0$; if $y \notin F$, then varying $s \in S$, $s(y)$ fills out the fibre of $X$ over $y$ since $S$ spans all fibres, so $h$ is a submersion at $y$.

(7) Apply the transversality lemma and take $f' = h_s$ for a given $s$. The required homotopy is $G : Y \times I \longrightarrow X$, $(y, t) \mapsto th_s(y)$; $G_0 = f$, $G_1 = h_s$ and $G|_{F \times I} = 0$.

□

# Chapter 5

# Cobordism

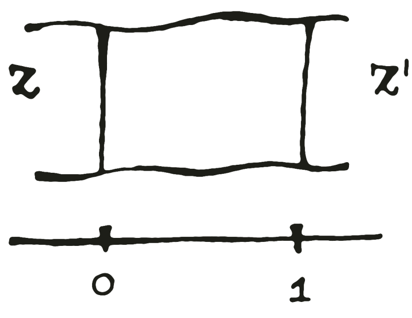
Hereafter we assume that all manifolds have bounded dimension and also that all vector bundles have bounded dimension. One consequence of this assumption is that every manifold is contained, as a closed submanifold, in some $\mathbb{R}^n$.

**Definition.** Two proper maps $g : Z \longrightarrow X$, $g' : Z' \longrightarrow X$ are *cobordant* if there is a proper map $h : W \longrightarrow X \times \mathbb{R}$ such that the following transversal cartesian squares exist

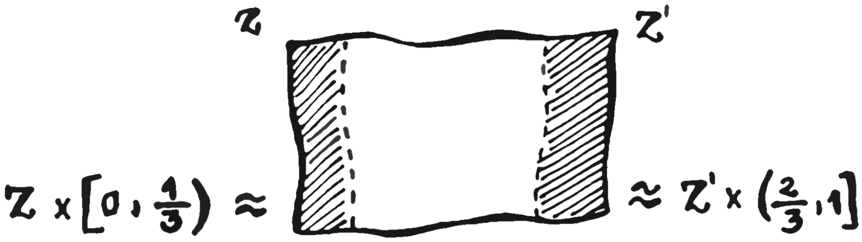
$$\begin{array}{ccccc} Z & \longrightarrow & W & \longleftarrow & Z' \\ \Big\downarrow g & & \Big\downarrow h & & \Big\downarrow g' \\ X & \xrightarrow[x \to (x,0)]{} & X \times \mathbb{R} & \xleftarrow[(x,1) \leftarrow x]{} & X \end{array}$$

**Examples:**

1. If $X = pt$. In this case $h^{-1}([0,1])$ is a manifold with boundary equal to $Z \cup Z'$.

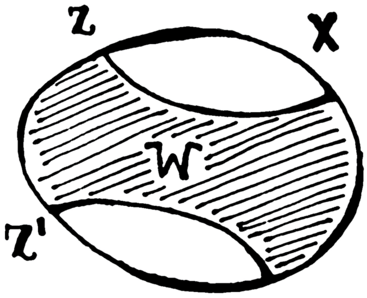



2. Let $W'$ be a manifold with boundary such that $Z \cup Z' = \partial W'$. Consider the picture:




By the Tietze extension theorem there is $u : W' \longrightarrow \mathbb{R}$ extending the map on the shaded areas, so take $X = pt$ and $u : W' \longrightarrow \mathbb{R}$ to get a cobordism between $Z \longrightarrow pt$ and $Z' \longrightarrow pt$.

3. We consider now the following case. Let $Z, Z'$ be closed submanifolds of $X$ and $W$ closed in X.




**Lemma.** *Let $h : W \longrightarrow X \times S$ be a proper map and let $s_0, s_1 \in S$ be regular values of $pr_2 \circ h : W \longrightarrow S$ that are in the same component of $S$. Let $W_{s_i} = (pr_2 \circ h)^{-1}(s_i)$ then the maps $W_{s_0} \longrightarrow X$ and $W_{s_1} \longrightarrow X$ are cobordant.*

*Proof.* Note that $W_{s_i}$ is a manifold since $s_i$ is a regular value of $pr_2 \circ h$. Consider the diagram

$$\begin{array}{ccc} W_{s_i} & \longrightarrow & W \\ \downarrow & & \downarrow h \\ X & \longrightarrow & X \times S \\ \downarrow & & \downarrow pr_2 \\ pt & \xrightarrow[\mapsto s_i]{} & S \end{array}$$

All squares are transversal cartesian. Also $W_{s_i} \longrightarrow X$ is proper. Now choose a path $u : \mathbb{R} \longrightarrow S$ such that $u(0) = s_0$ and $u(1) = s_1$. Consider the diagram:

$$\begin{array}{ccc} \mathbb{R} \times_s W & \longrightarrow & W \\ \downarrow & & \downarrow{\scriptstyle h} \\ X \times \mathbb{R} & \longrightarrow & X \times S \\ \downarrow & & \downarrow{\scriptstyle pr_2} \\ \mathbb{R} & \xrightarrow{\;u\;} & S \end{array}$$

Now $u$ is transversal to $pr_2 \circ h$ on the closed set $\{0, 1\} \subset \mathbb{R}$. By the Transversality Theorem, we can make $u$ transversal to $pr_2$ with no change on $0, 1$. Thus $\mathbb{R} \times_S W$ can be formed and the map $\mathbb{R} \times_S W \longrightarrow X \times \mathbb{R}$ is proper and gives the desired cobordism. $\square$

**Corollary.** *For a fixed manifold $X$, cobordism of proper maps $Z \longrightarrow X$ is an equivalence relation.*

*Proof.* Think of a cobordism between $Z \xrightarrow{f} X, Z' \xrightarrow{f'} X$ as a family of manifolds $Z_t$ such that $Z_0 = Z$ and $Z_1 = Z'$ where $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is some function with $\phi(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x \geq 1 \end{cases}$ $\phi'(x) \neq 0$ for $0 < x < 1$. Then $W'$ is the pull back which exists by the lemma.

If $f : Z \longrightarrow X$, $f' : Z' \longrightarrow X$ are cobordant and $f' : Z' \longrightarrow X$ and $f'' : Z'' \longrightarrow X$ are cobordant, then choose cobordism $h : W \longrightarrow X \times \mathbb{R}$ of $f$ and $f'$, and $h' : W' \longrightarrow X \times \mathbb{R}$ as above, i.e.

$$h^{-1}(X \times 0) = Z \qquad h'^{-1}(X \times 0) = Z'$$

$$h^{-1}(X \times 1) = Z' \qquad h'^{-1}(X \times 1) = Z''$$

$$h^{-1}(X \times (1 - \epsilon, 1 + \epsilon)) = Z' \times (1 - \epsilon, 1 + \epsilon)$$

$$h'(X \times (-\epsilon, \epsilon)) = Z'' \times (-\epsilon, \epsilon)$$

and piece them together in the obvious fashion. $\square$

The *cobordism ring of $X$*, $N^*(X)$ is the set of cobordism classes of proper maps with target $X$; if $g : Z \longrightarrow X$ is such a map, let $[Z \xrightarrow{g} X]$ denote its class in $N^*(X)$.

The rest of this chapter is devoted to properties of $N^*$.

First we note that $N^*$ is a covariant functor from the category of manifolds and proper maps; its range will b determined later.

If $f : X \longrightarrow Y$ is proper, the the *Gysin Homomorphism* (or *trace* or *integration along the fiber*), $f_* : N^*(X) \longrightarrow N^*(Y)$ is defined by composition i.e.

$$f_*[Z \xrightarrow{g} X] = [Z \xrightarrow{fg} Y].$$

Secondly notice that $N^*$ is a contravariant functor from $\mathcal{M}$, for given $f : X \longrightarrow Y$, define $f^* : N^*(Y) \longrightarrow N^*(X)$ by the following procedure: Let $[Z \xrightarrow{g} Y] \in N^*(Y)$. Then by the transversality theorem there is $f' : X \longrightarrow Y$, $f \simeq f'$ and such that $f'$ is transversal to $g$. We have:

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & Z \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f'} & Y \end{array}$$

where $X \times_Y Z$ is a proper map. We put $f^*[Z \xrightarrow{g} Y] = [X \times_Y Z \longrightarrow X]$.

We check that this definition is independent of the choice of $f'$: let $f''$ be another deformation of $f$, transversal to $g$. Then there is a smooth homotopy $h : X \times \mathbb{R} \longrightarrow Y$ such that $h_0 = f', h_1 = f''$. Then $h$ is transversal to $g$ on the closed subset $X \times 0 \cup X \times 1$ of $X \times \mathbb{R}$ (also $h$ can be chosen to be a proper homotopy, i.e., the map $X \times \mathbb{R} \longrightarrow Y \times \mathbb{R} : (x, t) \longrightarrow (h(x, t), t)$ is proper; one can check the proof of the transversality theorem to see that this is possible); hence, by the transversality theorem we can move $h$ to be transversal to $g$ without changing it on $X \cup X \times 1$. Now form the transversal cartesian square:

$$\begin{array}{ccc} W & \longrightarrow & Z \\ \downarrow & & \downarrow g \\ X \times \mathbb{R} & \xrightarrow{h} & Y \end{array}$$

Hence, $W \longrightarrow X \times \mathbb{R}$ is a cobordism between $X \times_{f',Y} Z \longrightarrow X$ and $X \times_{f'',Y}$. Furthermore, it is independent of the choice of $Z \xrightarrow{g} Y$, for given $W \longrightarrow Y \times \mathbb{R}$ joining $Z' \longrightarrow Y$ and $Z'' \longrightarrow Y$ we have:

$$\begin{array}{ccc} \widetilde{W} & \longrightarrow & W \\ \downarrow & & \downarrow \\ X \times \mathbb{R} & \xrightarrow{f \times 1} & Y \times \mathbb{R} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

after $f$ is moved transversally to $W \longrightarrow Y \times \mathbb{R} \longrightarrow Y$. Then $\widetilde{W} \longrightarrow X \times \mathbb{R}$ is a cobordism between $X \times_{f',Y} Z' \longrightarrow X$ and $X \times_{f',Y} Z'' \longrightarrow X$.

**Properties.**

1. Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ then $(gf)_* = g_* f_*$ if both maps are proper and $(gf)^* = f^* g^*$ in general.

2. If $f_0 \simeq f_1 : X \longrightarrow Y$ then $f_0^* = f_1^* : N^*(Y) \longrightarrow N^*(X)$.

3. If the square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{f} & Y \end{array}$$

   is transversal cartesian then $f^* g_* = g'_* f'^*$.

*Proof.* 1 and 2 are clear. For 3, we have

$$\begin{array}{ccc} Q & \longrightarrow & Z \\ \downarrow & & \downarrow{\scriptstyle k \in \alpha \in N^*(X)} \\ X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{f} & Y \end{array}$$

$f^* g_*(\alpha) = f^*[Z \longrightarrow Y] = [Q \longrightarrow Y']$ since $f$ and $gk$ are transversal. Also $g'_* f'^*(\alpha) = g'_*[Q \longrightarrow X']$ since $f'$ and $k$ are transversal $= [Q \longrightarrow Y']$. □

**Example.** We would like properties 1,2,3 to characterize $N^*$ as a functor into the category of sets, but let $H(X) = H^*(X;\mathbb{Z}_2)$ which is a contravariant homotopy functor (i.e., it satisfies the second part of 1 and 2). By the Poincaré Duality, if $\dim X = n$, $H^{n-1}(X)$ is dual to $H^i_C(X)$ by a pairing

$$H^{n-1}(X) \otimes H^i(X) \xrightarrow{\cup} H^n_C(X) \xrightarrow{\cap[X]} \mathbb{Z}_2$$

so one can define $f_* : H^i(X) \longrightarrow H^{i+r}(X)$ $(r = \dim Y - \dim X)$ as the transpose of $f^r_C : H^*_C(Y) \longrightarrow H^*_C(X)$ where $f$ is a proper map. Then 1,2,3 are satisfied. Property 3 is not trivial.

**Proposition.** *Given a contravariant functor $H$ on $\mathcal{M}$ which is also a covariant functor on $\mathcal{M}_{proper}$ such that 1,2,3 hold, then for all $a \in H(pt)$ there is a unique $\phi(\cdot) : N(\cdot) \longrightarrow H(\cdot)$ compatible with $f^*, f_*$ and such that $\phi[pt \xrightarrow{id} pt] = a$.*

*Proof.* Define $\phi[Z \xrightarrow{g} X] = g_*\pi_Z^*(a)$ where $\pi_Z : Z \longrightarrow pt$. Now $\phi$ is clearly unique since $[Z \xrightarrow{g} X] = g_*\pi_Z^*[pt \xrightarrow{id} pt]$ (this is important to remember):

$$\begin{array}{ccc} Z & \longrightarrow & pt \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\ Z & \xrightarrow{\pi_Z} & pt \\ \downarrow{\scriptstyle g} & & \\ X & & \end{array}$$

where the square is transversal cartesian.

For existence one needs only to check that $\phi$ is well defined: suppose that $h : W \longrightarrow X \times \mathbb{R}$ is cobordism of $g : Z \longrightarrow X$ and $g' : Z' \longrightarrow X$. Then the squares in

$$\begin{array}{ccccc} Z & \xrightarrow{j_0} & W & \xleftarrow{j_1} & Z' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g'} \\ X & \xrightarrow{i_0} & X \times \mathbb{R} & \xleftarrow{i_1} & X \end{array}$$

are transversal cartesian. Then

$$\begin{aligned} g_*\pi_Z^* a &= g_* j_0^* \pi_W^* a && \text{by 1} \\ &= i_0^* h_* \pi_W^* a && \text{by 2} \\ &= i_1^* h_* \pi_W^* a && \text{by 2, since } i_0 \simeq i_1 \\ &= g'_* j_1^* \pi_W^* a && \text{by 3} \\ &= g'_* \pi_{Z'}^* a && \text{by 1} \end{aligned}$$

the rest is trivial. □

**Example.** Let $H(X) = H^*(X; \mathbb{Z}_2)$. Let $a = 1 \in H^*(pt; \mathbb{Z}_2)$. Then by the proposition, there is a unique $\phi : N(X) \longrightarrow H(X)$ with $\phi[pt \xrightarrow{id} pt] = 1$ (compatible with $f^*$ and $f_*$). Moreover $\phi[Z \xrightarrow{g} X] = g_* 1_Z$ where $1_Z \in H^0(X)$ is the unit of the ring. If $Z$ is a submanifold of $X$, $g_* 1_Z$ is the cohomology dual of $Z$ in $X$. Steenrod proposed this question: Can any cohomology class be realized by a submanifold in this way?

We now look into the additive structure of $N(X)$.

If $g : Z \longrightarrow X$, $g' : Z' \longrightarrow X$, let $g + g' : Z \amalg Z' \longrightarrow X$ be defined by $g + g'|_Z = g$, $g + g'|_Z = g'$. Define $[Z \xrightarrow{g} X] + [Z' \xrightarrow{g} X] = [Z \amalg Z' \xrightarrow{g+g'} X]$. The zero element in $N(X)$ is $[\emptyset \longrightarrow X]$. Every element in $N(X)$ is of order 2: in fact, let $[Z \xrightarrow{g} X] \in N(X)$.

Let $h : Z \times \mathbb{R} \longrightarrow X \times \mathbb{R}$ be defined by $(z, \lambda) \mapsto (g(z), \lambda^2)$. The squares in the following diagram are transversal cartesian:

$$\begin{array}{ccccc} Z \amalg Z' & \longrightarrow & Z \times \mathbb{R} & \longleftarrow & \emptyset \\ {\scriptstyle g+g'}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow \\ X & \underset{x \to (x,1)}{\longrightarrow} & X \times \mathbb{R} & \underset{(x,-1) \leftarrow x}{\longleftarrow} & X \end{array}$$

and so $h : Z \times \mathbb{R} \longrightarrow X \times \mathbb{R}$ is a cobordism between $[g + g']$ and $[\emptyset \longrightarrow X]$. hence $N(X)$ is an abelian group and every element in $N(X)$ has order 2.

We give $N(X)$ a grading in the following way: let $N(X) = \oplus_{n \in \mathbb{Z}} N^q(X)$ where $[Z \xrightarrow{g} X] \in N^q(X)$ if $g$ *is everywhere of dimension* $q$, i.e. if $\dim_{g(z)} X - \dim_z Z = q$ for every $z \in Z$. Notice that $N^q(X) = 0$ for $q > \dim X$. (in general $N^q(X)$ is nonzero for infinitely many negative values of $q$).

**Properties.**

4. $f^*$ and $f_*$ are additive homomorphisms.

5. If $X = X_1 \coprod X_2$, $N(X) \simeq N(X_1) \oplus N(X_2)$.

6. If $g : Z \longrightarrow X$, $g' : Z' \longrightarrow X$, then

$$(g + g')_* 1_{Z \amalg Z'} = g_* 1_Z + g'_* 1_{Z'}.$$

**Proposition.** *Properties* 4, 5, 6 *characterize the addition in* $N(X)$.

We define an *external product* $N(X) \otimes N(Y) \longrightarrow N(X \times Y) : \alpha \otimes \beta \mapsto \alpha \boxtimes \beta$ by $[Z \xrightarrow{g} X], [Z' \xrightarrow{g'} Y] \longrightarrow [Z \times Z' \xrightarrow{g \times g'} X \times Y]$ ($\boxtimes$ is well-defined since a cobordism $Z_+ \longrightarrow X$ gives $Z_+ \times Z' \longrightarrow X \times Z' \longrightarrow X \times Y$).

Define an *internal product* $N(X) \otimes N(Y) \longrightarrow N(X) : \alpha \otimes \beta \longrightarrow \Delta^*(\alpha \boxtimes \beta)$. Geometrically, afeter moving $g : Z \longrightarrow X$ transversal to $g' : Z' \longrightarrow X$, $[Z' \xrightarrow{g} X] \cdot [Z' \xrightarrow{g'} X] = [Z^\alpha \times Z' \longrightarrow X]$ since the square

$$\begin{array}{ccc} Z^\alpha \times Z' & \longrightarrow & Z \times Z' \\ \downarrow & & \downarrow{\scriptstyle g \times g'} \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

is transversal cartesian.

With the internal product, $N^*(X)$ is a graded commutative ring with unit $1_X = [X \xrightarrow{id} X]$.

**Properties.**

7. $f^*$ is a ring homomorphism.

8. If $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ are proper, then $(f \times f')_*(\alpha \boxtimes \alpha') = f_*\alpha \boxtimes f'_*\alpha'$ and $(f \times f')^*(\beta \boxtimes \beta') = f^*\beta \boxtimes f'^*\beta'$.

9. If $f : X \longrightarrow Y$ is proper, then $f_*(xf^*(y)) = f_*(x)y$ (*projection formula*)

*Proof.* 7. is clear.

For 8., let $\alpha = [Z \longrightarrow X]$, $\alpha' = [Z' \longrightarrow X']$, then

$$\begin{aligned}(f \times f')_*([Z \longrightarrow X] \boxtimes [Z' \longrightarrow X']) &= (f \times f')_*([Z \times Z' \longrightarrow X \times X']) \\ &= [Z \times Z' \longrightarrow X \times X' \xrightarrow{f \times f'} Y \times Y'] \\ &= [Z \longrightarrow X \xrightarrow{f} Y] \boxtimes [Z' \longrightarrow X' \xrightarrow{f'} Y'] \\ &= f_*[Z \longrightarrow X] \boxtimes f'_*[Z' \longrightarrow X'].\end{aligned}$$

For 9., consider the square

$$\begin{array}{ccccc} X & \xrightarrow{(1,f)} & X \times Y & \xrightarrow{pr_1} & X \\ {\scriptstyle f}\downarrow & (1) & \downarrow{\scriptstyle f \times 1} & (2) & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\Delta_Y]{} & Y \times Y & \xrightarrow[pr_1]{} & Y \end{array}$$

The rectangle (1)(2) and the square (2) are transversal cartesian, so (1) is transversal cartesian. Hence:

$$\begin{aligned}f_*(xf^*(y)) &= f_*\Delta_X^*(x \boxtimes f^*(y)) \\ &= f_*\Delta_X^*(1 \times f^*)(x \boxtimes y) \\ &= f_*(1, f)^*(x \boxtimes y) = \Delta_Y^*(f \times 1)_*(x \boxtimes y) \\ &= \Delta_Y^*(f_*x \boxtimes y) = f_*(x)y.\end{aligned}$$

□

In the case of 9., we say $f_*$ *is a module homomorphism for* $f^*$.

**Property.**(Exactness)

10. Let $i : Y \longrightarrow X$ be the inclusion of a closed submanifold and $j : U \longrightarrow X$ the inclusion of $U = X - Y$. Then the sequence

$$N(Y) \xrightarrow{i_*} N(X) \xrightarrow{j^*} N(U)$$

is exact.

*Proof.* **Step 1:**

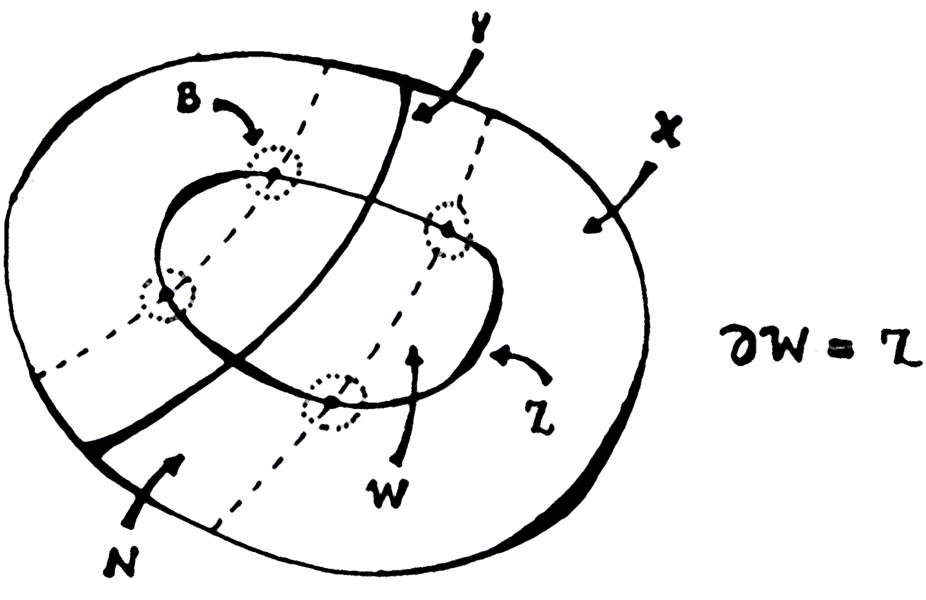
$$\begin{aligned} j^* i_*[Z \longrightarrow Y] &= j^*[Z \longrightarrow Y \xrightarrow{i} X] \\ &= [U \times_X Z \longrightarrow X] \\ &= [\emptyset \longrightarrow U] \\ &= 0. \end{aligned}$$

**Step 2:**

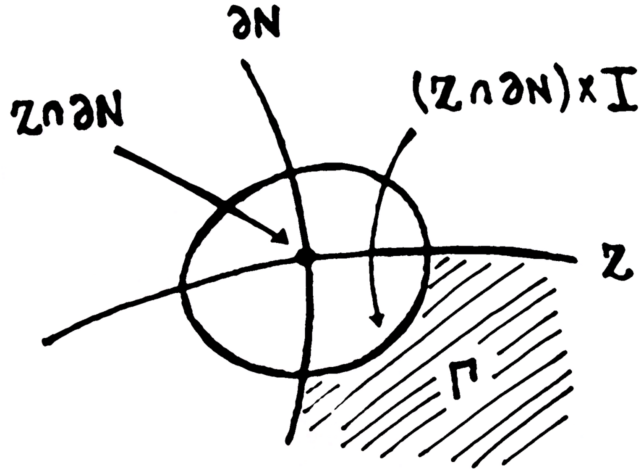
**Lemma** (Excision)**.** *Given a closed submanifold $i : Y \longrightarrow X$ and tubular neighbourhood $N$ of $Y$, then any cobordism class in $N(X)$ vanishing in $X - Y$ can be represented by a proper map $g : Z \longrightarrow X$ with image in $N$.*

**Step 3:** $j^*[Z \xrightarrow{g} X] = 0$, then by lemma $g$ is properly homotopic to the map $Z \xrightarrow{g} N \xrightarrow{pr} Y \xrightarrow{i} X$ so $[Z \xrightarrow{g} X] = [Z \longrightarrow Y \xrightarrow{i} X] = i_*[Z \longrightarrow Y]$, thus proving exactness modulo proof of the lemma. □

**Discussion of Lemma:**

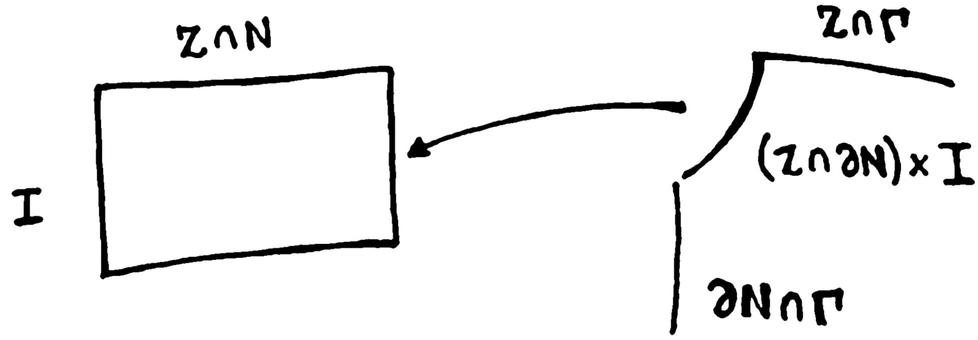



Choose a tubular neighbourhood $N$ of $Y$ in $X$. Then the sphere bundle concentric to $\partial N$ give a family $\partial N \times (0,1) \hookrightarrow X$ so by the transversality lemma, for almost all $\lambda \in (0,1)$, the sphere bundle $\partial N \times \lambda \hookrightarrow X$ is transversal to $Z$ and $W$. We claim $Z$ is cobordant to $(Z \cap N) \cup_{Z \cap \partial N} (W \cap \partial N)$ after all corners are suitably smoothed. Look at a small tubular neighbourhood $B$ of $Z \cap \partial N$. Let $\Gamma$ be the part of $W$ exterior to $N$ and to $B$.




$$\partial\Gamma = (Z \cap \Gamma) \amalg_{Z \cap \partial N} ((Z \cap \delta N) \times I) \amalg_{Z \cap \partial N} (\partial N \cap \Gamma)$$

Let $\Delta = \Gamma \cup_{(Z \cap \partial N) \times I} ((Z \cap N) \times I)$

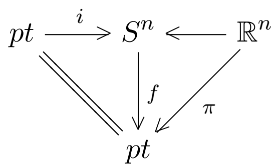



$\partial\Delta = Z \amalg ((Z \cap N) \cup_{Z \cap \partial N} (W \cap \partial N))$, which does what we want it to.

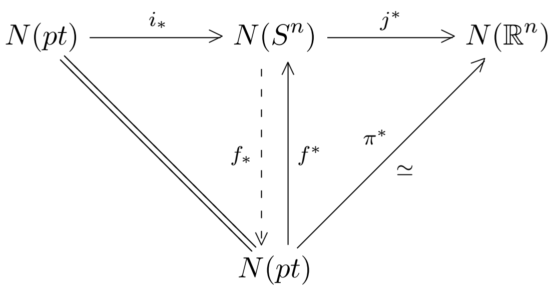
# Chapter 6

# Module structures in cobordism

For computing $N(S^n)$ as an $N(pt)$-module, consider the commutative diagram:

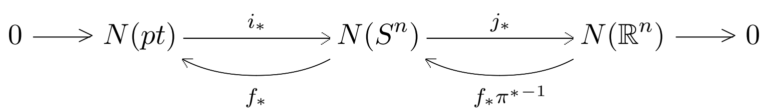



By the exactness property, and since $\pi$ is a homotopy equivalence, we have the commutative diagram




where the top line is exact, $i_*$ is injective since $f_* i_* = 1$, and $j^* f^*$ is an isomophism.

Hence the sequence




is split exact. We now regard $N(S^n)$ as a right $N(pt)-$module under $f^*$, and identify

$N(pt)$ with $N(\mathbb{R}^n)$ using $\pi^*$.

$$i_*(xy) = i_*(xi^*f^*y) = (i_*x)(f^*y)$$

since $fi = 1$, $j^*(xf^*y) = (j^*x)(j^*f^*y) = (j^*x)(\pi^*y)$ so $i_*$ and $j^*$ are $N(pt)-$module homomorphism.

Since the exact sequence splits, $N(S^n) \simeq \operatorname{Im}(i_*) \oplus \operatorname{Im}(f^*)$ as an $N(pt)-$module. Hence $N(S^n)$ is the free $N(pt)-$module with basis $1_{S^n}, i_*1_{pt}$. The grading is given by $N^i(S^n) \simeq N^{i-n}(pt) \oplus N^i(pt)$ for, if $[X \xrightarrow{g} S^n] \in N^i(S^n)$, $n - \dim X = i$, so if $g$ factors through a point, its degree is $i - n$. If $g$ is $f^*$ of something $n - \dim(S^n \times Z) = i$, so its degree is $i$.

## Euler Class

Let $E \longrightarrow X$ be a vector bundle, and let $i : X \longrightarrow E$ be the zero section; then the *Euler class of* $E$, $e(E)$ is

$$e(E) = i^*i_*1_X \in N(X).$$

If $\dim E = n$, $e(E) \in N^n(X)$. Let $s : X \longrightarrow E$ be the section obtained by moving $i$ transversally to itself, and let $Y = s^{-1}i(x)$. Then the square

$$\begin{array}{ccc} Y & \xrightarrow{j} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{s} & E \end{array}$$

is transversal cartesian. Hence

$$i^*i_*1_X = s^*i_*1_X = j_*j^*1_X = j_*1_Y,$$

so $e(E)$ is the class associated to the submanifold $Y$. If $X$ is a compact manifold of dimension $n$ and $E = TX$, then a section is a vector field on $X$, and $Y$ in this case is the set of zeroes of $s$. The reason for the terminology in the classical index theorem.

**Theorem** (Index Theorem). *The number of zeroes of a vector field is congruent with* $\chi(X) \mod 2$, *where* $\chi(X)$ *is the Euler characteristic of* $X$.

**Properties.**

1. $e(f^*E) = f^*e(E)$

2. If $i : X \longrightarrow Y$ is a closed imbedding with normal bundle $r$, then $i^*i_*(x) = e(r)x$ for all $x \in N(X)$.

3. $e(E' \oplus E'') = e(E')e(E'')$.

*Proof.* 1. Let $z, z'$ be the zero sections. Consider the diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \Big\downarrow z' & (2) & \Big\downarrow z \\ f^*E & \xrightarrow{\widetilde{f}} & E \\ \Big\downarrow & (1) & \Big\downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

Then (1) is transversal cartesian by definition, and the rectangle (1)(2) is clearly transversal cartesian. Hence (2) is so too. Thus

$$\begin{aligned} f^*e(E) &= f^*z^*z_*1_X && \text{by definition} \\ &= z'^*\widetilde{f}^*z_*1_X && \text{by functoriality} \\ &= z'^*z'_*f^*1_X && \text{since (2) is transversal cartesian} \\ &= z'^*z'_*1_Y && \text{since } f^* \text{ is a ring homomorphism} \\ &= e(f^*E) && \text{by definition} \end{aligned}$$

2. Let $U$ be a tubular neighbourhood of $X$ in $Y$, $U \simeq r$, where $r$ is the normal bundle with projection $\pi : r \longrightarrow X$. Then:

$$\begin{array}{ccccc} & & X & \xrightarrow{id} & X \\ & & \Big\downarrow z & & \Big\downarrow i \\ X & \xrightarrow{z} & U & \xhookrightarrow{j} & Y \end{array}$$

is a transversal cartesian square where $z$ is the zero section and $j$ is an open imbedding.

Because $i = j \circ z$, then:

$$\begin{aligned}
i^* i_* x &= z^* j^* i_* x && \text{by functoriality} \\
&= z^* z_* id^* x && \text{by transversality} \\
&= z^* z_* x && \\
&= z^* z_* z^* \pi^* x && \text{since } pi \circ z = id \\
&= z^*((z_* 1_*)(\pi^* x)) && \text{by projection formula} \\
&= (z^* z_* 1_X)(z^* \pi^* x) && \text{since } z^* \text{ is a ring homomoprphism} \\
&= e(r)x && \text{by definition and since } \pi \circ z = id.
\end{aligned}$$

3. We have imbeddings $X \overset{i}{\hookrightarrow} E'' \overset{j}{\hookrightarrow} E' \oplus E''$ where $i$ is the zero section and $j(e'') = 0 \oplus e''$. Obviously $ji$ is the zero section of $E' \oplus E''$. Moreover, $r_j = \pi^* E'$ where $\pi : E'' \longrightarrow X$ is the vector bundle projection.

$$\begin{aligned}
e(E' \oplus E'') &= i^* j^* j_* i_* 1_X && \text{by definition and functoriality} \\
&= i^*(e(r_j) i_* 1_X) && \text{by 2.} \\
&= i^*(e(\pi^* E') i_* 1_X) && \\
&= i^*(\pi^* e(E') i_* 1_X) && \text{by 1.} \\
&= e(E') i^* i_* 1_X && \text{since } i^* \text{ is a ring homomorphism} = e(E')e(E'') \quad \text{by definition.}
\end{aligned}$$

□

If $E \longrightarrow X$ is a vector bundle, the *associated projective bundle* $\mathbb{P}E \longrightarrow X$ consists of lines in the fibres of $E \longrightarrow X$. The *Hopf Bundle* $O(-1)$ is the canonical line bundle over $\mathbb{P}E$ whose fibre at $l \subset E_x$ is $l$, i.e.,

$$O(-1) = \{(l, t) : l \in \mathbb{P}E, t \in l\}.$$

Let $O$ be the trivial line bundle; define $O(1) = \mathrm{Hom}(O(-1), O)$. If $n > 0$, $O(n) = O(1) \otimes \cdots \otimes O(1)$, $n$ times; if $n < 0$, $O(n) = O(-1) \otimes \cdots \otimes O(-1)$, $-n$ times. Since we are working in the real case, any bundle is isomorphic to its dual, so $O(1) \simeq O(-1)$.

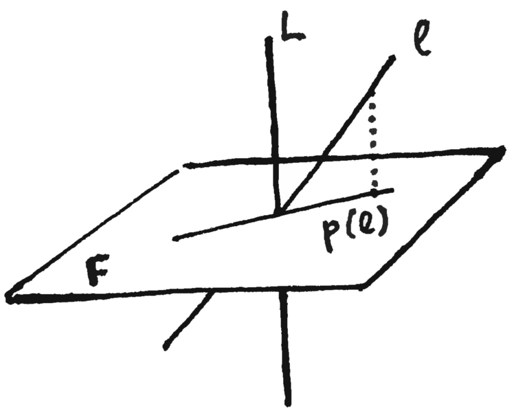
**Theorem.** *Let $E \longrightarrow X$ be an $n$-dimensional vector bundle over $X$, and let $f : \mathbb{P}E \longrightarrow X$ be the associated projective bundle. Let $\xi = e(O(-1)) \in N(\mathbb{P}E)$. Then $N(\mathbb{P}E)$ is a free $N(X)-$module (via $f^*$) with basis $1, \xi, \ldots, \xi^{n-1}$.*

*Proof.* We will start to prove the theorem in case $E$ is a sum of line bundles, but we will be forced to specialize to the case where $E$ is a trivial bundle; we will then develop the machinery to complete the proof.

Let $E = L_1 \oplus \ldots \oplus L_n$. Assume by induction that the theorem is true for a sum of less than $n$ line bundles (the case $n = 1$ is trivial). Set $L = L_1$, and $F = L_2 \oplus \cdots \oplus L_n$. We have the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{s} & \mathbb{P}E & \xleftarrow{j} & \mathbb{P}F \\ & \searrow^{=} & \downarrow f & \swarrow & \\ & & X & & \end{array}$$

where $s(x) = L_x \subset E_x$ and $j$ is the inclusion. $\mathbb{P}E - s(X)$ is a vector bundle over $\mathbb{P}F$, $p : \mathbb{P}E - s(X) \longrightarrow \mathbb{P}F$.




Then $p$ is a deformation retraction of $\mathbb{P}E - s(X) \longrightarrow \mathbb{P}F$; $pj = 1_{\mathbb{P}F}$ and $jp \simeq 1_{\mathbb{P}E - s(X)}$ so $j$ is a homotopy equivalence between $\mathbb{P}E - s(X)$ and $\mathbb{P}F$.

Hence by exactness:

$$\begin{array}{ccccc} N(X) & \xrightarrow{s_*} & N(\mathbb{P}E) & \xrightarrow{j^*} & N(\mathbb{P}F) \\ & \searrow^{=} & \downarrow f_* & \swarrow & \\ & & N(X) & & \end{array}$$

with the top row exact.

By induction $N(\mathbb{P}F)$ is the free $N(X)-$module with basis $1, \xi', \ldots, \xi'^{n-1}$ where $\xi' = e(O(-1) \longrightarrow \mathbb{P}F)$, but clearly $j^*(O(-1) \longrightarrow \mathbb{P}E) = (O(-1) \longrightarrow \mathbb{P}F)$. Thus $\xi' = e(O(-1) \longrightarrow \mathbb{P}F) = e(j^*(O(-1) \longrightarrow \mathbb{P}E)) = j^*e(O(-1) \longrightarrow \mathbb{P}E) = j^*\xi$ so in particular $\xi, \ldots, \xi^{n-2}$ are all non-zero, and $j^*$ is an epimorphism.

Thus we have a split short exact sequence of $N(X)-$modules:

$$0 \longrightarrow N(X) \underset{f_*}{\overset{s_*}{\rightleftarrows}} N(\mathbb{P}E) \xrightarrow{j^*} N(\mathbb{P}F) \longrightarrow 0$$

Hence $N(\mathbb{P}E)$ is the free $N(X)-$module with basis $1, \xi, \ldots, \xi^{n-2}, s_* 1_X$. We will show in a lemma that $s_* 1_X = u \prod_{i=2}^{n} (\xi - f^* e(L_i))$, where $u$ is a unit in $N(X)$. Hence $N(\mathbb{P}E)$ has the $N(X)-$basis $1, \xi, \ldots, \xi^{n-1}$.

**Lemma.** $s_* 1_X = u \prod_{i=2}^{n} (\xi - f^* e(L_i))$, *where $u$ is a unit in $N(X)$.*

*Proof.* Consider the vector bundle $O(1) \otimes f^* L_i = \mathrm{Hom}(O(-1), f^*(L_i))$ on $\mathbb{P}E$ whose fibre at a line $l \subset E_x$ is the space of linear maps from $l$ to $L_{i,x}$. This bundle has a canonical section $s_i$ given by $s_i(l)$ is the homomorphism $l \subset E_x \longrightarrow^{pr_i} L_{i,x} s_i^{-1}(0) = \mathbb{P}(L_1 \oplus \ldots \oplus \widehat{L_i} \oplus \ldots \oplus L_n) = H_i$ since $s_i$ is transversal to the zero section. Thus $e(O(1) \otimes f^* L_i) = [H_i \longrightarrow \mathbb{P}E]$.

Now the $H_i$ intersect transversally and $S(X) = \cap_{i=2}^{n} H_i$ so we have $s_* 1_X = \prod_{i=2}^{n} [H_i \longrightarrow \mathbb{P}E] = \prod_{i=2}^{n} e(O(1) \otimes f^* L_i)$. If the $L_i$ are trivial, $O(1) \otimes f^* L_i \simeq O(-1)$, so $s_* 1_X = \prod_{i=2}^{n} e(O(-1)) = \xi^{n-1}$.

Note that we have thus proved theorem 1 only for trivial bundles.

**Corollary.** $N(X \times \mathbb{P}^{n-1}) = N(X)[\xi]/(\xi^n)$, *where $\xi = e(O(-1))$.*

We actually proved the corollary only so we now develop the rest of the needed machinery.

□

□

Let $A$ be a commutative ring. A power series $F(x, y) \in A[[x, y]]$ is a *formal group law* if

i. $F(x, 0) = x = F(0, x)$ and

ii. $F(x, F(y, z)) = F(F(x, y), z)$.

It is commutative if

iii. F(x,y)=F(y,x).

**Examples.**

1. $F(x,y) = x + y$.

2. $F(x,y) = x + y + axy$. This law is a multiplicative law if $a = 1$ and we use the change of coordinates $x \longrightarrow b_{x-1}$. 1 and 2 are the only polynomials which are formal group laws.

3. If $F(x,y) \in A[[x,y]]$ is a formal group law and $u(x) = \sum_{n=0} u_n x^{n+1}$ with the $u_0 \in A^*$ (thus $u \in A[[x]]^*$), then $G(x,y) = u(F(u^{-1}(x), u^{-1}(y)))$ is a new formal group law. Note that if $F(x,x) = 0$, $G(x,x) = 0$, so in characteristic 2, the laws $x+y$ and $x+y+xy$ are not related by a change of coordinates.

**Proposition.** *There exists a unique commutative formal group law* $F(x,y) = \sum a_{kl} x^k y^l \in N(pt)[[x,y]]$ *such that*

*i.* $e(L_1 \otimes L_2) = F(e(L_1), e(L_2))$ *for all line bundles* $L_1, L_2$ *over* $X$*, and*

*ii.* $F(x,x) = 0$

*Proof.* Note $N(\mathbb{P}^m \times \mathbb{P}^m) = N(pt)[\xi_1, \xi_2]/(\xi_1^{m+1}, \xi_2^{m+1})$, where $\xi_j = pr_j^* e(O(1) \longrightarrow \mathbb{P}^m)$. Hence over $\mathbb{P}^m \times \mathbb{P}^m$, there exist unique $a_{kl}^m \in N^{1-k-l}(pt)$ such that

$$e(pr_1^* O(1) \otimes pr_2^* O(1)) = \sum_{0 \le k,l \le m} a_{kl}^m \xi_1^k \xi_2^l.$$

Imbedding $\mathbb{P}^m \times \mathbb{P}^m \longrightarrow \mathbb{P}^n \times \mathbb{P}^n$, $m < n$, one sees that $a_{kl}^m = a_{kl}^n$ if $k, l < m$. Let $a_{kl} = a_{kl}^m$ for $m > k + l$. Let $L_1, L_2$ be line bundles over $X$, there exist $f_j : X \longrightarrow \mathbb{P}^m$, $m >> \dim X$, such that

$$\begin{aligned} e(L_1 \otimes L_2) &= e(f_1^* O(1) \otimes f_2^* O(1)) \\ &= (f_1, f_2)^* (\sum a_{kl} \xi_1^k \xi_2^l) \\ &= \sum a_{kl} (f_1^* \xi_1)^k (f_2^* \xi_2)^l \\ &= \sum a_{kl} e(L_1)^k e(L_2)^l. \end{aligned}$$

This proves the existence and uniqueness of $F$ satisfying i. The commutative group law properties result formally from

1. $e(O \otimes L) = e(L) = e(L \otimes O)$, where $O$ is the trivial line bundle.
2. $e(L_1 \otimes (L_2 \otimes L_3)) = e((L_1 \otimes L_2) \otimes L_3)$.
3. $e(L_1 \otimes L_2) = e(L_2 \otimes L_1)$

   and ii. results from

4. $e(L \otimes L) = 0$, since $L \otimes L$ is trivial.

□

To interpret the $a_{kl}$, let $H_{m,n} \subset \mathbb{P}^m \times \mathbb{P}^m$ be a non-singular hypersurface of degree $(1,1)$, i.e. the zero submanifold of a generic section of $pr_1^* O(1) \otimes pr_2^* O(2)$,

$$H_{m,n} = \{(x,y) \in \mathbb{P}^m \times \mathbb{P}^n : \sum_{0 \le i \le m,\ o \le j \le n} \} \lambda_{ij} x_i y_j = o, \lambda_{ij} \text{ generic.}$$

$H_{m,0} = \mathbb{P}^{m-1}$. Now $[H_{m,n} \hookrightarrow \mathbb{P}^m \times \mathbb{P}^n] = \sum_{0 \le k,l \le m,n} a_{kl} \xi_1^k \xi_2^l$ where $\xi_1^k$ is the cobordism class of $[\mathbb{P}^{m-k} \longrightarrow \mathbb{P}^m \times \mathbb{P}^m]$ and $\xi_1^k \xi_2^l$ is the cobordism class of $[\mathbb{P}^{m-k} \times \mathbb{P}^{n-l} \longrightarrow \mathbb{P}^m \times \mathbb{P}^n]$. projecting into $N(pt)$ we have

$$[H_{m,n}] = \sum_{0 \le k \le m,\ 0 \le l \le n} a_{kl} [\mathbb{P}^{m-k}][\mathbb{P}^{n-l}] \ \in N(pt).$$

Hence $[H_{m,n}] = a_{mn} +$ lower $a$'s$[\mathbb{P}^i]$ so the subring of $N(pt)$ generated by $\{a_{kl}\}$ is the same as the subring generated by $[H_{m,n}]$ (remembering that $H_{m,0} = \mathbb{P}^{m-1}$).

By the identity property of a formal group law, we have $F(x,y) = x + y + xyG(x,y)$, so $F(x,y) - x = y(1 + xG(x,y))$. Let $x = e(L_1)$, $y = e(L_1^{-1} \otimes L_2)$, then the formula becomes

$$e(L_1 \otimes L_1^{-1} \otimes L_2) - e(L_1) = e(L_1^{-1} \otimes L_2)(1 + e(L_1))G(e(L_1), e(L_1^{-1} L_2))$$

but $e(L_1)$ is nilpotent so the latter term is a unit in $N(X)$. Thus we have proved the following:

**Lemma.** *$e(L_1) - e(L_2) = e(L_1^{-1} \otimes L_2) \cdot u$, where $u$ is a unit in $N(X)$.*

We now complete the proof of the Lemma 1.

$$\begin{aligned} s_*1_X &= e(O(1) \otimes f^*(L_2 \otimes \cdots \otimes L_n)) \\ &= \prod_{i=2}^{n} e(O(1) \otimes f^* L_i) \\ &= \prod_{i=2}^{n} (e(O(-1)) - f^* e(L_i)) \end{aligned}$$

$u$ is a unit in $N(\mathbb{P}E)$. But

$$\begin{aligned} (s_*1_X)u &= \prod_{i=2}^{n} (e(O(-1)) - f^* e(L_i)) \\ &= s_*(s^*u) \\ &= f^*(s^*u)s_*1_X \end{aligned}$$

since $s_*$ is a $N(X)-$module homomorphism, so we may assume the unit $u$ lies in $N(X)$. This completes the proof of Lemma 1 and of the Theorem 1 in the case where $E$ is a sum of line bundles.

To finish the proof of Theorem 1, we need the following construction. Let $Z$ be a closed submanifold of $X$; the *blow-up* of $Z$ in $X$, $\widetilde{X}$, is defined to be $(X - Z) \cup \mathbb{P}r$ where $r$ is the normal bundle of $Z$ in $X$. Think of $\widetilde{X}$ as an $X$ away from $Z$ and normal directions on $Z$. There is a natural map $\pi : \widetilde{X} \longrightarrow X$ such that $\pi|_{X-Z} = id$ and $\pi|_{\mathbb{P}r}$ is the bundle projection to $Z$. The map $\pi$ is sometimes called a *monoidal transformation.*

**Examples:**

1. Let $V$ be a vector space, $0 \in V$; the blow-up of $0 \in V$, $\widetilde{V} = V - 0 \cup \mathbb{P}V$. Note $\widetilde{V} = O_{\mathbb{P}V}(-1) = \{(l, v) : l \text{ a line in } V \text{ through } 0,\ v \in l\}$ and $\pi : \widetilde{V} \longrightarrow V : (l, v) \longrightarrow v$ is a diffeomorphism over $V - 0$ and $\pi^{-1}(0) = \mathbb{P}V$.

2. Let $E \longrightarrow X$ be a vector bundle over $X$. Let $\widetilde{E}$ be the blow-up of the zero section of $E$. Then $\widetilde{E} = O_{\mathbb{P}E}(-1) = \{(l, v) : l \subset E_x \text{ line through } 0, v \in L\}$. The square:

$$\begin{array}{ccc} \mathbb{P}E & \longrightarrow & \widetilde{E} \\ \downarrow & & \downarrow \pi \\ X & \xhookrightarrow[\text{zero section}]{} & E \end{array}$$

is transversal cartesian; $\pi$ is a diffeomorphism on $E - X$.

3. In general, given $Z \subset X$ as a closed submanifold, we have by the Tubular Neighbourhood Theorem $z \overset{0-sec}{\longrightarrow} r \overset{open}{\hookrightarrow} X$. Then $\widetilde{X} = \widetilde{r} \cup_{r-z} X$.

**Problem.** Show that this formulation is independent of the choice of the tubular neighbourhood (and algebraic geometric approach is to use the universal property:

$$\mathrm{Hom}(T,\widetilde{X}) = \{(u : T \longrightarrow X, L, \phi : \mathcal{Y}_Z \longrightarrow u_*\underline{L}) : L \text{ is a line bundle on } T;$$

$$\mathcal{Y}_Z = \text{ germs of functions } f \text{ or } X \text{ such that } f|_Z = 0; \underline{L} = \text{ germs of sections of } L;$$

$$\phi \text{ is } O_x\text{−linear, and a section } \phi(f) \text{ should generate } L\})$$

.

**Problem.** What can be said about the various blow-ups for $Z \subset Y \subset X$?

**Proposition.** *Let $f : Y \longrightarrow X$ be transversal to $Z \subset X$, $\widetilde{X} = (X - Z) \cup \mathbb{P}r_{Z\subset X}$, and $\widetilde{Y} = (Y - f^{-1}Z) \cup \mathbb{P}r_{f^{-1}Z\subset Y}$. then the square*

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

*is transversal cartesian.*

**Theorem** (Hironaka-Kleinman)**.** *If $E \longrightarrow X$ is a vector bundle of dimension $n$, then there is a succession of blow-ups*

$$\widetilde{X}_{n-1} \longrightarrow \widetilde{X}_{n-2} \longrightarrow \cdots \longrightarrow \widetilde{X}_2 \longrightarrow \widetilde{X}_1 \longrightarrow X$$

*such that $E$ pulle back to $\widetilde{X}_{n-1}$ is a sum of line bundles.*

*Proof.* Let $s : X \longrightarrow E$ be a section transversal to $O$, and let $\widetilde{E}$ be the blow-up of the zero section of $E$. it is clear that we have the diagram

$$\begin{array}{ccccc} \mathbb{P}E & \longrightarrow & \widetilde{E} & \longrightarrow & \mathbb{P}E \\ \downarrow & & \downarrow & & \downarrow \\ X & \hookrightarrow & E & \longrightarrow & X \end{array}$$

Let $\widetilde{X}$ be the blow-up of $s^{-1}O$ in $X$. Then the diagram

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \widetilde{E} \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \xrightarrow{s} & E \end{array}$$

is transversal cartesian.

Hence, the diagram

$$\begin{CD} \widetilde{X} @>>> \widetilde{E} @>>> \mathbb{P}E \\ @V{\pi}VV @VVV @VVV \\ X @>{s}>> E @>>> X \end{CD}$$

commutes. Thus $\pi : \widetilde{X} \longrightarrow X$ factor through $\mathbb{P}E$ so $\pi^*E$ contains a line bundle. The process can be iterated. $\square$

We finish the proof of Theorem 1 with the following lemma:

**Lemma.** *If $\pi : \widetilde{X} \longrightarrow X$ is a blow-up, there exists $x \in N(\widetilde{X})$ such that $\pi_* x = 1$.*

*Proof.* The square

$$\begin{array}{ccccc} & & \widetilde{X} & \longleftarrow & X - Z \\ & & {\scriptstyle \pi}\big\downarrow & & \big\| \\ Z & \xhookrightarrow{\;i\;} & X & \xleftarrow{\;j\;} & X - Z \end{array}$$

is transversal cartesian, so $j^*(\pi_* 1 - 1) = 0$, which implies $pi_* 1 - 1 = i_* z$ for some $z \in N(Z)$. But $i_* z$ is nilpotent because of the way products are computed: choose $r$ such that $r$ codim $Z > \dim X$, then $[Y \longrightarrow Z \longrightarrow X]^r$ is 0. Hence $\pi_* 1 = 1 +$ nilpotent elemente , which must be a unit $u$. Let $x = \pi^* u^{-1}$, then $pi_*(\pi^* u^{-1}) = (\pi_* 1) u^{-1} = u u^{-1} = 1$. $\square$

**Lemma.** *Let $f : Y \longrightarrow X$ be a proper map and assume there exists $y \in N(Y)$ such that $f_* y = 1$. Then, if Theorem 1 is true for $f^*E$, it is true for $E$ over $X$.*

*Proof.* We have

$$\begin{CD} \mathbb{P}(f^*E) @>{f'}>> \mathbb{P}E \\ @V{g'}VV @VV{g}V \\ Y @>{f}>> X \end{CD}$$

is transversal cartesian.

**Step 1.** The $\xi^i$ are independent. if not, there exists $a_i \in N(X)$ sucht that $\sum_{i=0}^{n-1} g^*(a_i)\xi^i = 0$. Applying $f'^*$, we obtain $\sum_{i=0}^{n-1} g'^* f^*(a_i) \xi'^i = 0.$, where $\xi' = f'^*\xi$, which implies $f^*(a_i) = 0$. But $f^*$ is injective since $f_*(y f^*(a)) = (f_* y) a = a$. Hence $a_i = 0$ for all i.

**Step 2.** The $\xi^i$ generate. Given $z \in N(\mathbb{P}E)$, we have $f'^* z = \sum_{i=0}^{n-1} g'^*(b_i)\xi'^i = 0$ for some $b_i \in N(Y)$. Then

$$\begin{aligned}
\sum_{i=0}^{n-1} g^* f_*(yb_i)\xi^i &= \sum f'_* g'^*(yb_i)\xi^i \\
&= \sum f'_*(g'^*(yb_i)(f^*\xi)^i) \\
&= \sum f'_*(g'^*(yb_i)(\xi')^i) \\
&= f'_*(g'^* y \sum (g'^* b_i)(\xi')^i) \\
&= f'_*(g'^* y)(f'^* z) \\
&= (f'_* g'^* y)z \\
&= (g^* f_* y)z \\
&= z
\end{aligned}$$

□

This completes the proof of Theorem 1 since we have proved it for sums of line bundles, and we have the Hironaka-Kleinman theorem. We need only know that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that $f_* x = 1$ and $g^* y = 1$, then there exists $x'$ such that $(gf)_* x' = 1$. But $(gf)_*(xf^*y) = g_*(f_* xy) = 1$.

# Chapter 7

# Stiefel-Whitney classes

Let $E \longrightarrow X$ be an $n-$dimensional vector bundle. Then by the projective bundle theorem, $N(\mathbb{P}E) = \bigoplus_{i=0}^{n-1} N(X)\xi^i$ where $\xi = e(O(-1))$. Let $f : \mathbb{P}E \longrightarrow X$ there exist unique elements $w_i(E) \in N^i(X), i \geq 0$, such that

$$\xi^n + f^* w_1(E)\xi^{n-1} + \cdots + f^* w_n(E) = 0.$$

The $w_i(E)$ are the *cobordism Stiefel-Whitney classes of* $E$. We set $w_0(E) = 1$ and $w_i(E) = 0$ for $i > \dim E$. The *total Stiefel-Whitney class of* $E$, $w_\tau(E) \in N(X)[\tau]$, is defined to be $\sum_{i \geq 0} w_i(E)\tau^i$.

**Theorem.** *The Stiefel-Whitney classes are uniquely characterized by the properties:*

1. $g^* w_i(E) = w_i(g^* E)$

2. $w_\tau(E' \oplus E'') = w_\tau(E')w_\tau(E'')$

   $w_k(E' \oplus E'') = \sum_{i+j=k} w_i(E')w_j(E'')$ *(Whitney sum formula).*

3. $w_\tau(L) = 1 + e(L)\tau$

4. *If* $\dim E = n$, $w_n(E) = e(E)$.

*Proof.* 1. and 3. are clear.

To prove 2., since there exists a map $\pi : \widetilde{X} \longrightarrow X$ such that $\pi^* E'$ and $\pi^* E''$ are sums of line bundles with $\pi^*$ injective, it suffices to show $w_\tau(L_1 \oplus \ldots \oplus L_n) = \prod_{i=1}^{n} w_\tau(L_i)$; i.e., $w_i(L_1 \oplus \cdots \oplus L_n)$ is the $i$-th elementary symmetric function of the $e(L_j)$. From the proof of the projective bundle theorem, we have

$$\prod_{j=1}^{n} e(O(1) \oplus f^* L_j) = 0$$

and $e(O(1) \otimes f^*L_j) = (e(O(1) - f^*e(L_j)) \cdot unit$.

Therefore, $\prod_{j=1}^{n} (\xi - f^*e(L_j)) = 0$, which proves 2.

4. and uniqueness are similar. □

Let $C \subset N(pt)$ be the subring of $N(pt)$ generated by the coefficients of the formal group law $F$.

**Theorem.** *Let $E, E'$ be the vector bundles over $X$ of dimension $n$ and $m$, respectively. Then there exits a power series $\Phi_i \in C[[x_1, \ldots, x_n, y_1, \ldots, y_m]]$ such that*

$$w_i(E \otimes E') = \Phi_i(w_1(E), \ldots, w_n(E), w_1(E'), \ldots, w_m(E'))$$

*Proof.* Let $\pi : \widetilde{X} \longrightarrow X$ be a map sucht that $\pi^*E = \bigoplus_{i=1}^{n} L_i$ and $\pi^*E' = \bigoplus_{j=1}^{m} L'_j$. Then $\pi^*(E \otimes E') = \bigoplus_{1 \leq i \leq n \; 1 \leq j \leq m} L_i \otimes L'_j$. So

$$\begin{aligned} \pi^* w_\tau(E \otimes E') &= w_\tau(\oplus L_i \otimes L'_j) \\ &= \prod w_\tau(L_i \otimes L'_j) \\ &= \prod (1 + \tau e(L_i \otimes L'_j)) \\ &= \prod (1 + \tau F(e(L_i), e(L'_j))). \end{aligned}$$

The power series $\prod(1 + \tau F(\bar{x}_i, \bar{y}_i))$ is symmetric in $\bar{x}_1, \ldots, \bar{x}_n$, so by the symmetric function theorem, it can be written as $\sum_{i=o}^{mn} \tau^i \Phi'_i(x_1, \ldots, x_n, \bar{y}_i, \ldots, \bar{y}_m)$ where $x_i = \sum_{j_1 < \cdots < j_i} \bar{x}_{j_1} \cdots \bar{x}_{j_i}$. Similarly the power series is symmetric in $\bar{y}_1, \ldots, \bar{y}_m$, so it is $\sum_{i=0}^{mn} \Phi_i(x_1, \ldots, x_n, y_1, \ldots, y_m)$, where $y_i = \sum_{j_1 < \cdots < j_i} \bar{y}_{j_1} \cdots \bar{y}_{j_i}$. □

**Corollary.** *Let $E$ be an $n-$dimensional vector bundle over $X$. Then there exists a power series $\Psi_{i,j} \in C[[x_1, \ldots, x_n]]$ such that $w_i(\wedge^i E) = \Psi_{i,j}(w_1(E), \ldots, w_n(E))$.*

**Corollary.** *Let $E$ be an $n-$dimensional vector bundle over $X$, and $L$ a line bundle. Then there exists a formula $e(E \otimes L) = e(L)^n + \sum_{\alpha > 0} a_\alpha(e(L)) w(E)^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $w(E)^\alpha = w_1(E)^{\alpha_1} \ldots w_n(E)^{\alpha_n}$ and $a_\alpha \in C[[x]]$.*

The corollaries are proved similarly to the theorem.

# Chapter 8

# Steenrod Operations in Cobordism

If $X$ and $Y$ are two left $G-$spaces, let $X \times_G Y$ be the orbit space of the action $g(x,y) = (gx, gy)$. If $X$ is a right $G-$space, it can be made into a left $G-$space by $gx = xg^{-1}$. In this case $X \times_G Y = X \times Y/(xg, y) = (x, gy)$. If $G$ acts freely on $X \times Y$, then $X \times_G Y$ is a manifold.

Let $\mathbb{Z}_2$ act trivially on $X$, by antipodal action on $S^n$, and by interchange on $X^2 = X \times X$, i.e., $\sigma(x_1, x_2) = (x_2, x_1)$. note that $\mathbb{Z}_2$ acts freely on $S^n \times X^2$ so that $S^n \times_{\mathbb{Z}_2} X^2$ is a manifold. $S^n \times_{Z_2} X^2$ is a fibre bundle over $S^n/\mathbb{Z}_2$ with fibre $X^2$.

The square

$$\begin{array}{ccc} S^n \times X^2 & \longrightarrow & S^n \times_{\mathbb{Z}_2} X^2 \\ \downarrow & & \downarrow \\ S^n & \longrightarrow & S^n/\mathbb{Z}_2 = \mathbb{R}P^n \end{array}$$

is cartesian.

Define the *external Steenrod square* $P_{ext} : N^i(X) \longrightarrow N^{2i}(S^n \times_{\mathbb{Z}_2} X^2)$ by the following: if $[Z \xrightarrow{f} X] \in N^i(X)$, let $\widetilde{f}^2$ be $(id_{S^n}, f^2)/\mathbb{Z}_2 : S^n \times_{\mathbb{Z}_2} Z^2 \longrightarrow S^n \times \mathbb{Z}_2 X^2$. Let $P_{ext}[Z \xrightarrow{f} X] = [S^n \times_{\mathbb{Z}_2} Z^2 \xrightarrow{\widetilde{f}^2} S^n \times_{\mathbb{Z}_2} X^2]$.

To see that $P_{ext}$ is well-defined, think of $X \longrightarrow N(S^n \times_{\mathbb{Z}_2} X^2)$, $f \longrightarrow \widetilde{f}^2_*$ (for $f$ proper) as a theory with $N$ as the universal example.

If

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

is transversal cartesian, then

$$\begin{array}{ccc} X'^2 & \longrightarrow & X^2 \\ \downarrow & & \downarrow \\ Y'^2 & \longrightarrow & Y^2 \end{array}$$

is transversal cartesian, so

$$\begin{array}{ccc} S^n \times_{\mathbb{Z}_2} X'^2 & \longrightarrow & S^n \times_{\mathbb{Z}_2} X^2 \\ \downarrow & & \downarrow \\ S^n \times_{\mathbb{Z}_2} Y'^2 & \longrightarrow & S^n \times_{\mathbb{Z}_2} Y^2 \end{array}$$

is transversal cartesian. We can then check the axioms of chapter $V$ and conclude that there exists a unique map $P_{ext} : N(X) \longrightarrow N(S^n \times_{\mathbb{Z}_2} X^2)$ such that

i. $P_{ext} f_* = \widetilde{f}^2_* P_{ext}$

ii. $P_{ext} f^* = \widetilde{f}^2_* P_{ext}$

iii. $P_{ext} 1_X = 1_{S^n} \times_{\mathbb{Z}_2} 1_X$.

Now let $\Delta_X : X \times X^2$ be the diagonal which is $\mathbb{Z}_2-$equivariant so it induces $\widetilde{\Delta}_X : S^n \times_{\mathbb{Z}_2} X = \mathbb{R}P^n \times X \longrightarrow S^n \times_{\mathbb{Z}_2} X^2$. The *internal Steenrod square* $P : N^i(X) \longrightarrow N^{2i}(\mathbb{R}P^n \times X)$ is defined by $P_x = \widetilde{\Delta}^*_X P_{ext} x$. If we move $\widetilde{\Delta}_X$ to $d \simeq \widetilde{\Delta}_X$ and transversal to $\widetilde{f}^2$ (where $f : Z \longrightarrow X$) and form the pull-back $W$ :

$$\begin{array}{ccc} W & \longrightarrow & S^n \times_{\mathbb{Z}_2} Z^2 \\ \downarrow & & \downarrow \widetilde{f}_2 \\ \mathbb{R}P^n \times X & \overset{d}{\underset{\widetilde{\Delta_X}^2}{\rightrightarrows}} & S^n \times_{\mathbb{Z}_2} X^2 \end{array}$$

, then $P[Z \xrightarrow{f} X] = [w \longrightarrow \mathbb{R}P^n \times X]$. The map $d$ is analogous to the diagonal approximation in classical theory. Note $Pf^* = (1 \times f)^*$, but $Pf_* \neq (1 \times f)_* P$.

Recall $N(\mathbb{R}P^n \times X) = N(X)[v]/v^{n+1} = 0$, where $v = e_{\mathbb{R}P^n}(O(-1))$. Thus $P_x = \sum_{i=0}^{n} (S_{q_i} x) v^i$. This decomposition is compatible with $n$ as it approaches $\infty$, so we obtain well-defined operations $S_{q_i} : N^j(X) \longrightarrow N^{2j-1}(X)$ for $i \geq 0$. The $S_{q_i}$ are natural with respect to $f^*$. The *Steenrod operations in unoriented cobordism theory* $S_{q^i} : N^j(X) \longrightarrow N^{j+1}(X)$ are defined by $S_{q^i} x = S_{q_{n-i}} x$ if $x \in N^n(X)$. The $S_{q^i}$ satisfy the following properties:

1. $S_{q^i} f^* = f^* S_{q^i}$
2. $S_{q^i} x = 0$ for $i > \deg x$
3. $S_{q^i}$ is additive
4. The $S_{q^i}$ satisfy the Cartan formula:

$$S_{q^k}(xy) = \sum_{i+j=k} S_{q^i}(x) S_i(y).$$

In general, however, $S_{q^0} \neq 1$ and $S_{q^i} \neq 0$ for $i < 0$ in obvious contrast to the Steenrod squares in $\mathbb{Z}_2-$cohomology.

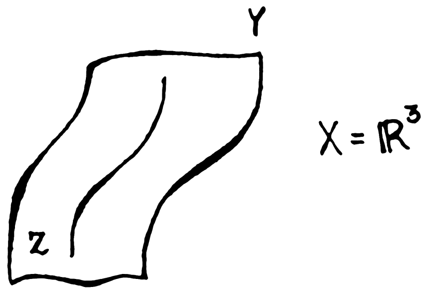
# Chapter 9

# Clean Intersections

Let $Y, Z \subset X$ be closed submanifold. $Y$ and $Z$ *intersect cleanly* if $W = Y \cap Z$ is a submanifold and, for all $w \in W$, $T_w W = T_w Y \cap T_w Z$.

Examples.

1. $Z \subset Y \subset X$.




2. Two curves in $\mathbb{R}^3$

   if we have a clean intersection

$$\begin{array}{ccc} W & \xrightarrow{g'} & Z \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & X \end{array}$$

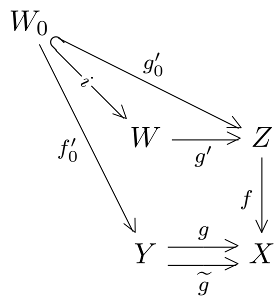
we define the *excess bundle* $E \longrightarrow W$ to be the vector bundle with fibres $E_w = T_w X / (T_w Y \oplus T_w Z)$.

**Theorem** (Clean intersection formula)**.** *If*

$$\begin{array}{ccc} W & \xrightarrow{g'} & Z \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & X \end{array}$$

*is a clean intersection with excess bundle* $E \longrightarrow W$*, then, for all* $z \in N(Z)$*,* $g^* f_* z = f'_*(e(E) g'^* z)$*.*

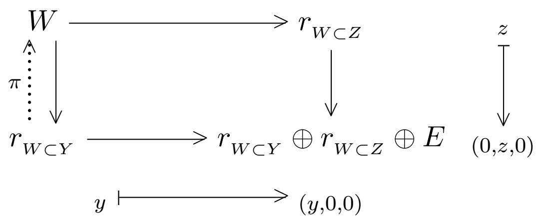
*Proof.* Let $s$ be a generic section of $E$, i.e. $s : W \longrightarrow E$ is transversal to the zero section, and let $w_0 = s^{-1}0$. Move $g$ to $\widetilde{g}$ transversal to $f$ to obtain the diagram (we construct $\widetilde{g}$ explicity later):



in which the triangles, the outer square and the inner square commute, and the outer square is transversal cartesian.

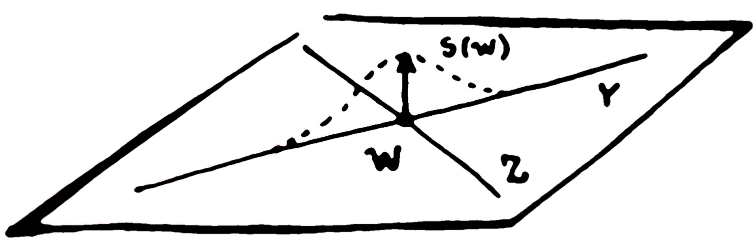
$$\begin{aligned} g^* f_* z &= \widetilde{g} f_* z && \text{since } g \simeq \widetilde{g} \\ &= f'_{0*} g'_{0*} z && \text{by tansversality} \\ &= f'_* i_* i^* g'^* z && \text{by commutativity} \\ &= f'_*(e(E) g'^* z). \end{aligned}$$

To construct $\widetilde{g}$, note that in a neighbourhood of $W$, the situation is diffeomorphic to



Let $\rho : r_{W \subset Y} \longrightarrow \mathbb{R}$ be 1 on $W$ and 0 outside of some neighbourhood of $W$. Define $\rho : r_{W \subset Y} \longrightarrow r_{W \subset Y} \oplus r_{W \subset Z} \oplus E$ by $\widetilde{g}(y) = (y, 0, \rho(y) s(\pi y))$. Then $\widetilde{g}(z) = W_0$.

□




**Example.** Let $i : Y \hookrightarrow X$ with normal bundle $r$. Then

$$\begin{array}{ccc} Y & = & Y \\ \| & & \downarrow i \\ Y & \xrightarrow[i]{} & X \end{array}$$

is a clean intersection with excess bundle $r$. Hence $i_* i^* y = e(r)y$.

We generalize the notion of a clean intersection by defining *clean squares.* A square (with $f$ proper)

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

is *clean* if it is cartesian and

$$\begin{array}{ccc} T_{x'}X' & \longrightarrow & T_{g'x'}X \\ \downarrow & & \downarrow \\ T_{f'x'}Y' & \longrightarrow & T_{fg'x'}Y \end{array}$$

is cartesian; i.e.,

$$0 \longrightarrow T_{x'}X' \longrightarrow T_{f'x'}Y' \oplus T_{g'x'}X \longrightarrow T_{fg'x'}Y$$

is an exact sequence of vector spaces. Define the *excess bundle* of a clean square, $E$, by $E_{x'} = T_{fg'x'}(Y)/(dgT_{f'x'}Y' + dfT_{g'x}X)$.

**Theorem.** *For clean square with excess bundle $E$, $g^* f_* x = f'_*(e(E)g'^* x), x \in N(X)$.*

*Proof.* Left as an exercise. Factor $g$ and $f$ into embeddings followed by submersions. $\square$

**Example.** The square

$$\begin{array}{ccc} \mathbb{P}E & \xrightarrow{j} & O(-1) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & E \end{array}$$

($i$ the zero section) si clean. $f^*i_*x = j_*(e(Q)g^*x)$, where $Q$ is the excess bundle. The tangent bundle along the fibres of $g$ is $\mathrm{Hom}(O(-1), E/O(-1))$. $Q$ is $g^*E/O(-1)$.

# Chapter 10

# Determination of $N^*(pt)$: I

$N^*(pt) = \bigoplus_{n \leq m} N^n(pt)$, where $N^n(pt)$ = cobordism classes of $(-n) - manifolds\ Z \longrightarrow pt$. Recall that $C \subset N^*(pt)$ is the subring generated by the coefficients of the formal group law (or equivalently by the cobordism classes $[\mathbb{P}^n \longrightarrow pt]$ and $[H_{mn} \longrightarrow pt]$, where $H_{mn} \subset \mathbb{P}^m \times \mathbb{P}^n$ is the hypersurface of degree (1,1) ).

**Theorem.** $C = N^*(pt)$.

*Proof.* By induction we can assume that $C^{-i} = N^{-i}(pt)$ for $i < n$. Let $[Z^n \xrightarrow{f} pt] \in N^{-n}(pt)$ and factor $f$ into a embedding followed by a projection i.e.

$$\begin{array}{ccc} Z^n & \xrightarrow{i} & X = S^{n+m} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p} & \\ & pt & \end{array}$$

$$\begin{array}{ccc} Z & \xrightarrow{\Delta_Z} & Z^2 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i^2} \\ X & \xrightarrow{\Delta_X} & X^2 \end{array}$$

is a clean intersection and the diagram:

$$\begin{array}{ccccccccc} 0 \longrightarrow & T_zZ & \xrightarrow{d\Delta_Z} & T_{(z,z)}(Z^2) = T_z(Z) \otimes T_z(Z) & \longrightarrow & T_z(Z) & \longrightarrow 0 \\ & \downarrow{\scriptstyle di} & & {\scriptstyle di^2}\downarrow & & \downarrow & \\ 0 \longrightarrow & T_zX & \xrightarrow{d\Delta_X} & T_{(z,z)}(X^2) = T_z(X) \otimes T_z(X) & \longrightarrow & T_z(X) & \longrightarrow 0 \\ & & & (\alpha,\beta) & \longmapsto & \alpha - \beta & \end{array}$$

has exact rows, injective columns, and cartesian first square, so the sequence

$$0 \longrightarrow T(Z) \longrightarrow T_z(X) \oplus T_z(Z) \oplus T_z(Z) \longrightarrow T_z(X) \oplus T_z(X) \longrightarrow T_z(X)/T_z(Z) \longrightarrow 0$$

□

is exact. Hence the excess bundle of the clean intersection above is just the normal bundle of $i, r_i$. Letting $\mathbb{Z}_2$ act on this intersection, we have the generator $\sigma$ of $\mathbb{Z}_2$ acts as $-1$ on $v_i$. The square

$$\begin{array}{ccc} \mathbb{R}P^n \times Z & \xrightarrow{\widetilde{\Delta}_Z} & S^n \times_{\mathbb{Z}_2} Z^2 \\ \downarrow i & & \downarrow \widetilde{i}^2 \\ \mathbb{R}P^n \times X & \xrightarrow{\widetilde{\Delta}_X} & S^n \times_{\mathbb{Z}_2} X^2 \end{array}$$

is clean with excess bundle $\widetilde{E} = O(-1) \boxtimes r_i = pr_1^* O(-1) \otimes pr_2^* r_i$. The clean intersection formula gives

$$\widetilde{\Delta}_X^* \widetilde{i}_*^2 \gamma = \widetilde{i}_*(e(O(-1) \boxtimes r_i) \widetilde{\Delta}_Z^* \gamma)$$

for $\gamma \in N(S^n \times_{\mathbb{Z}_2} Z^2)$. if $\gamma = 1$, $\widetilde{i}_*^2 1 = P_{ext}(i_* 1)$ and $\widetilde{\Delta}_X \widetilde{i}_*^2$, where $P_{ext}$ and $P$ are the external and internal Steenrod operations. Hence $P(i_* 1) = \widetilde{i}_* e(O(-1) \boxtimes r_i)$ where $\widetilde{i} = id \times i : \mathbb{R}P^n \times Z \longrightarrow \mathbb{R}P^n \times X$. If $\gamma = P_{Ext}(g_* 1) = \widetilde{g}_*^2 1$, for $g : W \longrightarrow Z$, we have

$$\begin{array}{ccc} & & S^n \times_{\mathbb{Z}_2} W^2 \\ & & \downarrow \widetilde{g}_2 \\ & & S^n \times_{\mathbb{Z}_2} Z^2 \\ & & \downarrow \widetilde{i}^2 \\ \mathbb{R}P^n \times X & \xrightarrow{\Delta_X} & S^n \times_{\mathbb{Z}_2} X^2 \end{array}$$

so

$$\begin{aligned} P i_*[W \xrightarrow{g} Z] &= P(i_* g_* 1) \\ &= \widetilde{\Delta}_X^* \widetilde{(ig)}_*^2 1 \\ &= \widetilde{\Delta}_X^* \widetilde{i}_*^2 \widetilde{g}_*^2 1 \\ &= \widetilde{i}_*(e(O(-1) \boxtimes r_i) \widetilde{\Delta}_Z^* \widetilde{g}_*^2 1) && \text{by clean intersection formula} \\ &= \widetilde{i}_*(e(O(-1) \boxtimes r_i) P[W \xrightarrow{g} Z]. \end{aligned}$$

Therefore, for all $z \in N(Z), P(i_* z) = i_*(e(O(-1) \boxtimes r_i) P z)$ (this formula is clearly valid for any embedding $i$).

Let $j : pt \longrightarrow X$. then $P(j_*\alpha) = \widetilde{j}_*(e()(-1) \boxtimes r_j)P_\alpha)$, $\alpha \in N(pt)$. Let $v = e((O-1)) \in N^1(\mathbb{R}P^n)$; $r_j$ si trivial of dimension $n + r$. Thus $e(O(-1) \boxtimes r_j) = e(pr_1 * O(-1)^{n+r}) = (pr_1^* v)^{n+r}$, where $pr_1 : \mathbb{R}P^n \times pt \longrightarrow \mathbb{R}P^n$. Hence $P(j_*\alpha) = (pr_1^*)^{n+r}\widetilde{j}_* P\alpha$, where $\widetilde{j} : \mathbb{R}P^n \times pt \longrightarrow \mathbb{R}P^n \times X$. Recall $p : X \longrightarrow pt$ so $\widetilde{p} = id_{\mathbb{R}P^n} \times p : \mathbb{R}P^n \times X \longrightarrow \mathbb{R}P^n \times pt$ and $\widetilde{p}\widetilde{j} = id$, so $\widetilde{p}_* P(\widetilde{j}_*\alpha) = (pr_1^* v)^{n+r} P\alpha$.

Let $\alpha = f_*1 = [Z \xrightarrow{f} pt]$. Note $Z \xrightarrow{f} pt \xrightarrow{j} X$ is homotopic to $i : Z \longrightarrow X$ if $r \geq 1$, so that $i$ is not onto. Then ( here $pr_1 : \mathbb{R}P^n \times pt \longrightarrow \mathbb{R}P^n$ is a diffeomorphism so we identify $\mathbb{R}P^n \times pt$ and $\mathbb{R}P^n$ under it):

$$
\begin{aligned}
(pr_1^* v)^{n+r} P(f_*1) = \widetilde{p}_* P(j_* f_* 1) & \\
&= \widetilde{p}_* P(i_* 1) \\
&= \widetilde{p}_* i_* e(O(-1) \boxtimes r_i) \\
&= \widetilde{f}_* e(O(-1) \boxtimes r_i)
\end{aligned}
$$

In the preceding, we could replace $pt$ by a manifold $Y$ and $X$ by $Y \times$ sphere. We state this generalization as a proposition an leave the proof as an exercise).

**Proposition.** *Let $f : Z \longrightarrow Y$ be a proper map factored as $Z \xrightarrow{i} Y \times S^{n+r} \xrightarrow{p} Y$, $r >\sim Z$. Then*

$$(pr_1^* v)^{n+r} P(f_* z) = \widetilde{f}_*(e(O(-1)) \otimes r_i) P_Z).$$

Recall that $e(L \otimes E^r) = e(L)^r + \sum_{\alpha > 0} a_\alpha(e(L)) w(E)^\alpha$, where $a_\alpha \in C[[x]]$. Then

$$
\begin{aligned}
e(O(-1) \boxtimes r_i) &= e(pr_1^* O(-1) \otimes pr_2^* r_i) \\
&= (pr_1^* v)^r + \sum_{\alpha > 0} pr_1^* a_\alpha(v) pr_2^*(w(r_i)^\alpha).
\end{aligned}
$$

where $pr_1 : \mathbb{R}P^n \times \mathbb{Z} \longrightarrow \mathbb{R}P^n \times pt$, we identify $\mathbb{R}P^n \times pt$ with $\mathbb{R}P^n$ to get $\widetilde{f} = pr_1$. Hence

$$\widetilde{f} e(O(-1) \boxtimes r_i) = v^r f_* 1 + \sum_{\alpha > 0} a_\alpha(v) f_*(w(r_i)^\alpha), \qquad (10.1)$$

because

$$
\begin{array}{ccc}
\mathbb{R}P^n \times Z & \xrightarrow{pr_2} & Z \\
{\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle f} \\
\mathbb{R}P^n & \xrightarrow{\pi} & pt
\end{array}
$$

is transversal cartesian and everything is considered as an $N(pt)-$module, so we omit $pi^*$ in (10.1).

Now we have $v^{n+r}P(f_*1) = v^r f_*1 + \sum_{\alpha>0} a_\alpha(v) f_*(w(r_i)^\alpha)$ in $N(\mathbb{R}P^n) = N(pt)[v]/(v^{n+1})$. Hence $v^{n+r}P(f_*1) = v^r f_*1 + \sum_{\alpha>0}\sum_{i\leq n} c_{\alpha i} v^i f_*(w(r_i)^\alpha)$, $c_{\alpha_i} \in C$. by the induction assumption, $f_*(w(r_i)^\alpha) \in N^{|\alpha|-n}(pt) = C^{|\alpha|-n}$. Setting $r = n$, $v^{n+n} = 0$, and, looking at the coefficients of $v^n$ we see $f_*1 \in C$. Completing the induction.

# Chapter 11

# Characteristic classes in cobordism

Let $X$ be a manifold with a finite number of components for simplicity. $\mathrm{Vect}(X)$ is the set of isomorphic classes of real vector bundles over $X$; it has the structure of an abelian monoid under direct sum. $\mathrm{Vect}(X) \longrightarrow KO(X)$ is the universal map to an abelian group. let $H^0(X;\mathbb{Z})$ b the ring of locally constant functions on $X$ with values in $\mathbb{Z}$. Define a map rank= $\mathrm{rk} : \mathrm{Vect}(X) \longrightarrow H^0(X;\mathbb{Z})$ by $\mathrm{rk}\, E(X) = \dim_{\mathbb{R}} E_X$.

Then rk extends to $KO(X)$ :

$$\begin{array}{ccc} \mathrm{Vect}(X) & \xrightarrow{\mathrm{rk}} & H^0(X;\mathbb{Z}) \\ & \searrow \qquad \nearrow_{\mathrm{rk}} & \\ & KO(X) & \end{array}$$

Let $\widetilde{KO}(X)$ be the kernel of rk, so

$$0 \longrightarrow \widetilde{KO}(X) \longrightarrow KO(X) \underset{\curvearrowleft}{\overset{\mathrm{rk}}{\longrightarrow}} H^0(X;\mathbb{Z}) \longrightarrow 0$$

since, given $f \in H^0(X;\mathbb{Z})$ written in the form $f^+ - f^-$ where $f^\pm(x) \in \mathbb{Z}_0^+$, the one can construct a trivial vector bundle over each component of $X$ with rank $f^+(x)$ at $x$ (similarly for $f^-$), which shows rk is surjective and the sequence splits. Hence $KO(X) = \widetilde{KO} \otimes H^0(X;\mathbb{Z})$.

Another interpretation of $\widetilde{KO}(X)$ when $X$ is a manifold (or more generally when every vector bundle can be imbedded in a trivial bundle) is the following: two vector bundles $E, E'$ are *stably isomorphic* if there exist bundles $F, F'$ trivial over each component of $X$ such that $E \otimes F \simeq E' \otimes F'$. Then $\widetilde{KO}(X)$ is the set of stable isomorphism classes. It is an easy exercise to show $\widetilde{KO}(X)$ is a group under $\otimes$ whit this definition.

A *characteristic class* is a natural transformation from Vect to $N^*$.

**Examples.**

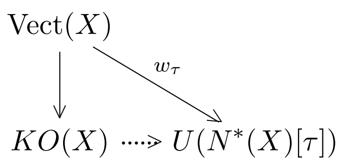
1. Stiefel-Whitney classes $w_i : \mathrm{Vect}(X) \longrightarrow N(X)$

$$w_i f^* E = f^* w_i E.$$

2. The total Stiefel-Whitney class $w_\tau : \mathrm{Vect}(X) \longrightarrow N^*(X)[t]$, where we take the degree of $\tau$ to be $-1$. $w_\tau(E) = \sum w_i(E)\tau^i$ is a polynomial, natural in $E$, with the following property:

$$w_\tau(E \otimes F) = w_\tau(E) w_\tau(F). \tag{11.1}$$

A characteristic class satisfying (11.1) is an *exponential characteristic class.* Since $w_i(E) \in N^i(X), i > 0$, in nilpotent ($N^i(X) = 0$ for $i > \dim X$), $w_\tau(E) = 1 +$ *nilpotent* so it is a unit.




$w_\tau$ is thus a homomorphism of $\mathrm{Vect}(X)$ into the units of $N^*(X)[\tau]$ so it extends to $KO(X)$. Since $w_\tau(E) = 1$ if $E$ si trivial on each component, $w_\tau : \widetilde{KO}(X) \longrightarrow U(N^*(X)[\tau])$.

3. Let $\underline{\tau} = \tau_1, \tau_2, \ldots$ be sequence of indeterminants. We will define a characteristic class $w_{\underline{\tau}} : \widetilde{KO}(X) \longrightarrow N^*(X)[\tau_1, \tau_2, \ldots] = N^*[\underline{\tau}]$ satisfying

   i. $w_{\underline{\tau}}(E \otimes F) = w_{\underline{\tau}}(E) w_{\underline{\tau}}(F)$, and
   
   ii. for a line bundle $L$, $w_{\underline{\tau}}(L) = \sum\limits_{i=0}^{\infty} \tau_i e(L)^i$, where $t_0 = 1$.

By the splitting principle, $w_\tau$ is uniquely characterized by these properties. Suppose $\dim E = n$, $E$ a vector bundle over $X$, then $N^*(\mathbb{P}E)[\underline{\tau}]$ is a free $N^*(X)[\underline{\tau}]-$module of rank $n$ with basis $1, \xi, \ldots, \xi^{n-1}$, where $\xi = e(O(-1))$. We define Norm : $N^*(\mathbb{P}E)[\underline{\tau}] \longrightarrow N^*(X)[\underline{\tau}]$ by: Norm$(X)$ is the determinant of the endomorphism $y \mapsto xy$ of $N(\mathbb{P}E)[\underline{\tau}]$ as an $N(X)[\underline{\tau}]-$module. Let $w_{\underline{\tau}}(E) = \mathrm{Norm}(\sum\limits_{i=0}^{\infty} \tau_i \xi^i)$ where $\xi = e(O_{\mathbb{P}E}(-1))$. Using $N(\coprod\limits_{i=1}^{n} X_i) = \prod\limits_{i=1}^{n} N(X_i)$, we extend the definition to vector bundles of arbitrary rank.

For property ii., $\mathbb{P}L = X$, $O(-1) = L$, so $w_\tau(L) = \sum_{i=0}^{\infty} \tau_i e(L)^i$.

For property i.,

$$\begin{array}{ccccc} \mathbb{P}F & \xrightarrow{i} & \mathbb{P}(E \oplus F) & \xleftarrow{j} & \mathbb{P}(E \oplus F) - \mathbb{P}F \\ & & & \nwarrow^{i'} & \downarrow{\scriptstyle v.b} \\ & & & & \mathbb{P}E \end{array}$$

gives an exact sequence,

$$0 \longrightarrow N(\mathbb{P}F) \xrightarrow{i_*} N(\mathbb{P}(E \oplus F)) \xrightarrow{i'^*} N(\mathbb{P}E) \longrightarrow 0$$

(for $\dim F = 1$, this was proved earlier; one can either reduce to this case or check that the earlier argument extends).

Let $\varphi_{\underline{\tau}}(\xi) = \sum \tau_i \xi^i$.

$$\left( \begin{array}{c|c} \text{multiply by } \varphi_{\underline{\tau}} \text{ in } N(\mathbb{P}(E \otimes F))[\underline{\tau}] & * \\ \hline 0 & 0 \end{array} \right)$$

$$\det \begin{pmatrix} \text{mult by } \varphi_{\underline{\tau}}(\xi) \\ \\ \text{in } N(\mathbb{P}(E \oplus F))[\underline{\tau}] \end{pmatrix} = \det \begin{pmatrix} \text{mult by } \varphi_{\underline{\tau}}(\xi) \text{ in } N(\mathbb{P}(E \oplus F))[\underline{\tau}] & * \\ 0 & \text{mult by} \varphi_{\underline{\tau}}(\xi) \text{ in } N(\mathbb{P}F)[\underline{\tau}] \end{pmatrix} =$$

$$= \det \begin{pmatrix} \text{mult by } \varphi_{\underline{\tau}}(\xi) \\ \\ \text{in } N(\mathbb{P}E)[\underline{\tau}] \end{pmatrix} \det \begin{pmatrix} \text{mult by } \varphi_{\underline{\tau}}(\xi) \\ \\ \text{in } N(\mathbb{P}F)[\underline{\tau}] \end{pmatrix}$$

which proves $w_{\underline{tau}}(E \oplus F) = w_\tau(E) w_\tau(F)$. (This passes over the work involved with $N(\mathbb{P}F) \longrightarrow N(\mathbb{P}(E \oplus F))$, but it is not too dificult).

Let $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\alpha_i \geq 0 \in \mathbb{Z}$ with almost all $\alpha_i = 0$. Let $|\alpha| = \sum_{i>1} i\alpha_i$. Give the $\tau_i$ degree $-i$ in $N(X)[\underline{\tau}]$, and let $\tau^\alpha = \tau_1^{\alpha_1} \tau_2^{\alpha_2} \ldots$. Then $w_{\underline{\tau}}(E) = \sum_{\alpha \geq 0} \tau^\alpha w_\alpha(E)$, $w_\alpha(E) \in N^{|\alpha|}(X)$.

**Remarks.**

1. $w_{(1,0,0,\ldots}(E) = w_i(E)$.

2. $w_\alpha(E \oplus F) = \sum_{\beta+\gamma=\alpha} w_\beta(E) w_\gamma(F)$

3. $w_\alpha(E) = P_\alpha(w_1E, w_2E, \ldots)$, where $P_\alpha \in \mathbb{Z}_2[\tau_1, \ldots]$

$P_\alpha$ is a universal polynomial since, if $E = L_1 \oplus \cdots \oplus L_n$, $w_{\underline{\tau}}(E) = \prod w_{\underline{\tau}}(L_i)$ is a symmetric function of the $e(L_i)$, so can be written

$$w_{\underline{\tau}}(E) = \sum_{\alpha \geq 0} \tau^\alpha P_\alpha(w_1 E, \ldots, w_n E).$$

# Chapter 12

# Landweber-Novikov operations

**Theorem.** *There exists a unique operation* $s_{\underline{\tau}} : N(X) \longrightarrow N(X)[\underline{\tau}]$ *such that:*

*i.* $s_{\underline{\tau}} f^* = f^* s_{\underline{\tau}}$.

*ii.* $s_{\underline{\tau}} f_* x = f_*(w_{\underline{\tau}}(v_f) s_{\underline{\tau}} x)$ *for* $f$ *proper,* $v_f = f^* T_y - T_x \in KO(X)$.

*iii.* $s_{\underline{\tau}}$ *is a ring homomorphism.*

*iv.* $s_{\underline{\tau}} e(L) = \sum_{j=0}^{\infty} \tau_j e(L)^{j+1}$, *for a line bundle* $L$.

Clearly i., ii., and iv. characterize $s_{\underline{\tau}}$ and thus impy iii..

Define $s_{\underline{\tau}}$ by $s_{\underline{\tau}}[Z \xrightarrow{g} X] = s_{\underline{\tau}} g_* w_{\underline{\tau}}(r_g)$. To show $s_{\underline{\tau}}$ is well-defined by this formula and to establish i. and ii., we use the universal property of $N$ [Quillen thinks of $N$ as this universal object; the significance of this viewpoint will be established by proofs using it and especially in later chapters].

Put $f^! = f^* : N(Y)[\underline{\tau}] \longrightarrow N(X)[\underline{\tau}]$ for arbitrary, $f : X \longrightarrow Y$, and $f_!(x) = f_*(w_{\underline{\tau}}(r_f)x)$, so $f_! : N(X)[\tau] \longrightarrow N(Y)[\tau]$ for proper $f : X \longrightarrow Y$.

We check the axioms: that $f^! g^! = (gf)^!$; $id^! = id$, $f \simeq g \implies f^! = g^!$ is clear.

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are proper maps, note $\nu_{gf} = f^* \nu_g + \nu_f$, so

$$\begin{aligned} g_! f_!(x) &= g_*(w_{\underline{\tau}}(\nu_g) f_*(w_{\underline{\tau}}(\nu_f)x)) \\ &= g_* f_*((f^* w_{\underline{\tau}}(\nu_g)) w_{\underline{\tau}}(\nu_f)x) \\ &= (gf)_*(w_{\underline{\tau}}(\nu_{gf})x) \\ &= (gf)_! x \end{aligned}$$

and clearly $id_! = id$. Given a transversal cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & {\scriptstyle f}\downarrow \\ Y' & \xrightarrow{g} & Y \end{array}$$

we have $(g')^*\nu_f = \nu_{f'}$: this is clear for $f$ a submersion and for $f$ an imbedding by transversality Any $f$ factors into a submersion followed by an imbedding. If $f$ is proper,

$$\begin{aligned} g^! f_! x &= g^* f_*(w_{\underline{T}}(\nu_f)x) \\ &= f'_* g'^*(w_{\underline{T}}(\nu_f)x) \\ &= f'_*(w_{\underline{T}}(\nu_{f'})g'^* x) \\ &= f'_! g'^! x. \end{aligned}$$

Since the axioms hold, there exist a unique $s_{\underline{T}} : N(X) \longrightarrow N(X)[\underline{T}]$ such that

i. $s_{\underline{T}} f^* = f^! s_{\underline{T}}$

ii. $s_{\underline{T}} f_* = f'_! s_{\underline{T}}$ and

iii. $s_{\underline{T}} 1_{pt} = 1_{pt}$,

given by $s_{\underline{T}}[Z \xrightarrow{g} X] = g_! 1_Z = g_*(w_{\underline{T}}(\nu_g))$.

To prove iii. it suffices to prove it for the external product $\boxtimes$.

Let $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ be proper maps; then

$$\begin{aligned} s_{\underline{T}}(f_* 1 \boxtimes f'_* 1) &= s_{\underline{T}}(f \times f')_* 1 \\ &= (f \times f')_*(w_{\underline{T}}(\nu_{f \times f'})) \\ &= (f \times f')_*(w_{\underline{T}}(pr_1^* \nu_f \otimes pr_2^* \nu_{f'})) \\ &= (f \times f')_*(pr_1^* w_{\underline{T}}(\nu_f) \cdot pr_2^* w_{\underline{T}}(\nu_{f'})) \\ &= f_* w_{\underline{T}}(\nu_f) \boxtimes f'_* w_{\underline{T}}(\nu_{f'}) \\ &= s_{\underline{T}} f_* 1 \boxtimes s_{\underline{T}} f'_* 1. \end{aligned}$$

For iv. let $i : X \longrightarrow L$ be the zero section, $\nu_i = L$ and $i^* i_* 1 = e(L)$.

$$\begin{aligned} s_{\underline{T}} e(L) &= i^* s_{\underline{T}} i_* 1 \\ &= i^*(i_* w_{\underline{T}}(\nu_i)) \\ &= i^* i_*(\sum_{j \geq 0} t_j e(L)^j) \\ &= \sum_{j \geq 0} t_j e(L)^{j+1}. \end{aligned}$$

We can write $s_{\underline{\tau}} = \sum_{\alpha \geq 0} t^{\alpha} s_{\alpha}$ as before, then $s_{\alpha} f_* 1 = f_* w_{\alpha}(\nu_f)$.

The $s_{\alpha} : N^j(X) \longrightarrow N^{j+|\alpha|}(X)$ are called the *Landweber-Novikov Operations.*

If we think of $\underline{\tau}$ of consisting of generic elements of $N(X)$, $s_{\underline{\tau}}$ becomes an endomorphism of $N(X)$. $s_{\underline{\tau}}$ can then be taken to transform a complicated formal group law into a simpler one. We do this in the next chapter.

# Chapter 13

# The Lazard ring and $N(x)$

Let $A_{ij}$ denote indeterminants, and let $F_\mu = \sum A_{ij}x^iy^j$ be a formal power series. The *Lazard ring*, Laz, is the $\mathbb{Z}_2$-algebra $\mathbb{Z}_2[A_{ij}]_{i,j\geq 0}$ modulo the ideal generated by the coefficient of the following power series:

$$\begin{cases} F_\mu(x, F_\mu(y,z)) - F_\mu(F_\mu(x,y),z) \\ F_\mu(x,y) - F_\mu(y,x) \\ F_\mu(x,0) - x \\ F_\mu(x,y) \end{cases}$$

Thus Laz is provided with a natural formal group law $F_\mu$ (the universal formal group law) such that given any $\mathbb{Z}_2$-algebra $\mathcal{R}$,

$$\mathrm{Hom}_{\mathbb{Z}_2-alg}(\mathrm{Laz}, \mathcal{R}) = \{F \in \mathcal{R}[[x,y]]/F \text{ is a commutative formal}$$

$$\text{group law of height } \infty\text{; i.e. } F(x,x) = 0\}.$$

We call the latter set $\mathcal{F}(\mathcal{R}$. Let $g(\mathcal{R})$ be the group of formal power series $\sum_{j\geq 0} r_j t^{j+1}$ with $r_0 = 1$, $r_j \in \mathcal{R}$, under composition $g(\mathcal{R})$ acts on $\mathcal{F}(\mathcal{R})$ by the formula: let $f \in g(\mathcal{R})$, $F \in \mathcal{F}(\mathcal{R})$, the $(f * F)(x,y) = f(F(f^{-1}x, f^{-1}y))$. We have a group acting on a set, so we can make te set into a category with morphisms elements of the group. This category dependes on $\mathcal{R}$.

Let $\mathcal{R}$ be a $Z_2$-algebra and $a : \mathrm{Laz} \longrightarrow \mathcal{R}$ a homomorphism (there is a unique $F_a \in \mathcal{F}(\mathcal{R})$ corresponding to $a$). Let $\alpha : \mathrm{Laz} \longrightarrow N(pt)$ be the homomorphism carrying $F_\mu$ into $F$, the formal group law of $N(pt)$.

$$
\begin{array}{ccc}
\mathrm{Laz} & \xrightarrow{a} & \mathcal{R} \\
\downarrow{\scriptstyle \alpha} & & \\
N(pt) & & \downarrow \\
\downarrow & & \\
N(X) & \longrightarrow & \mathcal{R} \otimes_{\mathrm{Laz}} N(X)
\end{array}
$$

Let $N_a(X)$ denote $\mathcal{R} \otimes_{\mathrm{Laz}} N(X)$. If $f : X \longrightarrow Y$, let

$$f^* : N_a(Y) \longrightarrow N_a(X) : r\otimes \mapsto r \otimes f^*y.$$

If $f$ is proper, let

$$f_* N_a(X) \longrightarrow N_a(Y) : r \otimes x \mapsto r \otimes f_*x.$$

Then $N_a$ is a functor on manifolds with Gysin homomorphism and products. In particular we have Euler classes for line bundles: $e_a(L) = 1 \otimes e(L)$.

The formal group law over $N_a(pt)$ describing $e_a(L \otimes L')$ is clearly $1 \otimes F = F_a \otimes 1$ so, identifying $r \otimes 1$ with $r$, we have $e_a(L \otimes L') = F_a(e_aL, e_aL')$.

Suppose we are given a power series $\varphi(t) = \sum\limits_{j\geq 0} r_j t^{j+1} \in g(\mathcal{R}), r_0 = 1$ and $r_j \in \mathcal{R}$.

**Theorem.** *Given $\varphi \in g(\mathcal{R})$, there exists a natural transformation $\hat{\varphi} : N_a(X) \longrightarrow N_b(X)$ such that:*

*i. $\varphi$ is an $\mathcal{R}$-linear ring homoprohism;*

*ii. $\varphi(e_aL) = \sum\limits_{j\geq 0} r_j(e_bL)^{jH} = \varphi(e_bL)$ for $L$ a line bundle, and*

*iii. $F_a(x, y) = \varphi(F_b(\varphi^{-1}x, \varphi^{-1}y))$.*

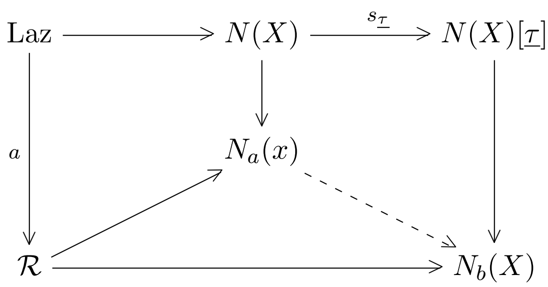
*Proof.* Let $\beta : N(X) \longrightarrow N_b(X)$ be the ring homomorphism such that $\beta(x) = \sum r^\alpha s_\alpha(x)$.

$$
\begin{array}{ccccc}
N(X) & \xrightarrow{s_{\underline{\tau}}} & N(X)[\underline{\tau}] & \qquad & t^\alpha x \\
& \searrow{\scriptstyle \beta} & \downarrow & & \downarrow \\
& & N_b(X) = \mathcal{R} \otimes_{\mathrm{Laz}} N(X) & & r^\alpha x
\end{array}
$$

$\beta(eL) = \sum_{j\geq 0} r_j e_b(L)^{j+1} = \varphi(e_bL)$, since $s_{\underline{\tau}}(eL) = \sum_{j\geq 0} t_j e(L)^{j+1}$.

$$
\begin{aligned}
varphi(F_b(e_bL, e_bL')) &= \varphi(e_b(L \otimes L')) \\
&= \beta(e(L \otimes L')) \\
&= \beta(F(eL, eL')) \\
&= (\beta F)(\beta(eL), \beta(eL')) = (\beta F)(\varphi e_bL, \varphi e_bL').
\end{aligned}
$$

Hence $(\beta F)(x,y) = \varphi(F_b(\varphi^{-1}x, \varphi^{-1}y))$. We have established that the diagram:




commutes. This implies there exists a unique homomorphism $\hat{\varphi} : \mathcal{R} \otimes_{\text{Laz}} N(X) \longrightarrow N_b(X)$ such that $\hat{\varphi}(r \otimes x) = r\beta x$. $\hat{\varphi}$ is a ring homomorphism since $\beta$ is, and $\mathcal{R} - linear$. $\hat{\varphi} e_a L = \hat{\varphi}(1 \otimes eL) = \beta(eL) = \varphi(eL)$

$$\begin{aligned}
\varphi(F_b(e_b L, e_b L')) &= \varphi(e_b(L \otimes L')) \\
&= \hat{\varphi}(e_b(L \otimes L')) \\
&= \hat{\varphi}(e_a(L \otimes L')) \\
&= \hat{\varphi}(F_a(e_a L, e_a L')) \\
&= F_a(\hat{\varphi} e_a L, \hat{\varphi} e_a L') \\
&= F_a(\varphi e_b L, \varphi e_b L')
\end{aligned}$$

Apply this to $L = pr_1^* O(1), L' = pr_2^* O(2)$ on $\mathbb{R}P^n \times \mathbb{R}P^n$ and use that $N_b(\mathbb{R}P^n \times \mathbb{R}P^n) = N_b(pt)[e_b L, e_b L']/(e_b(L)^n, e_b(L')^n)$. (freenes of $N_b(\mathbb{R}P^n)$) is guaranteed by $N_b(X) = (\mathcal{R} \otimes_{\text{Laz}} N(pt)) \otimes_{N(pt)} N(X)]$. Hence $F_a(x,y) = \varphi(F_b(\varphi^{-1}x, \varphi^{-1}y))$. $\square$

$$N(X) \xrightarrow{s_{\underline{t}}} N(X)[\underline{\tau}] \xrightarrow{s_{\underline{u}}} N(X)[\underline{t}, \underline{u}]$$

Define $v_i \in \mathbb{Z}_2[\tau_1, \ldots, \tau_i, u_1, \cdots, u_i]$ by

$$\begin{aligned}
\sum v_i T^{i+1} &= \sum \tau_j T^{j+1} \circ \sum u_k T^{k+1} \\
&= \sum t_j (\sum u_k T^{k+1})^{j+1}
\end{aligned}$$

Forming a few $v_i$ integrally we get

$$\begin{aligned}
v_0 &= 1 \\
v_1 &= u_1 + t_1 \\
v_2 &= u_2 + 2u_1 t_1 + t_2
\end{aligned}$$

**Proposition.** *With* $\underline{v} = (v_1, v_2, \cdots)$ *as above,* $s_{\underline{u}} s_{\underline{t}} = s_{\underline{v}}$*, i.e.*

$$\begin{aligned} s_{\underline{u}} s_{\underline{t}} x &= \sum u^\beta s_\beta (\sum t^\alpha s_\alpha x) \\ &= \sum u^\beta t^\alpha s_\beta s_\alpha x \qquad \textit{is equal to} s_{\underline{v}} x = \sum v(u,t)^\gamma s_\gamma(\alpha). \end{aligned}$$

*Proof.*

$$\begin{aligned} s_{\underline{u}} s_{\underline{t}} f_* 1 &= s_{\underline{u}} f_*(w_{\underline{t}}(\nu_f)) \\ &= f_*(w_{\underline{u}}(\nu_f) \cdot s_{\underline{u}} w_{\underline{t}}(\nu_f)), \\ s_{\underline{v}} f_* 1 &= f_*(w_{\underline{v}}(\nu_f)), \end{aligned}$$

so it suffices to show $w_{\underline{u}}(E) s_{\underline{u}} w_{\underline{t}}(E) = w_{\underline{v}(E)}$ for any vector bundle $E$. Since both sides are exponential characteristic classes, it suffices to check equality for line bundles $L$ by the splitting principle.

$$\begin{aligned} w_{\underline{u}}(L) s_{\underline{u}} w_{\underline{t}}(L) &= (\sum u_i e(L)^i) s_{\underline{u}} (\sum t_j e(L)^j) \\ &= (\sum u_i e(L)^i)(\sum t_j s_{\underline{u}}(e(L))^j) \\ &= (\sum u_i e(L)^i)(\sum t_j (\sum u_i e(L)^{i+1})^j), \\ w_{\underline{v}}(L) &= \sum v_k e(L)^k. \end{aligned}$$

We need only to show the equality of the power series $\sum v_k T^k$ and $(\sum u_i T^i)(\sum t_j (\sum u_i T^{i+1})^j)$.

Multiplying these by $T$, we get $\sum v_k T^{k+1}$ and $\sum t_j (\sum u_i T^{i+1})^{j+1}$, which are equal by the definition of $\underline{v}$. □

**Theorem.** *Given* $\varphi, \psi \in \mathcal{G}(\mathcal{R})$*,* $F_a, F_b, F_c \in \mathcal{F}(\mathcal{R})$*, such that* $F_a = \varphi * F_b$*,* $F_b = \psi * F_c$*, then the diagram*

$$\begin{array}{ccc} N_a(X) & \xrightarrow{\hat{\varphi}} & N_b(X) \\ & {\scriptstyle (\varphi\psi)\hat{}} \searrow \quad \swarrow {\scriptstyle \hat{\psi}} & \\ & N_c(X) & \end{array}$$

*commutes.*

*Proof.* Immediate from proposition. □

**Corollary.** $\hat{\varphi} : N_a(X) \longrightarrow N_b(X)$ *is an isomorphism.*

*Proof.* $(\varphi^{-1})\hat{} : N_b(X) \longrightarrow N_a(X)$ and $(\varphi^{-1})\hat{}\hat{\varphi} = (\varphi\hat{\varphi}^{-1}) = \hat{i}d = id$. □

# Chapter 14

# Determination of $N^*(pt)$: II

We assume, for the moment, the following proposition.

**Proposition.** *Given a formal group law $F(x,y)$ over a $\mathbb{Z}_2-$algebra, $\mathcal{R}$ such that $F(x,x) = 0$ there exists a opower series $l(x) = \sum_{i\geq 0} r_i x^{i+1}$, $r_0 = 1$, $r_i \in \mathcal{R}$, such that $l(F(x,y)) = l(x) + l(y)$.*

*Furthermore, $l$ is uniquely determined by the requirement that it have no terms of degree $2^r, r \geq 1$.*

The unique series of the proposition is the *canonical logarithm.* The proposition gives a 1-1 correspondence between formal group laws $F$ over $\mathcal{R}$ and their canonical logarithms because $F(x,y) = l^{-1}(l(x) + l(y))$.

**Theorem.** *Let $c$ : Laz $\longrightarrow N(pt)$ be the homomorphism sending $F_\mu$ to $F$, and let $l = \sum a_j T^{j+1}$ be the canonical logarithm of $F$. Then there is a natural isomorphism of rings*

$$\text{Laz} \otimes_{\mathbb{Z}_2} (\mathbb{Z}_2 \otimes_{\text{Laz}} N(X)) \xrightarrow{\simeq} N(x)$$

*given by*

$$m \otimes (1 \otimes x) \longrightarrow c(m) \sum_{\alpha \geq 0} a^\alpha s_\alpha(x).$$

*Proof.* Let $b = id$ : Laz $\longrightarrow$ Laz; $\varepsilon$ : Laz $\longrightarrow \mathbb{Z}_2$ be the augmentation; and $a$ : Laz $\longrightarrow$ Laz be the composition

$$\text{Laz} \xrightarrow{\varepsilon} \mathbb{Z}_2 \longrightarrow \text{Laz}\,.$$

Then $F_b = F_\mu$ and $F_a(x,y) = x + y$, and $l * F_b = F_a$. Hence $\hat{l} : N_a(X) :\xrightarrow{\simeq} N_b(X)$.

$$\begin{aligned} N_a(X) &= \mathrm{Laz} \otimes N(X) \\ &= (\mathrm{Laz} \otimes \mathbb{Z}_2) \otimes N(X) \\ &= \mathrm{Laz} \otimes (\mathbb{Z}_2 \otimes N(X)) \\ N_b(X) &= \mathrm{Laz} \otimes N(X) = N(X). \end{aligned}$$

More explicitly:

$$\begin{array}{ccc} N(X) & \xrightarrow{\quad s_{\underline{t}} \quad} & N[X][\underline{t}] \\ \Big\downarrow i & & \Big\downarrow j \\ \mathbb{Z}_2 \otimes_{\mathrm{Laz}} N(X) & \dashrightarrow & N(X) \end{array}$$

Where $i$ is universal for $F \longrightarrow x + y$ and $j : \sum t_j T^{j+1} \mapsto l(T)$, i.e. $t_j \mapsto a_j$.

$js_{\underline{t}}$ gives $N(X)$ new "additive" Euler classes. $\mathbb{Z}_2 \otimes_{\mathrm{Laz}} N(X)$ is $N(X)$ modulo coefficients of higher order terms of the formal group law (of degree ¿1)]. Now extend the dotted arrow linearly over Laz by means of $c : \mathrm{Laz} \longrightarrow N(X)$.

$$\begin{array}{ccc} \mathbb{Z}_2 \otimes_{\mathrm{Laz}} N(X) & \longrightarrow & N(X) \\ \Big\downarrow & \nearrow \simeq & \\ \mathrm{Laz} \otimes_{\mathbb{Z}_2} (\mathbb{Z}_2 \otimes_{\mathrm{Laz}} N(X)) & & \end{array}$$

By the above, it is an isomorphism, and, clearly,

$$m \otimes (1 \otimes x) \longrightarrow c(m) \sum a^\alpha s_\alpha(x).$$

□

**Theorem.** $\mathrm{Laz} \xrightarrow{\simeq} N(pt)$.

*Proof.* We have $\mathrm{Laz} \otimes_{\mathbb{Z}_2} (\mathbb{Z}_2 \otimes_{\mathrm{Laz}} N(pt)) \xrightarrow{\simeq} N(pt)$, but by Chapter X, $\mathrm{Laz} \longrightarrow N(pt)$ is surjective so $Z_2 \otimes_{\mathrm{Laz}} N(pt) = \mathbb{Z}_2$. This shows that the law $F$ over $N(pt)$ is the universal formal group law which is commutative of height $\infty$ □

The preceding can be carried through for graded rings with all isomorphisms preserving grading. By the proposition we assumed at the beginning of this chapter, Laz is actually a polynomial ring $\mathbb{Z}_2[a_n]$ with $n \geq 1, n \neq 2^j - 1, j > 0$, where $a_n$ are the coefficients of the canonical logarithm of $F$.

**Remark:** for even $n$, the $x_n$ are the classes of the $\mathbb{R}P^n$.

**Theorem.** $\mathbb{Z}_2 \otimes_{N(pt)} N(X) \xrightarrow{\simeq} H^*(X;\mathbb{Z})$.

*Proof.* We have a unique map $N(X) \longrightarrow H^*(X;\mathbb{Z}_2)$ commuting with $f_*, f^*$ because of the universal property of $N$. The isomorphism constructed above $\mathrm{Laz} \otimes_{\mathbb{Z}_2} (\mathbb{Z}_2 \otimes_{\mathrm{Laz}} N(X)) \xrightarrow{\simeq} N(X)$ is compatible with suspension and shows that the generalized cohomology theorey $N(X)$ is a direct sum of copies of $Z_2 \otimes_{N(pt)} N(X)$, so, since any direct summand of a generalized cohomology theory is again a cohomology theory, $\mathbb{Z}_2 \otimes_{N(pt)} N(X)$ is. But $X = pt$, $\mathbb{Z}_2 \otimes_{N(pt)} N(pt) = \mathbb{Z}_2$, so $\mathbb{Z}_2 \otimes_{N(pt)} N(X)$ satisfies the dimension axioms. Hence by the Eilenberg-Steenrod uniqueness dimension theorem $\mathbb{Z}_2 \otimes_{N(pt)} N(X) \longrightarrow H^*(X, \mathbb{Z}_2)$ is an isomorphism. $\square$