Name: ____________________________

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Math 16100 Midterm

Autumn Quarter 2025

Monday, October 10, 2025

Name:

Instructions:
Show *all* your work (unless otherwise noted). This test has two problems and four pages. Good luck!

Prob.	Possible points	Score
1	50	
2	25	
3	25	
TOTAL	100	

(P1)	(Associative law for addition)	$a + (b + c) = (a + b) + c.$
(P2)	(Existence of an additive identity)	$a + 0 = 0 + a = a.$
(P3)	(Existence of additive inverses)	$a + (-a) = (-a) + a = 0.$
(P4)	(Commutative law for addition)	$a + b = b + a.$
(P5)	(Associative law for multiplication)	$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$
(P6)	(Existence of a multiplicative identity)	$a \cdot 1 = 1 \cdot a = a; \quad 1 \neq 0.$
(P7)	(Existence of multiplicative inverses)	$a \cdot a^{-1} = a^{-1} \cdot a = 1$, for $a \neq 0$.
(P8)	(Commutative law for multiplication)	$a \cdot b = b \cdot a.$
(P9)	(Distributive law)	$a \cdot (b + c) = a \cdot b + a \cdot c.$

(P10) (Trichotomy law) For every number a, one and only one of the following holds:

(i) $a = 0$,
(ii) a is in the collection P,
(iii) $-a$ is in the collection P.

(P11) (Closure under addition) If a and b are in P, then $a + b$ is in P.
(P12) (Closure under multiplication) If a and b are in P, then $a \cdot b$ is in P.

These three properties should be complemented with the following definitions:

$$
\begin{aligned}
a > b &\quad \text{if} \quad a - b \text{ is in } P; \\
a < b &\quad \text{if} \quad b > a; \\
a \geq b &\quad \text{if} \quad a > b \text{ or } a = b; \\
a \leq b &\quad \text{if} \quad a < b \text{ or } a = b.*
\end{aligned}
$$

Theorem 1: [Proved in Class]

1. For any x, $0 \cdot x = x \cdot 0 = 0$.
2. We have $(-1)^2 = 1$.

Question 1. (25 + 25 points) Give proofs of the following claims, assuming the axioms above as well as Theorem 1. You may use axioms $P1$–$P9$ ***without mention***, but you should make ***explicit*** any use of $P10$–$P12$ or Theorem 1.

1. If $a > 0$ and $x \geq y$, then $ax \geq ay$.

Answer: The hypothesis $x \geq y$ means that either $x = y$ or $x > y$. We consider both cases in turn.

Case 1: $(x = y)$. If $x = y$, then $ax = ay$. This certainly implies that $ax \geq ay$.

Case 2: $(x > y)$. If $x > y$ then $x - y \in P$. By $P12$, and the assumption $a > 0$, we deduce that $a(x-y) > 0$. By $P9$, we have $a(x-y) = ax-ay$, and hence $ax-ay \in P$. By definition, this means that $ax > ay$, and so $ax \geq ay$.

In both cases, we conclude that $ax \geq ay$, and hence we are done. □

2. If x is a number, then $x^2 \geq 0$.

Answer: Either $x > 0$, $x = 0$, or $x < 0$ by $P10$. We consider each case in turn.

Case 1: $(x > 0)$. If $x > 0$, then $x^2 > 0$ by $P12$.

Case 2: $(x = 0)$. If $x = 0$, then $x^2 = 0^2 = 0$ by Theorem 1, and thus $x^2 \geq 0$.

Case 2: $(x < 0)$. If $x < 0$, then $-x > 0$ by $P10$. But nowe we have

$$\begin{aligned} x^2 &= 1 \cdot x^2 \\ &= (-1)^2 \cdot x^2 \text{ by Theorem 1} \\ &= -1 \cdot -1 \cdot x \cdot x = (-1 \cdot x) \cdot (-1 \cdot x) \text{ by } P5 \text{ and } P8 \\ &= (-x) \cdot (-x) = (-x)^2 \end{aligned}$$

But now $(-x)^2$ is positive by $P10$ (or by Case 1).

In all cases, we conclude that $x^2 \geq 0$, and hence we are done. $\square$

Question 2. (25 points) Let x and y be positive. Prove the inequality below. You do not need to argue from the axioms, but your proof should be rigorous.

$$\frac{x+y}{2} \geq \frac{2}{\frac{1}{x}+\frac{1}{y}}$$

Since x and y are positive, so is xy, and hence so is $(xy)^{-1}$. Thus

$$\begin{aligned}
(x-y)^2 &\geq 0, \text{ because squares are non-negative, e.g. Question 1} \\
\frac{(x-y)^2}{xy} &\geq 0, \text{ dividing by } xy \neq 0 \text{ which is positive} \\
\frac{x^2-2xy+y^2}{xy} &\geq 0, \text{ by expanding out} \\
\frac{x}{y}-2+\frac{y}{x} &\geq 0, \text{ by expanding out some more} \\
\frac{x}{y}+2+\frac{y}{x} &\geq 4, \text{ by adding 4 to both sides} \\
(x+y)\left(\frac{1}{x}+\frac{1}{y}\right) &\geq 4, \text{ since the LHS expands out to the previous line}
\end{aligned}$$

Now dividing both sides by $2(1/x+1/y)$ (which is positive), we get

$$\frac{x+y}{2} \geq \frac{2}{\frac{1}{x}+\frac{1}{y}} \quad \square$$

Remark: The AM-GM inequality is $\frac{x+y}{2} \geq \sqrt{xy}$. Applying this to $1/x$ and $1/y$, we get:

$$\frac{\frac{1}{x}+\frac{1}{y}}{2} \geq \sqrt{\frac{1}{xy}}$$

For positive a, b, $a > b$ implies that $1/b > 1/a$, so this implies

$$\sqrt{xy} \geq \frac{2}{\frac{1}{x}+\frac{1}{y}}$$

Combining this with AM-GM for x and y we get

$$\frac{x+y}{2} \geq \sqrt{xy} \geq \frac{2}{\frac{1}{x}+\frac{1}{y}}$$

which is sometimes called the AM-GM-HM inequality (arithmetic mean, geometric mean, and harmonic mean).

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Question 3. (25 points) Suppose that α and β are numbers such that $\alpha\beta$ and $\alpha+\beta$ are both integers. Prove that $\alpha^n+\beta^n$ is an integer for each $n \in \mathbf{N} = \{1,2,3,4,\dots,\}$.

Let $P(n)$ be the statement that $\alpha^n+\beta^n$ is an integer. We proceed by (strong) induction.

Base cases: Since $\alpha+\beta$ is an integer by assumption we have $P(1)$

We have $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$. By assumption both $\alpha+\beta$ and $\alpha\beta$ are integers, so $\alpha^2+\beta^2$ is also an integer, thus $P(2)$.

Induction : We show, for $k \geq 1$, that $P(k)$ and $P(k+1)$ together imply $P(k+2)$. Since we know $P(1)$ and $P(2)$, this suffices to prove the general result by induction.

By assumption and induction, we have $\alpha+\beta$, $\alpha\beta$, $\alpha^k+\beta^k$ and $\alpha^{k+1}+\beta^{k+1}$ are all integers. Thus certainly

$$X = (\alpha+\beta)(\alpha^{k+1}+\beta^{k+1}) - \alpha\beta(\alpha^k+\beta^k)$$

is also an integer. But expanding this out, we get

$$\begin{aligned}
X &= \alpha^{k+2}+\beta\alpha^{k+1}+\alpha\beta^{k+1}+\beta^{k+2} - \left(\alpha^{k+1}\beta+\alpha\beta^{k+1}\right). \\
&= \alpha^{k+2}+\beta^{k+2}+(\beta\alpha^{k+1}-\alpha^{k+1}\beta)+(\alpha\beta^{k+1}-\alpha\beta^{k+1}), \text{ rearranging terms} \\
&= \alpha^{k+2}+\beta^{k+2}.
\end{aligned}$$

Hence $\alpha^{k+2}+\beta^{k+2}$ is an integer, which is the claim $P(k+2)$ to be proven. Hence by induction we deduce that $P(n)$ is true for all $n \in \mathbf{N}$.

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