Some problems to think about

These problems are supposed to make you think a little bit more broadly about the course, and are supposed to be in addition rather than instead of problems in Spivak. I plan to add some more problems soon.

1. **Isolated zeros**. Let $f(x)$ be a function and suppose that $f(c) = 0$. Say that c is an *isolated* zero if there exists a $\delta > 0$ so that for $x \in (c-\delta, c+\delta)$ with $f(x) = 0$ then $x = c$.

   (a) Prove that if $f(x)$ is a polynomial and $f(c) = 0$, then c is an isolated zero of $f(x)$.

   (b) Show that it is possible for a differentiable function $f(x)$ to have a zero c which is not isolated.

2. Let $a(x)$ be a function of x and suppose that $a(0) = 0$ and $a(x)$ is continuous at $x = 0$.

   (a) If $f(x)$ is continuous at $x = 0$, prove that
   $$\lim_{h\to 0} f(a(h)) = f(0).$$

   (b) Suppose that $x = 0$ is an isolated zero of $a(x)$. If $f(x)$ is differentiable at $x = 0$, prove that
   $$\lim_{h\to 0} \frac{f(a(h)) - f(0)}{a(h)} = f'(0).$$

   (c) Suppose that $a(x)$ is differentiable at $x = 0$. Prove that
   $$\lim_{h\to 0} \frac{a(h)}{h} = a'(0).$$
   (Warning: this is easy.)

   (d) Suppose that $x = 0$ is an isolated zero of $a(x)$ and $a(x)$ and $f(x)$ are differentiable at $x = 0$. Prove that
   $$\lim_{h\to 0} \frac{f(a(h)) - f(0)}{h} = f'(0)a'(0).$$
   Show this is a case of the chain rule applied to the derivative of $f(a(x))$ at $x = 0$.

   (e) If $x = 0$ is not an isolated zero of $a(x)$ then things are more subtle. This is exactly what happens if one wants to prove the chain rule in general. Think about how one could try to extend the argument above then look at what Spivak does.

3. If $f(x)$ and $g(x)$ are differentiable then the derivative of $f(x)g(x)$ is $f'(x)g(x) + f(x)g'(x)$. Suppose that $1/f(x)$ is differentiable. Use the product rule to show that its derivative is
   $$-\frac{f'(x)}{f(x)^2}.$$

4. **Calculus in higher dimensions**. Let $f(x,y)$ be a function in two variables. What ever the "derivative(s)" of $f(x,y)$ might mean, we should hope it tells us something about linear approximations. In particular, around the point $(x,y) = (0,0)$, we might hope that for x and y small we have an approximation:
   $$f(x,y) \sim f(0,0) + ax + by.$$

   (a) How would you try to define a or b in terms of some limit involving $f(x,y)$?

(b) How would you try to define the continuity of $f(x,y)$? Give it a try.

(c) Suppose that $f(x,y)$ is continuous in your new definition on the region $[0,1]^2$, that is, $x,y \in [0,1]$. Does $f(x,y)$ have to have a maximum value in this region? If $f(x,y)$ is differentiable, how would you try to find such a maximum?

5. **Increasing functions**. A function $f(x)$ is increasing on the interval $[a,b]$ if, for any $x,y \in [a,b]$ with $x > y$ we have $f(x) > f(y)$.

   (a) If $f(x)$ is increasing on $[a,b]$ and differentiable, prove that the derivative $f'(x)$ is non-negative on this interval.

   (b) Find a function which is increasing on all of $\mathbf{R}$ and differentiable, but also has a critical point (that is, a c with $f'(c) = 0$. Can it have infinitely many critical points?

   (c) If $f(x)$ is differentiable and $f'(x) > 0$ for $x \in [a,b]$, prove that $f(x)$ is increasing on this interval. (Hint: use the mean value theorem.)

   (d) Suppose that $f(x)$ is increasing on $[a,b]$ and continuous, Suppose that y satisfies $f(a) \le y \le f(b)$. Prove that there is a unique $x \in [a,b]$ so that $f(x) = y$.

6. Suppose that $f(x)$ is a differentiable function which satisfies $f'(x) = f(x)$.

   (a) Prove that the derivative of $f(x^2)$ is $2xf(x^2)$.

   (b) Prove that there exists a polynomial $P_n(x)$ of degree n so that the nth derivative of $f(x^2)$ is $P_n(x)f(x^2)$, so $P_0(x) = 1$ and $P_1(x) = 2x$.

   (c) Find the general formula for $P_n(x)$. (Hint for this and many questions: compute lots of examples, try to find a pattern, then prove it by induction.)

7. Try to define the function $f(x) = 2^x$ and then prove it is continuous.

8. Try to convince yourself it is not possible to move on a line with position $p(t)$ at time t so that $p(t)$ is not differentiable. Then try to decide if it is possible that $p'(t)$ is not differentiable.