Week 8, Due Fri 11/21

1. Spivak, Chapter 9, Problem $5, 8, 19, 27$

2. Assume that $f(x)$ is a function with the following two properties:

   (a) $f(x+y) = f(x)f(y)$
   (b) $f(x)$ is differentiable at $x = 0$.

   Prove the following:

   (a) If $f(0) = 0$, then $f(x) = 0$ for all x.
   (b) If $f(0) \neq 0$, then $f(0) = 1$.
   (c) There is an equality $f'(x) = f'(0)f(x)$. Hint: use that

   $$f(x+h) - f(x) = f(x+h) - f(x)f(h) + f(x)f(h) - f(x).$$

3. Let $f(x)$ and $g(x)$ be two functions which are infinitely differentiable. Let $e(x) = f(x)g(x)$.

   (a) Prove that for all integers $n \geq 0$, we have

   $$e^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x). \tag{1}$$

   (b) Let $a, b \geq 0$ be integers. Specialize the identity (1) to $f(x) = x^a$ and $g(x) = x^b$ and $n = a + b$ and prove directly that both sides are the same. (You may use Spivak Chapter 9, Problem 27.)