Week 7, Due Fri 11/14

1. The function 2^x. We can now answer a question raised on the first day of class.

   (a) Let $a, b \in \mathbf{Z}$ with $b \neq 0$. Prove that x^b on $(0, \infty)$ is increasing if $b > 0$ and decreasing if $b < 0$.

   (b) Prove that there is a unique positive real number α with $\alpha^b = 2^a$. (This absolutely must use something like the intermediate value theorem — at this point the expression $2^{a/b}$ is not defined.)

   (c) Prove that the α of Part 1b only depends on $r = a/b$, not a and b themselves.

   (d) Deduce that one can define 2^r for a rational number r by the formula $2^r := \alpha$ with α constructed above for any choice of integers a, b, with $a/b = r$.

   (e) At this point, we can now define a function 2^x on the domain of rational numbers $\mathbf{Q}$. Prove that for any two rational numbers r and s, we have $2^{r+s} = 2^r \cdot 2^s$.

   (f) Prove that if $r > s$ are rational, then $2^r > 2^s$.

   (g) For a real number x, let $A_x = \{r \in \mathbf{Q} \mid r < x\}$. Let S_x be the set of values 2^r for $r \in A_x$. Prove that S_x is bounded above, and deduce that we may define a function $f(x)$ on $\mathbf{R}$ by the formula

   $$f(x) = \sup S_x.$$

   (h) If $x = r$ is a rational number, show that $f(r) = 2^r$.

   (i) ★ Prove that $f(x + y) = f(x)f(y)$ for all real numbers x and y.

   (j) ★ Prove that $f(x)$ is continuous for all x. Hence there exists a function 2^x for any real x which is increasing, satisfies $2^{x+y} = 2^x \cdot 2^y$, and such that for integers x, 2^x agrees with the usual definition.