Week 6, Due Fri 11/8

1. Spivak, Chapter 6, Problem $6, 7, 10$.

2. Suppose that $\lim_{x \to a} f(x)$ exists. Show that $\forall \epsilon > 0$, there $\exists \delta > 0$ such that, if x and y satisfy $0 < |x - a| < \delta$ and $0 < |y - a| < \delta$, then
$$|f(x) - f(y)| < \varepsilon.$$

3. Is the converse to the last statement true? Think about it — but don't submit anything.

4. (★) Suppose that $f(x)$ has domain $\mathbf{R}$ and that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$. Suppose that $f(1) = 0$ and $f(x)$ is continuous at $x = 0$. Prove that $f(x) = 0$.

5. Let $f(x)$ be continuous on the interval $(0, 1]$ which contains 1 but not 0. We say that $f(x)$ takes on its maximal (respectively minimal) value if there is an $a \in (0, 1]$ so that $f(a) \geq f(x)$ for all $x \in (0, 1]$, (respectively $f(x) \geq f(a)$ for all $x \in (0, 1]$).

   (a) Show that $f(x) = x$ does not take on a mimimal value and $g(x) = -x$ does not take on a maximal value.

   (b) (★) Must a continuous $f(x)$ on $(0, 1]$ with $|f(x)| \leq 1$ *either* take on its maximal or its minimal value?