Week 2, Due Fri 10/10

1. Spivak, Chapter 2, Problems: $3, 12, 19, 26$

2. Let $p_n = 1, 1, 3, 7, 17\dots$ and $q_n = 0, 1, 2, 5, 12, \dots$ be sequences defined by the initial values $p_0 = 1, p_1 = 1$ and $q_0 = 0$, $q_1 = 1$ and then

$$p_n = 2p_{n-1} + p_{n-2},$$
$$q_n = 2q_{n-1} + q_{n-2}.$$

   (a) Prove by induction that

$$p_n^2 - 2q_n^2 = (-1)^n.$$

   (b) Prove that, for $n \geq 1$, $\left|\sqrt{2} - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^2}$, and deduce that $\sqrt{2}$ is irrational.

3. Let $b > a$ be non-negative integers. Let

$$H_{a,b} = \frac{1}{a+1} + \frac{1}{a+2} + \dots + \frac{1}{b-1} + \frac{1}{b}$$

   be the sums of the reciprocals of all integers between $a+1$ and b, and let

$$H_n = H_{0,n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

   (a) Prove that $H_{a,b} \geq \frac{b-a}{b}$.

   (b) Prove by induction that $H_{2^n} \geq 1 + n \cdot \frac{1}{2}$ for all $n \in \mathbf{N}$.