Week 1, Due Fri 10/4

1. Let x be a number such that $x^2 = 2$ and assume that $x > 0$. Prove (using only the properties P1 to P12) that
$$\frac{17}{12} > x > \frac{17}{12} - \frac{1}{12^2}$$

2. Spivak, Chapter 1, Problems: $1, 7, 12, 19, 25$.

3. **Rational numbers are poorly approximated by other rational numbers . . . other than themselves.** Let α be a number (ultimately a real number but for now just in the sense of Spivak, Chapter 1.)

   (a) Suppose that $\alpha = a/b$ is a rational number. Suppose that p/q is another rational number which is different from α. Prove that
   $$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{|bq|}.$$

   (b) Suppose that $\alpha = a/b$ is a rational number. Suppose that p/q is another rational number which is different from α. Suppose that
   $$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^2}.$$
   Prove that $|b| \geq |q|$.

   (c) Drop the assumption that α is a rational number. Suppose there exist infinitely many rational numbers p_n/q_n with p_n and q_n and with $|q_n|$ increasing so that
   $$0 < \left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^2}.$$
   Prove that α is not a rational number.

   (d) (*) Let $p_n = 1, 1, 3, 7, 17 \ldots$ and $q_n = 0, 1, 2, 5, 12, \ldots$ be sequences defined by the initial values $p_0 = 1, p_1 = 1$ and $q_0 = 0$, $q_1 = 1$ and then
   $$p_n = 2p_{n-1} + p_{n-2},$$
   $$q_n = 2q_{n-1} + q_{n-2}.$$
   Prove that, for $n \geq 1$, $\left|\sqrt{2} - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^2}$, and deduce that $\sqrt{2}$ is irrational.