Week 1, Due Fri 10/3

1. Let x be a number such that $x^2 = 2$ and assume that $x > 0$. Prove (using only the properties P1 to P12) that
$$\frac{17}{12} > x$$

2. Spivak, Chapter 1, Problems: $1, 7, 12, 19, 25$.

3. **Rational numbers are poorly approximated by other rational numbers . . . other than themselves.** Let α be a number (ultimately a real number but for now just in the sense of Spivak, Chapter 1.)

   (a) Suppose that $\alpha = a/b$ is a rational number. Suppose that p/q is another rational number which is different from α. Prove that
   $$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{|bq|}.$$

   (b) Suppose that $\alpha = a/b$ is a rational number. Suppose that p/q is another rational number which is different from α. Suppose that
   $$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^2}.$$
   Prove that $|b| \geq |q|$.

   (c) Drop the assumption that α is a rational number. Suppose there exist infinitely many rational numbers p_n/q_n with p_n and q_n and with $|q_n|$ increasing so that
   $$0 < \left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^2}.$$
   Prove that α is not a rational number.