SOLUTIONS TO MIDTERM

Remarks. All proofs below use Spivak's axioms $P1$–$P12$ for the real numbers. We freely use the field axioms $P1$–$P9$ without comment, and we also assume the elementary facts $0 \cdot x = 0$ and $(-1)^2 = 1$ (these may be proved from the axioms but are taken here as available).

(1) (a) **Claim.** If $a > 0$ and $x \geq y$, then $ax \geq ay$.

Proof. By the meaning of $x \geq y$ we have two cases: either $x = y$ or $x > y$.
If $x = y$ then $ax = ay$ and hence $ax \geq ay$ holds.
Suppose $x > y$. Then $x - y > 0$. Since $a > 0$, the order-compatibility axiom for multiplication (Spivak's version of the statement "multiplying by a positive number preserves inequalities") gives

Comment: say $P12$

$$a(x - y) > 0.$$

But $a(x-y) = ax-ay$ by distributivity. Thus $ax-ay > 0$, i.e. $ax > ay$. Consequently $ax \geq ay$.
Combining the two cases yields the desired result. □

(b) **Claim.** For any number x we have $x^2 \geq 0$.

Proof. Either $x \geq 0$ or $x \leq 0$ (by trichotomy of the order).

Comment: say $P10$

If $x \geq 0$ then multiplying both sides of $x \geq 0$ by $x \geq 0$ (which is nonnegative) and using the order-compatibility of multiplication yields $x \cdot x \geq 0 \cdot x$. Since $0 \cdot x = 0$ by assumption, we obtain $x^2 \geq 0$.
If $x \leq 0$ then $-x \geq 0$. Applying the previous case to $-x$ gives $(-x)^2 \geq 0$. But $(-x)^2 = (-1)^2 x^2 = 1 \cdot x^2 = x^2$ (using the given fact $(-1)^2 = 1$ and field axioms). Hence $x^2 \geq 0$.

Comment: Say $P11$

Thus in either case $x^2 \geq 0$. □

Grade: This would probably get 45/50.

(2) Let $x, y > 0$. Define the arithmetic mean $A = \dfrac{x+y}{2}$ and the harmonic mean $H = \dfrac{2}{1/x + 1/y}$. Note that $H = \dfrac{2xy}{x+y}$. We prove the standard inequality $A \geq H$, with equality iff $x = y$. In particular, when $x \neq y$ this gives the strict inequality requested.

Comment: the question didn't ask about strict inequalities. This leads to some sloppiness below

Comment: This is circular, starting with the claim then woking backwards.

Proof. Starting from the algebraic equivalence,

$$A > H \iff \frac{x+y}{2} > \frac{2xy}{x+y}.$$

Since $x + y > 0$, multiply both sides by $2(x + y) > 0$ to obtain the equivalent inequality

$$(x + y)^2 > 4xy.$$

Expand the left-hand side:

$$(x + y)^2 - 4xy = x^2 + 2xy + y^2 - 4xy = x^2 - 2xy + y^2 = (x - y)^2.$$

Hence

$$(x + y)^2 - 4xy = (x - y)^2 \geq 0,$$

with equality exactly when $x - y = 0$, i.e. $x = y$. Therefore $(x+y)^2 \geq 4xy$, so $A \geq H$, and if $x \neq y$ then $(x-y)^2 > 0$ which gives $A > H$. This proves the claim. □

Grade: The ideas are here but the circular aspect is bad, and there is some sloppiness with the equality and inequality cases. 10/25.

(3) Suppose α, β are numbers with $\alpha\beta \in \mathbb{Z}$ and $\alpha + \beta \in \mathbb{Z}$. For each $n \in \mathbb{N}$ define

$$s_n := \alpha^n + \beta^n.$$

We show $s_n \in \mathbb{Z}$ for all positive integers n.

Proof. We first note initial values:

$$s_0 = \alpha^0 + \beta^0 = 1 + 1 = 2 \in \mathbb{Z}, \qquad s_1 = \alpha + \beta \in \mathbb{Z}$$

by hypothesis.

We establish the linear recurrence

$$s_{n+2} = (\alpha + \beta)\, s_{n+1} - (\alpha\beta)\, s_n \qquad \text{for all } n \geq 0.$$

To verify this identity, compute

$$(\alpha + \beta)s_{n+1} - (\alpha\beta)s_n = (\alpha + \beta)(\alpha^{n+1} + \beta^{n+1}) - (\alpha\beta)(\alpha^n + \beta^n).$$

Expanding,

$$= (\alpha^{n+2} + \alpha\beta\alpha^n + \alpha\beta\beta^n + \beta^{n+2}) - (\alpha\beta\alpha^n + \alpha\beta\beta^n) = \alpha^{n+2} + \beta^{n+2} = s_{n+2},$$

as required.

Now proceed by induction on n. The base cases s_0, s_1 are integers. Assume $s_k \in \mathbb{Z}$ for all $k \leq n + 1$. Then by the recurrence

$$s_{n+2} = (\alpha + \beta)s_{n+1} - (\alpha\beta)s_n,$$

the right-hand side is an integer because $\alpha + \beta$ and $\alpha\beta$ are integers by hypothesis and s_{n+1}, s_n are integers by the inductive hypothesis. Hence $s_{n+2} \in \mathbb{Z}$. This completes the induction, and therefore $s_n \in \mathbb{Z}$ for every nonnegative integer n. Restricting to positive n gives the desired conclusion. □

Grade: This is fine, 25/25.