Low-degree integral cohomology of finite groups

Approaches, partial results, and obstructions

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1 Conventions and the homological formulation

All group (co)homology has trivial coefficients unless another coefficient module is displayed. Thus $H^n(G)$ means $H^n(G;\mathbb{Z})$. Positive-degree integral homology groups of a finite group are finite.

Lemma 1.1 (Integral duality for finite groups)**.** *Let G be finite. Then $H^1(G;\mathbb{Z}) = 0$, and for every $n \geq 2$ there is a natural isomorphism*

$$H^n(G;\mathbb{Z}) \cong \mathrm{Hom}\big(H_{n-1}(G;\mathbb{Z}), \mathbb{Q}/\mathbb{Z}\big).$$

In particular, $H^n(G;\mathbb{Z}) = 0$ if and only if $H_{n-1}(G;\mathbb{Z}) = 0$.

Proof. The universal coefficient theorem gives

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(G;\mathbb{Z}),\mathbb{Z}) \longrightarrow H^n(G;\mathbb{Z}) \longrightarrow \mathrm{Hom}(H_n(G;\mathbb{Z}),\mathbb{Z}) \longrightarrow 0.$$

The last term vanishes because $H_n(G;\mathbb{Z})$ is finite. Applying $\mathrm{Hom}(H_{n-1}(G;\mathbb{Z}),-)$ to $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ identifies the first term with the displayed Pontryagin dual. The assertion in degree one follows directly from $H^1(G;\mathbb{Z}) = \mathrm{Hom}(G,\mathbb{Z})$. □

Corollary 1.2. *For a finite group G:*

(i) *$H^2(G;\mathbb{Z}) = 0$ if and only if G is perfect;*

(ii) *if G is perfect, then $H^3(G;\mathbb{Z})$ is the Pontryagin dual of the Schur multiplier $H_2(G;\mathbb{Z})$.*

Consequently, a group whose integral cohomology vanishes through degree three is superperfect.

Lemma 1.3 (Passage from mod p to integral cohomology)**.** *If $H^n(G;\mathbb{F}_p) \neq 0$, then at least one of $H^n(G;\mathbb{Z})_{(p)}$ and $H^{n+1}(G;\mathbb{Z})_{(p)}$ is nonzero.*

Proof. This follows from the universal coefficient exact sequence

$$0 \to H^n(G;\mathbb{Z}) \otimes \mathbb{F}_p \to H^n(G;\mathbb{F}_p) \to \mathrm{Tor}_1^{\mathbb{Z}}(H^{n+1}(G;\mathbb{Z}),\mathbb{F}_p) \to 0.$$

□

More generally, a proposed bound N is equivalent to the assertion that every nontrivial finite group has nonzero integral homology in one of the degrees $1,\dots,N-1$.

2 The sharp known lower obstruction

Theorem 2.1 (Milgram)**.** *For the Mathieu group M_{23},*

$$H_j(M_{23};\mathbb{Z}) = 0 \quad (1 \leq j \leq 4), \qquad H_5(M_{23};\mathbb{Z}) \cong \mathbb{Z}/7.$$

Consequently

$$H^i(M_{23};\mathbb{Z}) = 0 \quad (1 \leq i \leq 5), \qquad H^6(M_{23};\mathbb{Z}) \cong \mathbb{Z}/7.$$

At the prime two, the first nonzero integral homology group is $H_6(M_{23};\mathbb{Z})_{(2)} \cong \mathbb{Z}/2$.

Proof. The homology calculation is contained in Milgram's calculation of the cohomology of M_{23} [38]. The cohomological statement follows from Lemma 1.1. □

Corollary 2.2. *If an integer N has the property that every nontrivial finite group G has $H^i(G;\mathbb{Z}) \neq 0$ for some $1 \leq i \leq N$, then $N \geq 6$.*

This makes $N = 6$ the smallest possible bound. It does not suggest that any nearby value should be optimized before existence of a finite bound is proved.

3 Cyclic Sylow detection

Theorem 3.1 (Cyclic Sylow detection)**.** *Let $P \in \mathrm{Syl}_p(G)$ be cyclic and put*

$$e = |N_G(P) : C_G(P)|.$$

Then

$$H^{2e}(G;\mathbb{Z})_{(p)} \neq 0.$$

If P has order p, this group contains a subgroup isomorphic to $\mathbb{Z}/p$.

Proof. Write $H^*(P;\mathbb{Z}) \cong \mathbb{Z}[u]/(|P|u)$, where $|u| = 2$. The group $N_G(P)/C_G(P)$ is a p'-subgroup of $\mathrm{Aut}(P)$, hence is cyclic of order e. If an automorphism sends a generator of P to its a-th power, it sends u to au. It follows that $u^e \in H^{2e}(P;\mathbb{Z})$ is nonzero and invariant under the normalizer. Fusion in a cyclic Sylow subgroup is controlled by its normalizer, and the stable-elements theorem therefore identifies u^e with the restriction of a p-primary class of G; see [12, Chapter XII] or [9, Chapter VII]. □

Corollary 3.2 (Cyclic-Sylow vanishing below the first invariant)**.** *In the notation of Theorem 3.1, if $k \geq 1$ and $e \nmid k$, then*

$$H^{2k}(G;\mathbb{Z})_{(p)} = 0.$$

In particular, if $e \nmid 2$, then

$$H_3(G;\mathbb{Z})_{(p)} = 0.$$

For every cyclic Sylow p-subgroup, independently of e, one also has

$$H_2(G;\mathbb{Z})_{(p)} = 0. \tag{1}$$

Consequently, if G is perfect, $\widetilde{G}$ is its universal central extension, and $e \nmid 2$, then

$$H^4(\widetilde{G};\mathbb{Z})_{(p)} = 0. \tag{2}$$

Proof. Restriction to P is injective on p-primary integral cohomology, because its composite with corestriction is multiplication by the p-adic unit $[G : P]$. Its image is fixed by $N_G(P)$. A generator of the cyclic group $N_G(P)/C_G(P)$ acts on $H^{2k}(P;\mathbb{Z}) = \langle u^k \rangle$ as multiplication by a^k, where the reduction of a modulo p has order e: the kernel of $\mathrm{Aut}(P) \to \mathrm{Aut}(\Omega_1(P))$ is a p-group, whereas $N_G(P)/C_G(P)$ is a p'-group. If $e \nmid k$, then $a^k - 1$ is prime to p, so the invariant subgroup is zero. Injectivity of restriction proves the first assertion. The second follows from Lemma 1.1 with $2k = 4$.

For (1), use coefficients $\mathbb{Q}/\mathbb{Z}$. Restriction

$$H^2(G;\mathbb{Q}/\mathbb{Z})_{(p)} \longrightarrow H^2(P;\mathbb{Q}/\mathbb{Z})$$

is injective by restriction–corestriction, while the target is zero: for cyclic P it is $(\mathbb{Q}/\mathbb{Z})/|P|(\mathbb{Q}/\mathbb{Z}) = 0$. Since $\mathbb{Q}/\mathbb{Z}$ is injective, the universal coefficient theorem gives

$$H^2(G;\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(H_2(G;\mathbb{Z}),\mathbb{Q}/\mathbb{Z}),$$

which proves (1). The kernel $H_2(G;\mathbb{Z})$ of $\widetilde{G} \to G$ therefore has order prime to p. Lemma 5.1, together with the already proved vanishing of $H^4(G;\mathbb{Z})_{(p)}$, gives (2). □

4 Finite simple groups

Theorem 4.1 (Finite simple groups)**.** *Let S be a nontrivial finite simple group. Then*

$$H^i(S;\mathbb{Z}) \neq 0 \qquad \textit{for some } 2 \leq i \leq 8.$$

More precisely:

(i) *if $S \cong C_p$, one may take $i = 2$;*

(ii) *if $S \cong A_n$, $n \geq 5$, one may take $i = 3$;*

(iii) *for every simple group of Lie type other than ${}^2F_4(2)'$, one may take $i = 3$ when the Schur multiplier is nontrivial and $i = 4$ when it is trivial;*

(iv) $H^4({}^2F_4(2)';\mathbb{Z}) \neq 0$*;*

(v) *for sporadic simple groups one may take $i \leq 4$, except that the three groups M_{23}, J_4, Ly have the respective witnesses*

$$H^6(M_{23};\mathbb{Z})_{(7)} \neq 0, \quad H^6(J_4;\mathbb{Z})_{(7)} \neq 0, \quad H^8(\mathrm{Ly};\mathbb{Z})_{(2)} \textit{ contains } \mathbb{Z}/8.$$

The proof is divided into the classification families.

4.1 Cyclic and alternating groups

For C_p, the standard periodic resolution gives $H^2(C_p;\mathbb{Z}) \cong \mathbb{Z}/p$. For A_n, $n \geq 5$, the group is perfect and its Schur multiplier is nontrivial: it has order two for $n = 5$ and $n \geq 8$, and order six for $n = 6, 7$. Corollary 1.2 therefore gives $H^3(A_n;\mathbb{Z}) \neq 0$.

Proposition 4.2 (A Pontryagin class for alternating groups)**.** *For every $n \geq 5$,*

$$H^4(A_n;\mathbb{Z})_{(3)} \neq 0.$$

Proof. Let $\rho_n : A_n \to \mathrm{SO}(n)$ be the real permutation representation and consider $p_1(\rho_n) \in H^4(A_n;\mathbb{Z})$. Restrict to the subgroup C_3 generated by a 3-cycle. The restricted real representation is the sum of trivial lines and the underlying oriented real plane $L_{\mathbb{R}}$ of a nontrivial complex character L of C_3. If $u = c_1(L)$ is a generator of $H^2(C_3;\mathbb{Z}) \cong \mathbb{Z}/3$, then

$$\mathrm{res}^{A_n}_{C_3}\, p_1(\rho_n) = p_1(L_{\mathbb{R}}) = c_1(L)^2 = u^2.$$

The standard calculation $H^*(C_3;\mathbb{Z}) \cong \mathbb{Z}[u]/(3u)$ shows that $u^2 \neq 0$. Hence $p_1(\rho_n)$ is nonzero and has nonzero 3-primary component. □

Proposition 4.3 (Three-prime support for alternating groups)**.** *For every $n \geq 5$, the group A_n has nonzero integral cohomology at the three distinct primes $2, 3, 5$ in degrees at most eight.*

Proof. The two-primary Schur multiplier gives a class in degree three, and Proposition 4.2 gives a three-primary class in degree four. For the third prime, let π_n be the complex permutation representation and restrict it to a subgroup C_5 generated by a 5-cycle. The restriction is the regular representation of C_5 plus trivial lines. If u generates $H^2(C_5; \mathbb{Z})$, then

$$c(\mathrm{res}^{A_n}_{C_5} \pi_n) = \prod_{j=0}^{4} (1 + ju) = 1 - u^4, \qquad \mathrm{res}^{A_n}_{C_5} c_4(\pi_n) = -u^4 \neq 0.$$

Thus $c_4(\pi_n) \in H^8(A_n; \mathbb{Z})$ has nonzero five-primary component. □

4.2 Rank-one linear groups

Proposition 4.4 (Fourth cohomology of $\mathrm{PSL}_2(q)$)**.** *Let q be a prime power for which $\mathrm{PSL}_2(q)$ is nonabelian simple. Then*

$$H^4(\mathrm{PSL}_2(q); \mathbb{Z}) \neq 0.$$

Proof. Write $q = p^d$. Mirzaii and Torres Pérez prove that, when $(p-1)d > 6$, there is an exact sequence whose first map is an injection

$$0 \longrightarrow \mathrm{Tor}^{\mathbb{Z}}_1\big(\widetilde{\mu}(\mathbb{F}_q), \widetilde{\mu}(\mathbb{F}_q)\big) \longrightarrow H_3(\mathrm{PSL}_2(q); \mathbb{Z}),$$

where $\widetilde{\mu}(\mathbb{F}_q) = \mathbb{F}_q^{\times}/\mu_2(\mathbb{F}_q)$ [39, Corollary 3.6]. This cyclic group has order $q-1$ when q is even and $(q-1)/2$ when q is odd. In every simple case satisfying the displayed numerical hypothesis it is nontrivial, and $\mathrm{Tor}^{\mathbb{Z}}_1(C_m, C_m) \cong C_m$. Hence $H_3(\mathrm{PSL}_2(q); \mathbb{Z}) \neq 0$.

The simple cases not satisfying $(p-1)d > 6$ are

$$q \in \{4, 5, 7, 8, 9, 16, 27, 32, 64\}.$$

The groups at $q = 4, 5$ are A_5, and the group at $q = 9$ is A_6; Proposition 4.2 applies. For the remaining values, choose respectively the primes

q	7	8	16	27	32	64
ℓ	3	7	5	13	31	7.

In each case a Sylow ℓ-subgroup is the indicated prime-order subgroup of a split torus, and its normalizer acts by inversion. Thus it is cyclic with automizer of order two. Theorem 3.1 gives a nonzero class in degree four. Lemma 1.1 converts the nonzero third homology in the first set of cases to the same conclusion. □

4.3 Groups of Lie type

The following homotopy-fixed-point argument supplies a uniform class in degree four.

Proposition 4.5 (Simply connected fixed-point groups)**.** *Let $\mathbf{H}$ be a simple simply connected algebraic group in characteristic p, let F be an ordinary or graph-twisted Steinberg endomorphism with field parameter q, and put $H = \mathbf{H}^F$. If H is perfect, then, for every prime $\ell \mid q^2 - 1$,*

$$H_3(H; \mathbb{Z})_{(\ell)} \cong H^4(H; \mathbb{Z})_{(\ell)} \cong \mathbb{Z}/\ell^{v_\ell(q^2-1)}.$$

Proof. Let K be the simply connected compact Lie group with the root datum of $\mathbf{H}$, put $X = (BK)^\wedge_\ell$, and let σ be the self-equivalence induced by F. The Quillen–Friedlander–Mislin comparison gives a homotopy equivalence

$$(BH)^\wedge_\ell \simeq X^{h\sigma}$$

for $\ell \neq p$; this applies because every prime dividing $q^2 - 1$ is different from p [16, 17, 27]. The space X is 3-connected and $\pi_4(X) \cong \mathbb{Z}_\ell$. The Frobenius component of σ acts on this group as multiplication by q^2, while a graph automorphism fixes its generator because it preserves the basic invariant bilinear form. The homotopy-equalizer exact sequence therefore shows that

$$Y = X^{h\sigma} \quad \text{is 2-connected,} \qquad \pi_3(Y) \cong \operatorname{coker}(1 - q^2 : \mathbb{Z}_\ell \longrightarrow \mathbb{Z}_\ell).$$

Pass to the plus construction BH^+ before completing. Since H is perfect, BH^+ is simply connected and has the same integral homology as BH. Finite-group homology is finite in every positive degree, so ℓ-completion identifies its $\mathbb{Z}_\ell$-homology with that of Y [11, Chapter VI]. The ℓ-adic Hurewicz theorem, applied to the 2-connected space Y, therefore gives

$$H_3(H;\mathbb{Z})_{(\ell)} \cong H_3(Y;\mathbb{Z}_\ell) \cong \pi_3(Y) \cong \mathbb{Z}_\ell/(q^2-1) \cong \mathbb{Z}/\ell^{v_\ell(q^2-1)}.$$

Lemma 1.1 identifies the ℓ-primary fourth cohomology with the Pontryagin dual of this cyclic group, proving the second isomorphism. □

Proposition 4.6 (The degree-four class)**.** *Let S be a finite simple group of Lie type in characteristic p, not the Tits group. If the Schur multiplier of S is trivial, then $H^4(S;\mathbb{Z}) \neq 0$.*

Proof. First consider an ordinary or graph-twisted group. Outside the usual small fixed-point exceptions, the simply connected finite fixed-point group H is perfect and maps centrally onto S. Since the Schur multiplier of S is trivial, the homology five-term sequence forces the kernel of $H \to S$ to vanish. Proposition 4.5 therefore applies with $H = S$. The small exceptions are either covered by an exceptional isomorphism with another Lie family or have nontrivial multiplier [25, Theorems 2.2.6 and 6.1.4, and Table 6.1.3].

For the Suzuki and Ree endomorphisms, the square of the endomorphism is an ordinary Frobenius. Its action on $\pi_4(X)$ is therefore multiplication by a number whose square is q^2, hence by q or $-q$. Choosing a prime divisor of $q - 1$ or $q + 1$, as appropriate, gives the same conclusion. The small derived exceptions reduce to the standard exceptional isomorphisms, apart from ${}^2F_4(2)'$, which is treated separately below. The comparison theorem and the description of unstable Adams operations used here are recorded in [16, 17, 27]. □

If the Schur multiplier is nontrivial, $H^3(S;\mathbb{Z}) \neq 0$ by Corollary 1.2. Proposition 4.6 handles all remaining simple groups of Lie type except the Tits group.

Proposition 4.7 (Tits group)**.** *For $T = {}^2F_4(2)'$,*

$$H_3(T;\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8.$$

Consequently $H^4(T;\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8$.

Proof. This is a reproducible HAP calculation from the degree-1600 ATLAS permutation representation. With GAP and the packages `AtlasRep` and `HAP` installed, the commands are

```
SetUserPreference("AtlasRep","AtlasRepAccessRemoteFiles",true);
LoadPackage("atlasrep");;
LoadPackage("hap");;
G := AtlasGroup("2F4(2)'",IsPermGroup);;
Size(G);                          # 17971200
GroupHomology(G,3);              # [ 2, 8 ]
```

Here HAP lists the invariant factors of the homology group. The last assertion follows from Lemma 1.1. The order, trivial Schur multiplier, and the permutation representation are independently recorded in the ATLAS [1]. □

4.4 Sporadic groups

The thirteen sporadic groups with nontrivial Schur multiplier have nonzero $H^3(-;\mathbb{Z})$. For the thirteen groups with trivial multiplier, the following table records a low-degree witness. The degree-four entries are the Pontryagin duals of the third-homology calculations in [29]; the M_{23} and J_4 entries are from [38, 20], and the Ly entry is from the mod-two cohomology calculation in [2].

Group	nonzero witness	source or reason
M_{11}	$H^4 \cong \mathbb{Z}/8$	third homology
M_{24}	$H^4 \cong \mathbb{Z}/12$	third homology
J_1	$H^4 \cong \mathbb{Z}/30$	third homology
Co_2	$H^4 \cong \mathbb{Z}/4$	third homology
Co_3	$H^4 \cong \mathbb{Z}/6$	third homology
He	$H^4 \cong \mathbb{Z}/12$	third homology
HN	$H^4_{(3)} \cong \mathbb{Z}/3$	characteristic class
Th	$H^4_{(2)} \cong \mathbb{Z}/8$	characteristic class
Fi_{23}	$H^4_{(2)} \cong \mathbb{Z}/2$	restriction calculation
M	H^4 contains $\mathbb{Z}/24$	Moonshine anomaly
M_{23}	$H^6_{(7)} \cong \mathbb{Z}/7$	Milgram
J_4	$H^6_{(7)} \cong \mathbb{Z}/7$	cyclic 7-local class
Ly	$H^8_{(2)}$ contains $\mathbb{Z}/8$	mod-two Bockstein calculation

Proposition 4.8 (Low-degree audit for the Lyons group)**.** *For the Lyons group Ly one has*

$$H^i(Ly;\mathbb{Z}) = 0 \quad (1 \le i \le 4), \qquad H^i(Ly;\mathbb{Z})_{(2)} = 0 \quad (1 \le i \le 7),$$

whereas

$$H^8(Ly;\mathbb{Z})_{(2)} \text{ contains } \mathbb{Z}/8, \qquad H^{10}(Ly;\mathbb{Z})_{(11)} \neq 0.$$

Thus degree eight is the first nonzero two-primary *integral degree. The odd-primary parts of $H^5(Ly;\mathbb{Z})$, $H^6(Ly;\mathbb{Z})$, and $H^7(Ly;\mathbb{Z})$ are not determined here; in particular, degree eight is not asserted to be the first nonzero integral degree without qualification.*

Proof. The calculation in [2] determines $H^*(Ly;\mathbb{F}_2)$. Its description has no positive-degree class below degree seven and identifies a degree-seven class d_7 whose $\mathbb{Z}/8$-Bockstein is the degree-eight class d_8. The coefficient exact sequence gives

$$0 \longrightarrow H^j(Ly;\mathbb{Z})/2 \longrightarrow H^j(Ly;\mathbb{F}_2) \longrightarrow H^{j+1}(Ly;\mathbb{Z})[2] \longrightarrow 0.$$

Since positive-degree integral cohomology of a finite group is finite, the mod-two vanishing through degree six implies that the two-primary integral cohomology vanishes through degree seven. The stated Bockstein then gives the class of order eight in degree eight.

Independently, 11 divides $|Ly|$ exactly once, and the two conjugacy classes of elements of order 11 show that the automizer of a Sylow 11-subgroup has order five. Theorem 3.1 therefore gives a nonzero eleven-primary class in degree ten. Concretely, if $u \in H^2(C_{11};\mathbb{Z})$ is the periodic generator, the stable monomial is

$$u^5 \in H^{2\cdot 5}(C_{11};\mathbb{Z}) = H^{10}(C_{11};\mathbb{Z}).$$

No further power and no second doubling occurs. Finally, the trivial Schur multiplier and the calculation $H_3(Ly;\mathbb{Z}) = 0$ give integral vanishing through degree four by Lemma 1.1 [29]. □

This finishes the proof of Theorem 4.1.

5 Basic extension reductions

Lemma 5.1 (Prime-to-kernel inflation)**.** *Let*

$$1 \longrightarrow K \longrightarrow E \xrightarrow{\pi} Q \longrightarrow 1$$

be an extension and let $\ell \nmid |K|$. Inflation is an isomorphism

$$H^*(Q;\mathbb{F}_\ell) \xrightarrow{\cong} H^*(E;\mathbb{F}_\ell)$$

and likewise with coefficients $\mathbb{Z}_{(\ell)}$.

Proof. For either coefficient module, $H^j(K;-) = 0$ for $j > 0$, since the order of K is invertible in the coefficient ring. The Lyndon–Hochschild–Serre spectral sequence therefore has only its bottom row and collapses to inflation. □

Lemma 5.2 (Split extensions)**.** *If $E \to Q$ has a group-theoretic section, inflation $H^*(Q;R) \to H^*(E;R)$ is injective for every trivial coefficient ring R.*

Proof. Restriction along the section is a left inverse to inflation. □

Proposition 5.3 (Universal centralization of a superperfect epimorphism)**.** *Let*

$$1 \longrightarrow K \longrightarrow E \longrightarrow Q \longrightarrow 1$$

be an extension in which E is superperfect. Then Q is perfect and

$$1 \longrightarrow K/[E,K] \longrightarrow E/[E,K] \longrightarrow Q \longrightarrow 1$$

is the universal central extension of Q. In particular,

$$H_2(Q;\mathbb{Z}) \cong K/[E,K], \qquad E/[E,K] \cong \widetilde{Q}$$

over Q.

Proof. The group Q is perfect because it is a quotient of the perfect group E. The displayed quotient extension is central, and its source remains perfect. The homology five-term sequence for the original extension is

$$0 = H_2(E;\mathbb{Z}) \longrightarrow H_2(Q;\mathbb{Z}) \xrightarrow{\tau} K/[E,K] \longrightarrow H_1(E;\mathbb{Z}) = 0,$$

so τ is an isomorphism.

Let $\widetilde{Q} \to Q$ be the universal central extension. Its universal property gives a homomorphism

$$\widetilde{Q} \longrightarrow E/[E,K]$$

over Q, and the induced map on central kernels is τ. Hence this homomorphism is surjective. Both finite groups have order $|H_2(Q;\mathbb{Z})|\,|Q|$, so it is an isomorphism. □

Corollary 5.4 (Reduced kernel after universal centralization)**.** *In Proposition 5.3, put*

$$D = [E,K].$$

Then

$$D = [E,D].$$

Equivalently, the coinvariants of D_{ab} under $E/D \cong \widetilde{Q}$ vanish. Consequently

$$H^2(D;\mathbb{Z})^{\widetilde{Q}} = 0.$$

Proof. The quotient $E/D \cong \widetilde{Q}$ is superperfect. The homology five-term sequence for $D \to E \to \widetilde{Q}$ therefore contains

$$0 = H_2(E;\mathbb{Z}) \longrightarrow H_2(\widetilde{Q};\mathbb{Z}) = 0 \longrightarrow D/[E,D] \longrightarrow H_1(E;\mathbb{Z}) = 0,$$

so $D = [E,D]$. Thus $(D_{\mathrm{ab}})_{\widetilde{Q}} = 0$. By Lemma 1.1, $H^2(D;\mathbb{Z})$ is the Pontryagin dual of D_{ab}; taking invariants on the dual is dual to taking coinvariants, which proves the last assertion. □

Proposition 5.5 (Integral inflation-edge filtration)**.** *Let*

$$1 \longrightarrow K \longrightarrow E \longrightarrow Q \longrightarrow 1$$

be a finite extension, let $n \geq 2$, and put

$$\kappa_n = \ker\bigl(\inf : H^n(Q;\mathbb{Z}) \longrightarrow H^n(E;\mathbb{Z})\bigr).$$

The group κ_n has a filtration with at most $n-2$ successive quotients, indexed by $3 \leq r \leq n$, such that the rth quotient is a subquotient of

$$H^{n-r}\bigl(Q; H^{r-1}(K;\mathbb{Z})\bigr).$$

In particular,

$$|\kappa_n| \leq \prod_{r=3}^{n} \left|H^{n-r}\bigl(Q; H^{r-1}(K;\mathbb{Z})\bigr)\right|. \tag{1}$$

Consequently,

$$|K|^{\,n-2}\kappa_n = 0. \tag{2}$$

If K is elementary abelian of characteristic p, then the sharper bound

$$p^{\,n-2}\kappa_n = 0 \tag{3}$$

holds. Thus a class in $H^n(Q;\mathbb{Z})$ whose order does not divide the right-hand annihilator in (2)*, or in* (3) *when applicable, has nonzero inflation to E.*

Proof. Use the integral Lyndon–Hochschild–Serre spectral sequence

$$E_2^{a,b} = H^a\big(Q; H^b(K;\mathbb{Z})\big) \Longrightarrow H^{a+b}(E;\mathbb{Z}).$$

No differential leaves the bottom-row term

$$E_2^{n,0} = H^n(Q;\mathbb{Z}).$$

Its d_2-source is zero because $H^1(K;\mathbb{Z}) = 0$. For $3 \le r \le n$, the only remaining possible incoming differential has source

$$E_r^{n-r,r-1},$$

a subquotient of the group displayed in the statement. Successively taking the images of these differentials gives the asserted filtration of the kernel of the bottom edge homomorphism, which is inflation. The order of a filtered finite group is the product of the orders of its successive quotients, and each such quotient has order at most that of the displayed E_2-source. This proves (1).

Positive-degree integral cohomology of the finite group K is annihilated by $|K|$, by restriction and corestriction to the trivial subgroup. Therefore each displayed source, and hence each filtration quotient, is annihilated by $|K|$. There are at most $n-2$ such quotients, proving (2). If K is elementary abelian, the integral Kunneth theorem, applied inductively to products of C_p, shows that every $H^j(K;\mathbb{Z})$ with $j > 0$ is an elementary abelian p-group. The same argument then proves (3). □

Corollary 5.6 (Persistence from a universal centralization)**.** *Let $E \twoheadrightarrow Q$ be an epimorphism of finite groups with Q perfect, put $K = \ker(E \to Q)$, and let $c \in H^d(\widetilde{Q};\mathbb{Z})$ with $d \ge 2$. Suppose that E is superperfect and put $D = [E,K]$. Then*

$$D = [E,D]$$

and

$$\Big|\ker\big(H^d(\widetilde{Q};\mathbb{Z}) \longrightarrow H^d(E;\mathbb{Z})\big)\Big| \le \prod_{r=3}^{d} \Big|H^{d-r}\big(\widetilde{Q}; H^{r-1}(D;\mathbb{Z})\big)\Big|. \tag{1}$$

In particular, if

$$\operatorname{ord}(c) \nmid |[E,K]|^{\,d-2},$$

then $H^d(E;\mathbb{Z}) \neq 0$. This condition certainly holds whenever some prime dividing the order of c does not divide $|[E,K]|$.

Proof. Proposition 5.3 identifies $\widetilde{Q}$ with $E/[E,K]$. Apply Proposition 5.5 to the resulting extension with kernel $D = [E,K]$; its order bound gives (1), and its annihilator bound gives the stated persistence criterion. The equality $D = [E,D]$ is Corollary 5.4. □

Lemma 5.7 (Chern classes on a cyclic subgroup)**.** *Let ρ be a complex representation of a finite group E, let $C = \langle g\rangle \le E$ have order m, and write the eigenvalues of $\rho(g)$ as*

$$\zeta_m^{a_1}, \dots, \zeta_m^{a_n}, \qquad a_i \in \mathbb{Z}/m.$$

If $u \in H^2(C;\mathbb{Z})$ is the first Chern class of the character sending g to ζ_m, then

$$\operatorname{res}^E_C c_j(\rho) = e_j(a_1,\dots,a_n)u^j \quad \text{in} \quad H^{2j}(C;\mathbb{Z}) = \mathbb{Z}/m\{u^j\}.$$

In particular, a coefficient $e_j(a_1,\dots,a_n)$ that is nonzero modulo m proves $H^{2j}(E;\mathbb{Z}) \neq 0$.

Proof. Every complex representation of the cyclic group C splits as a sum of one-dimensional characters. The summand on which g acts by $\zeta_m^{a_i}$ has total Chern class $1 + a_i u$. Hence the total Chern class of the restriction is

$$\prod_{i=1}^{n}(1 + a_i u),$$

and comparison of degree-$2j$ terms gives the formula. The last assertion follows by functoriality of Chern classes. □

Proposition 5.8 (The Rudvalis Schur cover)**.** *For the universal central extension* 2.Ru*,*

$$H^4(2.\mathrm{Ru};\mathbb{Z}) \text{ contains a class whose order is divisible by } 4.$$

Moreover,

$$H^8(2.\mathrm{Ru};\mathbb{Z})_{(5)} \neq 0. \tag{2}$$

Proof. The ordinary character table of 2.Ru in CTblLib 1.3.11 contains an irreducible character of degree 28 whose value data on the ATLAS class $4A$ give fourth-root eigenvalue multiplicities

$$[\zeta_4 : 14], \qquad [\zeta_4^3 : 14].$$

Let ρ afford this character and let g lie in class $4A$.

These data are recovered exactly by `EigenvaluesChar`; the complete reproducible calculation is in `scripts/check_cover_chern.g` [8]. Lemma 5.7 gives

$$\mathrm{res}^{2.\mathrm{Ru}}_{\langle g\rangle} c_2(\rho) = \left(\binom{14}{2} + 14^2 \cdot 3 + \binom{14}{2}3^2\right) u^2 = 2u^2$$

in $H^4(C_4;\mathbb{Z}) = \mathbb{Z}/4\{u^2\}$. This restriction is nonzero, so the global second Chern class is nonzero. The stronger order assertion comes from the same character on class $8A$. The exact eigenvalue-multiplicity calculation gives

$$\mathrm{res}^{2.\mathrm{Ru}}_{C_8} c_2(\rho) = 6u^2 \quad \text{in} \quad H^4(C_8;\mathbb{Z}) = \mathbb{Z}/8\{u^2\}.$$

This restriction has order four, so the order of $c_2(\rho)$ is divisible by four. The script records both cyclic detections and audits every irreducible character for the largest order detected in this way.

For the prime-five assertion use the same character on class $5B$. The eigenvalue weights modulo five have multiplicities

$$[0 : 4], \qquad [1 : 6], \qquad [2 : 6], \qquad [3 : 6], \qquad [4 : 6].$$

Consequently its restricted total Chern class is

$$\prod_{a\in\mathbb{F}_5^\times} (1 + au)^6 = (1 - u^4)^6 \quad \text{in } H^*(C_5;\mathbb{Z}),$$

where positive-degree coefficients are read modulo five. Its degree-eight term is therefore

$$\mathrm{res}^{2.\mathrm{Ru}}_{C_5} c_4(\rho) = -6u^4 = 4u^4 \neq 0 \quad \text{in } H^8(C_5;\mathbb{Z}) = \mathbb{Z}/5\{u^4\}.$$

This proves (2). The same script exhaustively scans every irreducible character and every cyclic prime-power class through c_5; it records this as the unique odd-primary cyclic detection on 2.Ru in that range. □

Proposition 5.9 (The largest Fischer Schur cover)**.** *For the universal central extension* $3.\mathrm{Fi}'_{24}$*,*

$$H^4(3.\mathrm{Fi}'_{24};\mathbb{Z})_{(2)} \neq 0.$$

Proof. Johnson–Freyd and Treumann prove that $H^4(\mathrm{Fi}'_{24};\mathbb{Z}[1/3])$ has order two or four [29, Theorem 8.4(4)]. In particular, $H^4(\mathrm{Fi}'_{24};\mathbb{Z})_{(2)}$ is nonzero. The kernel of $3.\mathrm{Fi}'_{24} \to \mathrm{Fi}'_{24}$ has order three, so Lemma 5.1, with $\ell = 2$, identifies this group with $H^4(3.\mathrm{Fi}'_{24};\mathbb{Z})_{(2)}$ by inflation. □

Proposition 5.10 (The Baby Monster Schur cover)**.** *For the universal central extension* $2.B$*,*

$$H^4(2.B;\mathbb{Z})_{(2)} \neq 0.$$

Proof. Let $\mathbb{M}$ be the Monster and let $\omega^\natural \in H^4(\mathbb{M};\mathbb{Z})$ be the Moonshine anomaly. Johnson–Freyd and Treumann recall that its restriction to a cyclic subgroup generated by a Monster class $4B$ element has order two [29, proof of Theorem 8.4].

It remains to exhibit such an element inside $2.B = C_{\mathbb{M}}(2A)$. The stored CTblLib class fusions for

$$U = (S_3 \times 2.\mathrm{Fi}_{22}).2 \longrightarrow 2.B \longrightarrow \mathbb{M}$$

compose to the stored fusion $U \to \mathbb{M}$. Under these maps, for example, the CTblLib class $4b$ of U maps first to class $4e$ of $2.B$ and then to class $4b$ of $\mathbb{M}$. The exact check, including all twelve classes of U with this image, is reproduced by `scripts/check_baby_monster_anomaly.g` [8]. Thus $U \leq 2.B$ contains an element x in Monster class $4B$, and

$$\mathrm{res}^{2.B}_{\langle x\rangle}\big(\mathrm{res}^{\mathbb{M}}_{2.B}\,\omega^\natural\big) = \mathrm{res}^{\mathbb{M}}_{\langle x\rangle}\,\omega^\natural \neq 0.$$

The restricted anomaly is therefore a nonzero 2-primary class in $H^4(2.B;\mathbb{Z})$. □

Proposition 5.11 (Minimal normal subgroup dichotomy)**.** *Fix a proposed bound* $N_0 \geq 3$*, and let* G *be a counterexample of least order. If* $A \lhd G$ *is a minimal normal subgroup, then:*

(i) G *is superperfect;*

(ii) A *is either elementary abelian or a direct power* S^r *of a nonabelian simple group* S*;*

(iii) *if* A *is elementary abelian and the extension* $1 \to A \to G \to G/A \to 1$ *is not split, then* $A \leq \Phi(G)$*.*

Proof. Part (i) follows from the vanishing in degrees two and three and Corollary 1.2. Part (ii) is the standard description of finite characteristically simple groups. For (iii), suppose $A \not\leq \Phi(G)$, and choose a maximal subgroup $M < G$ not containing A. Then $G = AM$. Since A is abelian, $A \cap M$ is normalized by both A and M, hence by G. Minimality of A and $A \not\leq M$ give $A \cap M = 1$, so M is a complement. □

Proposition 5.12 (Elementary kernels in a superperfect group)**.** *Let* G *be a finite superperfect group, let* $V \lhd G$ *be a minimal normal elementary abelian subgroup, and put* $Q = G/V$*. Then* Q *is perfect, and exactly one of the following holds:*

(i) *the* Q*-action on* V *is nontrivial and* Q *is superperfect;*

(ii) *the action is trivial,* $V \cong C_p$*, the transgression gives an isomorphism* $H_2(Q;\mathbb{Z}) \cong V$*, and* $G \to Q$ *is the universal central extension of* Q*. In this case the natural map* $H_3(G;\mathbb{Z}) \to H_3(Q;\mathbb{Z})$ *is surjective.*

Proof. The quotient Q is perfect because it is a quotient of the perfect group G. The five-term homology sequence for $1 \to V \to G \to Q \to 1$ contains

$$H_2(G;\mathbb{Z}) \longrightarrow H_2(Q;\mathbb{Z}) \xrightarrow{\tau} V_Q \longrightarrow H_1(G;\mathbb{Z}) \longrightarrow H_1(Q;\mathbb{Z}) \longrightarrow 0,$$

where $V_Q = V/[Q,V]$ denotes coinvariants. Since G is superperfect, τ is an isomorphism.

Minimal normality says that V is an irreducible $\mathbb{F}_pQ$-module. If its action is nontrivial, then $[Q,V] = V$, so $V_Q = 0$ and hence $H_2(Q;\mathbb{Z}) = 0$. Thus Q is superperfect. If the action is trivial, irreducibility gives $V \cong C_p$, and the displayed isomorphism becomes $H_2(Q;\mathbb{Z}) \cong V$. The extension is central, and G is perfect, so it is a stem central extension with full Schur-multiplier kernel. Since Q is perfect, this is its universal central extension.

For the last assertion, consider the homological Lyndon–Hochschild–Serre spectral sequence. The bottom-row term $E^2_{3,0} = H_3(Q;\mathbb{Z})$ has no incoming differential. Its only possible nonzero outgoing differential is

$$d_2 : E^2_{3,0} \longrightarrow E^2_{1,1} = H_1(Q;V),$$

and the target vanishes because Q is perfect and acts trivially on V. Thus $E^\infty_{3,0} = H_3(Q;\mathbb{Z})$, which is exactly the asserted surjectivity of the homology edge map. □

Proposition 5.13 (Homology model for a central cover)**.** *Let*

$$1 \longrightarrow A \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be a central extension, and let $e : BQ \to K(A,2)$ classify it. If

$$\widetilde{H}_j(G;\mathbb{Z}) = 0 \qquad (1 \le j \le m),$$

then

$$e_* : H_j(Q;\mathbb{Z}) \xrightarrow{\cong} H_j(K(A,2);\mathbb{Z}) \qquad (0 \le j \le m).$$

Proof. The homotopy fiber of e is BG. Indeed, the long exact homotopy sequence of the fiber realizes on fundamental groups the central extension classified by e, and all higher homotopy groups of the fiber vanish. Apply the homology Serre spectral sequence to

$$BG \longrightarrow BQ \xrightarrow{e} K(A,2).$$

The base is simply connected, and the hypothesis makes every row with $0 < t \le m$ vanish. In total degrees at most m, only the bottom row $H_s(K(A,2);\mathbb{Z})$ remains, and no differential can enter or leave it in that range. The edge map is e_*, which proves the assertion. □

Proposition 5.14 (First layer above a central monolith)**.** *Let G be a finite perfect group whose unique minimal normal subgroup is $A \cong C_p$, and suppose that $A \le Z(G)$. Put $Q = G/A$, and let $\overline{N}$ be a minimal normal subgroup of Q.*

(i) *If $\overline{N}$ is elementary abelian of characteristic ℓ, then $\ell = p$.*

(ii) *If $\overline{N} = S_1 \times \cdots \times S_r$ with the S_i isomorphic to a nonabelian simple group S, and N is its full inverse image in G, then N is perfect and the transgression is a nonzero Q-equivariant epimorphism*

$$\tau : H_2(\overline{N};\mathbb{Z}) \cong H_2(S;\mathbb{Z})^r \longrightarrow A.$$

In particular, p divides $|H_2(S;\mathbb{Z})|$. More precisely, let $K = N_Q(S_1)$ and let B be the image of K in $\mathrm{Aut}(S_1)$. Then the coinvariants

$$H_2(S;\mathbb{Z})_{B/S}$$

have a quotient isomorphic to C_p.

Proof. Let N be the full inverse image of $\overline{N}$. If $\overline{N}$ is an elementary abelian ℓ-group with $\ell \neq p$, then the cohomology groups $H^j(\overline{N}; A)$ vanish for $j > 0$, since multiplication by $|\overline{N}|$ is invertible on A. Hence the central extension

$$1 \longrightarrow A \longrightarrow N \longrightarrow \overline{N} \longrightarrow 1$$

splits. Its complement is unique: two complements differ by an element of $H^1(\overline{N}; A) = 0$. Conjugation by G preserves N and A, so it preserves this unique complement. The complement is therefore a minimal normal subgroup of G distinct from A, a contradiction. This proves (i).

Now suppose that $\overline{N} = S^r$. Since $\overline{N}$ is perfect, the universal coefficient description of central extensions identifies

$$H^2(\overline{N}; A) \cong \mathrm{Hom}(H_2(\overline{N}; \mathbb{Z}), A),$$

and sends the restricted extension to its homology transgression τ. If $\tau = 0$, the extension splits. Its complement is again unique, now because $\mathrm{Hom}(\overline{N}, A) = 0$, and is consequently a minimal normal subgroup of G distinct from A. Thus $\tau \neq 0$. Since A has prime order, τ is surjective. The homology five-term sequence then gives $H_1(N; \mathbb{Z}) = 0$, so N is perfect.

Naturality under conjugation makes τ a Q-module map, where Q acts trivially on the central group A. The product formula and $H_1(S; \mathbb{Z}) = 0$ identify $H_2(S^r; \mathbb{Z})$ with the direct sum of the r multiplier groups. Minimal normality of $\overline{N}$ makes the Q-action transitive on the factors. Therefore the restriction of τ to the first summand is a nonzero K-invariant homomorphism $H_2(S; \mathbb{Z}) \to C_p$; if it were zero, transitivity would make every coordinate restriction zero. Inner automorphisms act trivially on group homology, so this homomorphism is B/S-invariant. Equivalently, it factors through a surjection $H_2(S; \mathbb{Z})_{B/S} \twoheadrightarrow C_p$. This proves (ii). □

Proposition 5.15 (Central kernels over a simple quotient)**.** *Let Γ be a nonabelian finite simple group and let*

$$1 \longrightarrow K \longrightarrow E \longrightarrow \Gamma \longrightarrow 1$$

be a central extension.

(i) *If $H_2(\Gamma; \mathbb{Z}) = 0$, then $H^i(E; \mathbb{Z}) \neq 0$ for some $2 \leq i \leq 8$.*

(ii) *If $H^4(\Gamma; \mathbb{Z}) \neq 0$ and either K or $H_2(\Gamma; \mathbb{Z})$ is cyclic, then $H^i(E; \mathbb{Z}) \neq 0$ for some $2 \leq i \leq 4$.*

(iii) *Let $\widetilde{\Gamma} \to \Gamma$ be the universal central extension. If $H^4(\widetilde{\Gamma}; \mathbb{Z}) \neq 0$, then $H^i(E; \mathbb{Z}) \neq 0$ for some $2 \leq i \leq 4$.*

Proof. If E is not perfect, then $H^2(E; \mathbb{Z}) \neq 0$. If E is perfect but not superperfect, then $H^3(E; \mathbb{Z}) \neq 0$. It remains to suppose that E is superperfect.

The homology five-term sequence then contains

$$0 = H_2(E; \mathbb{Z}) \longrightarrow H_2(\Gamma; \mathbb{Z}) \longrightarrow K \longrightarrow H_1(E; \mathbb{Z}) = 0,$$

so $K \cong H_2(\Gamma; \mathbb{Z})$. In part (i), this isomorphism gives $K = 0$, so $E \cong \Gamma$ and Theorem 4.1 supplies a witness in degree at most eight. This proves part (i). It remains to prove parts (ii) and (iii). In part (iii), the superperfect central extension $E \to \Gamma$ is universal, so $E \cong \widetilde{\Gamma}$ and the stated hypothesis gives the result. For part (ii), K is cyclic either by the original hypothesis on K or, using the displayed isomorphism, by the alternative multiplier hypothesis.

Consider the homological Lyndon–Hochschild–Serre spectral sequence for the displayed central extension. The term

$$E^2_{3,0} = H_3(\Gamma; \mathbb{Z})$$

has no incoming differential. Its d_2-target is

$$E^2_{1,1} = H_1(\Gamma; K) = 0$$

because the action is trivial and Γ is perfect. Its only other possible outgoing differential has target $E^3_{0,2}$, a subquotient of $H_2(K;\mathbb{Z})$, which vanishes because K is cyclic. Thus the homology edge map

$$H_3(E;\mathbb{Z}) \longrightarrow H_3(\Gamma;\mathbb{Z})$$

is surjective. The hypothesis and Lemma 1.1 say that the target is nonzero, so $H_3(E;\mathbb{Z}) \neq 0$ and hence $H^4(E;\mathbb{Z}) \neq 0$. □

Proposition 5.16 (Central epimorphisms in the standard-cover case)**.** *Let Γ be a nonabelian simple group of ordinary or graph-twisted Lie type over $\mathbb{F}_q$, and suppose that its universal central extension is the natural map*

$$H = \mathbf{H}^F \longrightarrow H/Z(H) = \Gamma,$$

where $\mathbf{H}$ is simple and simply connected. Then every finite central epimorphism $E \twoheadrightarrow \Gamma$ has

$$H^i(E;\mathbb{Z}) \neq 0 \qquad \text{for some } 2 \leq i \leq 4.$$

Proof. If E is not perfect, then $H^2(E;\mathbb{Z}) \neq 0$; if it is perfect but not superperfect, then $H^3(E;\mathbb{Z}) \neq 0$. Suppose that E is superperfect, and write K for the central kernel. The homology five-term sequence gives an isomorphism

$$H_2(\Gamma;\mathbb{Z}) \xrightarrow{\cong} K.$$

Thus $E \to \Gamma$ is the universal central extension, so $E \cong H$. Proposition 4.5, applied at any prime dividing $q^2 - 1$, now gives $H^4(E;\mathbb{Z}) \neq 0$. □

Thus a minimal-counterexample argument has two genuinely difficult branches: elementary abelian Frattini kernels and nonabelian minimal normal subgroups. The next sections record reductions in both branches.

6 Finite Frattini covers

Definition 6.1. A finite epimorphism $\pi : E \twoheadrightarrow Q$ is a *Frattini cover* if $\ker \pi \leq \Phi(E)$, where $\Phi(E)$ is the intersection of the maximal subgroups of E. It is a p-Frattini cover if its kernel is a p-group.

Theorem 6.2 (Primewise decomposition)**.** *Let $\pi : E \twoheadrightarrow Q$ be a finite Frattini cover with kernel K. Write*

$$K = \prod_{p \mid |K|} K_p$$

for its characteristic Sylow decomposition, and set

$$E_p = E \Big/ \prod_{q \neq p} K_q.$$

Then each $E_p \to Q$ is a p-Frattini cover and the natural map is an isomorphism

$$E \xrightarrow{\cong} \prod_{Q} E_p.$$

Moreover

$$H^*(E;\mathbb{F}_p) \cong H^*(E_p;\mathbb{F}_p), \qquad H^*(E;\mathbb{Z}_{(p)}) \cong H^*(E_p;\mathbb{Z}_{(p)}).$$

Proof. The Frattini subgroup of a finite group is nilpotent, so K is nilpotent and has the displayed direct product decomposition. Every K_p is characteristic in K, hence normal in E. The kernel of the natural map from E to the fiber product is the intersection of the subgroups $\prod_{q\neq p} K_q$, which is trivial. Both source and target have order $|Q|\prod_p |K_p| = |E|$, proving the isomorphism. The Frattini property passes to quotients.

The map $E \to E_p$ has p'-kernel. The two cohomology isomorphisms now follow from Lemma 5.1. □

Theorem 6.3 (Persistence of a cyclic Sylow class)**.** *Let $E \twoheadrightarrow Q$ be a finite Frattini cover. Suppose that a Sylow p-subgroup $\overline{P}$ of Q is cyclic, and put*

$$e = |N_Q(\overline{P}) : C_Q(\overline{P})|.$$

Then

$$H^{2e}(E;\mathbb{Z})_{(p)} \neq 0.$$

Proof. By Theorem 6.2, it is enough to consider a p-Frattini cover. Let P be the full inverse image of $\overline{P}$. It is a Sylow p-subgroup of E. The standard Frattini intersection lemma gives $\ker\pi \leq \Phi(E) \cap P \leq \Phi(P)$. Since $P/\ker\pi \cong \overline{P}$ is cyclic, the Burnside basis theorem implies that P is cyclic.

The natural map

$$N_E(P)/C_E(P) \longrightarrow N_Q(\overline{P})/C_Q(\overline{P})$$

is surjective: every lift of a normalizer element normalizes the full inverse image P. Its kernel acts trivially on the cyclic quotient $P/\ker\pi$, and is therefore a p-group. On the other hand $N_E(P)/C_E(P)$ is a p'-group, because $P \leq C_E(P)$ is Sylow in $N_E(P)$. The kernel is consequently trivial, so the two automizers have the same order e. The result follows from Theorem 3.1. □

Corollary 6.4. *Every finite Frattini cover of M_{23}, J_4, or Ly has nonzero integral cohomology in degree at most ten. More precisely, witnesses occur in degrees six, six, and ten, at the primes $7, 7, 11$, respectively.*

Proof. The relevant cyclic Sylow automizers have orders $3, 3, 5$, respectively. For J_4, the degree-six 7-local generator is also displayed directly in Green's calculation [20]. Apply Theorem 6.3. For Ly it directly gives the degree-ten class; no squaring is required. The sharper degree-eight two-primary class on Ly is not asserted here to persist through an arbitrary Frattini cover. □

6.1 Ledger for simple quotients

For a finite simple group Γ and an integer $N \geq 1$, write $\mathcal{Q}_N(\Gamma)$ for the assertion

$$E \twoheadrightarrow \Gamma, \quad E \text{ finite} \qquad \Longrightarrow \qquad H^i(E;\mathbb{Z}) \neq 0 \quad \text{for some } 1 \leq i \leq N.$$

The following table is the live finite-bound ledger. A condition in the second column is part of the scope of the row; only the first row is currently unconditional for every epimorphism. Every finite entry is genuine progress toward the existence theorem, whether or not its displayed degree is sharp.

Simple quotient Γ	Epimorphisms covered	witness degree	Reason
C_p	every $E \twoheadrightarrow C_p$	2	$E_{\mathrm{ab}} \twoheadrightarrow C_p$
every simple Γ	every epimorphism with prime-power kernel	140	Theorem 6.29
every nonabelian simple Γ	$\lvert\Lambda(\Gamma) \cap \pi(\ker)\rvert \leq 1$; in particular, every prime-power kernel	$D^{(3)}_{\mathrm{simp}}$	two distinct surviving primes, Corollary 6.27
every nonabelian simple Γ	$\Lambda(\Gamma) \not\subseteq \pi(\ker)$; in particular, $\lvert\pi(\ker)\rvert \leq 2$	$D^{(3)}_{\mathrm{simp}}$	one surviving prime, Theorem 6.28
every simple Γ	split epimorphisms	at most 8	Theorem 4.1 and Lemma 5.2
every simple Γ	$\ell \nmid \lvert\ker(E \to \Gamma)\rvert$, with an ℓ-local witness on Γ in any finite degree d	d	Lemma 5.1
every A_n, $n \geq 5$	every epimorphism with prime-power kernel	at most 4	Corollary 6.13
ordinary or graph-twisted Lie type over $q \neq 2, 3$ with perfect standard cover	every epimorphism with prime-power kernel	at most 4	Corollary 6.13
simple classical Γ	every epimorphism with prime-power kernel	at most 24	Corollary 6.19
ordinary or graph-twisted exceptional type over $q \in \{2, 3\}$	every epimorphism with prime-power kernel	at most 60	Corollary 6.22
simple Suzuki or Ree $\Gamma \neq {}^2F_4(2)'$	every epimorphism with prime-power kernel	at most 24	Corollary 6.21
sporadic Γ	every epimorphism with prime-power kernel	at most 140	Corollary 6.14
${}^2F_4(2)'$	every epimorphism with prime-power kernel	at most 24	Corollary 6.14
every nonabelian simple Γ	E superperfect, $K = \ker(E \to \Gamma)$, and $c \in H^d(\widetilde{\Gamma}; \mathbb{Z})$ with $\mathrm{ord}(c) \nmid \lvert[E, K]\rvert^{d-2}$	d	Corollary 5.6

Simple quotient Γ	Epimorphisms covered	witness degree	Reason
M_{22}, Suz, Fi_{22}	putative elementary central-descent branches with nontrivial multiplier (proved empty)	—	Proposition 9.21
Ru	either elementary central-descent layer	8	Propositions 5.8 and 9.21
M_{12}, HS, J_2, $McL, O'N, J_3$, Co_1, Fi'_{24}	elementary central descent with nontrivial multiplier	at most 10	Proposition 9.21
every simple Γ with $H_2(\Gamma;\mathbb{Z}) = 0$	every central epimorphism	at most 8	Proposition 5.15(i)
every simple Γ with $H^4(\Gamma;\mathbb{Z}) \neq 0$	central epimorphisms when the kernel or $H_2(\Gamma;\mathbb{Z})$ is cyclic	at most 4	Proposition 5.15(ii)
every simple Γ with $H^4(\widetilde{\Gamma};\mathbb{Z}) \neq 0$	every central epimorphism	at most 4	Proposition 5.15(iii)
every nonabelian simple Γ with $H_2(\Gamma;\mathbb{Z}) \cong C_{p^a}$, $a \geq 1$	every central epimorphism	at most 10	Theorem 6.6

Simple quotient Γ	Epimorphisms covered	witness degree	Reason
every simple Γ	finite Frattini covers, when a cyclic Sylow p-subgroup has automizer order $e \leq 5$	at most 10	Theorem 6.3
M_{23}, J_4, Ly	finite Frattini covers	$6, 6, 10$	Corollary 6.4
sporadic Γ	every central epimorphism	at most 10	Corollary 6.5
every simple Γ	elementary abelian kernel satisfying the rank inequality in Theorem 8.1	3 or 4	Theorem 8.1
every simple Γ	nonsplit odd-characteristic kernel whose scalar extension is nontrivial absolutely irreducible orthogonal, with $H^4(\Gamma;\overline{\mathbb{F}}_p) = 0$	3 or 4	Theorem 8.2

For the companion finite-bound ledger, the entire nonzero-multiplier simple-R portion of either elementary central descent is now closed. The alternating, Lie-type, and sporadic bounds are respectively 4, 9, and 20, by Propositions 9.19, 9.20, and 9.21. Thus Corollary 9.22 gives a common bound 20 in this scope. These are scope-sensitive companion entries, not unconditional assertions $\mathcal{Q}_{20}(R)$ for arbitrary epimorphisms. Every row of this subledger is therefore complete for the existence problem.

The first central row covers every trivial-multiplier simple group, including the Tits group and all thirteen trivial-multiplier sporadic groups. The sharper degree-four central row applies, from the calculations in Section 4, to every trivial-multiplier simple group of Lie type and to the sporadic groups

$$M_{11},\ M_{24},\ J_1,\ Co_2,\ Co_3,\ He,\ HN,\ Th,\ Fi_{23},\ M.$$

For each of these Γ, every central epimorphism onto Γ has a witness in degree at most four, because its Schur multiplier is trivial. For M_{23}, J_4, and Ly, every central epimorphism still has a witness in degree at most eight; in the superperfect case it is an isomorphism and the respective displayed degrees are six, six, and eight. Proposition 4.2, together with the cyclic Schur multipliers of the alternating groups, also puts every central epimorphism onto every A_n, $n \geq 5$, in the sharper degree-at-most-four row. Proposition 4.4 and the cyclic Schur multipliers of the simple groups $\mathrm{PSL}_2(q)$ do the same for every central epimorphism onto a simple $\mathrm{PSL}_2(q)$. Here $\widetilde{\Gamma}$ denotes the universal central extension. The new universal-cover row includes every ordinary or graph-twisted Lie-type group for which this is the associated perfect simply connected fixed-point group; in particular it includes every nonexceptional Schur-multiplier case in the standard classification [25, Theorem 6.1.4 and Table 6.1.3]. The Johnson–Freyd–Treumann calculations also put

$$M_{12},\quad M_{22},\quad HS,\quad J_2,\quad Co_1,\quad Suz,\quad O'N,\quad J_3,\quad Fi_{22}$$

in this row [29, table on p. 2]. Propositions 5.8, 5.9, and 5.10 put Ru, Fi'_{24}, and B, respectively, in the same row. The universal-centralization row is different: it permits a noncentral kernel K. For a superperfect source it identifies $E/[E,K]$ with $\widetilde{\Gamma}$, so persistence depends only on the smaller kernel $D = [E,K]$, not on all of K. Corollary 5.4 further gives $D = [E,D]$, so $H^2(D;\mathbb{Z})^{\widetilde{\Gamma}} = 0$; and formula (1) of Corollary 5.6 records the exact finite list of coefficient-cohomology groups that can kill a class on $\widetilde{\Gamma}$. The cyclic-Sylow specialization of the scalar-endomorphism criterion does not add another row. Indeed, Corollary 3.2 shows that a cyclic Sylow p-subgroup forces both $H_2(\Gamma;\mathbb{Z})_{(p)} = 0$ and, when $e \nmid 2$, $H^4(\widetilde{\Gamma};\mathbb{Z})_{(p)} = 0$. Every remaining fourth-degree primary component has prime different from p and is already covered by prime-to-layer persistence. This scope audit prevents the valid injectivity criterion from being counted as a new simple-quotient case. There is an earlier multiplier filter in the actual central descent: Proposition 9.13 forces $H_2(R;\mathbb{Z})$ to be a cyclic p-group. In alternating type this excludes A_6, A_7 and makes $p = 2$ in every remaining case, so the three-primary Pontryagin class survives in degree four. In Lie type, the degree-four fixed-point class works unless every prime of $q^2 - 1$ equals p; the residual arithmetic is $(q,p) = (3,2)$ or $(2,3)$, and the signed-fundamental-degree audit in Proposition 9.20 closes it in degree at most thirteen. In sporadic type the mixed-prime multipliers of M_{22}, Suz, and Fi_{22} exclude those putative branches before their modules are considered. Proposition 9.21 audits the remaining groups with nontrivial prime-power multiplier. Prime-to-layer classes close all but the characteristic-two case $R = B$; its eleven-primary class closes it in degree twenty. For $R = \mathrm{Ru}$, the five-primary fourth Chern class in degree eight from Proposition 5.8 survives every characteristic-two elementary layer. Altogether Corollary 9.22 closes every nonzero-multiplier simple R in this scope with bound twenty. It does not address simple R with trivial multiplier, nor a nonsimple R. Proposition 9.25 retains a finer, supplementary description of the two-primary degree-four edge. The M_{22} invariant-form calculation and the Suz–Fi_{22} Brauer-table calculation are retained as supplementary representation-theoretic certificates, not as central-descent ledger rows. For the narrower central-minimal-kernel branch of a least counterexample, Corollary 9.2 gives an additional Lie-type ledger constraint: every prime dividing $q^2 - 1$ must occur in the Schur multiplier of the simple quotient.

Corollary 6.5 (Central epimorphisms onto sporadic groups)**.** *Let* Γ *be a sporadic simple group. Every finite central epimorphism* $E \twoheadrightarrow \Gamma$ *has*

$$H^i(E;\mathbb{Z}) \neq 0 \qquad \text{for some } 2 \leq i \leq 10.$$

Proof. If E is not perfect, degree two detects it; if it is perfect but not superperfect, degree three does. Thus suppose that E is superperfect. The proof of Proposition 5.15 identifies E with the universal central extension $\widetilde{\Gamma}$.

If Γ has trivial multiplier, then $E \cong \Gamma$ and Theorem 4.1 applies. Among the thirteen sporadic groups with nontrivial multiplier, the introductory table of [29] gives nonzero third homology for the universal covers of

$$M_{12},\ M_{22},\ HS,\ J_2,\ Co_1,\ Suz,\ O'N,\ J_3,\ Fi_{22}.$$

Lemma 1.1 therefore gives nonzero fourth cohomology for these nine covers. Propositions 5.8, 5.9, and 5.10 give the same conclusion for the covers of Ru, Fi'_{24}, and B.

The remaining group is $\Gamma = \mathrm{McL}$. Its universal cover is 3.McL. The central kernel C_3 lies in the Frattini subgroup: otherwise, since it is a minimal normal subgroup, a maximal subgroup not containing it would be a complement, giving $3.\mathrm{McL} \cong C_3 \times \mathrm{McL}$, contrary to perfectness. A Sylow 11-subgroup P of McL is cyclic. The ATLAS has exactly two classes $11A, 11B$, each with centralizer P [1]. Fusion of elements of P is controlled by its normalizer, so its nonidentity elements form two orbits and $|N_{\mathrm{McL}}(P) : C_{\mathrm{McL}}(P)| = 5$. Theorem 6.3 gives a class restricting to the first normalizer-invariant power of the periodic generator, and hence $H^{10}(3.\mathrm{McL};\mathbb{Z})_{(11)} \neq 0$. No squaring is needed. □

Theorem 6.6 (Simple groups with cyclic prime-power multiplier)**.** *Let* Γ *be a nonabelian finite simple group such that*

$$H_2(\Gamma;\mathbb{Z}) \cong C_{p^a} \qquad (a \geq 1).$$

Its universal central extension $\widetilde{\Gamma}$ *has*

$$H^i(\widetilde{\Gamma};\mathbb{Z}) \neq 0 \qquad \text{for some } 3 \leq i \leq 10.$$

Consequently, every finite central epimorphism $E \twoheadrightarrow \Gamma$ *has nonzero integral cohomology in some degree at most ten.*

Proof. For alternating groups, Proposition 4.2 and the cyclic-multiplier case of Proposition 5.15(ii) give the result in degree at most four. For sporadic groups it is Corollary 6.5. It remains to consider groups of Lie type.

If the universal central extension is the standard perfect simply connected fixed-point group, Proposition 5.16 gives a nonzero degree-four class. The complete exceptional-multiplier table [25, Theorem 6.1.4 and Table 6.1.3], after removing the alternating and PSL_2 exceptional isomorphisms, shows that the nonstandard cases whose *full* multiplier is a cyclic prime-power group are exactly

Γ	$\mathrm{PSU}_4(2)$	$\mathrm{PSp}_6(2)$	$G_2(3)$	$G_2(4)$	$F_4(2)$
$H_2(\Gamma;\mathbb{Z})$	C_2	C_2	C_3	C_2	C_2
ℓ	3	3	2	3	3.

Here $\ell \mid q^2 - 1$ in each column and $\ell \nmid |H_2(\Gamma;\mathbb{Z})|$. Proposition 7.8, applied with $S = B = \Gamma$, gives

$$H^4(\Gamma;\mathbb{Z})_{(\ell)} \neq 0.$$

The kernel of $\widetilde{\Gamma} \to \Gamma$ has order prime to ℓ, so Lemma 5.1 gives

$$H^4(\widetilde{\Gamma};\mathbb{Z})_{(\ell)} \cong H^4(\Gamma;\mathbb{Z})_{(\ell)} \neq 0.$$

The Suzuki–Ree families have trivial multiplier outside the exceptional noncyclic multiplier of ${}^2B_2(8)$; the Tits multiplier is trivial. Thus they add no case under the hypothesis.

Finally let $E \twoheadrightarrow \Gamma$ be central. If E is not perfect, then $H^2(E;\mathbb{Z}) \neq 0$; if it is perfect but not superperfect, then $H^3(E;\mathbb{Z}) \neq 0$. If E is superperfect, the homology five-term sequence identifies it with $\widetilde{\Gamma}$, to which the first part applies. □

Proposition 6.7 (Status of the quotient ledger)**.** *Every entry in the ledger has the stated scope. In particular, $\mathcal{Q}_2(C_p)$ holds for every prime p. The draft does not yet produce one finite N_0 for which $\mathcal{Q}_{N_0}(\Gamma)$ holds for every nonabelian simple Γ without a condition on the epimorphism.*

Proof. If $E \twoheadrightarrow C_p$, then $E_{\mathrm{ab}} \twoheadrightarrow C_p$, so $H_1(E;\mathbb{Z}) \neq 0$ and Lemma 1.1 gives $H^2(E;\mathbb{Z}) \neq 0$. The split and prime-to-kernel rows are exactly Lemmas 5.2 and 5.1, applied to the witnesses in Theorem 4.1. The general prime-power-kernel row is Theorem 6.29; its explicit family rows are Corollaries 6.13, 6.19, and 6.14, together with Corollary 6.21. The universal-centralization row is Corollary 5.6. The central and Frattini rows are Proposition 5.15, Proposition 5.16, Theorem 6.3, and Corollaries 6.4 and 6.5, together with Theorem 6.6. The final two rows are Theorems 8.1 and 8.2. None of these arguments removes the displayed restrictions for an arbitrary nonsplit epimorphism onto a nonabelian simple group, which proves the final status assertion. □

Proposition 6.8 (The first canonical p-Frattini layer)**.** *Let G be a finite perfect group, let $p \mid |G|$, put $k = \mathbb{F}_p$, and let*

$$1 \longrightarrow M \longrightarrow G_1 \longrightarrow G \longrightarrow 1 \tag{6.1}$$

be the universal elementary abelian p-Frattini cover. Suppose that

$$H^2(G;k) = H^3(G;k) = 0. \tag{6.2}$$

Then G_1 is perfect and, in the LHS spectral sequence of (6.1)*,*

$$H^2(G_1;k) \cong \ker\Big(d_2^{0,2} : H^2(M;k)^G \longrightarrow H^2(G;M^*)\Big). \tag{6.3}$$

More explicitly, Gaschütz's identification

$$M \cong \Omega^2_{kG}k \tag{6.4}$$

gives

$$(M^*)^G = 0, \qquad H^1(G;M^*) \cong H^3(G;k) = 0. \tag{6.5}$$

Consequently the source in (6.3) *is*

$$\begin{cases} (\Lambda^2 M^*)^G, & p \text{ odd}, \\ (\mathrm{Sym}^2 M^*)^G, & p = 2. \end{cases} \tag{6.6}$$

In particular,

$$H^3(G_1;\mathbb{Z})_{(p)} \neq 0 \quad \Longleftrightarrow \quad \ker d_2^{0,2} \neq 0. \tag{6.7}$$

The hypotheses (6.2) *hold at every prime p if $H^i(G;\mathbb{Z}) = 0$ for $1 \leq i \leq 4$.*

Proof. The Frattini module identification (6.4) is Gaschütz's theorem; see [41, Theorem 2.4]. The group G_1 is perfect. Indeed, $[G_1, G_1]$ maps onto $[G, G] = G$; if it were proper, a maximal subgroup containing it would still map onto G, contrary to the Frattini property.

The five-term sequence, perfectness of G and G_1, and $H^2(G; k) = 0$ give

$$0 = H^1(G_1; k) \longrightarrow (M^*)^G \longrightarrow H^2(G; k) = 0,$$

so $(M^*)^G = 0$. Since the group algebra kG is symmetric, duality takes syzygies to cosyzygies: M^* is stably $\Omega^{-2}_{kG} k$. Tate dimension shifting therefore gives

$$H^1(G; M^*) \cong \widehat{H}^1(G; \Omega^{-2} k) \cong \widehat{H}^3(G; k) = H^3(G; k) = 0. \tag{6.8}$$

Notice the direction of the shift: the group in (6.8) is $H^3(G; k)$, not a negative Tate cohomology group.

In total degree two, the LHS filtration has associated terms $E_\infty^{2,0}$, $E_\infty^{1,1}$, and $E_\infty^{0,2}$. The first two vanish by (6.2) and (6.8). No differential enters $E_r^{0,2}$; its only possible outgoing differentials are $d_2^{0,2}$ and $d_3^{0,2}$, and the latter has target a subquotient of $H^3(G; k) = 0$. This proves (6.3).

For an elementary abelian group M,

$$H^2(M; k) \cong \begin{cases} \Lambda^2 M^* \oplus \beta(M^*), & p \text{ odd}, \\ \operatorname{Sym}^2 M^*, & p = 2. \end{cases}$$

At odd p, equivariance and injectivity of the Bockstein identify the invariants in its image with $\beta((M^*)^G) = 0$, proving (6.6). Finally G_1 is perfect, so the universal coefficient theorem identifies $H^2(G_1; k)$ with $\operatorname{Hom}(H_2(G_1; \mathbb{Z}), k)$, while Lemma 1.1 identifies the p-part of $H^3(G_1; \mathbb{Z})$ with the p-part of $H_2(G_1; \mathbb{Z})$. This proves (6.7). The last assertion follows from the integral and field-coefficient universal coefficient theorems. □

Corollary 6.9 (Two barriers in a fixed-prime Frattini tower)**.** *Let*

$$\cdots \longrightarrow G_2 \longrightarrow G_1 \longrightarrow G_0$$

be the finite characteristic quotients of the universal p-Frattini cover. If G_0 is perfect, then every G_n is perfect. Moreover:

(i) *if $\ell \neq p$ and $H^d(G_0; \mathbb{Z})_{(\ell)} \neq 0$, then $H^d(G_n; \mathbb{Z})_{(\ell)} \neq 0$ for every n;*

(ii) *if G_0 has a cyclic Sylow ℓ-subgroup with automizer of order e, then $H^{2e}(G_n; \mathbb{Z})_{(\ell)} \neq 0$ for every n.*

Thus neither kind of fixed low-degree witness can be removed by climbing the tower.

Proof. Perfectness follows inductively from the Frattini argument in the proof of Proposition 6.8. Each composite $G_n \twoheadrightarrow G_0$ is a Frattini cover with p-group kernel. Part (i) follows by iterating Lemma 5.1, and part (ii) follows from Theorem 6.3. □

Remark 6.10 (Rapid dimension growth)**.** Write $M_n = \ker(G_{n+1} \to G_n)$ and $d_n = \dim_{\mathbb{F}_p} M_n$. The Schreier formula gives

$$d_{n+1} = 1 + p^{d_n}(d_n - 1)$$

[41, Section 4]. Except in the rare cases $d_0 \leq 1$, the dimensions grow by recursive exponentiation. For example, the first 2-Frattini module of A_5 has dimension $d_0 = 5$, so the next has dimension $d_1 = 129$. This does not exhibit homology vanishing, but it explains why a direct enumeration of several canonical levels is not a realistic substitute for a structural theorem.

Proposition 6.11 (Two-prime persistence through a prime-power kernel)**.** *Let Q be a finite group. Suppose that there are distinct primes ℓ_1, ℓ_2 and classes*

$$0 \neq x_j \in H^{d_j}(Q; \mathbb{Z})_{(\ell_j)}, \qquad d_j \leq D \quad (j = 1, 2).$$

Then every finite epimorphism $E \twoheadrightarrow Q$ whose kernel is a p-group has

$$H^i(E; \mathbb{Z}) \neq 0 \qquad \text{for some } i \leq D.$$

Proof. At least one of ℓ_1, ℓ_2 is different from p. For that prime, Lemma 5.1 identifies the corresponding localized cohomology of E and Q, so the chosen class survives inflation. □

For a finite group Q and $D \geq 1$, define its bounded-degree prime support by

$$\mathrm{Supp}_D(Q) = \{\ell : \ H^i(Q; \mathbb{Z})_{(\ell)} \neq 0 \text{ for some } 1 \leq i \leq D\}.$$

Proposition 6.12 (Prime-support loss under inflation)**.** *Let $E \twoheadrightarrow Q$ be a finite epimorphism with kernel K. Then, for every D,*

$$\mathrm{Supp}_D(E) \supseteq \mathrm{Supp}_D(Q) \setminus \pi(K),$$

where $\pi(K)$ is the set of prime divisors of $|K|$. Consequently, if

$$|\,\mathrm{Supp}_D(Q)| \geq |\pi(K)| + a,$$

then E has bounded-degree witnesses at at least a distinct primes. In particular, three distinct-prime witnesses on Q yield two on every extension of Q by a p-group.

Proof. For every $\ell \notin \pi(K)$, Lemma 5.1 identifies the ℓ-localized cohomology of E and Q in every positive degree. This proves the displayed inclusion and the cardinality statements. □

Corollary 6.13 (Large simple families with arbitrary p-group kernel)**.** *Let $E \twoheadrightarrow S$ be a finite epimorphism with nontrivial simple quotient and p-group kernel. Then $H^i(E; \mathbb{Z}) \neq 0$ for some $i \leq 4$ in either of the following cases:*

(i) *$S = A_n$ with $n \geq 5$;*

(ii) *S is an ordinary or graph-twisted group of Lie type over $\mathbb{F}_q$, $q \notin \{2, 3\}$, and its standard finite simply connected fixed-point group H is perfect and maps centrally onto S.*

Proof. For A_n, Proposition 4.2 supplies a three-primary class in degree four, while the two-primary part of its Schur multiplier supplies a class in degree three. Apply Proposition 6.11.

Now consider case (ii) and put $K = \ker(H \to S)$. For every prime $\ell \mid q^2 - 1$, there is a nonzero ℓ-primary class on S in degree three or four. Indeed, if $\ell \mid |K|$, the homology five-term sequence and perfectness of H give a surjection

$$H_2(S; \mathbb{Z})_{(\ell)} \twoheadrightarrow K_{(\ell)},$$

so $H^3(S; \mathbb{Z})_{(\ell)} \neq 0$. If $\ell \nmid |K|$, Proposition 4.5 and Lemma 5.1 give

$$H^4(S; \mathbb{Z})_{(\ell)} \cong H^4(H; \mathbb{Z})_{(\ell)} \neq 0.$$

It remains only to observe that $q^2 - 1$ has at least two distinct prime divisors. If $q > 2$ is even, the coprime odd integers $q - 1, q + 1$ are both greater than one. If $q > 3$ is odd and $q^2 - 1$ had only one prime divisor, that prime would be two and both $q - 1$ and $q + 1$ would be powers of two. Two powers of two differing by two are 2 and 4, which would give $q = 3$. Thus two distinct detecting primes are available, and Proposition 6.11 finishes the proof. □

Corollary 6.14 (Sporadic and Tits quotients with p-group kernel)**.** *Let $E \twoheadrightarrow S$ be a finite epimorphism with p-group kernel. If S is sporadic simple, then*

$$H^i(E;\mathbb{Z}) \neq 0 \qquad \text{for some } i \leq 140.$$

If $S = {}^2F_4(2)'$ is the Tits group, one may replace 140 by 24.

Proof. For each sporadic group, the table gives its largest prime divisor r and a different prime ℓ_0 at which the group has a nonzero integral class in degree at most eight:

S	r	ℓ_0	S	r	ℓ_0
M_{11}	11	2	M_{12}	11	2
M_{22}	11	2	M_{23}	23	7
M_{24}	23	2	J_1	19	2
J_2	7	2	J_3	19	3
J_4	43	7	HS	11	2
McL	11	3	Suz	13	2
Ru	29	2	$O'N$	31	3
Co_1	23	2	Co_2	23	2
Co_3	23	2	He	17	2
Fi_{22}	13	2	Fi_{23}	23	2
Fi'_{24}	29	3	HN	19	3
Ly	67	11	Th	31	2
B	47	2	M	71	2

(6.9)

For groups with nontrivial multiplier, the ℓ_0-class is in degree three. For groups with trivial multiplier, the choices follow from the table in Section 4; for J_1 use the two-primary component of its $\mathbb{Z}/30$ in degree four. The three exceptional degrees are the seven-primary degree-six classes of M_{23} and J_4 and the two-primary degree-eight class of Ly.

The ATLAS orders show that r occurs to the first power in every row [1]. Hence a Sylow r-subgroup is cyclic of order r. Its automizer has order $e \mid r-1$, so Theorem 3.1 gives a nonzero r-primary class in degree

$$2e \leq 2(r-1) \leq 140.$$

Since $r \neq \ell_0$, Proposition 6.11 applies.

For the Tits group, Proposition 4.7 gives a two-primary class in degree four. Its largest prime divisor is 13, occurring to the first power, so cyclic Sylow detection gives a thirteen-primary class in degree at most 24. Apply the same two-prime criterion. □

Lemma 6.15 (Prime powers with two-prime q^2-1)**.** *Let $q > 3$ be a prime power. If q^2-1 has fewer than three distinct prime divisors, then*

$$q \in \{4, 5, 7, 8, 9, 17\}.$$

Conversely, every value in this list has exactly two distinct prime divisors in q^2-1.

Proof. First suppose that $q = 2^f$ is even. The odd numbers $q-1$ and $q+1$ are coprime, so the hypothesis makes each a power of one prime. Write $q+1 = s^b$. If $b > 1$, Mihăilescu's theorem on consecutive perfect powers gives $(q, q+1) = (8, 9)$ [37]. If $b = 1$, the primality of $2^f + 1$ forces f

to be a power of two: otherwise an odd proper divisor of f gives the usual factorization of x^d+1. For $f \geq 4$ the two coprime odd factors

$$2^{f/2}-1, \qquad 2^{f/2}+1$$

show that $q-1$ is not a prime power. Hence $q=4$ or 8.

Now suppose that q is odd. The consecutive coprime integers

$$A=\frac{q-1}{2}, \qquad B=\frac{q+1}{2}$$

have prime divisors in a set $\{2,r\}$ for one odd prime r. Such an r exists: otherwise A and B would be consecutive powers of two, forcing $\{A,B\}=\{1,2\}$ and $q=3$. Exactly one of A,B is even, so

$$\{A,B\}=\{2^a,r^b\}.$$

If $a,b>1$, Mihăilescu's theorem gives $\{A,B\}=\{8,9\}$ and $q=17$. If $a=1$, then the other number is three and $q=5$.

It remains to take $b=1$. If $r=2^a+1$ is prime, then a is a power of two. For $a>1$, the number

$$q=2^{a+1}+1$$

is divisible by three. Since it is a prime power, it is 3^c; another application of Mihăilescu's theorem gives $q=9$. The case $a=1$ again gives $q=5$. If $r=2^a-1$ is prime, then a is prime. For odd a the factorization

$$2^{a+1}-1=(2^{(a+1)/2}-1)(2^{(a+1)/2}+1)$$

has two coprime nontrivial factors, contradicting the assumption that q is a prime power. Thus $a=2$ and $q=7$. Direct factorization checks the six displayed values. □

Proposition 6.16 (A projective-permutation Chern class)**.** *Let V be an n-dimensional vector space over $\mathbb{F}_Q$, let $\overline{G} \leq \mathrm{PGL}(V)$, and suppose that an element $g \in \overline{G}$ of prime order r has a lift whose action decomposes as*

$$V=U\oplus W, \qquad g|_U=1, \qquad \dim W=f,$$

where W has no one-dimensional g-invariant $\mathbb{F}_Q$-subspace. Put

$$m=Q^{n-f}\frac{Q^f-1}{r(Q-1)}.$$

If m is an integer not divisible by r, then the complex permutation representation of $\overline{G}$ on $\mathbb{P}(V)$ has

$$0\neq c_{r-1}\in H^{2(r-1)}(\overline{G};\mathbb{Z})_{(r)}.$$

Proof. The only projective points fixed by g are those in $\mathbb{P}(U)$. Every other orbit has length r, and their number is

$$\frac{|\mathbb{P}(V)|-|\mathbb{P}(U)|}{r}=Q^{n-f}\frac{Q^f-1}{r(Q-1)}=m.$$

On restriction to $C_r=\langle g\rangle$, the permutation representation is therefore a sum of trivial lines and m copies of the regular complex representation. If u generates $H^2(C_r;\mathbb{Z})$, the positive-degree part of its total Chern class is computed modulo r by

$$\prod_{j=0}^{r-1}(1+ju)=1-u^{r-1}.$$

Consequently

$$\operatorname{res}^{\overline{G}}_{C_r} c_{r-1} = -mu^{r-1} \neq 0,$$

which proves the assertion. □

Proposition 6.17 (Uniform three-prime support for classical groups)**.** *There is a finite integer $D^{(3)}_{\mathrm{cl}}$ such that every nonabelian finite simple classical group S has nonzero integral cohomology supported at three distinct primes in degrees at most $D^{(3)}_{\mathrm{cl}}$.*

Proof. Suppose first that the defining parameter q does not belong to

$$\mathcal{E} = \{2, 3, 4, 5, 7, 8, 9, 17\}. \tag{6.10a}$$

Lemma 6.15 gives three distinct primes dividing $q^2 - 1$. In every standard-cover case, the argument in the proof of Corollary 6.13 gives, at each such prime, a nonzero class in degree three or four. The exceptional nonperfect fixed-point groups occur over the fields in $\mathcal{E}$, up to the usual low-rank isomorphisms. Thus this case has three-prime support in degrees at most four.

It remains to keep $q \in \mathcal{E}$ fixed and let the rank grow. The following table gives enough additional primes. The entries in the last two columns are the multiplicative orders used for linear, symplectic, and orthogonal groups, and for unitary groups, respectively.

q	r	$\operatorname{ord}_r(q)$	$\operatorname{ord}_r(q^2)$
2	5, 7	4, 3	2, 3
3	5, 13	4, 3	2, 3
4	17	4	2
5	13	4	2
7	19	3	3
8	5	4	2
9	41	4	2
17	29	4	2

(6.10b)

In every entry, $r \nmid q^2 - 1$, and if f is either displayed order, then

$$v_r(q^f - 1) = 1 \quad \text{or} \quad v_r((q^2)^f - 1) = 1, \tag{6.10c}$$

as appropriate. These assertions follow immediately by factoring the small numbers $q^3 - 1$ or $q^2 + 1$; the table is therefore an exact finite check, not an appeal to unbounded primitive divisors.

For a linear group, take an irreducible order-r block W_0 of dimension $f = \operatorname{ord}_r(q)$. Its determinant is one because $r \nmid q - 1$. For a symplectic or orthogonal group, use the hyperbolic block $W_0 \oplus W_0^*$; its order is odd, so in the orthogonal case it lies in the derived isometry group. For a unitary group, work over $\mathbb{F}_{q^2}$ and pair an irreducible block of dimension $f = \operatorname{ord}_r(q^2)$ with its conjugate dual. Its determinant is one because $r \nmid q^2 - 1$. In each family, once the natural dimension is large enough, extend this block by the identity on a nondegenerate complement of the required type. The total nontrivial block has dimension at most eight and contains no invariant line over the field used for the projective-point action. Since $r \nmid q^2 - 1$, it is also prime to every relevant scalar center, so the projective image still has order r.

Apply Proposition 6.16 with $Q = q$ in the first three families and $Q = q^2$ in the unitary family. If b is the total block dimension, (6.10c) and the oddness of r give

$$v_r(Q^b - 1) = 1,$$

so the orbit number in that proposition is prime to r. We obtain an r-primary class in degree $2(r-1) \le 80$. For $q = 2, 3$, the two primes in the corresponding row, together with the prime dividing $q^2 - 1$, give three-prime support. For every other $q \in \mathcal{E}$, the one displayed prime supplements the two primes dividing $q^2 - 1$.

Only finitely many natural dimensions are too small for these fixed blocks. The nonabelian simple groups in that finite residue have at least three group primes by Burnside's $p^a q^b$ theorem [10]. Lemma 6.20 supplies a finite common degree ceiling for them. Taking the maximum of that ceiling and 80 proves the proposition. □

Corollary 6.18 (Linear simple quotients with p-group kernel)**.** *Let $S = \mathrm{PSL}_n(q)$ be simple. Every finite epimorphism $E \twoheadrightarrow S$ with p-group kernel has*

$$H^i(E; \mathbb{Z}) \neq 0 \qquad \text{for some } i \le 24.$$

For $q \notin \{2, 3\}$ one may use the bound four, and for $q \in \{2, 3\}$ with $n \ge 4$ one may use the bound eight.

Proof. The case $q \notin \{2, 3\}$ is Corollary 6.13. Suppose $q = 2$ or 3 and $n \ge 4$. The fifth cyclotomic polynomial is irreducible over $\mathbb{F}_q$, because q has order four modulo five. Its companion matrix C has order five, determinant one, and no invariant line. Thus

$$\mathrm{diag}(C, I_{n-4}) \in \mathrm{SL}_n(q)$$

defines an element of order five in S. Proposition 6.16 applies with $r = 5$, $f = 4$, and

$$m = \begin{cases} 3 \cdot 2^{n-4}, & q = 2, \\ 8 \cdot 3^{n-4}, & q = 3. \end{cases}$$

Hence $H^8(S; \mathbb{Z})_{(5)} \neq 0$. The argument in the proof of Corollary 6.13, applied to the standard central map $\mathrm{SL}_n(q) \to S$, gives a second class in degree three or four at the prime three when $q = 2$ and at the prime two when $q = 3$. Proposition 6.11 gives the bound eight.

The only remaining simple rank-three groups are $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$ and $\mathrm{PSL}_3(3)$. The former has its two-primary multiplier class in degree three and the three-primary degree-four class of Proposition 4.4. For the latter, the standard fixed-point argument gives a two-primary class in degree four, while

$$|\mathrm{PSL}_3(3)| = 2^4 \cdot 3^3 \cdot 13.$$

Its Sylow 13-subgroup is cyclic, so Theorem 3.1 supplies a thirteen-primary class in degree at most 24. Apply Proposition 6.11 once more. □

Corollary 6.19 (Classical simple quotients with p-group kernel)**.** *Let S be a finite simple classical group: linear, unitary, symplectic, or orthogonal. Every finite epimorphism $E \twoheadrightarrow S$ with p-group kernel has*

$$H^i(E; \mathbb{Z}) \neq 0 \qquad \text{for some } i \le 24.$$

Except for the low-rank group $\mathrm{PSU}_3(3)$ and groups covered through a standard low-rank isomorphism, the small-field cases over $\mathbb{F}_2$ and $\mathbb{F}_3$ have bound eight.

Proof. Corollary 6.18 treats the linear groups. By Corollary 6.13, it remains to treat the unitary, symplectic, and orthogonal families with defining parameter $q \in \{2, 3\}$.

First suppose that S is symplectic or orthogonal. There is a four-dimensional nondegenerate symplectic, respectively minus-type orthogonal, space W over $\mathbb{F}_q$ with an irreducible isometry of

order five. One way to see this is to use the order-five subgroup of the standard torus of order q^2+1; equivalently, its characteristic polynomial is the fifth cyclotomic polynomial, which is irreducible over $\mathbb{F}_q$. Extend this isometry by the identity on a nondegenerate orthogonal complement. It lies in the relevant derived isometry group: in the orthogonal case this also follows because its image in the two-group quotient of the special orthogonal group has odd order. The induced action of the simple projective group on all projective points satisfies Proposition 6.16 with $r = 5$, $f = 4$, and

$$m = \begin{cases} 3 \cdot 2^{\dim(V)-4}, & q = 2, \\ 8 \cdot 3^{\dim(V)-4}, & q = 3. \end{cases}$$

Thus it gives a nonzero five-primary class in degree eight.

For a unitary group, work over $\mathbb{F}_{q^2}$. The two irreducible two-dimensional order-five modules are interchanged by conjugate duality. Their direct sum therefore carries a nondegenerate invariant Hermitian form; the resulting order-five isometry has determinant one and no invariant $\mathbb{F}_{q^2}$-line. Extend it by the identity. In the projective point action, Proposition 6.16 now has $Q = q^2$, $r = 5$, $f = 4$, and

$$m = \begin{cases} 17 \cdot 4^{\dim(V)-4}, & q = 2, \\ 164 \cdot 9^{\dim(V)-4}, & q = 3. \end{cases}$$

Again $5 \nmid m$, so $H^8(S;\mathbb{Z})_{(5)} \neq 0$.

In every one of these standard-cover cases, the argument in the proof of Corollary 6.13 supplies a second integral class in degree three or four: it is three-primary for $q = 2$ and two-primary for $q = 3$. Proposition 6.11 gives the bound eight.

The only simple unitary group not reached by the four-dimensional block or a linear low-rank isomorphism is $\mathrm{PSU}_3(3)$. Its standard-cover argument supplies a two-primary class in degree four, and

$$|\,\mathrm{PSU}_3(3)| = 2^5 \cdot 3^3 \cdot 7 \cdot 13.$$

The Sylow 13-subgroup is cyclic, so Theorem 3.1 gives a thirteen-primary class in degree at most 24. The two-prime criterion completes this case. The familiar low-rank coincidences reduce the remaining simple exceptions to linear, alternating, or already treated symplectic groups. □

Lemma 6.20 (Every group prime occurs in positive integral cohomology)**.** *Let G be finite and let $\ell \mid |G|$. Then*

$$H^d(G;\mathbb{Z})_{(\ell)} \neq 0$$

for some positive integer d. Consequently, if $\mathcal{F}$ is a finite set of nontrivial finite groups and $a \geq 1$ is such that every member has at least a distinct prime divisors, there is a finite $D_a(\mathcal{F})$ such that every $G \in \mathcal{F}$ has witnesses at a distinct primes in degrees at most $D_a(\mathcal{F})$.

Proof. The mod-ℓ cohomology of a finite group whose order is divisible by ℓ is nonzero in arbitrarily high degrees [9, Chapter VII]. If the positive-degree integral cohomology had no ℓ-primary part, multiplication by ℓ would be an automorphism of every positive-degree integral cohomology group. The coefficient exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ell} \mathbb{Z} \longrightarrow \mathbb{F}_\ell \longrightarrow 0$$

would then force $H^d(G;\mathbb{F}_\ell) = 0$ for every $d > 0$, a contradiction. For a finite family, choose a group primes for each member and take the maximum of their first nonzero degrees. □

Corollary 6.21 (Suzuki–Ree quotients with p-group kernel)**.** *Let $E \twoheadrightarrow S$ be a finite epimorphism with p-group kernel. If S is a simple Suzuki or Ree group other than the Tits group, then*

$$H^i(E;\mathbb{Z}) \neq 0 \qquad \text{for some } i \leq 24.$$

More precisely, one may use 8 for ${}^2B_2(q)$, 12 for ${}^2G_2(q)$, and 24 for ${}^2F_4(q)$.

Proof. Babai–Pálfy–Saxl, Theorem 3.1 and Table I [3], give two nonconjugate cyclic sharp maximal tori T_+ and T_- in each of these families. Their orders are the two semicyclotomic factors $\Phi_d^+(q)$ and $\Phi_d^-(q)$, where

S	${}^2B_2(q)$	${}^2G_2(q)$	${}^2F_4(q)$
d	4	6	12
$\lvert N_S(T_\pm) : T_\pm \rvert$	4	6	12.

(6.10)

The two torus orders are coprime [3, Theorem 3.4 and Table II].

A sharp subgroup is a Hall subgroup [3, Proposition 1.14]. Choose primes $r_\pm \mid |T_\pm|$. They are distinct, and a Sylow $r_\pm$-subgroup $P_\pm$ is the cyclic $r_\pm$-part of $T_\pm$. Sharpness gives $C_S(P_\pm) = T_\pm$. Since a normalizer of $P_\pm$ normalizes its centralizer,

$$|N_S(P_\pm) : C_S(P_\pm)| \mid |N_S(T_\pm) : T_\pm|.$$

Theorem 3.1 therefore supplies witnesses at the two distinct primes in degrees at most 8, 12, and 24, respectively. Proposition 6.11 finishes the proof. □

Corollary 6.22 (Exceptional small-field quotients with p-group kernel)**.** *Let S be a simple ordinary or graph-twisted exceptional group of Lie type over $\mathbb{F}_2$ or $\mathbb{F}_3$, and let $E \twoheadrightarrow S$ have p-group kernel. Then*

$$H^i(E;\mathbb{Z}) \neq 0 \qquad \text{for some } i \leq 60.$$

Proof. First, S has a class of degree at most four supported at the prime

$$\ell_0 = \begin{cases} 3, & q = 2, \\ 2, & q = 3. \end{cases} \tag{6.11}$$

In the standard-cover cases this is the argument in the proof of Corollary 6.13, applied to the unique prime dividing q^2-1. The complete exceptional-multiplier audit [25, Theorem 6.1.4 and Table 6.1.3] leaves only three comments at the required prime. The multiplier of ${}^2E_6(2)$ has a three-primary part and hence gives a degree-three class. The proof of Theorem 6.6 gives a two-primary degree-four class on $G_2(3)$ and a three-primary degree-four class on $F_4(2)$. Finally $G_2(2)' \cong \mathrm{PSU}_3(3)$ is already classical.

Except in type E_7, Table I of Babai–Pálfy–Saxl [3] gives a cyclic sharp torus T all of whose prime divisors are primitive for the displayed exponent. In particular they are different from ℓ_0. The automizer ceilings are

type	${}^3D_4$	G_2	F_4	E_6	${}^2E_6$	E_8
$\lvert N_S(T) : T \rvert$	4	6	12	9	9	30.

(6.12)

As in the proof of Corollary 6.21, any prime $r \mid |T|$ has cyclic Sylow r-subgroup and yields an r-primary integral class in degree at most twice the displayed automizer. This is a second-prime witness in degree at most 60.

For $E_7(q)$, choose a primitive prime divisor r of q^7-1; there is no Zsigmondy exception for $q \in \{2,3\}$ and exponent seven [45]. The r-part occurs only in the factor $q^{14}-1$ of the group order, so a Sylow r-subgroup P is cyclic. The proof of Theorem 3.4 in [3] gives a cyclic torus T of order $(q^7-1)/(2,q-1)$ containing P, with

$$C_S(P) = T, \qquad |N_S(T) : T| = 14.$$

Thus $|N_S(P) : C_S(P)| \le 14$ and Theorem 3.1 gives a second-prime witness in degree at most 28. In all types Proposition 6.11 now applies. □

Proposition 6.23 (Three-prime support in ordinary exceptional type)**.** *There is a finite integer $D^{(3)}_{\mathrm{exc}}$ such that every nonabelian finite simple group of ordinary or graph-twisted exceptional Lie type has nonzero integral cohomology supported at three distinct primes in degrees at most $D^{(3)}_{\mathrm{exc}}$.*

Proof. If the field parameter q is outside the finite set $\mathcal{E}$ in (6.10a), Lemma 6.15 supplies three distinct primes dividing q^2-1. At every one of them, the standard central-cover argument in the proof of Corollary 6.13 gives a nonzero integral class in degree three or four. There are only finitely many exceptional types and only finitely many groups over the fields in $\mathcal{E}$. Each simple group in this residue has at least three group primes by Burnside's theorem, and Lemma 6.20 gives a finite common ceiling for three of them. The maximum of this finite ceiling and four proves the assertion. □

Proposition 6.24 (Three-prime support in Suzuki–Ree type)**.** *There is a finite integer $D^{(3)}_{\mathrm{SR}}$ such that every simple Suzuki or Ree group, including the Tits group, has nonzero integral cohomology supported at three distinct primes in degrees at most $D^{(3)}_{\mathrm{SR}}$. Outside a finite small-field residue, the ceiling may be taken to be* 24*.*

Proof. For a group outside the finite fixed-point and multiplier exceptions listed in the standard exceptional-multiplier tables [25, Theorem 6.1.4 and Table 6.1.3], choose the two primes r_+, r_- from the coprime sharp tori T_+, T_- in the proof of Corollary 6.21. They give integral witnesses in degrees at most 8, 12, and 24 in the Suzuki, small Ree, and large Ree families, respectively.

For these nonexceptional groups the standard finite fixed-point group is perfect and maps centrally onto S. The very-twisted homotopy-fixed-point calculation in the proof of Proposition 4.6, followed by the same central-kernel dichotomy as in Corollary 6.13, gives a third class in degree three or four at a prime ℓ dividing $q-1$ or $q+1$, according to the sign of the action on π_4. This prime is different from r_+ and r_-. Indeed, the products of the two semicyclotomic torus orders are, respectively,

$$q^2+1, \qquad q^2-q+1, \qquad q^4-q^2+1. \tag{6.12a}$$

The first and third are coprime to both $q-1$ and $q+1$ for even q. For q a power of three, the middle expression is also coprime to both: the only possible extra common divisor with $q+1$ is three, whereas $3 \nmid q+1$. Thus the three detecting primes are distinct.

There are only finitely many excluded small fixed-point groups, including the Tits group. Burnside's theorem and Lemma 6.20 give a finite three-prime ceiling on this residue. Taking its maximum with 24 proves the proposition. □

Theorem 6.25 (Two-prime simple base)**.** *For every nonabelian finite simple group S, there are distinct primes ℓ_1, ℓ_2 and classes*

$$0 \neq x_j \in H^{d_j}(S; \mathbb{Z})_{(\ell_j)}, \qquad d_j \le 140 \quad (j = 1, 2).$$

The family ceilings for both classes may be taken as follows:

family	$\max(d_1, d_2)$
A_n	4
classical	24
ordinary or graph-twisted exceptional, $q \notin \{2, 3\}$	4
ordinary or graph-twisted exceptional, $q \in \{2, 3\}$	60
Suzuki–Ree, including the Tits group	24
sporadic	140.

(6.13)

Proof. For alternating groups and for the ordinary or graph-twisted standard fixed-point groups over $q \notin \{2, 3\}$, the two classes are constructed in the proof of Corollary 6.13. The proof of Corollary 6.19 supplies the small-field classical pairs. Corollary 6.22 supplies the remaining ordinary and graph-twisted exceptional pairs, and Corollary 6.21 supplies the Suzuki–Ree pairs. For the Tits group and every sporadic group, the two primes and their degrees are exhibited in the proof of Corollary 6.14. The largest ceiling among these rows is 140. □

Theorem 6.26 (Uniform three-prime simple base)**.** *There is a finite integer* $D^{(3)}_{\mathrm{simp}}$ *such that every nonabelian finite simple group* S *has three distinct primes* ℓ_1, ℓ_2, ℓ_3 *and classes*

$$0 \neq x_j \in H^{d_j}(S; \mathbb{Z})_{(\ell_j)}, \qquad d_j \leq D^{(3)}_{\mathrm{simp}} \quad (1 \leq j \leq 3).$$

The bound is deliberately not optimized. Away from a finite list of simple groups, one may use degree at most 8 *for alternating groups, degree at most* 80 *for classical groups, degree at most* 4 *for ordinary or graph-twisted exceptional groups, and degree at most* 24 *for Suzuki–Ree groups.*

Proof. Proposition 4.3 treats all alternating groups. Propositions 6.17, 6.23, and 6.24 treat all groups of Lie type. The sporadic groups and any finite residues in those propositions form a finite set. Every member has at least three group primes by Burnside's theorem, so Lemma 6.20 supplies a common finite ceiling. Taking the maximum of the family ceilings proves the theorem. □

Corollary 6.27 (Prime support above a simple quotient)**.** *There is a finite integer* $D^{(3)}_{\mathrm{simp}}$ *and, for every nonabelian finite simple group* S*, a three-element set of primes* $\Lambda(S)$ *such that every epimorphism* $E \twoheadrightarrow S$ *with kernel* K *satisfies*

$$\left|\mathrm{Supp}_{D^{(3)}_{\mathrm{simp}}}(E)\right| \geq 3 - |\Lambda(S) \cap \pi(K)|. \tag{6.13a}$$

In particular, if $\pi(K)$ *meets* $\Lambda(S)$ *in at most one prime, then* E *has bounded-degree witnesses at two distinct primes. This applies to every* p*-group kernel.*

Proof. Take for $\Lambda(S)$ the three detecting primes in Theorem 6.26 and apply Proposition 6.12. Only primes in $\Lambda(S) \cap \pi(K)$ can be lost under inflation, which proves (6.13a) and its consequences. □

Theorem 6.28 (Simple quotients with two-prime-support kernel)**.** *There is a finite integer* $D^{(\leq 2)}_{\mathrm{simp}}$ *such that, for every nontrivial finite simple group* S *and every finite epimorphism*

$$E \twoheadrightarrow S$$

whose kernel K *has order divisible by at most two distinct primes,*

$$H^i(E; \mathbb{Z}) \neq 0 \qquad \text{for some } i \leq D^{(\leq 2)}_{\mathrm{simp}}.$$

More generally, for nonabelian S the same conclusion holds with $D_{\mathrm{simp}}^{(\le 2)} = D_{\mathrm{simp}}^{(3)}$ whenever

$$\Lambda(S) \not\subseteq \pi(K), \tag{6.13b}$$

where $\Lambda(S)$ is the three-prime detecting set in Corollary 6.27.

Proof. If S is cyclic of prime order, the epimorphism makes E_{ab} nontrivial, so $H^2(E;\mathbb{Z}) \neq 0$. Suppose that S is nonabelian. Under (6.13b), choose $\ell \in \Lambda(S) \setminus \pi(K)$. The corresponding class on S from Theorem 6.26 inflates nontrivially to E by Lemma 5.1. If $|\pi(K)| \le 2$, the three-element set $\Lambda(S)$ cannot be contained in $\pi(K)$, so this applies. Taking the maximum of 2 and $D_{\mathrm{simp}}^{(3)}$ gives the asserted common constant. □

Theorem 6.29 (Simple quotients with arbitrary prime-power kernel)**.** *For every nontrivial finite simple group S and every finite epimorphism $E \twoheadrightarrow S$ with prime-power kernel,*

$$H^i(E;\mathbb{Z}) \neq 0 \qquad \text{for some } i \le 140.$$

Thus one may take $D_{\mathrm{simp}} = 140$.

Proof. The cyclic simple groups already have the unconditional degree-two witness. If S is nonabelian, Theorem 6.25 and Proposition 6.11 give the assertion directly. □

7 Nonabelian minimal normal subgroups

7.1 Factor-stabilizer transfer

Theorem 7.1 (Factor-stabilizer transfer)**.** *Let $N = S_1 \times \cdots \times S_r \lhd G$, where the S_i are isomorphic nonabelian simple groups and G permutes them transitively. Put*

$$K = N_G(S_1), \qquad \theta : K \longrightarrow \mathrm{Aut}(S_1)$$

for the conjugation homomorphism. Let R be a commutative coefficient ring with trivial action. If there is a class

$$c \in H^d(\theta(K); R)$$

whose restriction to $S_1 = \mathrm{Inn}(S_1)$ is nonzero, then $H^d(G;R) \neq 0$.

Proof. Set $x = \theta^* c \in H^d(K;R)$ and $y = \mathrm{cor}_K^G(x)$. Since $N \le K$, the Mackey formula gives

$$\mathrm{res}_N^G(y) = \sum_{g \in G/K} g^* \,\mathrm{res}_N^K(x).$$

Under the Kunneth decomposition for $N = S_1 \times \cdots \times S_r$, this sum has one nonzero term pulled back from each factor. Restricting the sum to any one factor kills all other summands and leaves an automorphic image of $\mathrm{res}_{S_1}^{\theta(K)} c$, which is nonzero. Hence $\mathrm{res}_N^G(y) \neq 0$, so $y \neq 0$. □

The theorem converts the nonabelian-kernel problem into the following almost-simple question: for every $S \le A \le \mathrm{Aut}(S)$ arising as a factor stabilizer, find a low-degree class on A whose restriction to S is nonzero.

Lemma 7.2 (The norm criterion)**.** *Let $S \lhd B$ have finite index, let R be a commutative coefficient ring with trivial action, and let $x \in H^d(S;R)$. If*

$$N_{B/S}(x) := \sum_{bS \in B/S} b^* x$$

is nonzero, then the class $\operatorname{cor}_S^B(x) \in H^d(B;R)$ restricts nontrivially to S.

Proof. The restriction–corestriction formula gives

$$\operatorname{res}_S^B \operatorname{cor}_S^B(x) = N_{B/S}(x).$$

□

Lemma 7.3 (Prime-to-outer extension and sign-squares)**.** *Let $S \lhd B$, put $C = B/S$, and let $\ell \nmid |C|$. Then restriction is an isomorphism*

$$H^*(B;\mathbb{Z}_{(\ell)}) \xrightarrow{\cong} H^*(S;\mathbb{Z}_{(\ell)})^C.$$

Consequently, if $x \in H^4(S;\mathbb{Z}_{(\ell)})$ satisfies $x^2 \neq 0$ and every element of C sends x to x or $-x$, then a class in $H^8(B;\mathbb{Z}_{(\ell)})$ restricts nontrivially to S. The hypothesis $x^2 \neq 0$ holds, in particular, if x restricts nontrivially to a cyclic subgroup of order ℓ.

Proof. In the Lyndon–Hochschild–Serre spectral sequence, multiplication by $|C|$ is invertible on every $\mathbb{Z}_{(\ell)}C$-module. Hence $H^a(C;M) = 0$ for $a > 0$ and every such module M. The spectral sequence has only its zeroth column and the edge map is the asserted restriction isomorphism. If the action on x is by signs, then x^2 is invariant, so the first assertion lifts it to B. Finally, writing $H^*(C_\ell;\mathbb{Z}_{(\ell)}) \cong \mathbb{Z}_{(\ell)}[u]/(\ell u)$ with $|u| = 2$, a nonzero restriction of x is a nonzero multiple of u^2, whose square is nonzero. □

7.2 A cyclic-Sylow restriction criterion

Lemma 7.4 (Restriction from a shared cyclic Sylow subgroup)**.** *Let $S \leq B$, and suppose that P is a cyclic Sylow p-subgroup of both S and B. Put*

$$e_B = |N_B(P) : C_B(P)|.$$

There is a class $c \in H^{2e_B}(B;\mathbb{Z})_{(p)}$ such that

$$\operatorname{res}_S^B(c) \neq 0.$$

If P has order p, then $e_B \mid p-1$ and the degree is at most $2(p-1)$.

Proof. The class constructed in the proof of Theorem 3.1 restricts to the nonzero invariant $u^{e_B} \in H^{2e_B}(P;\mathbb{Z})$. Since restriction from B to P factors through S, its restriction to S cannot vanish. The last assertion follows because $N_B(P)/C_B(P)$ embeds in $\operatorname{Aut}(P) \cong (\mathbb{Z}/p)^\times$. □

Corollary 7.5 (The A_6 restriction problem)**.** *For every group*

$$A_6 \leq B \leq \operatorname{Aut}(A_6)$$

there is a class $c \in H^d(B;\mathbb{Z})$, with $d \leq 8$, whose restriction to A_6 is nonzero. Consequently, the nonabelian minimal-normal-subgroup branch with simple factor A_6 has an integral witness in degree at most eight.

Proof. The group A_6 has order $2^3 \cdot 3^2 \cdot 5$, while $|\operatorname{Out}(A_6)| = 4$ [1]. Hence a Sylow 5-subgroup $P \cong C_5$ of A_6 is also Sylow in every such B. Apply Lemma 7.4; its automizer has order dividing $|\operatorname{Aut}(C_5)| = 4$, so the resulting degree is at most eight. The final assertion follows from Theorem 7.1. □

7.3 The first nonzero row and the Tits group

Lemma 7.6 (Extension from the first nonzero row)**.** *Let*

$$1 \longrightarrow S \longrightarrow B \longrightarrow Q \longrightarrow 1$$

be an extension, let R be a commutative coefficient ring with trivial action, and suppose that, for some $d > 0$,

$$H^j(S; R) = 0 \quad (0 < j < d), \qquad H^{d+1}(Q; R) = 0.$$

Then restriction has image

$$\operatorname{im}\bigl(\operatorname{res}^B_S : H^d(B; R) \longrightarrow H^d(S; R)\bigr) = H^d(S; R)^Q.$$

Proof. In the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{a,b} = H^a(Q; H^b(S; R)) \Longrightarrow H^{a+b}(B; R),$$

the term $E_2^{0,d} = H^d(S; R)^Q$ has no incoming differential. For $2 \le r \le d$, the target of its outgoing differential is a subquotient of

$$E_2^{r,d-r+1} = H^r(Q; H^{d-r+1}(S; R)) = 0.$$

The only remaining possible differential is

$$d_{d+1} : E_{d+1}^{0,d} \longrightarrow E_{d+1}^{d+1,0},$$

whose target is a subquotient of $H^{d+1}(Q; R) = 0$. Thus $E_\infty^{0,d} = E_2^{0,d}$, and the edge homomorphism is restriction. □

Corollary 7.7 (The Tits-group restriction problem)**.** *Let $T = {}^2F_4(2)'$. For every group*

$$T \le B \le \operatorname{Aut}(T)$$

there is a class $c \in H^4(B; \mathbb{Z})$ whose restriction to T is nonzero. Consequently, the nonabelian minimal-normal-subgroup branch with simple factor T has an integral witness in degree four.

Proof. The group T is perfect and has trivial Schur multiplier, so

$$H^j(T; \mathbb{Z}) = 0 \qquad (1 \le j \le 3).$$

Proposition 4.7 gives

$$H^4(T; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8.$$

The outer automorphism group of T has order two [1]; hence it is enough to consider $B = T$ and $B = \operatorname{Aut}(T)$. The first case is immediate. In the second case, $Q = B/T \cong C_2$ and $H^5(C_2; \mathbb{Z}) = 0$. Lemma 7.6 therefore says that the image of restriction is $H^4(T; \mathbb{Z})^{C_2}$. This invariant subgroup is nonzero: the involution acts on a set of sixteen elements, and the fifteen nonzero elements cannot all lie in orbits of size two. The final assertion follows from Theorem 7.1. □

Proposition 7.8 (A Lie-type outer quotient criterion)**.** *Let S be a finite simple group of ordinary or graph-twisted Lie type over $\mathbb{F}_q$, where $q = p^m$, and let*

$$S \le B \le \mathrm{Aut}(S), \qquad C = B/S.$$

Suppose that the field-automorphism image of C is generated by the p^a-power Frobenius, allowing $a = 0$ when that image is trivial. If there is a prime $\ell \neq p$ such that

$$\ell \mid q^2 - 1, \qquad \ell \nmid |H_2(S;\mathbb{Z})|, \qquad p^{2a} \equiv 1 \pmod{\ell},$$

and a Sylow ℓ-subgroup of C is cyclic, then a class in $H^4(B;\mathbb{Z})_{(\ell)}$ restricts nontrivially to S.

Proof. Because the Schur multiplier has order prime to ℓ, passage from the simply connected finite group of rational points to S is an ℓ-local equivalence. The homotopy-fixed-point calculation in the proof of Proposition 4.6 therefore supplies an order-ℓ subgroup in $H^4(S;\mathbb{Z})_{(\ell)}$ arising from

$$\mathrm{coker}\big(1 - q^2 : \mathbb{Z}_\ell \longrightarrow \mathbb{Z}_\ell\big).$$

The comparison is natural for the standard outer automorphisms. On the basic π_4 generator, a p^a-field automorphism acts as multiplication by p^{2a}; a graph automorphism fixes the generator because it preserves the basic invariant bilinear form; and a diagonal automorphism acts through an inner automorphism of the associated compact group and hence acts trivially. The congruence hypothesis thus makes the order-ℓ subgroup fixed by C.

Moreover

$$H^j(S;\mathbb{Z}_{(\ell)}) = 0 \qquad (1 \le j \le 3),$$

because S is perfect and its Schur multiplier has order prime to ℓ. Let P be a Sylow ℓ-subgroup of C. Restriction on ℓ-primary cohomology from C to P is injective, because restriction followed by transfer is multiplication by the ℓ-local unit $[C : P]$. Since positive odd-degree integral cohomology of a cyclic group vanishes, $H^5(C;\mathbb{Z}_{(\ell)}) = 0$. Lemma 7.6, with $d = 4$ and $R = \mathbb{Z}_{(\ell)}$, shows that the fixed order-ℓ class lies in the image of restriction from B. □

Proposition 7.9 (Regular-embedding inner-diagonal classes)**.** *Let $H = \mathbf{H}^F$ be a perfect finite group of fixed points of a simple simply connected algebraic group, put $S = H/Z(H)$, and suppose that S is nonabelian simple. Let $\mathbf{H} \hookrightarrow \widetilde{\mathbf{H}}$ be an F-stable regular embedding with connected center, and put $\widetilde{H} = \widetilde{\mathbf{H}}^F$. Thus*

$$\mathrm{Inndiag}(S) = \widetilde{H}/Z(\widetilde{H}), \qquad H \cap Z(\widetilde{H}) = Z(H).$$

If $Z(H) \neq 1$, then for every

$$S \le B \le \mathrm{Inndiag}(S)$$

there is a class in $H^3(B;\mathbb{Z})$ whose restriction to S is nonzero.

Proof. Put $A = Z(\widetilde{H})$ and let $\widehat{B}$ be the inverse image of B in $\widetilde{H}$. There is a central extension

$$1 \longrightarrow A \longrightarrow \widehat{B} \longrightarrow B \longrightarrow 1.$$

Choose a nonzero character $\theta : Z(H) \to \mathbb{Q}/\mathbb{Z}$. Since $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group, θ extends across $Z(H) \le A$ to a character $\chi : A \to \mathbb{Q}/\mathbb{Z}$. Pushout gives a class $z \in H^2(B;\mathbb{Q}/\mathbb{Z})$.

The inverse image of S in $\widehat{B}$ is HA. For the central extension $1 \to Z(H) \to H \to S \to 1$, the five-term homology sequence shows that the connecting map

$$H_2(S;\mathbb{Z}) \longrightarrow Z(H)$$

is surjective, because H is perfect. The connecting map for $1 \to A \to HA \to S \to 1$ is its composite with the inclusion $Z(H) \hookrightarrow A$. Hence the restriction of z corresponds under

$$H^2(S;\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(H_2(S;\mathbb{Z}),\mathbb{Q}/\mathbb{Z})$$

to the nonzero composite with χ, and is nonzero. Finally $H^2(B;\mathbb{Q}/\mathbb{Z}) \cong H^3(B;\mathbb{Z})$ naturally for finite B. □

Proposition 7.10 (Equivariant central-cover criterion)**.** *Let $S \lhd B$ be finite groups and put $C = B/S$. Suppose that there is an extension*

$$1 \longrightarrow A \longrightarrow \widehat{B} \longrightarrow B \longrightarrow 1$$

with A abelian, together with a perfect subgroup $H \leq \widehat{B}$ that maps onto S and satisfies

$$[H,A] = 1, \qquad A \cap H = Z(H), \qquad H/Z(H) \cong S.$$

The conjugation action on A then factors through C. If the image of $Z(H)$ under

$$Z(H) \longrightarrow A \longrightarrow A_C$$

is nonzero, then a class in $H^3(B;\mathbb{Z})$ restricts nontrivially to S.

Proof. Because $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator, a nonzero element in the image of $Z(H) \to A_C$ is detected by a character $\overline{\chi} : A_C \to \mathbb{Q}/\mathbb{Z}$. Its composite $\chi : A \to \mathbb{Q}/\mathbb{Z}$ is C-invariant, hence a B-module map from the conjugation module A to the trivial module $\mathbb{Q}/\mathbb{Z}$. Pushing out the extension of B along χ gives a class in $H^2(B;\mathbb{Q}/\mathbb{Z})$.

As in Proposition 7.9, the homology connecting map $H_2(S;\mathbb{Z}) \to Z(H)$ is surjective because H is perfect. The restricted pushed-out class is its composite with $Z(H) \to A_C \xrightarrow{\overline{\chi}} \mathbb{Q}/\mathbb{Z}$, and is therefore nonzero. The natural isomorphism $H^2(B;\mathbb{Q}/\mathbb{Z}) \cong H^3(B;\mathbb{Z})$ gives the result. □

Corollary 7.11 (All inner-diagonal factor stabilizers)**.** *Let S be a finite nonabelian simple group of Lie type and let*

$$S < B \leq \mathrm{Inndiag}(S).$$

Then a class in $H^3(B;\mathbb{Z})$ restricts nontrivially to S.

Proof. For a regular embedding, standard fixed-point theory gives

$$|\,\mathrm{Inndiag}(S) : S| = |Z(H)|$$

for the simply connected fixed-point group H, and realizes $\mathrm{Inndiag}(S)$ as in Proposition 7.9; see [26, 36]. Thus $B > S$ forces $Z(H) \neq 1$, and the proposition applies whenever H is perfect. The nonperfect simply connected fixed-point groups are

$$\mathrm{SL}_2(2),\ \mathrm{SL}_2(3),\ \mathrm{SU}_3(2),\ \mathrm{Sp}_4(2),\ \mathrm{G}_2(2),\ {}^2\mathrm{B}_2(2),\ {}^2\mathrm{G}_2(3),\ {}^2\mathrm{F}_4(2)$$

[36, Theorem 24.17]. Their nonabelian simple derived sections are either represented by the perfect groups $\mathrm{SL}_2(9)$, $\mathrm{SU}_3(3)$, or $\mathrm{SL}_2(8)$ through the standard exceptional isomorphisms, or are the Tits group. The former are covered by the proposition using those realizations, while the Tits group has no proper inner-diagonal overgroup. This exhausts the exceptions. □

Proposition 7.12 (Inner-diagonal overgroups in type A)**.** *Let S be a nonabelian simple group isomorphic to $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q)$. For every proper overgroup*

$$S < B \leq \mathrm{Inndiag}(S)$$

there is a class in $H^3(B;\mathbb{Z})$ whose restriction to S is nonzero.

Proof. Consider first $S = \mathrm{PSL}_n(q)$ and put $d = \gcd(n, q-1)$. Since $B > S$, we have $d > 1$. Let $\widehat{B}$ be the inverse image of B under

$$\mathrm{GL}_n(q) \longrightarrow \mathrm{PGL}_n(q).$$

This gives a central extension

$$1 \longrightarrow \mathbb{F}_q^\times \longrightarrow \widehat{B} \longrightarrow B \longrightarrow 1.$$

Choose a character $\chi : \mathbb{F}_q^\times \to \mathbb{Q}/\mathbb{Z}$ that is nonzero on

$$Z(\mathrm{SL}_n(q)) \cong C_d;$$

such a character exists because $\mathbb{F}_q^\times$ is cyclic. Pushout along χ gives a class $z \in H^2(B;\mathbb{Q}/\mathbb{Z})$. Its restriction to S is nonzero: under the universal-coefficient identification

$$H^2(S;\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(H_2(S;\mathbb{Z}),\mathbb{Q}/\mathbb{Z}),$$

the restricted class is the character obtained from χ on the kernel of the central cover $\mathrm{SL}_n(q) \to \mathrm{PSL}_n(q)$, and this character is nonzero by construction. Since B is finite, the coefficient sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ identifies z with a class in $H^3(B;\mathbb{Z})$, compatibly with restriction.

For $S = \mathrm{PSU}_n(q)$ the same proof uses

$$\mathrm{GU}_n(q) \longrightarrow \mathrm{PGU}_n(q), \qquad Z(\mathrm{GU}_n(q)) \cong C_{q+1},$$

and a character nonzero on $Z(\mathrm{SU}_n(q)) \cong C_{\gcd(n,q+1)}$. □

Proposition 7.13 (Quadratic scalar characters)**.** *Assume that q is odd.*

(i) *Let $S = \mathrm{PSL}_n(q)$ be simple, put $d = \gcd(n, q-1)$, and let $S \leq B$ be contained in the standard automorphism group generated by inner-diagonal, field, and inverse-transpose automorphisms. If*

$$d \nmid \frac{q-1}{2},$$

then a class in $H^3(B;\mathbb{Z})$ restricts nontrivially to S.

(ii) *Let $S = \mathrm{PSU}_n(q)$ be simple, put $d = \gcd(n, q+1)$, and let $S \leq B$ be contained in its standard inner-diagonal, field, and graph automorphism group. If*

$$d \nmid \frac{q+1}{2},$$

then a class in $H^3(B;\mathbb{Z})$ restricts nontrivially to S.

Proof. In the linear case, pull B back to the group generated by $\mathrm{GL}_n(q)$, field automorphisms, and inverse-transpose. Its scalar kernel is $A = \mathbb{F}_q^\times$. The unique character

$$\chi : A \longrightarrow \mathbb{Q}/\mathbb{Z}$$

of order two is fixed by every standard outer automorphism: a field automorphism raises scalars to the odd power p^a, while inverse-transpose sends a scalar to its inverse. Thus χ is a B-module map to the trivial module $\mathbb{Q}/\mathbb{Z}$, and pushing out gives a class in $H^2(B;\mathbb{Q}/\mathbb{Z})$. The order-$d$ scalar kernel of $\mathrm{SL}_n(q) \to S$ is not contained in $\ker(\chi)$ exactly when $d \nmid (q-1)/2$. Under that hypothesis the pushed-out class restricts nontrivially to S.

The unitary proof is identical with scalar kernel C_{q+1}. Its quadratic character is fixed by odd-power field actions and inversion, and it is nonzero on $Z(\mathrm{SU}_n(q))$ exactly when $d \nmid (q+1)/2$. In both cases the natural isomorphism $H^2(B;\mathbb{Q}/\mathbb{Z}) \cong H^3(B;\mathbb{Z})$ finishes the proof. □

Proposition 7.14 (Inner-diagonal overgroups in symplectic type)**.** *Let S be a nonabelian simple group isomorphic to $\mathrm{PSp}_{2n}(q)$. For every proper overgroup*

$$S < B \leq \mathrm{Inndiag}(S)$$

there is a class in $H^3(B;\mathbb{Z})$ whose restriction to S is nonzero.

Proof. A proper inner-diagonal overgroup can occur only when q is odd; in that case $\mathrm{Inndiag}(S) = \mathrm{PGSp}_{2n}(q)$ and its quotient by S has order two. Thus $B = \mathrm{PGSp}_{2n}(q)$. The central extension

$$1 \longrightarrow \mathbb{F}_q^\times \longrightarrow \mathrm{GSp}_{2n}(q) \longrightarrow \mathrm{PGSp}_{2n}(q) \longrightarrow 1$$

may be pushed out along a character $\chi : \mathbb{F}_q^\times \to \mathbb{Q}/\mathbb{Z}$ that is nonzero on $Z(\mathrm{Sp}_{2n}(q)) = \{\pm I\}$. The resulting class in $H^2(B;\mathbb{Q}/\mathbb{Z})$ restricts nontrivially to S: its pullback to the central cover $\mathrm{Sp}_{2n}(q) \to \mathrm{PSp}_{2n}(q)$ is precisely the character $\chi|_{\{\pm I\}}$. The coefficient sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ now gives the required class in $H^3(B;\mathbb{Z})$, compatibly with restriction. □

Proposition 7.15 (A semilinear scalar-kernel criterion)**.** *Let $q = p^m$. Suppose that the image of B in the field automorphism group is generated by the p^a-power map, and put*

$$e = \gcd(m,a), \qquad L = \frac{q-1}{p^e-1},$$

with $a = 0$ and $e = m$ when the field image is trivial.

(i) *Let $S = \mathrm{PSL}_n(q)$ be simple and $S \leq B \leq \mathrm{P\Gamma L}_n(q)$. Put $d = \gcd(n, q-1)$. If $d \nmid L$, then a class in $H^3(B;\mathbb{Z})$ restricts nontrivially to S.*

(ii) *Let q be odd, let $S = \mathrm{PSp}_{2n}(q)$ be simple, and let $S \leq B \leq \mathrm{P\Gamma Sp}_{2n}(q)$. If L is odd, then a class in $H^3(B;\mathbb{Z})$ restricts nontrivially to S.*

(iii) *Let $S = \mathrm{PSU}_n(q)$ be simple and $S \leq B \leq \mathrm{P\Gamma U}_n(q)$. Suppose that the induced action on the scalar group C_{q+1} is generated by $x \mapsto x^{p^a}$, and put*

$$d = \gcd(n, q+1), \qquad L_+ = \frac{q+1}{\gcd(q+1, p^a-1)}.$$

If $d \nmid L_+$, then a class in $H^3(B;\mathbb{Z})$ restricts nontrivially to S.

Proof. First consider the linear and symplectic cases and put $A = \mathbb{F}_q^\times$. In the linear case, pull B back along

$$\Gamma\mathrm{L}_n(q) \longrightarrow \mathrm{P\Gamma L}_n(q);$$

in the symplectic case use $\Gamma\mathrm{GSp}_{2n}(q) \to \mathrm{P\Gamma Sp}_{2n}(q)$. This gives an extension with kernel A, on which a generator of the field image acts by $x \mapsto x^{p^a}$.

Choose a generator ξ of A. Its characters to $\mathbb{Q}/\mathbb{Z}$ are

$$\chi_k(\xi) = \frac{k}{q-1} \pmod{\mathbb{Z}} \qquad (k \in \mathbb{Z}/(q-1)).$$

The character χ_k is invariant under the field image exactly when $(p^a-1)k \equiv 0 \pmod{q-1}$. Since

$$\gcd(q-1, p^a-1) = p^e - 1,$$

the invariant characters are precisely those for which $L \mid k$. An invariant character is a B-module homomorphism from A to the trivial module $\mathbb{Q}/\mathbb{Z}$, so pushing out the pulled-back extension gives a class in $H^2(B;\mathbb{Q}/\mathbb{Z})$.

In the linear case, the scalar kernel of $\mathrm{SL}_n(q) \to \mathrm{PSL}_n(q)$ is the order-d subgroup of A. The character χ_L is nonzero on this subgroup exactly when $d \nmid L$. Consequently the pushed-out class restricts nontrivially to S. In the symplectic case the scalar kernel is $\{\pm I\}$, and χ_L is nonzero on it exactly when L is odd.

In the unitary case, pull B back along $\Gamma\mathrm{U}_n(q) \to \mathrm{P}\Gamma\mathrm{U}_n(q)$. Its scalar kernel $A_+ = C_{q+1}$ has a generator η, and write

$$\psi_k(\eta) = \frac{k}{q+1} \pmod{\mathbb{Z}} \qquad (k \in \mathbb{Z}/(q+1)).$$

The same calculation shows that ψ_k is invariant precisely when $L_+ \mid k$. The scalar kernel of $\mathrm{SU}_n(q) \to \mathrm{PSU}_n(q)$ has order d, so ψ_{L_+} is nonzero on it exactly when $d \nmid L_+$. Pushing out along this invariant character gives the required nonzero restriction. Finally $H^2(B;\mathbb{Q}/\mathbb{Z}) \cong H^3(B;\mathbb{Z})$ for finite B, naturally in restriction. □

Corollary 7.16 (Residual semilinear prime support)**.** *Retain the notation of Proposition 7.15.*

(i) *For $S = \mathrm{PSL}_n(q)$, a class of degree three or four on B restricts nontrivially to S if either $d \nmid L$ or some prime $\ell \mid p^{2e}-1$ does not divide $|H_2(S;\mathbb{Z})|$.*

(ii) *For $S = \mathrm{PSp}_{2n}(q)$ with q odd, the same conclusion holds if either L is odd or some odd prime $\ell \mid p^{2e}-1$ does not divide $|H_2(S;\mathbb{Z})|$.*

(iii) *For $S = \mathrm{PSU}_n(q)$, put $e = \gcd(m,a)$. The same conclusion holds if either $d \nmid L_+$ or some prime $\ell \mid p^{2e}-1$ does not divide $|H_2(S;\mathbb{Z})|$.*

Proof. The first alternative in each part is Proposition 7.15. Consider a prime in the second alternative. Since e divides both a and m, it satisfies

$$\ell \mid q^2-1, \qquad p^{2a} \equiv 1 \pmod{\ell}.$$

In the linear and unitary cases the diagonal outer group has order dividing d, and d divides the order of the Schur multiplier because the natural special-to-projective central cover has kernel of order d. Hence $\ell \nmid d$. The ℓ-Sylow subgroup of B/S therefore injects into the cyclic semilinear quotient and is cyclic. In the symplectic case the diagonal outer group has order at most two, so the same conclusion holds for the stipulated odd prime. Proposition 7.8 now supplies the degree-four class. □

Corollary 7.17 (The residual symplectic field tower)**.** *Let $q = p^m$ be odd, let $S = \mathrm{PSp}_{2n}(q)$ be simple, and let $S \le B \le \mathrm{P\Gamma Sp}_{2n}(q)$. Suppose that the field image is generated by the p^a-power map, put $e = \gcd(m, a)$, and let $f = m/e$ be the order of that field image. If neither Proposition 7.15 nor Corollary 7.16 supplies a restricting class, then*

$$p = 3, \qquad e = 1, \qquad 2 \mid f.$$

Thus the unresolved symplectic semilinear cases have fixed field $\mathbb{F}_3$ and an even-order field component.

Proof. Write

$$L = \frac{p^m - 1}{p^e - 1} = 1 + p^e + \cdots + p^{(f-1)e}.$$

Failure of the scalar criterion says that L is even. Every summand is odd, so f is even.

For a simple symplectic group in odd characteristic, the Schur multiplier has no odd prime divisor [1]. Hence failure of the prime-support criterion says that $p^{2e} - 1$ has no odd prime divisor. Put $x = p^e$. The two consecutive even numbers $x - 1$ and $x + 1$ are therefore powers of two. Their greatest common divisor is two and their difference is two, forcing $x - 1 = 2$ and $x + 1 = 4$. Thus $x = 3$, whence $p = 3$ and $e = 1$. □

Corollary 7.18 (Residual fixed fields in type E_6)**.** *Let $S \le B \le \mathrm{Aut}(S)$, put $C = B/S$, and suppose that $B \to C$ does not split. Write $q = p^m$, suppose that the field image of C is generated by the p^a-power map, and put*

$$e = \gcd(m, a), \qquad x = p^e, \qquad f = m/e.$$

If Proposition 7.8 supplies no restricting degree-four class, then the following necessary conditions hold.

(i) *If $S = E_6(q)$, then*

$$x \in \{2, 5, 7, 17\}, \qquad 2 \mid f,$$

and, if $x \neq 2$, the image of C in the graph-automorphism group is nontrivial.

(ii) *If $S = {}^2E_6(q)$, then $x = 2$. In this case $e = 1$ and the nontrivial diagonal multiplier forces $m = f$ to be odd.*

Thus all other nonsplit factor stabilizers of type E_6 or ${}^2E_6$ already have a restricting class in degree four.

Proof. For a nonsplit factor stabilizer in either family the diagonal part is nontrivial. Apart from the small twisted groups whose full automorphism extensions split, the Schur multiplier therefore has prime support $\{3\}$; the standard multiplier and outer-automorphism tables give this directly [1, 34]. The diagonal outer group has order three, the field group is cyclic, and only the untwisted family has an additional graph involution.

Let $\ell \neq 3$ divide $x^2 - 1$. Then

$$\ell \mid q^2 - 1, \qquad p^{2a} \equiv 1 \pmod{\ell}, \qquad \ell \nmid |H_2(S; \mathbb{Z})|.$$

For odd ℓ, a Sylow ℓ-subgroup of C lies in the cyclic field group, so Proposition 7.8 applies. Consequently, failure of that proposition forces

$$x^2 - 1 = 2^r 3^s$$

for some $r, s \geq 0$. If x is even, the coprime odd integers $x-1$ and $x+1$ are powers of three; one of them must be one, and hence $x = 2$. If x is odd, put

$$u = (x-1)/2, \qquad v = (x+1)/2.$$

The coprime consecutive integers u, v have no prime divisors other than two and three. Unless one is one, they are a power of two and a power of three. The elementary equations

$$|2^i - 3^j| = 1$$

have, for $i, j \geq 1$, only the adjacent pairs $(2,3), (3,4), (8,9)$: if $3^j - 2^i = 1$ with $i \geq 3$, reduction modulo eight makes j even and factoring $3^j - 1$ forces the adjacent powers of two to be two and four; if $2^i - 3^j = 1$ with $i \geq 3$, reduction modulo eight is impossible. The cases with a unit or a small exponent are immediate. Hence

$$x \in \{2, 3, 5, 7, 17\}.$$

In untwisted type, the prime two also satisfies all hypotheses of Proposition 7.8 when x is odd, unless a Sylow two-subgroup of C is noncyclic. The latter requires both a nontrivial graph image and an even-order field image, so $2 \mid f$. The same two conditions need not follow when $x = 2$, because two is then the defining prime. Nonsplitting nevertheless forces the diagonal factor of order three, so $3 \mid q - 1$; this excludes $x = 3$ and, for $x \in \{2, 5, 17\}$, forces f to be even. The preceding Sylow-two condition supplies the same conclusion for $x = 7$ and supplies the graph condition for every odd x on the list. This proves (i).

In twisted type there is no separate graph involution. Thus a Sylow two-subgroup of C is cyclic, and any odd x would make the prime two available in Proposition 7.8. Therefore $x = 2$. The diagonal factor has order three only when $3 \mid q + 1$; since $q = 2^m$ and $e = 1$, this says that $m = f$ is odd, proving (ii). □

Theorem 7.19 (Untwisted E_6 factor stabilizers)**.** *Let $S = E_6(q)$ be nonabelian simple and let*

$$S \leq B \leq \mathrm{Aut}(S).$$

If $B \to B/S$ is nonsplit, then a class of degree at most eight on B restricts nontrivially to S. Consequently, every nonabelian minimal-normal-subgroup branch with simple factor of untwisted type E_6 has an integral witness in degree at most eight.

Proof. If Proposition 7.8 applies, it supplies the required class in degree four. Otherwise Corollary 7.18(i) gives

$$q = p^m, \qquad p \in \{2, 5, 7, 17\}, \qquad 2 \mid m. \tag{19c}$$

Indeed its fixed field $\mathbb{F}_{p^e}$ has prime order, so $e = 1$, $p = x$, and the field-image order f is m. Put

$$\ell = \begin{cases} 13, & p = 5, \\ 5, & p \in \{2, 7, 17\}. \end{cases}$$

Then

$$p^2 \equiv -1 \pmod{\ell}, \qquad \ell \mid q^2 - 1. \tag{19d}$$

The standard simply connected cover $\widehat{S} \to S$ has 3-group kernel and is therefore an ℓ-local equivalence. Proposition 4.5 supplies an order-ℓ class

$$x \in H^4(S; \mathbb{Z}_{(\ell)})$$

from the basic degree-two line. A long-root $\mathrm{SL}_2(q)$ has Dynkin index one, so x restricts nontrivially to a cyclic subgroup of order ℓ. Lemma 7.3 gives

$$x^2 \neq 0 \qquad \text{in } H^8(S;\mathbb{Z}_{(\ell)}). \tag{19e}$$

Write $C = B/S$. Its field image is generated by a p^a-power map with $\gcd(a,m) = 1$. Both a and $m/\ell^{v_\ell(m)}$ have the parity forced by (19c): the first is odd and the second is even. On the order-ℓ subgroup generated by x, the field generator therefore acts as

$$p^{2a} \equiv -1 \pmod{\ell}.$$

The graph involution fixes the basic degree-two line, and diagonal automorphisms act trivially on it. Thus every element of C acts on x by a sign and fixes x^2.

Choose $P \in \mathrm{Syl}_\ell(C)$ and let B_P be its inverse image. Since $\ell \in \{5, 13\}$ while the diagonal and graph groups have orders dividing three and two, respectively, P lies in the cyclic field group. It is cyclic and acts trivially on x: when $P \neq 1$, a generator is a power of the field generator whose exponent is divisible by $m/\ell^{v_\ell(m)}$, and hence acts by an even power of -1.

The group S is perfect and its multiplier is 3-primary, so

$$H^j(S;\mathbb{Z}_{(\ell)}) = 0 \qquad (1 \leq j \leq 3),$$

while $H^5(P;\mathbb{Z}_{(\ell)}) = 0$. Lemma 7.6 lifts x to $y \in H^4(B_P;\mathbb{Z}_{(\ell)})$. Restriction–corestriction now gives

$$\mathrm{res}^B_S \,\mathrm{cor}^B_{B_P}(y^2) = \sum_{cP \in C/P} c^*(x^2) = [C:P]x^2 \neq 0, \tag{19f}$$

because $[C:P]$ is an ℓ-local unit. The factor-stabilizer conclusion is Theorem 7.1. □

Theorem 7.20 (Twisted E_6 factor stabilizers)**.** *Let $S = {}^2E_6(q)$ be nonabelian simple and let*

$$S \leq B \leq \mathrm{Aut}(S).$$

Suppose that $B \to B/S$ is nonsplit. Then a class in $H^d(B;\mathbb{F}_3)$ restricts nontrivially to S, where

$$d \leq 12, \qquad d \leq 6 \quad \text{if } |B/S| \text{ is odd.}$$

Consequently every nonabelian minimal-normal-subgroup branch with simple factor of twisted type E_6 has an integral witness in degree at most thirteen. Thus this entire family is complete for the finite-bound problem.

Proof. If Proposition 7.8 applies, it gives the required integral class in degree four. Otherwise Corollary 7.18(ii) gives

$$S = {}^2E_6(2^m), \qquad m \text{ odd},$$

and $q+1$ is divisible by three. Put $d = (3, q+1) = 3$. If $3 \nmid m$, then

$$\gcd\big((q+1)/d, d, m\big) = 1,$$

so Theorem 7.33 says that $\mathrm{Aut}(S) \to \mathrm{Out}(S)$ splits. A section restricts to every intermediate outer subgroup, contrary to the hypothesis on B. Hence $3 \mid m$. Since m is odd, lifting the exponent gives

$$v_3(q+1) = v_3(2^m + 1) = 1 + v_3(m) \geq 2,$$

and therefore $9 \mid q+1$.

Let

$$1 \longrightarrow C_3 \longrightarrow \widehat{S} \longrightarrow S \longrightarrow 1$$

be the standard triple cover and let $w \in H^2(S;\mathbb{F}_3)$ classify it. The class has nonzero powers through degree twelve. To see this directly, use the standard F-stable maximal-rank subsystem of type A_2^3 in the simply connected algebraic group; its fixed points give a central product of three copies of $\mathrm{SU}_3(q)$ in $\widehat{S}$ [25, Table 4.7.3A]. The center of $\widehat{S}$ lies in the image of the three factor centers. If $\zeta \in C_{q+1}$ has order three, choose $\lambda \in C_{q+1}$ with $\lambda^3 = \zeta$, possible because $9 \mid q+1$. Then

$$\mathrm{diag}(\lambda, \lambda, \lambda^{-2}) \in \mathrm{SU}_3(q), \qquad \mathrm{diag}(\lambda, \lambda, \lambda^{-2})^3 = \zeta I.$$

Taking such a cube root in each required factor shows that a generator of $Z(\widehat{S})$ is the cube of an element of order nine. Its image in S generates a subgroup $T \cong C_3$ over which the triple cover is $C_9 \to C_3$. Hence, for the polynomial generator $u \in H^2(C_3;\mathbb{F}_3)$,

$$\mathrm{res}^S_T(w) = u, \qquad \mathrm{res}^S_T(w^k) = u^k \neq 0 \quad (k \geq 1).$$

Put $C = B/S$. In twisted type E_6 the outer group has a canonical odd-order subgroup, generated by diagonal and pure-field automorphisms, of index two; the remaining coset is graph–field. Let C_0 be the intersection of C with this subgroup and let B_0 be its inverse image. Thus $[C : C_0] \leq 2$ and C_0 has odd order. Its action on the one-dimensional space $\langle w \rangle \leq H^2(S;\mathbb{F}_3)$ is trivial, because $\mathbb{F}_3^\times \cong C_2$. Let F_0 be the intersection of C_0 with the pure-field outer group, and let A be its inverse image. The diagonal outer group has order three, so

$$[B_0 : A] = [C_0 : F_0] \leq 3.$$

The standard field representatives split $A = S \rtimes F_0$ and lift to $\widehat{S}$; because F_0 has odd order, it fixes the central kernel. Thus w extends to a class $\widetilde{w} \in H^2(A;\mathbb{F}_3)$.

If $A = B_0$, put $z_0 = \widetilde{w}^3$. Otherwise the index is three and put

$$z_0 = \mathcal{N}_A^{B_0}(\widetilde{w}) \in H^6(B_0;\mathbb{F}_3).$$

The Evens restriction formula gives in either case

$$\mathrm{res}^{B_0}_S(z_0) = w^3 \neq 0.$$

If $B = B_0$, this proves the degree-six assertion. Otherwise $[B : B_0] = 2$; apply the Evens norm once more:

$$z = \mathcal{N}_{B_0}^B(z_0) \in H^{12}(B;\mathbb{F}_3).$$

Its restriction to S is $w^3 g^*(w^3) = \pm w^6 \neq 0$ for $g \in B \setminus B_0$. This proves the restriction statement. Theorem 7.1 and the mod-three universal-coefficient sequence give the stated integral degrees. □

Proposition 7.21 (A twisted-E_6 degree-nine lifting diagnostic)**.** *Let*

$$S = {}^2E_6(2^m), \qquad 3 \mid m, \qquad m \text{ odd},$$

and let $w \in H^2(S;\mathbb{F}_3)$ be the class of the standard triple cover. If β denotes the mod-three Bockstein, then

$$c = \beta(w)w^3 \in H^9(S;\mathbb{F}_3)$$

is nonzero and is fixed by every automorphism of S.

In the residual notation of Theorem 7.20, lifting $\beta(w)$ to B_0 gives a lower-degree refinement: if

$$b \in H^3(B_0;\mathbb{F}_3), \qquad \operatorname{res}_S^{B_0}(b) = \beta(w),$$

then a class in $H^9(B;\mathbb{F}_3)$ restricts nontrivially to S. Thus the optional refinement has a specific degree-three edge-lifting target; neither nonvanishing nor graph–field invariance of the degree-nine class remains to be proved. This refinement is not needed for existence of a finite universal bound.

Proof. Use the same A_2^3 central product in $\widehat{S}$ as in the proof of Theorem 7.20. In one $\mathrm{SU}_3(q)$ factor, the order-nine element already constructed has cube equal to a generator of $Z(\widehat{S})$. In a second factor choose a primitive cube root $\zeta \in C_{q+1}$ and put

$$d = \operatorname{diag}(1,\zeta,\zeta^2), \qquad p = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Both matrices lie in $\mathrm{SU}_3(q)$, have order three, and satisfy $pdp^{-1} = \zeta^2 d$. They therefore generate an extraspecial group 3_+^{1+2} of order twenty-seven and exponent three whose center is the factor center. The two chosen subgroups commute, and their factor centers have the same image in the central product. Consequently S contains an elementary abelian subgroup

$$V \cong C_3^3$$

over which the triple cover is the central product $C_9 \circ 3_+^{1+2}$.

Write

$$H^*(V;\mathbb{F}_3) = \mathbb{F}_3[u_0,u_1,u_2] \otimes \Lambda(t_0,t_1,t_2), \qquad |t_i| = 1, \quad u_i = \beta(t_i).$$

After rescaling the coordinates, the restricted extension class is

$$e = \operatorname{res}_V^S(w) = u_0 + t_1 t_2.$$

Here u_0 records $C_9 \to C_3$ and t_1t_2 records the extraspecial extension. Hence

$$e^3 = u_0^3, \qquad \beta(e) = u_1 t_2 - t_1 u_2,$$

and therefore

$$\operatorname{res}_V^S(c) = (u_1 t_2 - t_1 u_2) u_0^3 \neq 0. \tag{19g'}$$

This proves $c \neq 0$.

Since S is perfect and its multiplier has three-part C_3, $H^2(S;\mathbb{F}_3) = \langle w \rangle$ is one-dimensional. Thus every $\alpha \in \operatorname{Aut}(S)$ sends w to ϵw for some $\epsilon \in \mathbb{F}_3^\times$. Naturality of the Bockstein gives

$$\alpha^*(c) = \epsilon^4 c = c,$$

so c is automorphism-fixed.

Finally let $z_0 \in H^6(B_0;\mathbb{F}_3)$ be the class constructed in Theorem 7.20; it restricts to w^3. Under the stated lifting hypothesis, bz_0 restricts to c. If $B = B_0$ there is nothing more to prove. Otherwise $[B : B_0] = 2$, and the restriction–corestriction formula gives

$$\operatorname{res}_S^B \operatorname{cor}_{B_0}^B(bz_0) = c + g^*c = 2c \neq 0$$

for $g \in B \setminus B_0$. This supplies the asserted degree-nine class. □

Proposition 7.22 (The remaining twisted-E_6 Bockstein obstruction)**.** *Retain the residual notation of Proposition 7.21, assume that $B \neq B_0$, and put*

$$C_0 = B_0/S, \qquad \kappa = d_3(w) \in H^3(C_0; \mathbb{F}_3)$$

in the mod-three Lyndon–Hochschild–Serre spectral sequence for $S \to B_0 \to C_0$. Then

$$d_4(\beta w) = [-\beta\kappa] \in \frac{H^4(C_0; \mathbb{F}_3)}{H^1(C_0; \mathbb{F}_3)\kappa}, \tag{19g$''$}$$

and, in this twisted-E_6 situation,

$$\beta(w) \in \operatorname{im}\big(H^3(B_0; \mathbb{F}_3) \longrightarrow H^3(S; \mathbb{F}_3)\big) \quad \Longleftrightarrow \quad \beta(\kappa) = 0. \tag{19g$''$a}$$

Let $P \in \mathrm{Syl}_3(C_0)$. The condition in (19g$''$a) *is automatic if P is cyclic. If P is not cyclic, then*

$$P = D \times F \cong C_3 \times C_{3^b} \qquad (b \geq 1),$$

where D is diagonal and F is pure field. Choose generators so that

$$H^*(P; \mathbb{F}_3) = \mathbb{F}_3[u, v] \otimes \Lambda(x, y), \qquad |x| = |y| = 1, \quad |u| = |v| = 2,$$

with x, u diagonal and y, v field. There are scalars $A, B \in \mathbb{F}_3$ such that

$$\mathrm{res}^{C_0}_P(\kappa) = Axv + Byu. \tag{19g$'''$}$$

If $b = 1$, then $\beta x = u$, $\beta y = v$, and

$$\mathrm{res}^{C_0}_P\big(\beta\kappa\big) = (A + B)uv. \tag{19g$''''$}$$

If $b \geq 2$, then $\beta x = u$, $\beta y = 0$, and

$$\mathrm{res}^{C_0}_P\big(\beta\kappa\big) = Auv. \tag{19g$'''''$}$$

Thus the degree-three lift required in Proposition 7.21 is already proved unless the outer Sylow three-subgroup has rank two; in that last case its entire obstruction is the single scalar displayed in (19g$''''$) *or* (19g$'''''$).

Proof. Because S is perfect, $H^1(S; \mathbb{F}_3) = 0$. Hence w has no d_2 and its only possible differential is the displayed transgression $d_3(w)$. The Bockstein form of the Kudo transgression theorem gives

$$d_2\big(\beta w\big) = d_3\big(\beta w\big) = 0, \qquad d_4\big(\beta w\big) = [-\beta\big(d_3 w\big)];$$

see [31] or [35, Section 6]. The brackets matter: the only incoming differential to bidegree $(4, 0)$ before page four is

$$d_3 : H^1(C_0; \mathbb{F}_3)w \longrightarrow H^4(C_0; \mathbb{F}_3), \qquad aw \longmapsto \pm a\kappa.$$

Thus its fourth-page target is exactly the quotient in (19g$''$). There is no later differential from bidegree $(0, 3)$ and no incoming differential to that bidegree. The edge-image description of $E_\infty^{0,3}$ therefore says that βw lifts if and only if

$$\beta\kappa \in H^1(C_0; \mathbb{F}_3)\kappa. \tag{19g$''$b}$$

Restriction to a Sylow three-subgroup is injective on mod-three cohomology: restriction followed by corestriction is multiplication by $[C_0 : P]$, a unit in $\mathbb{F}_3$. It is consequently enough to compute $\beta(\operatorname{res}_P \kappa)$. If $P \cong C_{3^b}$ with $b \geq 2$, write

$$H^*(P; \mathbb{F}_3) = \mathbb{F}_3[v] \otimes \Lambda(y).$$

The homomorphism $y : P \to \mathbb{F}_3$ lifts to $P \to \mathbb{Z}/9$, so $\beta y = 0$; the polynomial class v is the reduction of an integral class, so $\beta v = 0$. The Bockstein therefore vanishes on $H^3(P; \mathbb{F}_3)$.

It remains to treat a cyclic group P of order three and the rank-two case. Choose $g \in B \setminus B_0$. The graph–field image of g inverts the diagonal outer group and centralizes the pure-field outer group. On the standard triple cover it acts on the central kernel μ_3 by

$$z \longmapsto z^2 = z^{-1};$$

equivalently, $g^* w = -w$. Naturality of transgression gives

$$g^* \kappa = -\kappa. \tag{19g$''''''$}$$

The Sylow group P is characteristic in the abelian group C_0, hence is normalized by g. On the three-torsion of the ambient outer group, g has eigenvalues -1 on the diagonal line and $+1$ on the field line. A g-stable subgroup of order three is therefore one of these two lines. On the field line, w extends across the split pure-field subgroup, so κ restricts to zero. On the diagonal line, $H^3(C_3; \mathbb{F}_3) = \langle xu \rangle$ and inversion fixes xu; the anti-invariance (19g$''''''$) again forces the restriction to vanish. This proves the cyclic order-three case.

If P is noncyclic, its intersection F with the cyclic pure-field group is nontrivial and P/F is diagonal of order three. An element of order three outside F can be multiplied by the unique element of order three in F to give a pure diagonal generator. Thus $P = D \times F$ as asserted. The extension of w across the pure-field preimage gives

$$\operatorname{res}_F^P \operatorname{res}_P^{C_0}(\kappa) = 0. \tag{19g$''''''$}$$

Before imposing this condition, a general degree-three class is a linear combination of

$$xu, \quad xv, \quad yu, \quad yv.$$

Condition (19g$''''''$) removes the yv term. Conjugation by g sends

$$x \mapsto -x, \quad u \mapsto -u, \qquad y \mapsto y, \quad v \mapsto v.$$

Thus xu is fixed, whereas xv and yu are negated. Anti-invariance removes the xu term and proves (19g$'''$). When $b = 1$, the derivation rule for the Bockstein gives

$$\beta(Axv + Byu) = (A + B)uv.$$

When $b \geq 2$, the same lifting argument as in the cyclic case gives $\beta y = 0$, and hence

$$\beta(Axv + Byu) = Auv.$$

The monomial uv is nonzero. Moreover

$$H^1(P; \mathbb{F}_3) \operatorname{res}_P(\kappa) \subseteq \langle xyu, xyv \rangle,$$

whereas uv has exterior degree zero. Hence a nonzero displayed Bockstein cannot belong to $H^1(P; \mathbb{F}_3) \operatorname{res}_P(\kappa)$. Restricting (19g$''$b) to P now proves that, in all of the cases above, the edge condition is equivalent to $\beta\kappa = 0$, and proves the two stated scalar criteria. □

Remark 7.23 (A unitary rank-two diagnostic)**.** The analogous scalar need not vanish for formal reasons. Let $S_u = \mathrm{PSU}_3(8)$. Its outer three-subgroup is $P_u = C_3 \times C_3$, with pure diagonal and pure field lines and a mixed line. A GAP 4.15.1 computation with PrimGrp 4.0.1 and Cohomolo 1.6.12 gives the following three-primary Schur multipliers:

primitive identifier	outer image	$H_2(B;\mathbb{Z})_{(3)}$
$(513,3)$	pure line I	C_3
$(513,4)$	pure line II	C_3
$(513,5)$	mixed line	0
$(513,8)$	$C_3 \times C_3$	C_3.

($19\mathrm{g}'''''''$)

The calculation is reproduced by `scripts/check_unitary_outer_multiplier.g`.

To interpret it, let w_u generate $H^2(S_u;\mathbb{F}_3)$ and put $\kappa_u = d_3(w_u)$. For a line $L \le P_u$, the preimage B_L has abelianization C_3. The universal coefficient theorem and the LHS five-term sequence show that $\mathrm{res}_L\,\kappa_u = 0$ exactly when the three-part of $H_2(B_L;\mathbb{Z})$ is nonzero. Thus κ_u vanishes on both pure lines but not on the mixed line. In the coordinates of Proposition 7.22, this says

$$\mathrm{res}_{P_u}\,\kappa_u = Axv + Byu, \qquad A + B \neq 0,$$

and hence $\beta\kappa_u = (A+B)uv \neq 0$. The C_3 in the last row of ($19\mathrm{g}'''''''$) is consistent with inflation of the quotient multiplier $H_2(P_u;\mathbb{Z}) = C_3$; it does not make w_u lift.

This is only a diagnostic: the unitary and twisted-E_6 root data are different, so it does not determine the coefficient in Proposition 7.22. It does prove that the cyclic-line vanishings and graph anti-invariance alone cannot force that coefficient to be zero. A twisted-E_6 calculation, a root-theoretic comparison theorem, or a different class would be required to obtain the optional lower-degree refinement.

Corollary 7.24 (Residual fixed fields in even orthogonal type)**.** *Let $S = D_n(q)$ or ${}^2D_n(q)$ be nonabelian simple, where $n \ge 4$, let $S \le B \le \mathrm{Aut}(S)$, and suppose that $B \to B/S$ does not split. Use the notation*

$$q = p^m, \qquad e = \gcd(m,a), \qquad x = p^e, \qquad f = m/e$$

for the field image of B/S. If Proposition 7.8 supplies no restricting degree-four class, then either

$$x = 3,$$

or all of the following hold:

$$S = D_4(q), \qquad x \in \{5,7,17\}, \qquad 3 \mid f,$$

and the outer image of B has nontrivial triality projection. In particular, outside untwisted triality the only residual fixed field is $\mathbb{F}_3$.

Proof. The Schur multiplier in the nonsplit even-orthogonal families is 2-primary [1]. Their diagonal outer groups are 2-groups, their field groups are cyclic, and their graph groups are 2-groups except for the triality group of untwisted D_4. Moreover a nonsplit case has odd defining characteristic: for untwisted D_n, even characteristic gives $d = 1$ in Theorem 7.33, while the twisted theorem explicitly splits when $p = 2$.

Let ℓ be an odd prime divisor of $x^2 - 1$. As in the proof of Corollary 7.18, all the numerical hypotheses of Proposition 7.8 hold. Unless $S = D_4(q)$ and $\ell = 3$, a Sylow ℓ-subgroup of the outer

quotient lies in the cyclic field group. Thus, outside that triality possibility, failure of the degree-four criterion forces $x^2 - 1$ to be a power of two. The two even integers $x - 1$ and $x + 1$ are then adjacent powers of two, so they are two and four and $x = 3$.

In the triality possibility, failure forces $x^2 - 1 = 2^r 3^s$. The elementary calculation in the proof of Corollary 7.18 gives $x \in \{3, 5, 7, 17\}$, since x is odd. If $x \neq 3$, the prime three is available, so Proposition 7.8 can fail only when a Sylow three-subgroup of the outer quotient is noncyclic. This requires both a nontrivial triality projection and a field image of order divisible by three, the latter being $3 \mid f$. □

Theorem 7.25 (Even orthogonal fixed-$\mathbb{F}_3$ stabilizers)**.** *Let $S = D_n(q)$ or ${}^2D_n(q)$ be nonabelian simple, where $n \geq 4$ and $q = 3^m$, and let*

$$S \leq B \leq \mathrm{Aut}(S).$$

Suppose that the field image of B/S has fixed field $\mathbb{F}_3$ and that, when $S = D_4(q)$, its graph image has no element of order three. Then there is a class in $H^8(B; \mathbb{Z})_{(5)}$ whose restriction to S is nonzero. Consequently, every such nonabelian minimal-normal-subgroup branch has an integral witness in degree eight.

Proof. Let $\widehat{S}$ be the standard simply connected fixed-point group and let $K = \mathrm{Spin}(2n)$ be the corresponding compact simply connected group. The kernel of $\widehat{S} \to S$ is a 2-group, so this map is a 5-local equivalence. Put $X = (BK)^\wedge_5$. The low homotopy of the compact Spin group gives

$$\pi_4(X) = \mathbb{Z}_5, \qquad \pi_i(X) = 0 \quad (5 \leq i \leq 7), \qquad \pi_8(X) = \begin{cases} \mathbb{Z}_5, & n \geq 5, \\ \mathbb{Z}_5^2, & n = 4. \end{cases} \tag{19g}$$

These are the generators corresponding to the first two fundamental degrees 2 and 4 of type D_n. An unstable Adams operation of degree u acts on them as multiplication by u^2 and u^4, respectively [27, Appendix A]. The graph involution fixes the degree-two generator. It also fixes the degree-four generator when $n \geq 5$. For D_4, the graph group S_3 acts on $\pi_8(X) \cong \mathbb{Z}_5^2$ as its reflection representation. Every subgroup with no triality has order at most two and fixes a primitive line. In the twisted case the defining graph involution and every graph automorphism centralizing it fix the same line. Denote the resulting graph-stable degree-four line by $L \cong \mathbb{Z}_5$.

The Friedlander–Mislin comparison identifies the 5-completion of $B\widehat{S}$ with the homotopy fixed points $Y = X^{h\sigma}$, where on $\pi_4(X)$ the Steinberg map σ acts by q^2 and on L it acts by q^4 [16, 17, 27].

Suppose first that m is odd. Then

$$q^2 \equiv -1 \pmod 5, \qquad q^4 \equiv 1 \pmod 5.$$

The homotopy-equalizer exact sequence and (19g) show that Y is 6-connected and that $\pi_7(Y)$ contains an order-five subgroup

$$\{z \in \mathrm{coker}(1 - q^4 : L \longrightarrow L) : 5z = 0\} \cong C_5.$$

The 5-adic Hurewicz theorem and the comparison therefore give

$$H^j(S; \mathbb{Z}_{(5)}) = 0 \quad (0 < j < 8), \qquad H^8(S; \mathbb{Z}_{(5)}) \neq 0. \tag{19h}$$

Choose the order-five subgroup generated by the class arising from L and call a generator x. Diagonal automorphisms act trivially on the compact root datum, graph automorphisms in the

stipulated subgroup fix L, and a 3^a-field automorphism acts on L as $3^{4a} \equiv 1$ modulo five. Thus every element of B/S fixes x.

Now suppose that m is even. Since the fixed field is $\mathbb{F}_3$, the field image is generated by a 3^a-power map with $\gcd(a, m) = 1$, so a is odd. Proposition 4.5, followed by the 5-local equivalence $\widehat{S} \to S$, supplies an order-five class

$$x \in H^4(S; \mathbb{Z}_{(5)}).$$

It may be chosen from the basic degree-two line. A long-root $\mathrm{SL}_2(q)$ fixed by the defining graph involution has Dynkin index one, so naturality restricts x nontrivially to a cyclic subgroup of order five. Consequently $x^2 \neq 0$ in $H^8(S; \mathbb{Z}_{(5)})$. Every element of B/S sends x to x or $-x$: graph and diagonal automorphisms fix the basic line, while a field automorphism acts on its order-five subgroup by a power of $3^{2a} \equiv -1 \pmod 5$.

It remains in both cases to lift the class across the outer quotient. Put $C = B/S$, choose $P \in \mathrm{Syl}_5(C)$, and let B_P be its inverse image. The diagonal and nontriality graph groups are 2-groups, so P lies in the cyclic field group and is cyclic. If m is odd, (19h), the equality $H^9(P; \mathbb{Z}_{(5)}) = 0$, and Lemma 7.6 lift x to a class $y \in H^8(B_P; \mathbb{Z}_{(5)})$. Corestriction gives

$$\mathrm{res}^B_S \, \mathrm{cor}^B_{B_P}(y) = [C : P]x \neq 0. \tag{19i}$$

If m is even, write $5^v = |P|$. The generator of P is a power of the field generator whose exponent is divisible by $m/5^v$, an even number (when $P = 1$ this assertion is vacuous). Hence P acts on x trivially: when $P \neq 1$, a generator acts by

$$3^{2a(m/5^v)} = 1 \pmod 5.$$

Since S is perfect and its multiplier is 2-primary, $H^j(S; \mathbb{Z}_{(5)}) = 0$ for $1 \leq j \leq 3$; also $H^5(P; \mathbb{Z}_{(5)}) = 0$. Lemma 7.6 lifts x to $y \in H^4(B_P; \mathbb{Z}_{(5)})$. The square y^2 restricts to x^2, and the restriction–corestriction formula gives

$$\mathrm{res}^B_S \, \mathrm{cor}^B_{B_P}(y^2) = \sum_{cP \in C/P} c^*(x^2) = [C : P]x^2 \neq 0, \tag{19j}$$

because every outer automorphism acts on x by a sign and $[C : P]$ is a 5-local unit. Thus in either parity a degree-eight class on B restricts nontrivially to S. The final assertion is Theorem 7.1. □

Theorem 7.26 (Residual even-orthogonal triality stabilizers)**.** *In the nonabelian minimal-normal-subgroup branch, every nonsplit factor stabilizer of even-orthogonal type has a degree-at-most-eight class whose restriction to the simple factor is nonzero. More precisely, after Corollary 7.24 and Theorem 7.25, a putative residual factor stabilizer has*

$$S = D_4(q), \qquad q = p^m, \qquad p \in \{3, 5, 7, 17\}, \qquad 2 \mid m,$$

and has genuine triality in its graph image. Put

$$\ell = \begin{cases} 13, & p = 5, \\ 5, & p \in \{3, 7, 17\}. \end{cases}$$

Then a class in $H^8(B; \mathbb{Z}_{(\ell)})$ restricts nontrivially to S. Consequently, every even-orthogonal nonabelian minimal-normal-subgroup branch has an integral witness in degree at most eight.

Proof. Let B be a factor stabilizer left after the two cited results. Its field image has fixed field one of $\mathbb{F}_3, \mathbb{F}_5, \mathbb{F}_7, \mathbb{F}_{17}$, and because these fields have prime order the notation of Corollary 7.24 has $e = 1$ and $f = m$. Thus p belongs to the displayed list. The map $B \to B/S$ is nonsplit by Theorem 7.32. For odd q, the diagonal order in type D_4 is $d = 4$, and

$$4 \mid \frac{q^4 - 1}{4}.$$

The criterion in Theorem 7.33 therefore says that $\mathrm{Aut}(S) \to \mathrm{Out}(S)$ splits whenever m is odd. A section restricts to every intermediate outer subgroup, so $B \to B/S$ would then split as well. Hence m is even.

For the displayed choice of ℓ one has

$$p^2 \equiv -1 \pmod{\ell}. \tag{19k}$$

In particular $\ell \mid q^2 - 1$. The standard simply connected cover $\widehat{S} \to S$ has 2-group kernel, so Proposition 4.5 supplies an order-ℓ class

$$x \in H^4(S; \mathbb{Z}_{(\ell)})$$

from the basic degree-two line. A long-root $\mathrm{SL}_2(q)$ has Dynkin index one, and restriction to a cyclic subgroup of order ℓ in that rank-one group is nonzero. Lemma 7.3 therefore gives

$$x^2 \neq 0 \qquad \text{in } H^8(S; \mathbb{Z}_{(\ell)}). \tag{19l}$$

Write $C = B/S$. Because the field image has fixed field $\mathbb{F}_p$, it is generated by a p^a-power map with $\gcd(a, m) = 1$; since m is even, a is odd. On the order-ℓ subgroup generated by x, this field generator acts as

$$p^{2a} \equiv -1 \pmod{\ell}.$$

Graph automorphisms, including triality, fix the basic degree-two line, and diagonal automorphisms act trivially on it. Thus every element of C sends x to x or $-x$ and fixes x^2.

Choose $P \in \mathrm{Syl}_\ell(C)$ and let B_P be its inverse image. Since $\ell \in \{5, 13\}$, neither the diagonal group nor the graph group has ℓ-torsion. Hence P lies in the cyclic field group and is cyclic. If $\ell^v = |P|$, then m/ℓ^v is even. When $P \neq 1$, a generator of P is a power of the field generator with exponent divisible by m/ℓ^v, so it acts on x by

$$(-1)^{m/\ell^v} = 1.$$

The group S is perfect and its multiplier is 2-primary, whence

$$H^j(S; \mathbb{Z}_{(\ell)}) = 0 \qquad (1 \leq j \leq 3).$$

Also $H^5(P; \mathbb{Z}_{(\ell)}) = 0$. Lemma 7.6 lifts x to a class $y \in H^4(B_P; \mathbb{Z}_{(\ell)})$. Its square restricts to x^2, and restriction–corestriction gives

$$\mathrm{res}^B_S \, \mathrm{cor}^B_{B_P}(y^2) = \sum_{cP \in C/P} c^*(x^2) = [C : P] x^2 \neq 0, \tag{19m}$$

because $[C : P]$ is an ℓ-local unit. This proves the restriction claim. Theorem 7.1 gives the final assertion. □

Proposition 7.27 (The squared symplectic multiplier class)**.** *Let q be odd and let $S = \mathrm{PSp}_{2n}(q)$ be nonabelian simple. The character of the kernel of*

$$1 \longrightarrow C_2 \longrightarrow \mathrm{Sp}_{2n}(q) \longrightarrow S \longrightarrow 1$$

defines a class $x \in H^3(S;\mathbb{Z})$ such that

$$x^2 \neq 0 \quad \text{in } H^6(S;\mathbb{Z}).$$

If $S \lhd B$ and $C = B/S$, then x^2 survives through the E_7-page of the Lyndon–Hochschild–Serre spectral sequence. In particular, if $H^7(C;\mathbb{Z})_{(2)} = 0$, a class in $H^6(B;\mathbb{Z})_{(2)}$ restricts nontrivially to S.

Proof. There is a quaternion subgroup of $\mathrm{Sp}_{2n}(q)$ whose central involution is $-I$. Explicitly, choose $a, b \in \mathbb{F}_q$ with $a^2 + b^2 = -1$ and put

$$i_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad j_0 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

Such a, b exist by the norm map from $\mathbb{F}_{q^2}$ when -1 is nonsquare, and trivially when it is square. The matrices lie in $\mathrm{SL}_2(q) = \mathrm{Sp}_2(q)$ and satisfy $i_0^2 = j_0^2 = -I$ and $i_0 j_0 = -j_0 i_0$. Taking the orthogonal direct sum of n copies gives a subgroup $Q_8 \leq \mathrm{Sp}_{2n}(q)$ with center $\{\pm I\}$. Its image in S is a subgroup $V \cong C_2^2$.

The restriction of x to V is the nonzero class classifying the stem extension $Q_8 \to V$. Write $H^*(V;\mathbb{F}_2) = \mathbb{F}_2[r,s]$ with $|r| = |s| = 1$. Under the integral Bockstein, this class is the generator

$$\beta(rs) \in H^3(V;\mathbb{Z}) \cong C_2.$$

Its mod-two reduction is

$$\rho\beta(rs) = \mathrm{Sq}^1(rs) = r^2 s + r s^2.$$

Therefore the reduction of its square is

$$(r^2 s + r s^2)^2 = r^4 s^2 + r^2 s^4 \neq 0.$$

Thus $\mathrm{res}^S_V(x^2) \neq 0$, proving $x^2 \neq 0$.

Now use the integral Lyndon–Hochschild–Serre spectral sequence for $1 \to S \to B \to C \to 1$. The group generated by x has order two, so every automorphism fixes it: the 2-primary part of the Schur multiplier of S is cyclic of order two [1]. Since $H^1(S;\mathbb{Z}) = H^2(S;\mathbb{Z}) = 0$, the class x reaches $E_4^{0,3}$. Multiplicativity and the Leibniz rule give $d_4(x^2) = 0$; the d_2 and d_3 differentials on x^2 vanish for the same reason. The targets of d_5 and d_6 from bidegree $(0,6)$ involve $H^2(S;\mathbb{Z})$ and $H^1(S;\mathbb{Z})$ and vanish. Hence the only remaining outgoing differential is

$$d_7 : E_7^{0,6} \longrightarrow E_7^{7,0}.$$

There are no incoming differentials. If the 2-primary part of $H^7(C;\mathbb{Z})$ vanishes, x^2 survives and the edge map lifts it to B. □

Theorem 7.28 (Symplectic factor stabilizers)**.** *Let q be odd, let $S = \mathrm{PSp}_{2n}(q)$ be nonabelian simple, and let*

$$S \leq B \leq \mathrm{Aut}(S).$$

There is a class $z \in H^4(B;\mathbb{F}_2)$ whose restriction to S is nonzero. Consequently, every nonabelian minimal-normal-subgroup branch with simple factor of symplectic type has an integral witness in degree four or five.

Proof. Let $w \in H^2(S;\mathbb{F}_2)$ classify the double cover $\mathrm{Sp}_{2n}(q) \to S$. The matrix i_0 in the proof of Proposition 7.27, repeated in n orthogonal blocks, has square $-I$. Its cyclic group of order four maps onto a subgroup $T \cong C_2$ of S, and the double cover restricts over T to $C_4 \to C_2$. Thus, if $H^*(T;\mathbb{F}_2) = \mathbb{F}_2[t]$ with $|t| = 1$, then

$$\mathrm{res}^S_T(w) = t^2, \qquad \mathrm{res}^S_T(w^2) = t^4 \neq 0.$$

In particular $w^2 \neq 0$. The 2-primary part of the Schur multiplier is C_2, so every automorphism of S fixes w and w^2.

Put $C = B/S$, choose $P \in \mathrm{Syl}_2(C)$, and let B_P be its inverse image in B. For odd q, the 2-part of the outer automorphism group of S is contained in the direct product of the diagonal group D, of order at most two, and the cyclic field-automorphism group F; there is no graph automorphism in odd symplectic type [1].

Put $F_0 = P \cap F$ and let A be its inverse image in B_P. Projection to D shows that $[B_P : A] = [P : F_0] \leq 2$, and

$$A = S \rtimes F_0.$$

Entrywise field automorphisms lift to $\mathrm{Sp}_{2n}(q)$ and fix its central involution. Thus the double cover of S extends to a central extension

$$1 \longrightarrow C_2 \longrightarrow \mathrm{Sp}_{2n}(q) \rtimes F_0 \longrightarrow A \longrightarrow 1.$$

It therefore defines $\widetilde{w} \in H^2(A;\mathbb{F}_2)$ restricting to w. If $A = B_P$, set $z_P = \widetilde{w}^2$. Otherwise $[B_P : A] = 2$; apply the Evens norm [15]

$$\mathcal{N}^{B_P}_A : H^2(A;\mathbb{F}_2) \longrightarrow H^4(B_P;\mathbb{F}_2).$$

For $g \in B_P \setminus A$, its restriction formula gives

$$\mathrm{res}^{B_P}_A \mathcal{N}^{B_P}_A(\widetilde{w}) = \widetilde{w}\, g^*\widetilde{w}.$$

After restricting farther to S, the right-hand side is w^2, because $g^*w = w$. Hence in both cases there is $z_P \in H^4(B_P;\mathbb{F}_2)$ with $\mathrm{res}^{B_P}_S(z_P) = w^2$.

Finally put $z = \mathrm{cor}^B_{B_P}(z_P)$. The Mackey formula and normality of S give

$$\mathrm{res}^B_S(z) = [C : P]w^2 = w^2,$$

because $[C : P]$ is odd and the coefficient field is $\mathbb{F}_2$. This proves the restriction assertion. Theorem 7.1 now gives nonzero $H^4(G;\mathbb{F}_2)$ for the corresponding S^r branch, and passage from mod two to integral cohomology gives a nonzero integral class in degree four or five. For even q, the symplectic automorphism extension is in the split list of Theorem 7.33, so no additional case remains. □

Theorem 7.29 (E_7 factor stabilizers)**.** *Let q be odd, let $S = E_7(q)$ be simple, and let*

$$S \leq B \leq \mathrm{Aut}(S).$$

There is a class $z \in H^4(B;\mathbb{F}_2)$ whose restriction to S is nonzero. Consequently, every nonabelian minimal-normal-subgroup branch with simple factor of type E_7 has an integral witness in degree four or five.

Proof. Let $\widehat{S} = E_7(q)_{\mathrm{sc}}$. Its center has order two and $\widehat{S}/Z(\widehat{S}) = S$. Moreover, $\widehat{S}$ contains a subgroup $L \cong \mathrm{SL}_2(q^7)$ with $Z(L) = Z(\widehat{S})$ [28, p. 927, item (3)]. The matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in L$$

has order four and its square is the nontrivial element of $Z(\widehat{S})$. Therefore the class $w \in H^2(S; \mathbb{F}_2)$ of the double cover $\widehat{S} \to S$ restricts to t^2 on a cyclic subgroup of order two, and $w^2 \neq 0$. The 2-primary Schur multiplier of S has order two, so every automorphism of S fixes w.

The outer automorphism group has its 2-part in the direct product of a diagonal group $D \cong C_2$ and the cyclic field-automorphism group F; type E_7 has no graph automorphism [1]. For $C = B/S$, choose $P \in \mathrm{Syl}_2(C)$, put $F_0 = P \cap F$, and let A and B_P be the inverse images of F_0 and P. Then $[B_P : A] \leq 2$ and $A = S \rtimes F_0$. Field automorphisms lift to $\widehat{S}$ and fix its central involution, so w extends to a class $\widetilde{w} \in H^2(A; \mathbb{F}_2)$.

If $A = B_P$, the class $\widetilde{w}^2$ restricts to w^2. Otherwise the Evens norm

$$\mathcal{N}_A^{B_P}(\widetilde{w}) \in H^4(B_P; \mathbb{F}_2)$$

restricts to w^2, by the same restriction formula used in Theorem 7.28. Corestriction from B_P to B has odd index and still restricts to w^2. The factor-stabilizer and mod-two-to-integral conclusions follow exactly as in that theorem. When q is even, the E_7 automorphism extension is in the split list of Theorem 7.33. □

Theorem 7.30 (Odd orthogonal factor stabilizers)**.** *Let q be odd, let $S = B_n(q) = \mathrm{P}\Omega_{2n+1}(q)$ be simple with $n \geq 3$, and let*

$$S \leq B \leq \mathrm{Aut}(S).$$

There is a class $z \in H^4(B; \mathbb{F}_2)$ whose restriction to S is nonzero. Consequently, every nonabelian minimal-normal-subgroup branch with simple factor of type B_n has an integral witness in degree four or five.

Proof. Let V be the natural $(2n+1)$-dimensional nondegenerate quadratic space over $\mathbb{F}_q$ and let

$$\widehat{S} = \mathrm{Spin}(V) \longrightarrow S$$

be the central double cover. Choose a nondegenerate three-dimensional subspace $U \leq V$. The inclusion of Clifford algebras embeds

$$\mathrm{Spin}(U) \cong \mathrm{SL}_2(q) \quad \text{in} \quad \mathrm{Spin}(V),$$

and the two Spin groups have the same scalar central involution. The order-four matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(q)$$

squares to that involution. Hence the class $w \in H^2(S; \mathbb{F}_2)$ of the Spin cover restricts to t^2 on a cyclic subgroup of order two, and $w^2 \neq 0$.

The diagonal automorphism is induced by conjugation in the relevant Clifford group, while field automorphisms act coefficientwise. Both lift to $\widehat{S}$ and fix its scalar central involution, so every standard outer automorphism fixes w. The 2-part of $\mathrm{Out}(S)$ is contained in the direct product of a diagonal group of order at most two and the cyclic field-automorphism group; type B_n has no graph automorphism [1].

Put $C = B/S$, choose $P \in \mathrm{Syl}_2(C)$, and let F_0 be the intersection of P with the field-automorphism group. If A and B_P are the inverse images of F_0 and P, then $[B_P : A] \leq 2$, $A = S \rtimes F_0$, and the field action on $\widehat{S}$ extends w to $\widetilde{w} \in H^2(A; \mathbb{F}_2)$. If $A = B_P$, use $\widetilde{w}^2$. Otherwise the Evens norm

$$\mathcal{N}_A^{B_P}(\widetilde{w}) \in H^4(B_P; \mathbb{F}_2)$$

restricts to w^2, because its two factors restrict to w and its diagonal conjugate, which is again w. Corestriction from B_P to B has odd index and restricts to the odd multiple $w^2 = w^2$. Thus it supplies the required z. Factor-stabilizer transfer and the mod-two-to-integral passage give the final assertion. In even characteristic, $B_n(q) \cong C_n(q)$ and the symplectic result applies. □

7.4 When automorphism groups split

Lemma 7.31 (Centerless pullback description)**.** *Let $1 \to N \to G \to Q \to 1$ be an extension with $Z(N) = 1$, and let $\alpha : Q \to \mathrm{Out}(N)$ be its outer action. Then*

$$G \cong \mathrm{Aut}(N) \times_{\mathrm{Out}(N)} Q.$$

If $\mathrm{Aut}(N) \to \mathrm{Out}(N)$ has a section, then $G \to Q$ has a section.

Proof. Send $g \in G$ to the pair consisting of conjugation by g on N and the image of g in Q. Its kernel is $Z(N)$, and every compatible pair differs from a chosen lift by a unique inner automorphism. This proves the fiber-product description. Pulling back a section of $\mathrm{Aut}(N) \to \mathrm{Out}(N)$ gives a section over Q. □

Theorem 7.32 (Factor-stabilizer splitting criterion)**.** *Let $N = S_1 \times \cdots \times S_r \lhd G$, where the S_i are isomorphic nonabelian simple groups and G permutes them transitively. Put $Q = G/N$, let $H \leq Q$ be the stabilizer of S_1, and let*

$$\alpha_1 : H \longrightarrow \mathrm{Out}(S_1)$$

be the induced outer action. Form the pullback

$$E_1 = \mathrm{Aut}(S_1) \times_{\mathrm{Out}(S_1)} H.$$

Then $G \to Q$ splits if and only if $E_1 \to H$ splits. Consequently, if $K = N_G(S_1)$ and B is the image of $K \to \mathrm{Aut}(S_1)$, a nonsplit extension $G \to Q$ forces $B \to B/S_1$ to be nonsplit.

Proof. By Lemma 7.31, G is the pullback of

$$\mathrm{Aut}(S_1^r) \longrightarrow \mathrm{Out}(S_1^r)$$

along the outer action of Q. A section of $G \to Q$, restricted to the stabilizer and then to the first coordinate, gives a section of $E_1 \to H$.

Conversely, suppose that $E_1 \to H$ has a section, equivalently that α_1 lifts to a homomorphism $\widehat{\alpha}_1 : H \to \mathrm{Aut}(S_1)$. Identify the set of factors with the transitive Q-set Q/H, choose representatives t_i with $t_i H = i$, and put

$$h(q, i) = t_{qi}^{-1} q t_i \in H.$$

The identity

$$h(q_1 q_2, i) = h(q_1, q_2 i) h(q_2, i)$$

shows that the permutation $i \mapsto qi$, decorated in the i-th coordinate by $\widehat{\alpha}_1(h(q, i))$, defines a homomorphism

$$Q \longrightarrow \mathrm{Aut}(S_1) \wr \mathfrak{S}_r.$$

Its projection to $\mathrm{Out}(S_1) \wr \mathfrak{S}_r$ is the given outer action after changing the chosen identifications of the factors. Such a change is conjugation by an element of $\mathrm{Out}(S_1)^r$ and may be lifted coordinatewise to $\mathrm{Aut}(S_1)^r$. After this conjugation, the displayed homomorphism lifts the given outer action, hence gives a section of its pullback $G \to Q$.

For the consequence, the image B contains $S_1 = \mathrm{Inn}(S_1)$ and maps onto $D = \alpha_1(H)$, so it is the full inverse image of D in $\mathrm{Aut}(S_1)$. If $B \to D$ split, its pullback $E_1 \to H$ would split, and the first part would make $G \to Q$ split. □

If $N = S^r$, then

$$\mathrm{Aut}(N) \cong \mathrm{Aut}(S) \wr \mathfrak{S}_r, \qquad \mathrm{Out}(N) \cong \mathrm{Out}(S) \wr \mathfrak{S}_r.$$

Consequently, a complement for S in $\mathrm{Aut}(S)$ induces a complement for N in $\mathrm{Aut}(N)$. Lemmas 7.31 and 5.2 then show that cohomology of G/N injects into that of G.

The precise classification of the cases where this argument applies is as follows.

Theorem 7.33 (Lucchini–Menegazzo–Morigi)**.** *Let S be a nonabelian finite simple group.*

(i) *If $S = A_n$, then $\mathrm{Aut}(S)$ splits over S exactly when $n \neq 6$.*

(ii) *If S is one of the twenty-six sporadic groups, then $\mathrm{Aut}(S)$ splits over S.*

(iii) *Let $S = {}^sL(q)$ be of Lie type over $q = p^m$, and put $d = [\mathrm{Inndiag}(S) : S]$. Then $\mathrm{Aut}(S)$ splits over S exactly in the following cases:*

  (a) *S is untwisted and not of type D_n, and $\gcd((q-1)/d, d, m) = 1$;*

  (b) *$S = D_n(q)$, and $\gcd((q^n-1)/d, d, m) = 1$;*

  (c) *S is twisted and is neither ${}^2D_n(q)$ nor ${}^2F_4(2)'$, and $\gcd((q+1)/d, d, m) = 1$;*

  (d) *$S = {}^2D_n(q)$, and either n is odd or $p = 2$.*

The Tits group does not split in its full automorphism group.

Proof. This is the classification in [34]; the statement about the Tits group is also recorded in [14]. □

It follows that the nonabelian minimal-normal-subgroup branch is eliminated whenever the simple factor occurs in the split list: a class from the proper quotient survives by a retraction. Corollary 7.5 and Corollary 7.7 also eliminate the two isolated nonsplit factors A_6 and the Tits group. The remaining nonabelian branch is therefore confined to the explicit arithmetic Lie-type families excluded by Theorem 7.33. In those cases, Theorem 7.1 remains available if an automorphism-stable low-degree class can be constructed; more generally, Lemma 7.2 applies whenever the outer-automorphism norm of a class on the simple factor is nonzero. Proposition 7.8 settles factor-stabilizer quotients satisfying its explicit prime condition when the detecting-prime Sylow subgroup of the outer quotient is cyclic. Corollary 7.11 settles every proper inner-diagonal factor stabilizer; Propositions 7.12 and 7.14 give elementary matrix-group proofs in the linear, unitary, and symplectic cases. Proposition 7.13 handles standard odd- characteristic linear and unitary overgroups whenever the central-cover kernel contains the full scalar 2-part. Proposition 7.15 also handles mixed diagonal–field stabilizers in all three families, and some unitary diagonal–graph stabilizers, whenever its explicit divisibility condition holds. Proposition 7.10 gives the corresponding degree-three criterion in every Lie type: the simply connected center must have nonzero image in the coinvariants of the regular-embedding center. Corollary 7.16 confines the residue further to

cases in which all primes in $p^{2e}-1$ already occur in the Schur multiplier (apart from the possible prime two in symplectic type). Corollary 7.17 identifies the residue left by those two tests, but Theorem 7.28 eliminates it, and indeed every symplectic factor stabilizer, in mod-two degree four. Theorem 7.29 applies the same multiplier-square and Evens-norm argument to eliminate every factor stabilizer of type E_7. Theorem 7.30 similarly eliminates every factor stabilizer of type B_n. Corollary 7.18 reduces the two E_6 families to explicit small fixed fields, and Theorem 7.19 uses a signed square to eliminate every residual untwisted E_6 stabilizer in degree eight. Theorem 7.20 gives every residual twisted stabilizer an integral witness in degree at most thirteen, and degree at most seven when its outer image is odd. Proposition 7.21 retains an optional lower-degree diagnostic based on the automorphism-fixed class $\beta(w)w^3 \in H^9(S;\mathbb{F}_3)$; it is not part of the finite-bound gap. Corollary 7.24 reduces every even-orthogonal family outside untwisted triality to fixed field $\mathbb{F}_3$, and Theorem 7.25 eliminates that residue in degree eight. Theorem 7.26 uses a signed degree-four class to eliminate the remaining untwisted D_4 triality stabilizers in degree eight. Thus every even-orthogonal factor stabilizer is settled. Theorem 7.34 below settles every linear and unitary factor whose Schur multiplier has one-prime support. Theorem 7.35 then settles the first genuinely mixed case, namely multiplier support $\{2,3\}$. Thus the residual type-A problem begins only when the multiplier has at least two distinct prime divisors and one of them is at least five. Theorem 7.36 reduces this to two explicit even-field-image residues, and Theorem 7.37 eliminates the unitary residue altogether. It also sharpens the remaining linear residue to (19ag). Theorem 7.38 closes every bounded-rank part of that residue and gives an explicit hierarchy of further necessary cyclotomic absorption conditions. Proposition 7.39 shows why any fixed finite list of those conditions still leaves genuine nonsplit examples of unbounded rank. Proposition 7.40 then constructs the expected degree-four projective class at an absorbed prime on the full inner-diagonal group. Lemma 7.41 proves that the subsequent field–graph LHS edge differentials vanish once an invariant preimage has been found, and Theorem 7.42 supplies such a preimage throughout the strict valuation range. Proposition 7.43 identifies the equality boundary in the explicit absorption family and proves that no correction within the projective cyclic line is possible. Theorem 7.44 then changes classes: its depth-s multiplier norm gives degree $2r^s(r-1)$ whenever one additional r-scalar layer is present. The $s=1, r=3$ case gives the entire explicit family a degree-twelve mod-three witness. Corollary 7.47 also quantifies every deeper layer; the full-depth boundary requires a different detector. The remaining off-line central-kernel invariant-preimage obstruction is isolated in Conjecture 11.8, but it is no longer needed for that family's exclusion.

Theorem 7.34 (Prime-power-multiplier type-A stabilizers)**.** *Let S be a nonabelian simple group of linear or unitary type,*

$$S = \mathrm{PSL}_n(q) \qquad \text{or} \qquad S = \mathrm{PSU}_n(q),$$

and suppose that its Schur multiplier is a nontrivial ℓ-group for some prime ℓ. Then, for every

$$S \le B \le \mathrm{Aut}(S),$$

either $B \to B/S$ splits or there is a prime r and a class

$$z \in H^d(B;\mathbb{F}_r), \qquad d \le 12,$$

whose restriction to S is nonzero. Except in the residual family $S = \mathrm{PSU}_3(2^m)$, one may take $d \le 8$.

Consequently every nonabelian minimal-normal-subgroup branch with such a simple factor has nonzero integral cohomology in degree at most thirteen.

Proof. The exceptional isomorphisms that replace a type-A presentation by an alternating or symplectic one are already covered by Corollary 7.5 and Theorem 7.28; we use the standard linear or unitary outer automorphisms in the remaining cases. Suppose that $B \to B/S$ is nonsplit. Theorem 7.33 then implies that $\mathrm{Aut}(S) \to \mathrm{Out}(S)$ is nonsplit. In particular the standard diagonal order

$$d_0 = \gcd(n, q-1) \quad \text{or} \quad d_0 = \gcd(n, q+1) \tag{19n}$$

is nontrivial. Since d_0 divides the order of the Schur multiplier, it is an ℓ-power.

Let B^+ be the intersection of B with the standard inner-diagonal– field subgroup. Thus $[B : B^+] \le 2$, with the possible quotient generated by a graph coset. Write $q = p^m$. If the field image of B^+/S is generated by the p^a-power map, put

$$e = \gcd(m, a), \qquad x = p^e, \qquad f = m/e; \tag{19o}$$

use $a = 0, e = m, f = 1$ when the field image is trivial.

First suppose that some prime $r \mid x^2 - 1$ differs from ℓ. The second alternative in Corollary 7.16, or equivalently its proof using Proposition 7.8, gives a class

$$c \in H^4(B^+; \mathbb{Z}_{(r)})$$

whose restriction to S is the order-r class on the basic degree-two line. A long-root rank-one subgroup has Dynkin index one, so this restricted class is nonzero on a cyclic subgroup of order r; its square is therefore nonzero. Reduce c modulo r. If $B = B^+$, this already gives the required restricting class. If $[B : B^+] = 2$, apply the Evens norm with $\mathbb{F}_r$ coefficients [15]:

$$\mathcal{N}^B_{B^+}(\bar{c}) \in H^8(B; \mathbb{F}_r).$$

The standard outer automorphisms preserve the one-dimensional basic line. For $g \in B \setminus B^+$, the restriction formula is therefore

$$\mathrm{res}^B_S \mathcal{N}^B_{B^+}(\bar{c}) = \mathrm{res}^{B^+}_S(\bar{c})\, g^* \,\mathrm{res}^{B^+}_S(\bar{c}),$$

a nonzero scalar multiple of the nonzero square. This closes the case.

We may consequently assume that every prime divisor of $x^2 - 1$ equals ℓ. If x is even, the coprime odd integers $x - 1$ and $x + 1$ cannot both be nontrivial powers of the same prime. Hence $x = 2$ and $\ell = 3$. If x is odd, both $x - 1$ and $x + 1$ are powers of two differing by two; hence $x = 3$ and $\ell = 2$. Thus

$$(x, \ell) = (2, 3) \quad \text{or} \quad (3, 2). \tag{19p}$$

In either case $e = 1$, so $f = m$ and the pure-field subgroup in B^+ is the full field group.

Consider first $S = \mathrm{PSL}_n(q)$. If $(x, \ell) = (2, 3)$, then $3 \mid d_0 \mid q - 1 = 2^f - 1$, so f is even. If $(x, \ell) = (3, 2)$ and f were odd, then $v_2(3^f - 1) = 1$, whence $d_0 = 2$ and

$$\gcd\big((q-1)/d_0, d_0, m\big) = 1.$$

Theorem 7.33 would make the full automorphism extension split, a contradiction. Thus f is again even. In both cases

$$q^2 \equiv 1 \pmod 5, \qquad x^2 \equiv -1 \pmod 5. \tag{19q}$$

The standard simply connected cover is a 5-local equivalence, and Proposition 4.5 supplies an order-five class

$$u \in H^4(S; \mathbb{Z}_{(5)})$$

on the basic line. Its restriction to a cyclic order-five subgroup in a long-root $\mathrm{SL}_2(q)$ is nonzero, so $u^2 \neq 0$.

Put $C = B/S$, choose $P \in \mathrm{Syl}_5(C)$, and let B_P be its inverse image. The diagonal group is an ℓ-group and the graph group has order at most two, so P lies in the cyclic field group. The field generator acts on u as $p^{2a} = x^{2a}$ modulo five. Since f is even and $\gcd(a, f) = 1$, this action is by -1. A generator of P is a power whose exponent is divisible by $f/|P|$, which is even; hence P fixes u. The multiplier is prime to five, so

$$H^j(S; \mathbb{Z}_{(5)}) = 0 \qquad (1 \leq j \leq 3),$$

and $H^5(P; \mathbb{Z}_{(5)}) = 0$. Lemma 7.6 lifts u to $v \in H^4(B_P; \mathbb{Z}_{(5)})$. Every element of C fixes u^2, and restriction–corestriction gives

$$\mathrm{res}^B_S \, \mathrm{cor}^B_{B_P}(v^2) = [C : P]u^2 \neq 0. \tag{19r}$$

This gives degree eight in the linear case.

Now let $S = \mathrm{PSU}_n(q)$. The alternative $(x, \ell) = (3, 2)$ cannot occur. Indeed, if f is odd then the third argument in

$$\gcd\big((q+1)/d_0, d_0, m\big)$$

is odd, whereas d_0 is a power of two. If f is even, then $q + 1 \equiv 2 \pmod 4$, so $d_0 = 2$ and $(q+1)/d_0$ is odd. In both cases the displayed gcd is one, contrary to Theorem 7.33. Therefore

$$q = 2^f, \qquad \ell = 3. \tag{19s}$$

Since $3 \mid d_0 \mid q + 1$, the integer f is odd. Nonsplitting further forces $3 \mid f$: otherwise the lifting-the-exponent formula gives $v_3(q + 1) = 1$, so $d_0 = 3$ and the same splitting gcd is again one. In particular $9 \mid q + 1$.

Suppose first that $n \geq 4$; since $3 \mid d_0 \mid n$, in fact $n \geq 6$. Let $K = \mathrm{SU}(n)$ and put $X = (BK)^\wedge_5$. Its first three polynomial lines have fundamental degrees $2, 3, 4$. On the corresponding homotopy groups the unitary Steinberg map acts by

$$(-q)^2, \qquad (-q)^3, \qquad (-q)^4,$$

respectively [17, 27]. Since f is odd, the first two scalars are not one modulo five and the third is one. The homotopy-equalizer sequence and the 5-adic Hurewicz theorem, exactly as in the proof of Theorem 7.25, therefore give

$$H^j(S; \mathbb{Z}_{(5)}) = 0 \quad (0 < j < 8), \qquad H^8(S; \mathbb{Z}_{(5)}) \neq 0. \tag{19t}$$

Choose the order-five class u arising from the fundamental degree-four line. Diagonal automorphisms act trivially on that line, a graph automorphism acts by $(-1)^4 = 1$, and every field automorphism acts on its order-five subgroup by $2^{4a} = 1$. Thus C fixes u. A Sylow five-subgroup P of C lies in the cyclic field group. Equation (19t), the equality $H^9(P; \mathbb{Z}_{(5)}) = 0$, and Lemma 7.6 lift u to its inverse image B_P. Corestriction to B restricts to $[C : P]u \neq 0$, giving degree eight.

It remains to treat $S = \mathrm{PSU}_3(2^f)$. Let

$$1 \longrightarrow C_3 \longrightarrow \widehat{S} = \mathrm{SU}_3(q) \longrightarrow S \longrightarrow 1$$

be the standard triple cover and let $w \in H^2(S; \mathbb{F}_3)$ classify it. Choose a generator ζ of its center and $\lambda \in C_{q+1}$ with $\lambda^3 = \zeta$, possible because $9 \mid q + 1$. Then

$$\mathrm{diag}(\lambda, \lambda, \lambda^{-2}) \in \mathrm{SU}_3(q)$$

has cube ζI. Hence there is $T \cong C_3$ in S over which the cover is $C_9 \to C_3$, and

$$\operatorname{res}_T^S(w^k) \neq 0 \qquad (k \geq 1). \tag{19u}$$

Let $C_0 \leq C$ be the intersection with the subgroup generated by diagonal and pure-field automorphisms, and let B_0 be its inverse image; then $[B : B_0] \leq 2$. Let F_0 be the pure-field part of C_0 and let A be its inverse image. Thus $A = S \rtimes F_0$ and $[B_0 : A] \leq 3$. The group F_0 has odd order, its standard representatives lift to $\widehat{S}$, and it fixes the central kernel. Consequently w extends to $\widetilde{w} \in H^2(A; \mathbb{F}_3)$. Put

$$z_0 = \begin{cases} \widetilde{w}^3, & A = B_0, \\ \mathcal{N}_A^{B_0}(\widetilde{w}), & [B_0 : A] = 3. \end{cases}$$

In either case $z_0 \in H^6(B_0; \mathbb{F}_3)$ and $\operatorname{res}_S^{B_0}(z_0) = w^3 \neq 0$. If $B = B_0$, take $z = z_0$. Otherwise the second Evens norm

$$z = \mathcal{N}_{B_0}^B(z_0) \in H^{12}(B; \mathbb{F}_3)$$

restricts to a nonzero scalar multiple of w^6, by (19u). This proves the restriction assertion in every case.

Finally apply Theorem 7.1. A nonzero mod-r class in degree d forces nonzero r-primary integral cohomology in degree d or $d+1$, so the common integral ceiling is thirteen. □

Theorem 7.35 (Two-small-prime-multiplier type-A stabilizers)**.** *Let S be a nonabelian simple group of linear or unitary type,*

$$S = \mathrm{PSL}_n(q) \qquad \text{or} \qquad S = \mathrm{PSU}_n(q),$$

and suppose that the prime support of its Schur multiplier is exactly $\{2, 3\}$. Then, for every

$$S \leq B \leq \mathrm{Aut}(S),$$

either $B \to B/S$ splits or there is a prime r and a class

$$z \in H^d(B; \mathbb{F}_r), \qquad d \leq 8,$$

whose restriction to S is nonzero. Consequently every nonabelian minimal-normal-subgroup branch with such a simple factor has nonzero integral cohomology in degree at most nine.

Proof. We first remove the exceptional multipliers. The complete multiplier table [25, Theorem 6.1.4 and Table 6.1.3] shows that the type-A entries for which the exceptional part makes the full multiplier divisible by both two and three are

$$\mathrm{PSL}_2(9) \cong A_6, \qquad \mathrm{PSL}_3(4), \qquad \mathrm{PSU}_4(3), \qquad \mathrm{PSU}_6(2). \tag{19v}$$

The first entry is covered by Corollary 7.5. For the other three, the standard diagonal orders d_0, field exponents m, and relevant splitting gcds from Theorem 7.33 are

S	d_0	m	$\gcd((q \mp 1)/d_0, d_0, m)$
$\mathrm{PSL}_3(4)$	3	2	1
$\mathrm{PSU}_4(3)$	4	1	1
$\mathrm{PSU}_6(2)$	3	1	1

(19w)

Here the minus sign is used in the linear row and the plus sign in the unitary rows. Thus their full automorphism extensions split, and the section restricts to every intermediate outer subgroup.

We may therefore assume that the multiplier is the standard cyclic multiplier. Put

$$d_0 = \gcd(n, q-1) \quad \text{or} \quad d_0 = \gcd(n, q+1),$$

as appropriate. Its prime support is $\{2, 3\}$, so $6 \mid d_0$ and in particular $n \geq 6$.

Suppose that $B \to B/S$ is nonsplit. Let B^+ be the intersection of B with the inner-diagonal–field subgroup. Thus $[B : B^+] \leq 2$, with the possible quotient represented by a graph coset. Write $q = p^m$ and, if the field image of B^+/S is generated by the p^a-power map, put

$$e = \gcd(m, a), \qquad x = p^e, \qquad f = m/e, \tag{19x}$$

using $a = 0, e = m, f = 1$ when the field image is trivial.

If a prime $r \mid x^2 - 1$ lies outside $\{2, 3\}$, Corollary 7.16 gives a class

$$c \in H^4(B^+; \mathbb{Z}_{(r)})$$

whose restriction to S is the nonzero order-r class on the basic degree-two line. Its restriction to a cyclic order-r subgroup in a long-root rank-one group is nonzero, and hence its square is nonzero. If $B = B^+$ we are done after reduction modulo r. Otherwise the Evens norm

$$\mathcal{N}_{B^+}^B(\overline{c}) \in H^8(B; \mathbb{F}_r)$$

restricts to the product of two nonzero scalar multiples of the basic class, and is nonzero. Thus it remains to suppose that

$$\pi(x^2 - 1) \subseteq \{2, 3\}.$$

The elementary 2–3-smooth calculation in the proof of Corollary 7.18 gives

$$x \in \{2, 3, 5, 7, 17\}. \tag{19y}$$

Every entry is prime, so $e = 1$ and $x = p$. Define

$$\rho = \begin{cases} 13, & x = 5, \\ 5, & x \in \{2, 3, 7, 17\}. \end{cases} \qquad \text{Then} \qquad x^2 \equiv -1 \pmod{\rho}. \tag{19z}$$

The prime ρ does not divide the multiplier.

Suppose first that f is even. Then $q^2 \equiv 1 \pmod{\rho}$, so Proposition 4.5, after passage through the standard prime-to-ρ central cover, supplies an order-ρ class

$$u \in H^4(S; \mathbb{Z}_{(\rho)})$$

on the basic line. It has $u^2 \neq 0$ by restriction to a long-root cyclic subgroup of order ρ. Since $\gcd(a, f) = 1$, the integer a is odd and a field generator acts on u as $x^{2a} = -1$ modulo ρ. Diagonal and graph automorphisms fix the basic line, so every element of B/S acts on u by a sign and fixes u^2.

Choose $P \in \mathrm{Syl}_\rho(B/S)$ and let B_P be its inverse image. The prime ρ divides neither the diagonal group nor the graph group, so P lies in the cyclic field group. A generator of P is a power whose exponent is divisible by $f/\rho^{v_\rho(f)}$, an even integer; hence P fixes u. Since S is perfect and its multiplier is prime to ρ,

$$H^j(S; \mathbb{Z}_{(\rho)}) = 0 \qquad (1 \leq j \leq 3),$$

and $H^5(P;\mathbb{Z}_{(\rho)})=0$. Lemma 7.6 lifts u to $v\in H^4(B_P;\mathbb{Z}_{(\rho)})$. Restriction–corestriction gives

$$\operatorname{res}^B_S \operatorname{cor}^B_{B_P}(v^2)=[B/S:P]u^2\neq 0. \tag{19aa}$$

It remains to suppose that f is odd. Now $q^2\equiv -1 \pmod{\rho}$. Let $K=\mathrm{SU}(n)$, and use the ordinary Steinberg map in the linear case and the unitary Steinberg map in the unitary case. On the first three fundamental polynomial lines, of degrees $2,3,4$, their scalars are respectively

$$q^2,q^3,q^4 \qquad \text{or} \qquad (-q)^2,(-q)^3,(-q)^4.$$

Modulo ρ, neither of the first two scalars is one and the last is one. The homotopy-equalizer sequence and the ρ-adic Hurewicz theorem, as in the proof of Theorem 7.25, therefore give

$$H^j(S;\mathbb{Z}_{(\rho)})=0 \quad (0<j<8), \qquad H^8(S;\mathbb{Z}_{(\rho)})\neq 0. \tag{19ab}$$

Here the fundamental degree-four line exists because $n\geq 6$, and the standard cover is a ρ-local equivalence. Choose its order-ρ class u. Diagonal automorphisms act trivially on this line, a graph automorphism acts by $(-1)^4=1$, and every field automorphism acts by $p^{4a}=1$ modulo ρ. Hence all of B/S fixes u.

For $P\in \mathrm{Syl}_\rho(B/S)$, the same outer-group argument puts P in the cyclic field group. Equation (19ab), the equality $H^9(P;\mathbb{Z}_{(\rho)})=0$, and Lemma 7.6 lift u to B_P. Its corestriction to B restricts to $[B/S:P]u\neq 0$. This completes every case. Reduction modulo ρ gives the stated restricting class; Theorem 7.1 and the mod-ρ universal coefficient sequence give the integral ceiling nine. □

Theorem 7.36 (Quadratic-prime escape in residual type A)**.** *Let*

$$S=\mathrm{PSL}_n(q) \qquad \textit{or} \qquad S=\mathrm{PSU}_n(q)$$

be nonabelian simple, and suppose that its Schur multiplier has at least two distinct prime divisors, one of them at least five. Let $S\leq B\leq \mathrm{Aut}(S)$ and suppose that $B\to B/S$ is nonsplit. Intersect B with the inner-diagonal–field subgroup to obtain B^+, write $q=p^m$, and, if the field image of B^+/S is generated by the p^a-power map, put

$$e=\gcd(m,a), \qquad x=p^e, \qquad f=m/e,$$

with $a=0, e=m, f=1$ for trivial field image. Then there is a prime ρ and a class

$$c\in H^8(B;\mathbb{Z}_{(\rho)})$$

whose restriction to S is nonzero, unless all of the following family-specific residual conditions hold:

(i) *for $S=\mathrm{PSL}_n(q)$, one has $4\mid f$, and every odd prime divisor of x^2+1 divides $d_0=\gcd(n,q-1)$;*

(ii) *for $S=\mathrm{PSU}_n(q)$, one has $f\equiv 2 \pmod 4$, and every odd prime divisor of x^2+1 divides $d_0=\gcd(n,q+1)$.*

Consequently, outside these residues, every associated S^r branch has $H^8(G;\mathbb{Z})\neq 0$.

Proof. The exceptional-multiplier audit in the proof of Theorem 7.35 shows that every type-A exception for which the full multiplier has more than one prime has support $\{2,3\}$. Hence the multiplier under the present hypothesis is the standard cyclic multiplier of order d_0. In particular,

d_0 and n are divisible by at least two distinct primes, so $n \geq 6$. Also $[B : B^+] \leq 2$, with the possible quotient represented by a graph coset. Put $b = a/e$ when the field image is nontrivial. Then

$$\gcd(b, f) = 1. \tag{19ac}$$

Choose an odd prime $\rho \mid x^2+1$. Such a prime always exists: if x is even then $x^2+1 > 1$ is odd, while if x is odd then $x^2 + 1 \equiv 2 \pmod 8$ and $(x^2 + 1)/2 > 1$ is odd. Thus

$$x^2 \equiv -1 \pmod \rho. \tag{19ad}$$

Suppose first that f is odd. Then $q^2 \equiv -1 \pmod \rho$, so $\rho \nmid q^2 - 1$ and in particular $\rho \nmid d_0$. The standard cover is a ρ-local equivalence. On the first three fundamental polynomial lines, of degrees $2, 3, 4$, the ordinary and unitary Steinberg scalars are

$$q^2, q^3, q^4 \qquad \text{and} \qquad (-q)^2, (-q)^3, (-q)^4,$$

respectively. Modulo ρ, the first two are not one and the last is one. The homotopy-equalizer and ρ-adic Hurewicz argument used in Theorem 7.35 therefore gives

$$H^j(S; \mathbb{Z}_{(\rho)}) = 0 \quad (0 < j < 8), \qquad H^8(S; \mathbb{Z}_{(\rho)}) \neq 0. \tag{19ae}$$

Let u be the order-ρ class on the fundamental degree-four line. Diagonal automorphisms act trivially, graph automorphisms act by $(-1)^4 = 1$, and a field generator acts by $p^{4a} = x^{4b} = 1$ modulo ρ. Thus B/S fixes u.

A Sylow ρ-subgroup P of B/S lies in the cyclic field group, because ρ divides neither the diagonal group nor the graph group. Let B_P be its inverse image. Equation (19ae), the equality $H^9(P; \mathbb{Z}_{(\rho)}) = 0$, and Lemma 7.6 lift u to B_P. Corestriction to B restricts to $[B/S : P]u \neq 0$. This proves the theorem whenever f is odd.

Now suppose that f is even and that some odd prime $\rho \mid x^2+1$ does not divide d_0. Then $q^2 \equiv 1 \pmod \rho$, so Proposition 4.5 gives an order-ρ basic-line class

$$u \in H^4(S; \mathbb{Z}_{(\rho)}).$$

It has $u^2 \neq 0$ by restriction to a long-root cyclic subgroup. By (19ac), b is odd, so the field generator acts on u as $x^{2b} = -1$ modulo ρ. Diagonal and graph automorphisms fix the basic line; hence all of B/S fixes u^2.

Again $P \in \mathrm{Syl}_\rho(B/S)$ lies in the cyclic field group. A generator of P is a power of the field generator whose exponent is divisible by $f/\rho^{v_\rho(f)}$, which remains even. Thus P fixes u. The vanishing of $H^j(S; \mathbb{Z}_{(\rho)})$ for $1 \leq j \leq 3$ and of $H^5(P; \mathbb{Z}_{(\rho)})$ lets Lemma 7.6 lift u to $v \in H^4(B_P; \mathbb{Z}_{(\rho)})$. Therefore

$$\mathrm{res}^B_S \, \mathrm{cor}^B_{B_P}(v^2) = [B/S : P]u^2 \neq 0. \tag{19af}$$

The only remaining possibility is that f is even and every odd prime divisor of $x^2 + 1$ divides d_0. In the linear case, if $f \equiv 2 \pmod 4$, then (19ad) gives $q \equiv -1 \pmod \rho$ for every such ρ, contradicting $\rho \mid d_0 \mid q - 1$. Hence $4 \mid f$. In the unitary case, if $4 \mid f$, then $q \equiv 1 \pmod \rho$, contradicting $\rho \mid d_0 \mid q + 1$; hence $f \equiv 2 \pmod 4$. These are precisely the two stated residues. The factor-stabilizer conclusion follows from Theorem 7.1. □

Theorem 7.37 (Unitary completion and cubic refinement in residual type A)**.** *Retain the hypotheses and notation of Theorem 7.36; in particular the full multiplier is the standard cyclic multiplier of order d_0.*

(i) *If $S = \mathrm{PSU}_n(q)$, then there are a prime ρ, an integer $d \le 20$, and a class*

$$c \in H^d(B; \mathbb{Z}_{(\rho)})$$

whose restriction to S is nonzero. Consequently every unitary S^r branch in this scope has nonzero integral cohomology in degree at most twenty.

(ii) *If $S = \mathrm{PSL}_n(q)$, then a class with coefficients $\mathbb{Z}_{(\rho)}$ or $\mathbb{F}_\rho$, in degree at most twelve, restricts nontrivially from B to S, unless*

$$12 \mid f, \qquad \ell \mid x^2 - 1 \Longrightarrow \ell \mid d_0, \qquad \substack{\ell \mid x^2+1 \\ \ell \text{ odd}} \Longrightarrow \ell \mid d_0. \tag{19ag}$$

Outside this residual linear case, the associated S^r branch has nonzero integral cohomology in degree at most thirteen.

Proof. We first add the prime-support test omitted from the statement of Theorem 7.36. Suppose that a prime $r \mid x^2 - 1$ does not divide d_0. Corollary 7.16, applied to B^+, gives

$$c_+ \in H^4(B^+; \mathbb{Z}_{(r)})$$

whose restriction is the nonzero basic-line class u on S. Its square is nonzero by restriction to a root cyclic subgroup. If $B = B^+$, this settles the case. If $[B : B^+] = 2$, reduction modulo r followed by the Evens norm gives a degree-eight class on B restricting to a nonzero scalar multiple of u^2. When r is odd one can remain integral: the field component fixes u because $x^2 = 1$ modulo r, graph and diagonal automorphisms fix the basic line, and

$$\mathrm{res}^B_S \, \mathrm{cor}^B_{B^+}(c_+) = 2u \neq 0. \tag{19ah}$$

Thus an unresolved case must satisfy the middle condition in (19ag).

There is a second uniform escape whenever $3 \nmid f$. By Zsigmondy's theorem, $x^3 - 1$ has a primitive prime divisor ρ; there is no exception at exponent three [45]. Hence

$$\mathrm{ord}_\rho(x) = 3.$$

The prime ρ is odd. Since $3 \nmid f$, the element $q = x^f$ also has order three modulo ρ, so ρ divides neither $q - 1$ nor $q + 1$ and in particular does not divide d_0.

In the linear case, the first fixed fundamental polynomial line has degree three. The homotopy-equalizer and ρ-adic Hurewicz argument used in Theorem 7.36 gives an order-ρ class

$$u \in H^6(S; \mathbb{Z}_{(\rho)}), \qquad H^j(S; \mathbb{Z}_{(\rho)}) = 0 \quad (0 < j < 6). \tag{19ai}$$

The degree-three line exists because $n \ge 6$. A field generator fixes u, since it acts by $x^{3b} = 1$; diagonal automorphisms also fix it. Let $P \in \mathrm{Syl}_\rho(B^+/S)$ and let B_P be its inverse image. The group P lies in the cyclic field group. Lemma 7.6, using $H^7(P; \mathbb{Z}_{(\rho)}) = 0$, lifts u to B_P, and corestriction gives a class $c_+ \in H^6(B^+; \mathbb{Z}_{(\rho)})$ restricting nontrivially to S. If $B \neq B^+$, a graph coset sends u to $-u$. The class u^2 is nonzero: on the cyclic subgroup of order ρ in a Coxeter torus, the degree-three line restricts as a nonzero multiple of the cube of the periodic generator. Therefore

$$\mathrm{res}^B_S \, \mathrm{cor}^B_{B^+}(c_+^2) = u^2 + (-u)^2 = 2u^2 \neq 0, \tag{19aj}$$

which gives degree twelve.

In the unitary case, $-q$ has exact order six modulo ρ. Thus the first fixed fundamental polynomial line has degree six, which again exists because $n \geq 6$. The same argument gives

$$H^j(S; \mathbb{Z}_{(\rho)}) = 0 \quad (0 < j < 12), \qquad H^{12}(S; \mathbb{Z}_{(\rho)}) \neq 0. \tag{19ak}$$

The order-ρ class on this line is fixed by field, diagonal, and graph automorphisms: their scalars are respectively $x^{6b} = 1$, 1, and $(-1)^6 = 1$. The first-nonzero-row argument over a Sylow ρ-subgroup of B/S, followed by corestriction, gives a restricting degree-twelve class on B. We have proved that every still-unresolved linear case has $3 \mid f$; together with $4 \mid f$ from Theorem 7.36, this gives $12 \mid f$ and proves part (ii).

It remains to close the unitary residue. There $f \equiv 2 \pmod 4$. If $x^2 - 1$ had an odd prime divisor r, then $q \equiv 1 \pmod r$, so $r \nmid q + 1$ and hence $r \nmid d_0$; the first paragraph would apply. Thus $x^2 - 1$ is a power of two. The two neighboring even integers $x - 1$ and $x + 1$ are then powers of two, which forces

$$x = 3. \tag{19al}$$

In particular $p = 3$ and $e = 1$. The quadratic residue already says $5 \mid d_0$, so the mixed-prime hypothesis gives $n \geq 10$.

Use the prime $\rho = 11$, for which 3 has exact order five. Suppose first that $5 \nmid f$. Then $q = 3^f$ has order five modulo eleven and $-q$ has order ten. The standard cover is an eleven-local equivalence, and the first fixed fundamental line is the degree-ten line. Hence

$$H^j(S; \mathbb{Z}_{(11)}) = 0 \quad (0 < j < 20), \qquad H^{20}(S; \mathbb{Z}_{(11)}) \neq 0. \tag{19am}$$

The line exists because $n \geq 10$. Its order-eleven class is fixed by all of B/S: field, diagonal, and graph scalars are $3^{10a} = 1$, 1, and $(-1)^{10} = 1$. Since eleven divides neither the diagonal nor graph group, the first-nonzero-row argument over a Sylow eleven-subgroup of B/S and corestriction produce the required class on B.

Finally suppose that $5 \mid f$. Now $q \equiv 1 \pmod{11}$, so the basic line supplies an order-eleven class

$$u \in H^4(S; \mathbb{Z}_{(11)}).$$

A field generator acts on u by 3^{2a}, which has exact order five because $\gcd(a, f) = 1$; diagonal and graph automorphisms fix u. Consequently u^5 is fixed by B/S and is nonzero on a root cyclic subgroup of order eleven. Let $P \in \mathrm{Syl}_{11}(B/S)$. It lies in the field group. If it is nontrivial, a generator is a power with exponent divisible by $f/11^{v_{11}(f)}$, hence by five, so P fixes u. Lemma 7.6 lifts u to $v \in H^4(B_P; \mathbb{Z}_{(11)})$. Restriction–corestriction now gives

$$\mathrm{res}^B_S \, \mathrm{cor}^B_{B_P}(v^5) = [B/S : P] u^5 \neq 0. \tag{19an}$$

This is the required degree-twenty class and completes part (i). The assertions for S^r follow from Theorem 7.1; in the one mod-r case, the universal coefficient sequence raises the integral ceiling by at most one. □

Theorem 7.38 (Cyclotomic escape and the absorption hierarchy)**.** *Retain the linear hypotheses and notation of Theorem 7.37; thus*

$$S = \mathrm{PSL}_n(q), \qquad q = x^f, \qquad d_0 = \gcd(n, q - 1),$$

and the standard multiplier has order d_0. Let $s \leq n$ be an odd prime and let ρ be a primitive prime divisor of $x^s - 1$. If $\rho \nmid d_0$, then there is a class

$$c \in H^d(B; \mathbb{Z}_{(\rho)}), \qquad d \leq 4s,$$

whose restriction to S is nonzero. Consequently:

(i) *if B has no restricting integral class in degree at most D, then every primitive prime divisor of $x^s - 1$ lies in d_0 for every odd prime*

$$s \le \min\{n, \lfloor D/4 \rfloor\}; \tag{19ao}$$

(ii) *some restricting integral class always occurs in degree at most $4n$;*

(iii) *for every fixed R, all residual linear factor stabilizers with $n \le R$ form a completed subfamily with common integral ceiling $4R$.*

These are rank-dependent conclusions; they do not give a universal ceiling as $n \to \infty$.

Proof. Zsigmondy's theorem has no exception at an odd prime exponent, so ρ exists and

$$\mathrm{ord}_\rho(x) = s, \qquad \rho \equiv 1 \pmod{s}. \tag{19ap}$$

In particular ρ is odd and $\rho \neq s$. Put $b = a/e$, as in (19ac), so $\gcd(b, f) = 1$.

Suppose first that $s \nmid f$. Then $q = x^f$ has order s modulo ρ. Because $\rho \nmid d_0$, the standard cover is a ρ-local equivalence. The first fixed fundamental polynomial line is the degree-s line, which exists because $s \le n$. The homotopy-equalizer and ρ-adic Hurewicz argument used in Theorem 7.37 gives

$$H^j(S; \mathbb{Z}_{(\rho)}) = 0 \quad (0 < j < 2s), \qquad 0 \neq u \in H^{2s}(S; \mathbb{Z}_{(\rho)}). \tag{19aq}$$

The field generator fixes u, since its scalar is $x^{bs} = 1$; diagonal automorphisms also fix it. The class u^2 is nonzero: on the order-ρ subgroup of an s-dimensional Coxeter torus, the degree-s line restricts as a nonzero multiple of the s-th power of the periodic generator.

Let B^+ be the inner-diagonal–field part and let $P \in \mathrm{Syl}_\rho(B^+/S)$. Since $\rho \nmid d_0$ and ρ is odd, P injects into the cyclic field group. Lemma 7.6, using (19aq) and $H^{2s+1}(P; \mathbb{Z}_{(\rho)}) = 0$, lifts u to the inverse image of P; corestriction gives a class c_+ on B^+ restricting nontrivially to u. If $B = B^+$, this is enough. Otherwise a graph coset sends u to $-u$, because s is odd, and

$$\mathrm{res}^B_S \, \mathrm{cor}^B_{B^+}(c_+^2) \quad \text{is a nonzero } \rho\text{-local multiple of } 2u^2. \tag{19ar}$$

This gives degree at most $4s$.

Now suppose that $s \mid f$. Then $q \equiv 1 \pmod{\rho}$, so the basic line gives an order-ρ class $u \in H^4(S; \mathbb{Z}_{(\rho)})$. A field generator acts on it by x^{2b}, which has exact order s because s is odd and $\gcd(b, s) = 1$; diagonal and graph automorphisms fix it. Thus u^s is B/S-invariant and is nonzero on a root cyclic subgroup of order ρ. For $P \in \mathrm{Syl}_\rho(B/S)$, the field exponent of a generator is divisible by $f/\rho^{v_\rho(f)}$, which is still divisible by s because $\rho \neq s$. Hence P fixes u. The first-nonzero-row lemma lifts u to $v \in H^4(B_P; \mathbb{Z}_{(\rho)})$, and

$$\mathrm{res}^B_S \, \mathrm{cor}^B_{B_P}(v^s) = [B/S : P] u^s \neq 0. \tag{19as}$$

This has degree $4s$ and proves the first assertion in every case.

Part (i) follows immediately. For part (ii), choose a primitive divisor ρ_3 of $x^3 - 1$. If $\rho_3 \nmid d_0$, take $s = 3$. Otherwise, Bertrand's postulate supplies an odd prime s with $n/2 < s < n$ (recall that the residual hypotheses give $n \ge 6$), and choose a primitive divisor ρ_s of $x^s - 1$. The primes ρ_3 and ρ_s are distinct, by their respective orders of x. If both divided d_0, then

$$n \ge d_0 \ge \rho_3 \rho_s \ge 7(s + 1) > n, \tag{19at}$$

a contradiction. Hence $\rho_s \nmid d_0$, and the first assertion gives degree at most $4s < 4n$. Part (iii) is immediate. □

Proposition 7.39 (A genuine cyclotomic-absorption family)**.** *Let $\mathcal{T}$ be any finite set of odd primes. For every $s \in \mathcal{T}$, choose a primitive prime divisor ρ_s of $2^s - 1$, choose f divisible by*

$$12 \prod_{s \in \mathcal{T}} s,$$

and put

$$q = 2^f, \qquad n = 15 \prod_{s \in \mathcal{T}} \rho_s, \qquad S = \mathrm{PSL}_n(q). \tag{19au}$$

Then

(i) *the standard multiplier has order $d_0 = \gcd(n, q-1) = n$, with at least the two prime divisors 3 and 5;*

(ii) *the extension $\mathrm{Aut}(S) \twoheadrightarrow \mathrm{Out}(S)$ is nonsplit;*

(iii) *with $x = 2$ and full field-image order f, the residue* (19ag) *holds and every selected ρ_s is absorbed by d_0.*

Thus no argument that fixes in advance finitely many cyclotomic orders and requires a primitive divisor outside the multiplier can finish the residual linear row. This family is not a counterexample to the main conjecture and does not prove that its first restricting degree grows. In fact Corollary 7.46 constructs an unrelated uniform degree-twelve mod-three witness. Proposition 7.43 shows that the same family also lies exactly on the valuation boundary where the simply connected comparison ceases to detect an invariant element of the projective basic line.

Proof. The orders $\mathrm{ord}_{\rho_s}(2) = s$ show that the selected primes are distinct and are different from 3 and 5. Every factor of n divides $q-1$: the orders of 2 modulo 3, 5, and ρ_s divide 2, 4, and s, respectively, and all of these divide f. Hence $d_0 = n$. The parameters are far outside the exceptional multiplier list, so this is the standard cyclic multiplier.

Since $12 \mid f$, the lifting-the-exponent formula gives

$$v_3(2^f - 1) = 1 + v_3(f/2) \geq 2.$$

The integer n is divisible by 3 exactly once, so 3 divides each of $(q-1)/d_0$, d_0, and f. The splitting criterion in Theorem 7.33 therefore makes the full automorphism extension nonsplit. Finally, $12 \mid f$, the primes of 2^2-1 and 2^2+1 are 3 and 5, and $s \mid f$ implies $\rho_s \mid q-1$. This proves every assertion. □

Proposition 7.40 (Projective basic line on the inner-diagonal group)**.** *Let*

$$S = \mathrm{PSL}_n(q), \qquad d_0 = \gcd(n, q-1), \qquad H = \mathrm{Inndiag}(S),$$

under the standard-multiplier hypotheses, and let r be an odd prime dividing d_0. Define the projective basic level

$$L(n, d_0) = \min \left\{ L \geq 1 : L \frac{n(n-1)}{d_0^2} \in 2\mathbb{Z} \right\}. \tag{19av}$$

If

$$v_r(q^2 - 1) > v_r(L(n, d_0)) = \max\{0, 2v_r(d_0) - v_r(n)\}, \tag{19aw}$$

then there is a class

$$z_H \in H^4(H; \mathbb{Z}_{(r)})$$

whose restriction to S is nonzero. Consequently its restriction to every $S \leq A \leq H$ is also nonzero on S. On the order-r subgroup of the projective basic line, a p^a-field automorphism acts by p^{2a} and a graph automorphism acts trivially.

Proof. Let $\mathbf{G} = \mathrm{SL}_n/\boldsymbol{\mu}_{d_0}$ over $\overline{\mathbb{F}}_p$ and let

$$K = \mathrm{SU}(n)/C_{d_0}$$

be the corresponding compact form. The fixed-point exact sequence for $\boldsymbol{\mu}_{d_0}$ identifies $\mathbf{G}^F$ with H, and the image of $\mathrm{SL}_n(q)$ in H is S. At $r \neq p$, the Friedlander–Mislin comparison identifies the completed classifying space of H with the homotopy fixed points

$$BH_r^\wedge \simeq (BK_r^\wedge)^{h\psi^q}[16,\ 17,\ 27]. \tag{19ax}$$

Let z_K generate $H^4(BK;\mathbb{Z})$ modulo its irrelevant sign. The pullback along $B\mathrm{SU}(n) \to BK$ is

$$L(n, d_0)c_2. \tag{19ay}$$

Indeed, the image is the basic-level lattice for $\mathrm{SU}(n)/C_{d_0}$, and (19av) is its type-A formula [42, Proposition 3.6.2]. The odd-primary valuation in (19aw) follows because $r \mid d_0 \mid n$ but $r \nmid n-1$.

Pull z_K back by evaluation from the homotopy fixed-point space in (19ax), and call the resulting finite-group class z_H. Naturality for the central isogeny compares its pullback to $\mathrm{SL}_n(q)$ with $L(n, d_0)u$, where the simply connected basic line contains

$$\langle u \rangle \cong \mathbb{Z}_r/(q^2-1)\mathbb{Z}_r \subseteq H^4(\mathrm{SL}_n(q);\mathbb{Z}_r). \tag{19az}$$

Condition (19aw) says exactly that $L(n, d_0)u \neq 0$. Since $\mathrm{SL}_n(q) \to H$ factors through S, the restriction of z_H to S is nonzero. Finite r-primary cohomology is unchanged on replacing $\mathbb{Z}_r$ by $\mathbb{Z}_{(r)}$.

Finally the comparison is natural for the standard outer maps. An unstable p^a-Adams operation multiplies the basic degree-four line by p^{2a}, while the graph involution preserves its invariant bilinear form. This gives the last assertion. □

Lemma 7.41 (The graph-sign edge lemma)**.** *Let $S = \mathrm{PSL}_n(q)$, with $n \geq 3$, have its standard cyclic multiplier, and put*

$$H = \mathrm{Inndiag}(S) = \mathrm{PGL}_n(q).$$

Let r be odd. Suppose that $H \lhd E \leq \mathrm{Aut}(S)$, that E contains a standard graph automorphism γ, and that E/H is generated by field automorphisms and γ. If

$$0 \neq w \in H^4(H;\mathbb{Z}_{(r)})$$

restricts nontrivially to S and is fixed by E/H, then some class in $H^4(E;\mathbb{Z}_{(r)})$ restricts nontrivially to S.

Proof. Put $d_0 = \gcd(n, q-1)$. We first record the two lower rows which could obstruct the edge. Under the standard-multiplier hypotheses, $\mathrm{SL}_n(q)$ is superperfect. The determinant splitting

$$\mathrm{GL}_n(q) = \mathrm{SL}_n(q) \rtimes \mathbb{F}_q^\times$$

therefore gives $H_2(\mathrm{GL}_n(q);\mathbb{Z}) = 0$. Apply the homology five-term sequence to

$$1 \longrightarrow \mathbb{F}_q^\times \longrightarrow \mathrm{GL}_n(q) \longrightarrow H \longrightarrow 1.$$

The map from the scalar kernel to the abelianization of $\mathrm{GL}_n(q)$ is $\lambda \mapsto \lambda^n$. Hence

$$H_1(H;\mathbb{Z}) \cong C_{d_0}, \qquad H_2(H;\mathbb{Z}) \cong C_{d_0}. \tag{19ba}$$

By finite-group duality, the r-primary parts of $H^2(H;\mathbb{Z}_{(r)})$ and $H^3(H;\mathbb{Z}_{(r)})$ are the duals of the two groups in (19ba). Inverse transpose inverts both the determinant quotient and the scalar kernel. Thus γ acts as -1 on both of these cohomology groups, whereas it fixes w.

Let $P \in \mathrm{Syl}_r(E/H)$ and let E_P be its inverse image. The group P is cyclic because the odd part of E/H lies in the cyclic field group. Since the field and graph images commute modulo H, conjugation by γ normalizes E_P and induces the identity on P. In the LHS spectral sequence

$$E_2^{i,j} = H^i(P; H^j(H;\mathbb{Z}_{(r)})) \Longrightarrow H^{i+j}(E_P;\mathbb{Z}_{(r)}), \tag{19bb}$$

the only possible differentials from $w \in E_2^{0,4}$ before the bottom row are

$$d_2(w) \in E_2^{2,3}, \qquad d_3(w) \in E_3^{3,2}.$$

Naturality under γ fixes each source and negates each target. Consequently both differentials are killed by two, and hence vanish because r is odd. The d_4 target is zero since $H^1(H;\mathbb{Z}_{(r)}) = 0$, and the d_5 target is zero since $H^5(P;\mathbb{Z}_{(r)}) = 0$ for cyclic P. Thus w lifts to E_P.

Corestrict that lift to E. The Mackey formula gives

$$\mathrm{res}^E_H \,\mathrm{cor}^E_{E_P}(\widetilde{w}) = [E/H : P]w, \tag{19bc}$$

because w is E/H-fixed. The coefficient is prime to r, so the right side still restricts nontrivially to S. □

Theorem 7.42 (The sharp valuation test for the projective line)**.** *Retain the setting of Proposition 7.40. Let $S \le B \le \mathrm{Aut}(S)$, and suppose that the field image of B/S is generated by the p^a-power map. Put*

$$t = v_r(q^2-1), \qquad \ell = v_r(L(n,d_0)) = \max\{0, 2v_r(d_0) - v_r(n)\}, \qquad v = v_r(p^{2a}-1), \tag{19bd}$$

with $v = \infty$ for a trivial field image. Assume $t > \ell$. If

$$v > \ell, \tag{19be}$$

then a class in $H^4(B;\mathbb{Z}_{(r)})$ restricts nontrivially to S.

The strict inequality in (19be) *is sharp for detection after pullback to the simply connected basic line. More precisely, the cyclic projective line is*

$$\mathcal{L}_H \cong \mathbb{Z}_r/(q^2-1)\mathbb{Z}_r,$$

field acts on it by p^{2a}, and its pullback to the simply connected basic line is multiplication by $L(n,d_0)$. Therefore

$$\mathrm{pullback}_{\mathrm{SL}_n(q)}(\mathcal{L}_H^{\langle p^a\rangle}) \neq 0 \quad \Longleftrightarrow \quad v > \ell. \tag{19bf}$$

Proof. On the r-primary cyclic line, choose a generator z. Its field-invariant subgroup is generated by

$$r^{\max\{0,t-v\}}z.$$

Restriction is multiplication by a number of valuation ℓ. The displayed invariant has nonzero restriction exactly when

$$\ell + \max\{0, t-v\} < t,$$

which, under $t > \ell$, is equivalent to $v > \ell$. This proves (19bf).

Under (19be), choose an invariant element w of the line with nonzero restriction. Adjoin to BH a standard graph automorphism and call the resulting group E. The field condition and graph invariance make w fixed by E/H. Lemma 7.41 produces a degree-four class on E restricting nontrivially to S; restriction from E to B proves the assertion. □

Proposition 7.43 (The sharp projective-line boundary in the absorption family)**.** *In the family* (19au)*, put* $k = v_3(f)$*. At* $r = 3$ *one has*

$$v_3(L(n,n)) = 1, \qquad v_3(q^2-1) = k+1, \qquad v_3(2^2-1) = 1. \tag{19bg}$$

For the full field generator, every invariant element of the projective basic line pulls back trivially to $\mathrm{SL}_n(q)$*. In particular the congruence* $2^2 \equiv 1 \pmod 3$ *alone is not enough for the simply connected comparison in Proposition 7.40 to certify an invariant restricting class across the field group.*

This does not make the groups in (19au) *counterexamples to the main conjecture. In fact the invariant order-three element also restricts trivially to* S*. More precisely, write* $\mathcal{L}_H$ *for the three-primary projective line, choose a generator* u*, and put*

$$K = \ker\left(H^4(H;\mathbb{Z}_{(3)}) \longrightarrow H^4(S;\mathbb{Z}_{(3)})\right), \qquad K_{\mathcal{L}} = K \cap \mathcal{L}_H.$$

Then

$$\mathcal{L}_H \cong C_{3^{k+1}}, \qquad K_{\mathcal{L}} = \langle 3^k u \rangle \cong C_3. \tag{19bh}$$

If c *is the full field generator and* $z = 3^{k-1}u$*, then* $x = \mathrm{res}^H_S(z)$ *is a nonzero order-three class fixed by* c*, while*

$$c^* z - z = 3^k u \tag{19bi}$$

generates $H^1(C_f; K_{\mathcal{L}}) \cong C_3$*. Thus no correction inside the projective cyclic line can produce an invariant preimage of* x*. A correction from the full kernel* K *exists exactly when the image of the class in* (19bi) *vanishes in* $H^1(C_f; K)$*.*

Proof. Here n is odd, $v_3(n) = 1$, and the type-A basic-level formula gives $L(n,n) = n$. Since $12 \mid f$, the lifting-the-exponent formula gives

$$v_3(q^2-1) = v_3(2^f-1) = 1 + v_3(f) = k+1.$$

The full field generator acts by four. Thus (19bg) holds, and Theorem 7.42 gives zero on the left side of (19bf) because the two valuations in question are equal.

We must also pass from the simply connected cover to S. For the central extension

$$1 \longrightarrow C_n \longrightarrow \mathrm{SL}_n(q) \longrightarrow S \longrightarrow 1,$$

the integral LHS spectral sequence shows that inflation

$$H^4(S;\mathbb{Z}_{(3)}) \longrightarrow H^4(\mathrm{SL}_n(q);\mathbb{Z}_{(3)}) \tag{19bj}$$

is injective. Indeed the only possible incoming sources at $E_2^{4,0}$ are the d_2 source with fiber degree one, the d_3 source $H^1(S; H^2(C_n;\mathbb{Z}_{(3)}))$, and the d_4 source with fiber degree three. The first is zero because $H^1(C_n;\mathbb{Z}_{(3)}) = 0$, the second because S is perfect, and the third because $H^3(C_n;\mathbb{Z}_{(3)}) = 0$ for cyclic C_n. Hence the invariant subgroup already shown to vanish after pullback also vanishes on S.

The kernel of multiplication by $L(n,n) = n$ on $\mathcal{L}_H \cong C_{3^{k+1}}$ has order three. By (19bj) it is exactly $K_{\mathcal{L}}$, proving (19bh). Finally $c^*u = 4u$, so for $z = 3^{k-1}u$ one obtains (19bi). The field action on $K_{\mathcal{L}}$ is trivial, $3 \mid f$, and therefore this cocycle generates $H^1(C_f; K_{\mathcal{L}}) = \mathrm{Hom}(C_f, C_3) \cong C_3$. □

Theorem 7.44 (The multiplier-depth norm)**.** *Let*

$$S = \mathrm{PSL}_n(q), \qquad d_0 = \gcd(n, q-1),$$

under the standard-multiplier hypotheses. Let r be an odd prime, put $s = v_r(d_0)$, and suppose that

$$s \ge 1, \qquad r^{s+1} \mid q-1. \tag{19bk}$$

Then, for every $S \le B \le \mathrm{Aut}(S)$, there is a class

$$y \in H^{2r^s(r-1)}(B; \mathbb{F}_r)$$

whose restriction to S is nonzero. In particular, when $r = 3$ and $s = 1$ the restricting degree is twelve.

Proof. Let

$$1 \longrightarrow C_r \longrightarrow \widehat{S}_r \longrightarrow S \longrightarrow 1$$

be the order-r quotient of the standard universal central extension, and let $w \in H^2(S; \mathbb{F}_r)$ classify it. We first prove that every power of w is nonzero. Condition (19bk) forces $v_r(n) = s$, so write $n = r^s m$ with $r \nmid m$. Choose $\lambda \in \mathbb{F}_q^\times$ of order r^{s+1} and put

$$\zeta = \lambda^r, \qquad \eta = \lambda^{r^s}.$$

Thus ζ generates the r-part of the scalar center and η has order r. Choose $a_1, \dots, a_n \in \mathbb{Z}/r$, not all equal, with $\sum_i a_i \equiv -m \pmod r$. Then

$$h = \mathrm{diag}(\lambda\eta^{a_1}, \dots, \lambda\eta^{a_n}) \in \mathrm{SL}_n(q), \qquad h^r = \zeta I. \tag{19bl}$$

Indeed $\det(h) = \eta^{m+\sum_i a_i} = 1$. The nonconstant choice makes h noncentral. The image of h in S generates a subgroup $T \cong C_r$ over which the displayed cover restricts to $C_{r^2} \to C_r$. If $u \in H^2(C_r; \mathbb{F}_r)$ is the polynomial generator, then

$$\mathrm{res}^S_T(w) = u, \qquad \mathrm{res}^S_T(w^j) = u^j \ne 0 \quad (j \ge 1). \tag{19bm}$$

Put $C = B/S$, choose $P \in \mathrm{Syl}_r(C)$, and let B_P be its inverse image. An odd-order subgroup has no graph projection. Let F denote the standard pure-field subgroup of $\mathrm{Out}(S)$. Since the r-part of the diagonal group has order r^s, the subgroup

$$F_0 = P \cap F$$

has index

$$d = [P : F_0] \le r^s \tag{19bn}$$

in P. The inverse image $A = B_P \cap (S \rtimes F)$ is $S \rtimes F_0$. The r-group F_0 acts trivially on the kernel C_r, since $\mathrm{Aut}(C_r)$ has order $r-1$. Consequently the cover extends over A and supplies $\widetilde{w} \in H^2(A; \mathbb{F}_r)$ restricting to w.

Apply the Evens norm, with the identity understood when $d = 1$:

$$z = \mathcal{N}_A^{B_P}(\widetilde{w}) \in H^{2d}(B_P; \mathbb{F}_r).$$

The r-group P acts trivially on the one-dimensional space $\langle w \rangle$, so the Evens restriction formula gives

$$\mathrm{res}^{B_P}_S(z) = w^d. \tag{19bo}$$

Every automorphism of S acts on $\langle w \rangle$ by an element of $\mathbb{F}_r^\times$. Since d divides r^s, the class

$$Z = z^{(r^s/d)(r-1)}$$

has degree $2r^s(r-1)$ and restricts to the nonzero, C-fixed class $w^{r^s(r-1)}$. Corestriction now gives

$$\mathrm{res}^B_S \, \mathrm{cor}^B_{B_P}(Z) = [C : P] w^{r^s(r-1)} \ne 0, \tag{19bp}$$

because $[C : P]$ is prime to r. This is the required class. □

Why the extra scalar layer matters. The hypothesis $r^{s+1} \mid q-1$ is what makes the polynomial detector in (19bm) available. At the opposite boundary

$$v_r(q-1) = v_r(d_0) = s,$$

put $R = r^s$ and choose a scalar ξ of order R. Repeated clock and shift matrices of size R give elements $X, Y \in \mathrm{SL}_n(q)$. More explicitly, take

$$X_0 = \mathrm{diag}(1, \xi, \dots, \xi^{R-1}), \qquad Y_0 e_i = e_{i+1}$$

with the subscripts read cyclically, and repeat both blocks n/R times. Since R is odd, both determinants are one, and

$$X^R = Y^R = 1, \qquad [X, Y] = \xi^{\pm 1} I.$$

Their projective images generate C_R^2. On this subgroup the order-r multiplier class is the alternating class $ab \in \Lambda^2 H^1(C_R^2; \mathbb{F}_r)$: the commutator is nontrivial in the order-r quotient, while the chosen lifts have unchanged order. Hence $(ab)^2 = 0$. This does not prove that w^2 vanishes on S; it proves that the cyclic nonnilpotence argument above has no analogue on this natural boundary detector. A uniform proof at full multiplier depth therefore needs a different class or operation, not merely the same norm calculation with the scalar-layer hypothesis deleted.

Theorem 7.45 (The finite projective torsion class)**.** *Let*

$$S = \mathrm{PSL}_n(q), \qquad H = \mathrm{Inndiag}(S) = \mathrm{PGL}_n(q)$$

under the standard cyclic-multiplier hypotheses, and let r be an odd prime dividing $d_0 = \gcd(n, q-1)$. There is a non-toral clock–shift subgroup

$$E_r \cong C_r^2 \le S$$

and a class

$$\theta_r \in H^{2(r+1)}(H; \mathbb{Z}_{(r)})$$

such that, for suitable polynomial generators $y_1, y_2 \in H^2(E_r; \mathbb{F}_r)$,

$$\overline{\mathrm{res}^H_{E_r}(\theta_r)} = y_1 y_2^r - y_1^r y_2 \neq 0. \tag{NT}$$

Consequently

$$\Theta_r = \theta_r^{r-1} \in H^{2(r^2-1)}(H; \mathbb{Z}_{(r)})$$

is nonzero, restricts modulo r to $(y_1 y_2^r - y_1^r y_2)^{r-1}$, and is fixed by every automorphism of S.

Let $S \le B \le \mathrm{Aut}(S)$, and let f be the order of its field image. If $r \nmid f$, then some class in

$$H^{2(r^2-1)}(B; \mathbb{Z}_{(r)})$$

restricts nontrivially to S.

Proof. Equation (19ba) gives

$$H_2(H; \mathbb{Z})_{(r)} \cong C_{r^s}, \qquad s = v_r(d_0).$$

Choose a generator $x \in H^3(H; \mathbb{Z}_{(r)}) \cong \mathrm{Hom}(H_2(H; \mathbb{Z}), \mathbb{Q}/\mathbb{Z})_{(r)}$, and write $\bar{x}$ for its mod-r reduction. If

$$\mathcal{B}_r : H^j(-; \mathbb{F}_r) \longrightarrow H^{j+1}(-; \mathbb{Z}_{(r)})$$

is the integral Bockstein and $\mathcal{P}^1$ is the first reduced power, put

$$\theta_r = \mathcal{B}_r\mathcal{P}^1(\bar{x}). \tag{NT$'$}$$

This is the finite-group version of the projective torsion operation studied for $BPGL_n$ by Gu [21, Proposition 2.5 and Corollary 9.6].

For completeness, its detection is elementary here. Choose a primitive rth root $\omega \in \mathbb{F}_q^\times$. On an r-dimensional block take

$$X_0 = \operatorname{diag}(1, \omega, \dots, \omega^{r-1}), \qquad Y_0 e_i = e_{i+1},$$

and repeat both blocks n/r times. Since r is odd, the resulting $X, Y \in \mathrm{SL}_n(q)$ have determinant one and

$$X^r = Y^r = 1, \qquad [X, Y] = \omega^{\pm 1} I.$$

Their projective images generate $E_r \cong C_r^2$. Under the homology map

$$H_2(E_r; \mathbb{Z}) \cong C_r \longrightarrow H_2(H; \mathbb{Z})_{(r)} \cong C_{r^s},$$

a generator maps to the element of order r represented by the scalar commutator. A generator of the dual cyclic group is nonzero on that element, so $\operatorname{res}^H_{E_r}(x)$ generates $H^3(E_r; \mathbb{Z}) \cong C_r$.

Write

$$H^*(E_r; \mathbb{F}_r) = \Lambda(a, b) \otimes \mathbb{F}_r[y_1, y_2], \qquad \beta(a) = y_1, \quad \beta(b) = y_2.$$

Up to a nonzero scalar, the mod-r reduction of this degree-three generator is $ay_2 - by_1$. The unstable and Cartan formulas give

$$\mathcal{P}^1(ay_2 - by_1) = ay_2^r - by_1^r,$$

and applying the Bockstein gives (NT), up to an irrelevant nonzero scalar. Its $(r-1)$st power is the nonzero polynomial displayed in the statement. This proves the detection and nonvanishing assertions.

A field automorphism given by the p^a-power map acts on the cyclic group $\langle x \rangle$ by p^a, while inverse transpose acts by -1. Reduced powers and the Bockstein are natural and $\mathbb{F}_r$-linear. Hence every outer automorphism acts on θ_r by a scalar in $\mathbb{F}_r^\times$, and fixes $\Theta_r = \theta_r^{r-1}$.

For the final assertion put $B^\sharp = BH$. The quotient $B^\sharp/H$ has order dividing $2f$, which is prime to r. Corestriction gives

$$\operatorname{res}^{B^\sharp}_H \operatorname{cor}^{B^\sharp}_H(\Theta_r) = [B^\sharp : H]\Theta_r \neq 0. \tag{NT$''$}$$

Restricting this class from $B^\sharp$ to B preserves its nonzero restriction to S, and proves the result. □

Corollary 7.46 (The absorption family has a bounded multiplier witness)**.** *For every group S in the family* (19au) *and every $S \leq B \leq \mathrm{Aut}(S)$, a class in $H^{12}(B; \mathbb{F}_3)$ restricts nontrivially to S. Consequently, if S^a is a minimal normal subgroup of a finite group G, then*

$$H^i(G; \mathbb{Z}) \neq 0 \qquad \textit{for some } i \leq 13. \tag{19bq}$$

Thus the family is an obstruction to a fixed cyclotomic-escape proof, but not a candidate family with increasing initial cohomological vanishing.

Proof. In (19au), one has $v_3(d_0) = 1$ and $9 \mid q - 1$ by the lifting-the-exponent calculation in Proposition 7.39. Apply Theorem 7.44 with $r = 3$. The factor transfer theorem and the mod-three universal coefficient sequence give (19bq). □

Corollary 7.47 (The depth-dependent factor bound)**.** *Retain the hypotheses of Theorem 7.44. If S^a is a minimal normal subgroup of a finite group G, then*

$$H^i(G;\mathbb{Z}) \neq 0 \qquad \text{for some } i \leq 2r^s(r-1)+1. \tag{19br}$$

Consequently, if G is an N_0-counterexample, then

$$2r^s(r-1)+1 > N_0. \tag{19bs}$$

Proof. The multiplier-depth theorem supplies the restricting mod-r class on every factor stabilizer. Apply Theorem 7.1; the mod-r universal coefficient sequence places a nonzero integral class in the same degree or the next one. This proves (19br), and (19bs) follows from the definition of an N_0-counterexample. □

Theorem 7.48 (Uniform finite bound for the completed S^r families)**.** *Let $N = S^r$ be a minimal normal subgroup of a finite group G, where S is nonabelian simple, and let*

$$S \leq B \leq \mathrm{Aut}(S)$$

be the factor-stabilizer image. Suppose that $B \to B/S$ is nonsplit and that S is one of the following:

$$\begin{gathered} A_6, \quad {}^2F_4(2)', \quad \text{symplectic type}, \quad B_n, \quad D_n \text{ or } {}^2D_n, \quad E_6 \text{ or } {}^2E_6, \quad E_7, \\ \mathrm{PSU}_n(q), \qquad \mathrm{PSL}_n(q) \quad \text{with either nontrivial prime-power Schur multiplier} \\ \text{or Schur-multiplier prime support } \{2,3\}, \quad \text{or satisfying} \\ v_3(\gcd(n,q-1)) = 1 \quad \text{and} \quad 9 \mid q-1. \end{gathered} \tag{7.24}$$

Then

$$H^i(G;\mathbb{Z}) \neq 0 \qquad \text{for some } i \leq 20. \tag{7.25}$$

Thus these are completed infinite S^r families for the existence problem, not merely partial calculations toward a prescribed degree.

Proof. The restricting classes on the factor stabilizer are supplied by Corollaries 7.5 and 7.7, and Theorems 7.28, 7.30, 7.25, 7.26, 7.19, 7.20, 7.29, and 7.34, together with Theorems 7.35 and 7.37, and Theorem 7.44. For a unitary factor, the exceptional multiplier audit in the latter two theorems leaves only the prime-power, $\{2,3\}$-support, and standard mixed-prime alternatives, so these results cover every nonsplit case. After passage from mod-p to integral coefficients, their largest ceiling is twenty. Theorem 7.1 transfers each restricting class from B to G. □

Corollary 7.49 (Concrete exclusion of completed S^r factors)**.** *Let $N_0 \geq 20$ and let G be a least-order N_0-counterexample. No minimal normal subgroup of G can have the form S^r if either*

(i) $\mathrm{Aut}(S) \to \mathrm{Out}(S)$ *splits; or*

(ii) *S belongs to the list* (7.24)*.*

In particular, for this fixed finite threshold the entire alternating and sporadic S^r branches, the displayed infinite Lie-type branches, and the unitary branches, together with the prime-power-multiplier and $\{2,3\}$-support linear branches and the displayed one-layer three-primary multiplier branch are absent from a least counterexample.

Proof. Theorem 4.1 excludes $G = N$, so G/N is a nontrivial proper quotient. In case (i), the splitting criterion gives a retraction $G \to G/N$; minimality supplies a class on G/N in degree at most N_0, and the retraction makes its inflation nonzero on G. In case (ii), if the factor-stabilizer map splits then Theorem 7.32 again makes $G \to G/N$ split. If it does not split, Theorem 7.48 gives a class on G in degree at most twenty. Every alternative contradicts the definition of G. □

8 An elementary abelian kernel lemma

Theorem 8.1 (A dimension criterion for elementary abelian kernels)**.** *Let*

$$1 \longrightarrow V_0 \longrightarrow E \longrightarrow Q \longrightarrow 1$$

be an extension with V_0 a nontrivial elementary abelian p-group, put $k = \overline{\mathbb{F}}_p$, and write

$$V^* = H^1(V_0; k).$$

If

$$\dim_k H^2(Q; V^*) > \dim_k H^2(V_0; k)^Q + \dim_k H^4(Q; k),$$

then $H^3(E; k) \neq 0$. Consequently, E has nonzero p-primary integral cohomology in degree three or four.

Proof. Consider the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{a,b} = H^a(Q; H^b(V_0; k)) \Longrightarrow H^{a+b}(E; k).$$

The term of interest is

$$E_2^{2,1} = H^2(Q; V^*).$$

The only differential entering it is

$$d_2 : E_2^{0,2} = H^2(V_0; k)^Q \longrightarrow E_2^{2,1},$$

and the only differential leaving it is

$$d_2 : E_2^{2,1} \longrightarrow E_2^{4,0} = H^4(Q; k).$$

It follows that

$$\dim_k E_3^{2,1} \geq \dim_k H^2(Q; V^*) - \dim_k H^2(V_0; k)^Q - \dim_k H^4(Q; k) > 0.$$

No differential on a later page can enter or leave bidegree $(2, 1)$. Thus $E_\infty^{2,1} \neq 0$, so $H^3(E; k) \neq 0$. The integral conclusion follows from the mod-p universal coefficient sequence. □

Theorem 8.2 (Orthogonal irreducible kernels)**.** *Let V_0 be a nontrivial finite-dimensional $\mathbb{F}_p Q$-module, where p is odd, and put $k = \overline{\mathbb{F}}_p$ and $V = k \otimes_{\mathbb{F}_p} V_0$. Assume that:*

(i) *V is absolutely irreducible and admits a Q-invariant nondegenerate symmetric bilinear form;*

(ii) *$H^4(Q; k) = 0$.*

If

$$1 \longrightarrow V_0 \longrightarrow E \longrightarrow Q \longrightarrow 1$$

is a nonsplit extension inducing the given action, then $H^3(E; k) \neq 0$. Hence E has nonzero p-primary integral cohomology in degree three or four.

Proof. Let $\alpha_0 \in H^2(Q; V_0)$ be the nonzero extension class. Extension of scalars identifies

$$H^2(Q; V_0) \otimes_{\mathbb{F}_p} k \cong H^2(Q; V),$$

so α_0 determines a nonzero class $\alpha \in H^2(Q; V)$. The invariant symmetric form identifies $V \cong V^*$, so α determines a nonzero element

$$x \in E_2^{2,1} = H^2(Q; H^1(V_0; k))$$

in the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{a,b} = H^a(Q; H^b(V_0; k)) \Longrightarrow H^{a+b}(E; k).$$

For the elementary abelian group V_0 in odd characteristic,

$$H^2(V_0; k) \cong V^* \oplus \Lambda^2 V^*.$$

The first summand has no fixed points because V is nontrivial irreducible. The second has no fixed points either: a nonzero invariant alternating form would be nondegenerate by irreducibility, and comparison with the invariant symmetric form would give, by Schur's lemma, a scalar endomorphism changing a symmetric form into an alternating one, impossible in odd characteristic. Thus $E_2^{0,2} = 0$, so no d_2 enters x. The only possible outgoing differential is

$$d_2 : E_2^{2,1} \longrightarrow E_2^{4,0} = H^4(Q; k),$$

whose target is zero. There are no higher incoming or outgoing differentials for bidegree $(2, 1)$. Therefore x survives to a nonzero class in $H^3(E; k)$. The integral conclusion follows from the mod-p universal coefficient sequence. □

9 Least-counterexample and central-descent reductions at a finite bound

For a positive integer N_0, call a nontrivial finite group an N_0-counterexample if its integral cohomology vanishes in every degree at most N_0. The results in this section are uniform finite-bound reductions: no particular value of N_0 is a conjectural target. Since all simple-base and completed factor-stabilizer results below have finite ceilings, their maximum is a finite constant.

Theorem 9.1 (Finite-bound minimal-counterexample reduction)**.** *There is a finite constant D_{red} such that, for every $N_0 \geq D_{\mathrm{red}}$ and for every least-order N_0-counterexample G, if A is a minimal normal subgroup of G, then exactly one of the following two unresolved configurations occurs:*

(i) *A is an elementary abelian p-group and $A \leq \Phi(G)$, so $G \twoheadrightarrow G/A$ is a nonsplit p-Frattini cover. If A is central, this is the universal central extension of G/A; otherwise G/A is superperfect. If A is central and G/A is simple, then*

$$H_2(G/A; \mathbb{Z}) \cong C_p, \qquad H^4(G/A; \mathbb{Z}) = 0.$$

In particular, the simple group G/A is neither alternating nor isomorphic to $\mathrm{PSL}_2(q)$. If A is noncentral and G/A is simple, then $H_2(G/A; \mathbb{Z}) = 0$ and the action of G/A on A is faithful.

(ii) $A = S^r$, *where*

$$S = \mathrm{PSL}_n(q),$$

$\mathrm{Aut}(S) \twoheadrightarrow \mathrm{Out}(S)$ *does not split, the Schur multiplier of* S *has at least two distinct prime divisors, and at least one of them is at least five. The quotient* G/A *is a nontrivial superperfect group.*

Moreover, if M *is any maximal proper normal subgroup of* G *and* $S = G/M$*, then* S *is nonabelian simple and*

$$\Lambda(S) \subseteq \pi(M).$$

In particular, every maximal-normal kernel in a least counterexample has order divisible by at least three distinct primes.

In case (ii)*, it is enough to solve the almost-simple restriction problem in Theorem 7.1 for every possible factor stabilizer* $S \le B \le \mathrm{Aut}(S)$ *for which* $B \to B/S$ *is nonsplit. For every unresolved such* B*, let* B^+ *be its inner-diagonal–field part, let its field image be generated by the* p^a*-power map, and put*

$$e = \gcd(m, a), \qquad x = p^e, \qquad f = m/e,$$

where $q = p^m$*. Then the field image is nontrivial and*

$$12 \mid f, \qquad \ell \mid x^2 - 1 \Longrightarrow \ell \mid \gcd(n, q-1), \qquad {}^{\ell \mid x^2+1}_{\ell \text{ odd}} \Longrightarrow \ell \mid \gcd(n, q-1). \tag{9A}$$

Moreover,

$$n > \frac{N_0}{4}, \tag{9B}$$

and, for every odd prime $s \le \min\{n, \lfloor N_0/4 \rfloor\}$*, every primitive prime divisor of* $x^s - 1$ *divides* $\gcd(n, q-1)$*. In addition, put* $d_0 = \gcd(n, q-1)$ *and*

$$\ell_r = \max\{0, 2v_r(d_0) - v_r(n)\}.$$

For every odd $r \mid d_0$*, the unresolved row necessarily satisfies*

$$v_r(q^2 - 1) > \ell_r \quad \Longrightarrow \quad v_r(p^{2a} - 1) \le \ell_r. \tag{9C}$$

It also satisfies

$$v_3(d_0) \ne 1. \tag{9D}$$

Consequently either $p = 3$ *and* $3 \nmid d_0$*, or* $9 \mid d_0$*. In the latter case put*

$$s = v_3(d_0), \qquad t = v_3(q-1).$$

Then

$$t = s \qquad \text{or} \qquad 4 \cdot 3^s + 1 > N_0. \tag{9E}$$

If instead $p = 3$*, put*

$$s_5 = v_5(d_0), \qquad t_5 = v_5(q-1).$$

Then $s_5 \ge 1$ *and*

$$t_5 = s_5 \qquad \text{or} \qquad 8 \cdot 5^{s_5} + 1 > N_0. \tag{9F}$$

Moreover, in this defining-characteristic-three row,

$$5 \mid f, \qquad \text{and hence} \qquad 60 \mid f. \tag{9G}$$

Finally, the relevant field-primary depth must itself grow with the asserted vanishing range:

$$\begin{aligned} p \neq 3: &\quad 16 \cdot 3^{v_3(f)} + 1 > N_0, \\ p = 3: &\quad 48 \cdot 5^{v_5(f)} + 1 > N_0. \end{aligned} \tag{9H}$$

In particular B is not contained in $\mathrm{Inndiag}(S)$*. No unitary factor, no prime-power-multiplier or $\{2,3\}$-support linear factor, no E_6 or ${}^2E_6$ factor, and no even-orthogonal factor remains, by Theorems 7.34, 7.35, 7.36, 7.37, 7.44, 7.45, 11.11, 7.19, 7.20, 7.25 and 7.26.*

Proof. Proposition 5.11 gives the two possible structures for A and shows that a nonsplit elementary abelian extension is Frattini. A split extension cannot occur: if G/A is nontrivial, minimality supplies a nonzero class of degree at most N_0 on G/A, and Lemma 5.2 injects it into the cohomology of G; if G/A is trivial, then G is a nontrivial elementary abelian group and already has nonzero $H^2(G;\mathbb{Z})$. This proves (i), including its final refinement by Proposition 5.12. If G/A is simple and A is central, that proposition gives $H_2(G/A;\mathbb{Z}) \cong A \cong C_p$. If $H^4(G/A;\mathbb{Z})$ were nonzero, Proposition 5.15(ii) would give a nonzero class on G in degree at most four, a contradiction. This proves the central simple-quotient refinement in (i). Propositions 4.2 and 4.4 give the two stated family exclusions. If instead A is noncentral, Proposition 5.12 makes G/A superperfect. Its action kernel on A is a normal subgroup of the simple group G/A and is not all of G/A, so it is trivial. This gives the noncentral refinement.

Now let $A = S^r$. If $G = A$, projection onto one factor and inclusion of that factor exhibit $H^*(S;\mathbb{Z})$ as a direct summand of $H^*(G;\mathbb{Z})$, in contradiction with Theorem 4.1. Hence G/A is nontrivial. It is superperfect: it is perfect as a quotient of G, while the five-term homology sequence contains

$$H_2(G;\mathbb{Z}) \longrightarrow H_2(G/A;\mathbb{Z}) \longrightarrow H_1(A;\mathbb{Z})_{G/A}.$$

The outer terms vanish because G is superperfect and A is perfect. If $\mathrm{Aut}(S) \to \mathrm{Out}(S)$ splits, then so does $\mathrm{Aut}(S^r) \to \mathrm{Out}(S^r)$; Lemma 7.31 makes $G \to G/A$ split, and the same minimality argument again gives a contradiction. If $S = A_6$, Corollary 7.5 and Theorem 7.1 give a contradiction; the same argument using Corollary 7.7 applies to the Tits group. The same conclusion for symplectic factors follows from Theorem 7.28, Theorem 7.1, and the mod-two-to-integral passage. Theorem 7.29 gives the identical conclusion for factors of type E_7, and Theorem 7.30 does so for type B_n. Theorem 7.19 does so for untwisted type E_6. Theorem 7.20 does so for twisted type E_6. Theorems 7.25 and 7.26 do so for every even-orthogonal factor. Theorem 7.34 does so for every linear or unitary factor with nontrivial prime-power Schur multiplier. Theorem 7.35 does so when the multiplier has prime support $\{2,3\}$. Theorem 7.36 forces every remaining factor stabilizer into its two even-field-image residues, and Theorem 7.37 eliminates the unitary row and sharpens the linear row to (9A). Theorem 7.38 gives the additional absorption conditions and (9B): if $n \leq N_0/4$, its rank-dependent witness has degree at most $4n \leq N_0$, a contradiction. Theorem 7.42 gives (9C): if its left side held while the right side failed, the resulting degree-four factor-stabilizer class would again contradict the vanishing of G. To prove (9D), suppose that $v_3(d_0) = 1$. Then $p \neq 3$ and $3 \mid x^2 - 1$. Since $12 \mid f$, lifting the exponent gives $v_3(x^f - 1) \geq 2$: use $x - 1$ when $x \equiv 1 \pmod 3$, and use $x^2 - 1$ together with $3 \mid f/2$ when $x \equiv -1 \pmod 3$. Thus $9 \mid q - 1$. Theorem 7.44 with $r = 3$ supplies a degree-twelve mod-three factor-stabilizer class and hence an integral witness by degree thirteen, again a contradiction after enlarging D_{red} if necessary. If $p = 3$, then $3 \nmid q - 1$ and hence $3 \nmid d_0$. If $p \neq 3$, then $3 \mid x^2 - 1$, so (9A) gives $3 \mid d_0$ and (9D) strengthens this to $9 \mid d_0$. For the last assertion assume $p \neq 3$ and use the displayed notation s, t. If $t > s$, then $3^{s+1} \mid q - 1$, so Corollary 7.47 with $r = 3$ gives a nonzero integral class on G in degree at most

$$2 \cdot 3^s(3 - 1) + 1 = 4 \cdot 3^s + 1.$$

The asserted vanishing through degree N_0 forces this number to exceed N_0. This proves (9E); when $t = s$, the entire three-primary part of $q-1$ is already absorbed by d_0 and the scalar-layer hypothesis of the multiplier-depth theorem is unavailable. Finally suppose $p = 3$. Since $12 \mid f$, if e is even then $5 \mid x^2 - 1$, while if e is odd then $5 \mid x^2 + 1$. Condition (9A) therefore gives $5 \mid d_0$, so $s_5 \geq 1$. If $t_5 > s_5$, Corollary 7.47 with $r = 5$ gives a nonzero integral class on G in degree at most

$$2 \cdot 5^{s_5}(5-1) + 1 = 8 \cdot 5^{s_5} + 1.$$

Vanishing through N_0 proves (9F). Thus the defining-characteristic-three row also has only two possibilities at its first forced odd multiplier prime: full five-primary absorption, or depth growing at least logarithmically with the asserted vanishing range. Theorem 7.45, applied with $r = 5$, now forces $5 \mid f$: if $5 \nmid f$, it supplies a restricting integral class on the factor stabilizer in degree

$$2(5^2 - 1) = 48,$$

and Theorem 7.1 contradicts the asserted vanishing after enlarging D_{red} to at least forty-eight. Since $12 \mid f$ by (9A), this proves (9G). For (9H), first suppose $p \neq 3$. Then $9 \mid d_0$ by (9D), and $v_3(f) \geq 1$ by (9A). Apply Theorem 11.11 with $r = 3$ and $a = v_3(f)$. It gives a nonzero integral class on G in some degree at most

$$2 \cdot 3^a(3^2 - 1) + 1 = 16 \cdot 3^a + 1.$$

Vanishing through N_0 forces the first inequality in (9H). If $p = 3$, then $5 \mid d_0$ and $v_5(f) \geq 1$ by (9F) and (9G). The same theorem with $r = 5$ and $a = v_5(f)$ gives the ceiling

$$2 \cdot 5^a(5^2 - 1) + 1 = 48 \cdot 5^a + 1,$$

which proves the second inequality. Thus not only the multiplier depth but also the actual field-primary depth is unbounded along any hypothetical sequence with increasing initial vanishing. The classification in Theorem 7.33 now leaves precisely the factors displayed in (ii). Finally, Theorem 7.32 shows that only nonsplit factor stabilizers can occur, and Theorem 7.1 gives the stated sufficient restriction problem. Corollary 7.11 disposes of every factor stabilizer contained in $\mathrm{Inndiag}(S)$.

Finally, let M be maximal normal in G. Since G is perfect, the simple quotient $S = G/M$ is nonabelian. If $\Lambda(S) \not\subseteq \pi(M)$, Theorem 6.28 supplies nonzero cohomology on G in degree at most $D^{(3)}_{\mathrm{simp}}$, a contradiction. This proves the displayed containment. Taking D_{red} to be the maximum of the finitely many displayed ceilings, including $D^{(3)}_{\mathrm{simp}}$, validates every contradiction used above. □

Corollary 9.2 (Prime support in the central simple-quotient branch)**.** *Let G be a least counterexample in Theorem 9.1, let A be a central minimal normal subgroup, and suppose that $\Gamma = G/A$ is a simple group of ordinary or graph-twisted Lie type over $\mathbb{F}_q$. Then*

$$\ell \mid q^2 - 1 \quad \Longrightarrow \quad \ell \mid |H_2(\Gamma; \mathbb{Z})|$$

for every prime ℓ. Consequently:

(i) *if the Schur multiplier of Γ is 2-primary, then $q = 3$;*

(ii) *if it is 3-primary, then $q = 2$;*

(iii) *if its prime support is contained in $\{2, 3\}$, then*

$$q \in \{2, 3, 5, 7, 17\}.$$

These are necessary conditions; they do not assert that the listed cases actually occur in a least counterexample.

Proof. If a prime $\ell \mid q^2-1$ did not divide the Schur multiplier, apply Proposition 7.8 with $S = B = \Gamma$ and trivial outer quotient. It would give $H^4(\Gamma;\mathbb{Z})_{(\ell)} \neq 0$, contrary to Theorem 9.1.

For (i), q^2-1 must be a power of two. The adjacent even factors $q-1$ and $q+1$ are then two and four, so $q = 3$. For (ii), q^2-1 must be a power of three. Hence q is even, the coprime odd factors $q-1, q+1$ are one and three, and $q = 2$. For (iii), the elementary 2–3-smooth calculation in the proof of Corollary 7.18, applied with $x = q$, gives the displayed five values. □

Corollary 9.3 (No central simple Lie-type quotient)**.** *In the central elementary-abelian branch of Theorem 9.1, the quotient by the minimal normal subgroup cannot be a finite simple group of Lie type. Consequently, if that quotient is simple, then it is sporadic.*

Proof. Write $\Gamma = G/A$. Theorem 9.1 gives

$$H_2(\Gamma;\mathbb{Z}) \cong C_p, \qquad H^4(\Gamma;\mathbb{Z}) = 0.$$

Suppose first that Γ is ordinary or graph-twisted and that its universal central extension is the standard simply connected fixed-point group. Since G is itself that universal central extension, Proposition 5.16 contradicts the assumed vanishing of the cohomology of G.

It remains to audit the exceptional Schur multipliers. The complete list and the exceptional isomorphisms are given in [25, Theorem 2.2.10, Theorem 6.1.4, and Table 6.1.3]. Entries whose multiplier is not cyclic of prime order are already incompatible with the displayed isomorphism. The entries $\mathrm{PSL}_2(4)$, $\mathrm{PSL}_3(2)$, and $\mathrm{PSL}_4(2)$ are respectively isomorphic to A_5, $\mathrm{PSL}_2(7)$, and A_8, and are excluded by Theorem 9.1. The remaining exceptional entries with cyclic prime-order multiplier are exactly

Γ	$\mathrm{PSU}_4(2)$	$\mathrm{PSp}_6(2)$	$G_2(3)$	$G_2(4)$	$F_4(2)$
$H_2(\Gamma;\mathbb{Z})$	C_2	C_2	C_3	C_2	C_2.

For the three groups over $\mathbb{F}_2$, Corollary 9.2 would require the prime 3 to divide the displayed multiplier. For $G_2(3)$ it would require the prime 2, and for $G_2(4)$ it would require both 3 and 5. Each is a contradiction.

Finally, the Suzuki–Ree multiplier tables are trivial apart from the exceptional multiplier $C_2 \times C_2$ of ${}^2B_2(8)$, and the Tits group has trivial multiplier. None can have multiplier C_p. Thus no simple Lie-type quotient occurs. A simple quotient of the perfect group G is nonabelian; after the alternating and Lie-type families have been removed, the classification of finite simple groups leaves only the sporadic groups. □

Corollary 9.4 (No central simple sporadic quotient)**.** *In the central elementary-abelian branch of Theorem 9.1, the quotient G/A cannot be simple and sporadic.*

Proof. Put $\Gamma = G/A$. The map $G \twoheadrightarrow \Gamma$ is central, so Corollary 6.5 gives nonzero integral cohomology of G in degree at most ten, contrary to the choice of G. □

Corollary 9.5 (No central simple quotient)**.** *In the central elementary-abelian branch of Theorem 9.1, the quotient G/A cannot be simple.*

Proof. The group G/A is perfect because it is a quotient of the superperfect group G, so a simple G/A would be nonabelian. By the classification of finite simple groups it would be alternating, of Lie type, or sporadic. The alternating case is excluded in Theorem 9.1, the Lie-type case by Corollary 9.3, and the sporadic case by Corollary 9.4. □

Corollary 9.6 (Low homology of the central quotient)**.** *In the central elementary-abelian branch of Theorem 9.1, write $A \cong C_p$ and $Q = G/A$, and let $e : BQ \to K(C_p, 2)$ classify the extension. Then*

$$e_* : H_j(Q;\mathbb{Z}) \xrightarrow{\cong} H_j(K(C_p,2);\mathbb{Z}) \qquad (0 \le j \le 9).$$

Consequently, for every prime $\ell \neq p$,

$$H^i(Q;\mathbb{Z})_{(\ell)} = 0 \qquad \textit{throughout the cohomological range corresponding to } 0 \le j \le 9.$$

Proof. The vanishing hypothesis on G and Lemma 1.1 give $\widetilde{H}_j(G;\mathbb{Z}) = 0$ for $1 \le j \le 9$. The first assertion is Proposition 5.13 with $m = 9$.

For $\ell \neq p$, the path–loop fibration of $K(C_p, 2)$ has fiber $K(C_p, 1) = BC_p$. Its homology Serre spectral sequence with $\mathbb{Z}_{(\ell)}$ coefficients has only the bottom row, because positive homology of C_p vanishes when p is invertible. Since the total path space is contractible, $\widetilde{H}_j(K(C_p,2);\mathbb{Z}_{(\ell)}) = 0$ for every j. Thus the displayed integral homology groups of Q are p-primary in positive degrees. Lemma 1.1 gives the cohomological conclusion. □

Corollary 9.7 (The first forced groups in the central quotient)**.** *Under the hypotheses of Corollary 9.6,*

$$H_2(Q;\mathbb{Z}) \cong C_p, \qquad H_3(Q;\mathbb{Z}) = 0, \qquad H_4(Q;\mathbb{Z}) \cong \begin{cases} C_p, & p \text{ odd}, \\ C_4, & p = 2. \end{cases}$$

Equivalently,

$$H^3(Q;\mathbb{Z}) \cong C_p, \qquad H^4(Q;\mathbb{Z}) = 0, \qquad H^5(Q;\mathbb{Z}) \cong \begin{cases} C_p, & p \text{ odd}, \\ C_4, & p = 2. \end{cases}$$

Proof. The Hurewicz theorem gives $H_2(K(C_p,2);\mathbb{Z}) \cong C_p$. Since the only nonzero homotopy group of $K(A, 2)$ is $\pi_2 = A$, Whitehead's certain exact sequence gives $H_3(K(A,2);\mathbb{Z}) = 0$ and identifies

$$H_4(K(A,2);\mathbb{Z}) \cong \Gamma(A),$$

where Γ is the universal quadratic functor [44]. Its cyclic-group calculation is $\Gamma(C_n) \cong C_n$ for odd n and C_{2n} for even n. Corollary 9.6 now gives the homology groups displayed above, and Lemma 1.1 gives their cohomological reformulation. □

Proposition 9.8 (The integral model through degree nine)**.** *For a prime p, the positive integral homology of $K(C_p, 2)$ through degree nine is as follows; an empty entry denotes zero.*

	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9
$p = 2$		C_2		C_4	C_2	C_2	C_2	$C_8 \oplus C_2$	C_2^2
$p = 3$		C_3		C_3		C_9	C_3	C_3	C_3
$p \ge 5$		C_p		C_p		C_p		C_p	

Proof. Cartan's integral calculation expresses $H_*(K(A,n);\mathbb{Z})$ as the homology of a tensor product of explicit elementary divided-power complexes [13, Exposé 11, Theorem 1]. For $A = C_p$ and $n = 2$, the initial complex has generators x and y of degrees two and three, with

$$dx = 0, \qquad dy = px.$$

The class $\gamma_k(x)$ has order kp before taking the primary component. Thus its p-primary order is $p^{1+v_p(k)}$. No further admissible p-word contributes below degree $2p + 1$. This already gives the row

$p \geq 5$, and gives the cyclic summands C_4 in degree four and C_8 in degree eight for $p = 2$, and C_9 in degree six for $p = 3$.

For completeness, the remaining exceptional-prime calculation is finite. Cartan's mod-p calculation gives, in the range needed for the table,

$$H^*(K(C_3, 2); \mathbb{F}_3) = \mathbb{F}_3[x_2, w_8] \otimes \Lambda(y_3, z_7), \qquad y_3 = \beta x_2, \qquad z_7 = P^1 y_3, \qquad w_8 = \beta z_7.$$

The first Bockstein is the derivation with $d_1 x_2 = y_3$ and $d_1 z_7 = w_8$. In particular,

$$d_1(x_2^4) = x_2^3 y_3, \qquad d_1(x_2 z_7) = y_3 z_7 + x_2 w_8.$$

These two nonzero first Booksteins give respectively the C_3 terms in the last two displayed homology degrees. The preceding first Booksteins give the displayed C_3 terms, while the second Bockstein on x_2^3 gives the C_9 in H^7, equivalently in H_6.

At the prime two, enumerating Cartan's admissible words through degree nine gives the first row. There is also an independent exact Smith-normal- form check in `scripts/check_em_homology.g`. It constructs the Dold–Kan simplicial group $K(C_p, 2)$ in HAP, forms its integral chain complex, and coreduces it. HAP returns

$p = 2,\ j = 2, \ldots, 9$	2	[ ]	4	2	2	2	[2, 8]	[2, 2]
$p = 3,\ j = 2, \ldots, 8$	3	[ ]	3	[ ]	9	3	3	

Here a positive integer denotes a cyclic invariant factor and [ ] denotes the zero group. The $p = 3$ Bockstein calculation above supplies degree nine without requiring one further, much larger, simplicial truncation. □

Corollary 9.9 (A displayed forced range in the central quotient)**.** *Under the hypotheses of Corollary 9.6, the integral cohomology groups of Q through degree nine are*

	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8	H^9
$p = 2$			C_2		C_4	C_2	C_2	C_2	$C_8 \oplus C_2$
$p = 3$			C_3		C_3		C_9	C_3	C_3
$p \geq 5$			C_p		C_p		C_p		

where every blank entry is zero.

Proof. Combine Corollary 9.6, Proposition 9.8, and Lemma 1.1. □

Corollary 9.10 (First quotient layer in the central branch)**.** *Let G be a least counterexample in the central elementary-abelian branch of Theorem 9.1, with central minimal subgroup $A \cong C_p$, and put $Q = G/A$. Then Q is nonsimple. Moreover, either G has a second minimal normal subgroup, to which Theorem 9.1 may also be applied, or A is the unique minimal normal subgroup and every minimal normal subgroup of Q has one of the following forms:*

(i) *an elementary abelian p-group;*

(ii) *S^r for a nonabelian simple S, where, for the induced factor-stabilizer image $S \leq B \leq \mathrm{Aut}(S)$, $H_2(S; \mathbb{Z})_{B/S}$ has a quotient C_p.*

Proof. Nonsimplicity is Corollary 9.5. If A is not the unique minimal normal subgroup, choose a second one. If it is unique, apply Proposition 5.14 to each minimal normal subgroup of Q. □

Proposition 9.11 (The quotient after the first central layer)**.** *Assume the unique-central-monolith alternative of Corollary 9.10. Let $\overline{N}$ be a minimal normal subgroup of Q, let N be its inverse image in G, and put $R = Q/\overline{N} = G/N$.*

(i) *If $\overline{N} = S^r$, then R is nontrivial and superperfect, and there is a natural short exact sequence*

$$0 \longrightarrow H_3(R;\mathbb{Z}) \xrightarrow{d_3} H_2(S;\mathbb{Z})_R^r \xrightarrow{\overline{\tau}} C_p \longrightarrow 0.$$

Here d_3 is the homology transgression and $\overline{\tau}$ is induced by the restricted central cover $N \to \overline{N}$.

(ii) *If $\overline{N} = V$ is elementary abelian, it has characteristic p. If N is nonabelian, then R is nontrivial and superperfect, and the commutator induces a nondegenerate R-invariant alternating form*

$$\bigwedge\nolimits^2_{\mathbb{F}_p} V \longrightarrow A \cong \mathbb{F}_p.$$

In particular, V is an irreducible symplectic $\mathbb{F}_pR$-module of even dimension. If N is abelian, exactly one of the following holds:

(a) *$N \cong C_{p^2}$ and $N \le Z(G)$;*

(b) *N is elementary abelian, the sequence of $\mathbb{F}_pR$-modules*

$$0 \longrightarrow A \longrightarrow N \longrightarrow V \longrightarrow 0$$

is nonsplit, V is nontrivial and irreducible, and R is nontrivial and superperfect.

Proof. The central extension $A \to G \to Q$ has homology transgression

$$\delta : H_2(Q;\mathbb{Z}) \xrightarrow{\cong} A,$$

because G is superperfect. Corollary 9.7 also gives $H_3(Q;\mathbb{Z}) = 0$.

Suppose first that $\overline{N} = S^r$. Since $\overline{N}$ is perfect, the low-degree exact sequence of the homology Lyndon–Hochschild–Serre spectral sequence for $\overline{N} \to Q \to R$ is

$$H_3(Q;\mathbb{Z}) \longrightarrow H_3(R;\mathbb{Z}) \xrightarrow{d_3} H_2(\overline{N};\mathbb{Z})_R \longrightarrow H_2(Q;\mathbb{Z}) \longrightarrow H_2(R;\mathbb{Z}) \longrightarrow 0.$$

By naturality, the composite

$$H_2(\overline{N};\mathbb{Z})_R \longrightarrow H_2(Q;\mathbb{Z}) \xrightarrow{\delta} A$$

is the epimorphism τ of Proposition 5.14. Hence the map into $H_2(Q;\mathbb{Z})$ is surjective. The displayed sequence, together with $H_3(Q;\mathbb{Z}) = 0$, is therefore the asserted short exact sequence and gives $H_2(R;\mathbb{Z}) = 0$. The group R is perfect because it is a quotient of the perfect group Q, so it is superperfect.

If $R = 1$, then $Q = S^r$, and

$$C_p \cong H_2(Q;\mathbb{Z}) \cong H_2(S;\mathbb{Z})^r.$$

This forces $r = 1$, making Q simple, contrary to Corollary 9.5. Thus R is nontrivial.

Now let $\overline{N} = V$. Its characteristic is p by Proposition 5.14. Since N/A is abelian, $[N,N] \le A$. If N is nonabelian, unique minimality gives $[N,N] = A$, and commutation defines a nonzero alternating R-equivariant map $\bigwedge^2 V \to A$. Its radical is an R-submodule of the irreducible module V, so the radical is zero. The form is therefore nondegenerate and V has even dimension. The

action on V is nontrivial: a trivial irreducible module would have order p, whereas every group of order p^2 is abelian. Hence $V_R = 0$.

For an elementary abelian group, $H_2(V;\mathbb{Z}) \cong \bigwedge^2 V$, and naturality again shows that

$$H_2(V;\mathbb{Z})_R \longrightarrow H_2(Q;\mathbb{Z}) \xrightarrow{\delta} A$$

is the commutator epimorphism. Thus the image of the bottom filtration piece in the homology spectral sequence is all of $H_2(Q;\mathbb{Z})$. The top filtration quotient is

$$E^{\infty}_{2,0} = H_2(R;\mathbb{Z}),$$

because the only outgoing differential from $E^2_{2,0}$ lands in $E^2_{0,1} = V_R = 0$. It follows that $H_2(R;\mathbb{Z}) = 0$. As above, R is perfect, hence superperfect; it is nontrivial because the nontrivial abelian group V cannot equal the perfect group Q.

Finally suppose that N is abelian. Since N/A has exponent p, $pN \le A$. The subgroup pN is characteristic in N and hence normal in G. Unique minimality of A now gives $pN = 1$ or $pN = A$.

If $pN = A$, multiplication by p induces a nonzero R-map

$$V = N/A \longrightarrow A, \qquad n + A \longmapsto pn.$$

Irreducibility makes this map an isomorphism, so $|V| = p$ and $N \cong C_{p^2}$. Its automorphism group is abelian. Conjugation by the perfect group G is therefore trivial, and $N \le Z(G)$.

Suppose instead that $pN = 1$. Then N is an $\mathbb{F}_p R$-module. The action on V cannot be trivial. Indeed, triviality and irreducibility would give $|V| = p$; conjugation on N would then have image in the abelian group of automorphisms acting trivially on both A and N/A. Perfectness of G would make N central, producing order-p central subgroups distinct from A. Thus V is a nontrivial irreducible module and $V_R = 0$.

The module sequence $0 \to A \to N \to V \to 0$ cannot split: an R-stable complement would be a nontrivial G-normal subgroup disjoint from A, contrary to unique minimality. Let

$$\xi \in \mathrm{Ext}^1_{\mathbb{F}_p R}(V, A) \cong H^1(R; V^*)$$

be its nonzero extension class. The homology long exact sequence contains

$$H_1(R;V) \xrightarrow{\partial_\xi} A \longrightarrow N_R \longrightarrow V_R = 0.$$

On the finite bar complex, the natural identification $H^1(R;V^*) \cong H_1(R;V)^*$ identifies ∂_ξ with evaluation against ξ. Hence ∂_ξ is nonzero, therefore surjective, and $N_R = 0$.

The homology five-term sequence for $N \to G \to R$ now gives

$$H_2(R;\mathbb{Z}) \cong N_R = 0.$$

The quotient R is perfect and nontrivial, so it is superperfect. This proves the two alternatives. □

Proposition 9.12 (Full-center absorption)**.** *Under the unique-central-monolith hypotheses above, the full center of G is a cyclic p-group*

$$C = Z(G) \cong C_{p^k} \qquad (k \ge 1),$$

whose unique subgroup of order p is A. Put $Q_C = G/C$. Then the classifying map

$$BQ_C \longrightarrow K(C, 2)$$

is an integral homology isomorphism through degree nine.

Let V be an elementary abelian minimal normal subgroup of Q_C, let L be its inverse image in G, and suppose that L is abelian. Put $R = G/L$ and $M = \Omega_1(L)$. Then V has characteristic p, and

$$0 \longrightarrow A \longrightarrow M \longrightarrow V \longrightarrow 0$$

is a nonsplit sequence of $\mathbb{F}_p R$-modules with V nontrivial and irreducible. Moreover,

$$U = G/M$$

is a nontrivial superperfect group, and

$$1 \longrightarrow C/A \longrightarrow U \longrightarrow R \longrightarrow 1$$

is the universal central extension of R. In particular, when $C = A$ the quotient R itself is superperfect.

Proof. Every subgroup of prime order in $Z(G)$ is a minimal normal subgroup of G. Unique minimality therefore excludes primes other than p from $Z(G)$ and says that its p-torsion has the unique subgroup A of order p. A finite abelian p-group with a unique subgroup of order p is cyclic, proving the first assertion. The homology statement is Proposition 5.13, applied to the central extension $C \to G \to Q_C$.

If V had characteristic different from p, the extension $C \to L \to V$ would have a unique complement, since its first and second cohomology with coefficients in C vanish. That complement would be a minimal normal subgroup of G distinct from A. Hence V has characteristic p.

The action of R on $V = L/C$ is nontrivial. Otherwise irreducibility would give $|V| = p$, and the conjugation action of G on the abelian group L would act trivially on both C and L/C. Its image would embed in the abelian group $\mathrm{Hom}(V, C)$, hence would be trivial because G is perfect. This would put L in $Z(G) = C$, a contradiction. Thus $V_R = 0$.

Multiplication by p induces an R-map

$$V \longrightarrow C/pC, \qquad l + C \longmapsto pl + pC.$$

The target is trivial and one-dimensional, while V is nontrivial and irreducible, so this map is zero. Consequently $pL \le pC$. Given $l \in L$, choose $c \in C$ with $pl = pc$; then $l - c \in \Omega_1(L)$. It follows that $M = \Omega_1(L)$ maps onto V, with kernel $M \cap C = \Omega_1(C) = A$. In particular,

$$L = CM, \qquad L/M \cong C/A.$$

As in the preceding proposition, the module extension $0 \to A \to M \to V \to 0$ cannot split. Its nonzero class in $H^1(R; V^*)$, paired with $H_1(R; V)$, makes the connecting map onto A surjective. Hence $M_R = 0$.

Set $U = G/M$. Since L is abelian, its action on M is trivial, so $M_U = M_R = 0$. The five-term sequence for $M \to G \to U$, together with the superperfectness of G, gives

$$H_1(U; \mathbb{Z}) = H_2(U; \mathbb{Z}) = 0.$$

Thus U is superperfect. The quotient R is perfect, and it is nontrivial because $R = 1$ would make $G = L$ abelian. Hence U is nontrivial as well.

Finally, $L/M \cong C/A$ is central in U, since it is the image of C. The five-term sequence for this central extension now identifies

$$H_2(R; \mathbb{Z}) \xrightarrow{\cong} C/A.$$

The perfect central extension $U \to R$ has full multiplier kernel and superperfect source, so it is the universal central extension of R. □

Proposition 9.13 (Full-center multiplier descent)**.** *Under the hypotheses of Proposition 9.12, put*

$$X = G/C.$$

Then X is perfect and centerless, and $G \to X$ is its universal central extension. In particular,

$$H_2(X;\mathbb{Z}) \cong C, \qquad H_3(X;\mathbb{Z}) = 0.$$

Suppose that X is not simple. Let $\overline{N}$ be a minimal normal subgroup of X, let L be its inverse image in G, and put $R = X/\overline{N} = G/L$. Then R is nontrivial and perfect, and the edge epimorphism

$$C \cong H_2(X;\mathbb{Z}) \longrightarrow H_2(R;\mathbb{Z})$$

factors through an epimorphism

$$C/A \twoheadrightarrow H_2(R;\mathbb{Z}).$$

Consequently $H_2(R;\mathbb{Z})$ is a cyclic p-group of order at most p^{k-1}. More precisely:

(i) *if $\overline{N} = S^r$, then*

$$0 \longrightarrow H_3(R;\mathbb{Z}) \longrightarrow H_2(S;\mathbb{Z})_R^r \xrightarrow{\tau} C \longrightarrow H_2(R;\mathbb{Z}) \longrightarrow 0$$

is exact, and $\operatorname{im}\tau$ is nonzero and therefore contains A;

(ii) *if $\overline{N} = V$ is elementary abelian, then V has characteristic p. If L is nonabelian, then $[L,L] = A$ and commutation induces a nondegenerate R-invariant alternating form*

$$\bigwedge\nolimits^2_{\mathbb{F}_p} V \longrightarrow A.$$

If L is abelian, then the last epimorphism above is an isomorphism

$$H_2(R;\mathbb{Z}) \cong C/A.$$

Put $D = [G,L]$. Then

$$G/D \cong \widetilde{R}$$

over R. In particular the smaller universal central extension is a quotient of G. Moreover,

$$1 \neq \widetilde{R}, \qquad |\widetilde{R}| = |H_2(R;\mathbb{Z})|\,|R| \leq \frac{|G|}{p|\overline{N}|} < |G|.$$

Thus minimality of G supplies nonzero integral cohomology of $\widetilde{R}$ in some degree at most N_0.

Proof. The group X is perfect because it is a quotient of the perfect group G. If $xC \in Z(X)$, then $[x,G] \leq C$. Since C is central, the map

$$G \longrightarrow C, \qquad g \longmapsto [x,g]$$

is a homomorphism. It is trivial because G is perfect, so $x \in C$. Thus $Z(X) = 1$.

The homology five-term sequence for $C \to G \to X$ and the superperfectness of G identify the transgression as an isomorphism

$$H_2(X;\mathbb{Z}) \xrightarrow{\cong} C.$$

This is the standard characterization of $G \to X$ as the universal central extension. Proposition 5.13 and Whitehead's calculation $H_3(K(C,2);\mathbb{Z}) = 0$ give $H_3(X;\mathbb{Z}) = 0$.

Now assume that X is not simple. The minimal normal subgroup $\overline{N}$ is proper, so R is nontrivial; it is perfect because it is a quotient of X.

Suppose first that $\overline{N} = S^r$. The low-degree homology sequence for $\overline{N} \to X \to R$, together with $H_3(X;\mathbb{Z}) = 0$, is

$$0 \longrightarrow H_3(R;\mathbb{Z}) \longrightarrow H_2(S;\mathbb{Z})_R^r \xrightarrow{\tau} H_2(X;\mathbb{Z}) \longrightarrow H_2(R;\mathbb{Z}) \longrightarrow 0.$$

Under $H_2(X;\mathbb{Z}) \cong C$, the middle map is the transgression of the restricted central extension $C \to L \to S^r$. If it were zero, the universal-coefficient sequence for the perfect group S^r would make that central extension split. Its complement would be unique, since $\mathrm{Hom}(S^r, C) = 0$, and hence characteristic under the conjugation action of G. This would give a nontrivial normal subgroup of G disjoint from A, contrary to unique minimality. Thus $\operatorname{im}\tau \neq 0$. Every nonzero subgroup of the cyclic p-group C contains A, proving both the exact sequence and the factorization through C/A.

Now let $\overline{N} = V$ be elementary abelian, of characteristic q. If $q \neq p$, then

$$H^1(V;C) = H^2(V;C) = 0.$$

The central extension $C \to L \to V$ would therefore have a unique complement. As above, that complement would be a nontrivial normal subgroup of G disjoint from A. Hence $q = p$.

Suppose that L is nonabelian. Since L/C is abelian, $[L,L] \leq C$. Moreover, $l^p \in C$ for every $l \in L$, so the class-two commutator identity gives $[l,m]^p = [l^p, m] = 1$. Thus $[L,L] \leq \Omega_1(C) = A$, while unique minimality gives the reverse inclusion. Hence $[L,L] = A$. The resulting nonzero alternating map $\bigwedge^2 V \to A$ has zero radical because V is irreducible. It also shows that the image of

$$H_2(V;\mathbb{Z})_R \longrightarrow H_2(X;\mathbb{Z}) \cong C$$

is exactly A. The action on V is nontrivial: if the irreducible module V were trivial, then V would be cyclic and the central extension $C \to L \to V$ would be abelian. Hence $V_R = 0$. The homology five-term sequence for $V \to X \to R$ now makes $H_2(X;\mathbb{Z}) \to H_2(R;\mathbb{Z})$ surjective, and its kernel contains A.

If L is abelian, Proposition 9.12 constructs a superperfect central extension

$$1 \longrightarrow C/A \longrightarrow G/\Omega_1(L) \longrightarrow R \longrightarrow 1$$

which is universal. Therefore $H_2(R;\mathbb{Z}) \cong C/A$.

Since G is superperfect and R is perfect, Proposition 5.3, applied to $L \to G \to R$, gives

$$G/[G,L] \cong \widetilde{R}$$

over R.

In every case $H_2(R;\mathbb{Z})$ is a quotient of C/A, so its order is at most $|C|/p$. Since R is nontrivial and perfect, its universal central extension $\widetilde{R}$ is nontrivial and

$$|\widetilde{R}| = |H_2(R;\mathbb{Z})|\,|R| \leq \frac{|C|}{p}|R| = \frac{|G|}{p|\overline{N}|} < |G|.$$

The final assertion is the defining minimal-order property of G. □

Proposition 9.14 (Prime-to-layer persistence)**.** *Retain the notation of Proposition 9.13, with X nonsimple, and let $\widetilde{R} \to R$ be the universal central extension. Put $D = [G, L]$, so that $G/D \cong \widetilde{R}$. For every prime $\ell \nmid |D|$ and every $n \geq 0$, inflation is an isomorphism*

$$H^n(\widetilde{R}; \mathbb{Z}_{(\ell)}) \xrightarrow[\cong]{\inf} H^n(G; \mathbb{Z}_{(\ell)}).$$

Consequently, a nonzero ℓ-primary integral cohomology class on $\widetilde{R}$ in any finite degree forces one on G in the same degree.

If $\overline{N} = V$ is elementary abelian, then L is a p-group, so the conclusion holds for every $\ell \neq p$. If $\overline{N} = S^r$, it holds for every

$$\ell \notin \{p\} \cup \pi(S),$$

where $\pi(S)$ is the set of prime divisors of $|S|$. Thus, in a least counterexample, every primary component on the smaller universal cover $\widetilde{R}$ within the vanishing range is supported on a prime divisor of $|[G, L]|$; this is p in the elementary case and lies in $\{p\} \cup \pi(S)$ in the nonabelian case.

Proof. The identification $G/[G, L] \cong \widetilde{R}$ is part of Proposition 9.13. Apply Lemma 5.1 directly to this quotient.

For an elementary layer, both C and $L/C = V$ are p-groups, so L is a p-group and so is $[G, L]$. For a nonabelian layer, $|L| = |C||S|^r$, and $[G, L] \leq L$. These observations give the two stated prime supports. The final assertion follows because the cohomology of G vanishes through degree N_0. □

Proposition 9.15 (Structure of the canonical descent kernel)**.** *Retain the notation of Proposition 9.13, and put*

$$D = [G, L], \qquad U = G/D \cong \widetilde{R}.$$

Then

$$D = [G, D]. \tag{1}$$

The three layer types have the following additional structure.

(i) *If $\overline{N} = S^r$, then*

$$D = [L, L], \qquad D = [D, D], \qquad Z(D) = D \cap C = \operatorname{im} \tau,$$

where τ is the map in Proposition 9.13(i)*. Thus*

$$1 \longrightarrow \operatorname{im} \tau \longrightarrow D \longrightarrow S^r \longrightarrow 1 \tag{2}$$

is a perfect central extension and

$$|D| = |S|^r |\operatorname{im} \tau|.$$

(ii) *If $\overline{N} = V$ is elementary abelian and L is nonabelian, put $Z_D = D \cap C$. Then*

$$Z(D) = Z_D, \qquad [D, D] = \Phi(D) = A,$$

D/A is elementary abelian with zero R-coinvariants, and

$$0 \longrightarrow Z_D/A \longrightarrow D/A \longrightarrow V \longrightarrow 0. \tag{3}$$

Moreover,

$$Z_D = A \quad \text{or} \quad Z_D \cong C_{p^2}, \qquad |D| = |V||Z_D| \leq p^2|V|. \tag{4}$$

If $Z_D > A$, the module extension (3) *is nonsplit. If p is odd, then necessarily $Z_D = A$ and D has exponent p. In either case the Schur multiplier $H_2(D; \mathbb{Z})$ is elementary abelian, and hence so is $H^3(D; \mathbb{Z})$.*

(iii) *If $\overline{N} = V$ is elementary abelian and L is abelian, then*

$$D = \Omega_1(L)$$

is elementary abelian.

Proof. Corollary 5.4, applied to $L \to G \to R$, gives (1). In either of the first two cases, the image of $D = [G, L]$ in $X = G/C$ is $[X, \overline{N}] = \overline{N}$: for S^r this follows from $Z(X) = 1$, and for V from the nontrivial irreducible action established in Proposition 9.13. Hence

$$DC = L, \qquad D/(D \cap C) \cong \overline{N}. \tag{5}$$

Suppose first that $\overline{N} = S^r$. Since the quotient in (5) is perfect, $D = [D, D](D \cap C)$. Thus conjugation by G on D_{ab} is trivial: every element of the abelianization is represented by an element of the central group $D \cap C$. Equation (1) now forces $D_{\mathrm{ab}} = 0$, so D is perfect. Therefore

$$D = [D, D] \le [L, L] \le [G, L] = D,$$

which gives $D = [L, L]$. An element of $Z(D)$ maps to an element centralizing all of S^r, so (5) and $Z(S^r) = 1$ give $Z(D) = D \cap C$. Finally, the homology five-term sequence for the central extension $C \to L \to S^r$ identifies the image of its transgression with

$$C \cap [L, L] = C \cap D.$$

The transgression factors through the R-coinvariants and is the map τ in Proposition 9.13; hence $D \cap C = \operatorname{im} \tau$. This proves (i).

Now suppose that $\overline{N} = V$ and L is nonabelian. The nondegenerate commutator form on V and the surjectivity in (5) show that

$$[D, D] = A.$$

They also show that an element central in D must have zero image in V, so $Z(D) = Z_D$. Put $W = D/A$. It is abelian, its G-action factors through $R = G/L$, and (1) gives $W_R = 0$. Equation (5) gives the exact sequence (3).

Since V has exponent p, multiplication by p defines an R-homomorphism

$$W \longrightarrow Z_D/A.$$

The target is trivial as an R-module, whereas $W_R = 0$, so this homomorphism is zero. Hence $W = D/A$ is elementary abelian. The cyclic group $Z_D \le C$ contains A, and Z_D/A is a subgroup of W; therefore $|Z_D/A| \le p$, proving (4). Also $D^p \le A$ and $[D, D] = A$, so

$$\Phi(D) = D^p[D, D] = A.$$

If (3) split when $Z_D > A$, its nonzero trivial summand Z_D/A would survive in the coinvariants of W, contradicting $W_R = 0$.

Suppose that p is odd. Because D has class two, $[D, D] = A$ has exponent p, and $D/A = W$ is elementary abelian, the power map

$$W \longrightarrow A, \qquad xA \longmapsto x^p$$

is a well-defined R-homomorphism. Its target is trivial as an R-module, so $W_R = 0$ forces this map to vanish. Hence D has exponent p. If $Z_D > A$, a generator of the cyclic group $Z_D \cong C_{p^2}$ would have nontrivial pth power, a contradiction. Thus $Z_D = A$ in odd characteristic.

It remains to prove the assertion about the multiplier. If $Z_D = A$, then

$$Z(D) = [D, D] = \Phi(D) = A \cong C_p,$$

so D is extraspecial. The Schur multiplier of every extraspecial p-group is elementary abelian: for groups of order p^{2m+1} with $m \geq 2$ this is [24, Corollary 1.1], and the order-p^3 cases, including D_8 and Q_8 when $p = 2$, are recorded in [30, Theorem 3.3.6].

Suppose instead that $Z_D \cong C_{p^2}$. The alternating commutator form on $W = D/A$ has radical Z_D/A. Choose a vector-space complement W_0 to this radical, and let E be its inverse image in D. The restriction of the commutator form to W_0 is nondegenerate, whence

$$Z(E) = [E, E] = \Phi(E) = A.$$

Thus E is extraspecial, and

$$D = EZ_D, \qquad E \cap Z_D = A. \tag{6}$$

Let

$$P = E \times Z_D \longrightarrow D$$

be the multiplication map. Its kernel is the diagonal copy $N = \{(a, a^{-1}) : a \in A\}$, while $[P, P] = A \times 1$; hence $N \cap [P, P] = 1$. In the homology five-term sequence for $N \to P \to D$, the map $N \to H_1(P; \mathbb{Z})$ is therefore injective, so $H_2(P; \mathbb{Z}) \to H_2(D; \mathbb{Z})$ is surjective. The integral Kunneth theorem gives

$$H_2(P; \mathbb{Z}) \cong H_2(E; \mathbb{Z}) \oplus H_2(Z_D; \mathbb{Z}) \oplus \big(H_1(E; \mathbb{Z}) \otimes H_1(Z_D; \mathbb{Z})\big).$$

Here $H_2(Z_D; \mathbb{Z}) = 0$, the first summand is elementary abelian by the extraspecial case, and the tensor summand is elementary abelian because $H_1(E; \mathbb{Z}) = E/A$ is. Thus $H_2(P; \mathbb{Z})$, and therefore its quotient $H_2(D; \mathbb{Z})$, is elementary abelian. The universal coefficient theorem identifies $H^3(D; \mathbb{Z})$ with the Pontryagin dual of $H_2(D; \mathbb{Z})$. This proves (ii).

In the abelian case Proposition 9.12 puts $M = \Omega_1(L)$, gives $M_R = 0$, and identifies $G/M \cong \widetilde{R}$. Since $M_R = 0$, one has $M = [G, M] \leq D$. The quotients G/M and G/D both have order $|\widetilde{R}|$, so $M = D$. This proves (iii). □

Proposition 9.16 (All-degree edge bound in the central descent)**.** *Retain the notation above and put $D = [G, L]$. For every $n \geq 2$,*

$$|D|^{\,n-2} \ker\Big(\inf : H^n(\widetilde{R}; \mathbb{Z}) \longrightarrow H^n(G; \mathbb{Z})\Big) = 0. \tag{1}$$

Consequently, any class on $\widetilde{R}$ whose order does not divide $|D|^{n-2}$ survives on G.

The layer structure gives the following sharper conclusions.

(i) *If $\overline{N} = S^r$, then inflation is injective in degree three and, for every $n \geq 4$,*

$$|D|^{\,n-3} \ker\Big(\inf : H^n(\widetilde{R}; \mathbb{Z}) \longrightarrow H^n(G; \mathbb{Z})\Big) = 0. \tag{2}$$

(ii) *If $\overline{N} = V$ and L is nonabelian, inflation is injective in degree three. For $n \geq 4$, put*

$$\beta_n = \begin{cases} 2, & n = 4, \\ 4 + \displaystyle\sum_{j=5}^{n-1} \left(\left\lfloor \frac{j}{2} \right\rfloor + 1 \right), & n \geq 5. \end{cases}$$

Then

$$p^{\beta_n} \ker\Big(\inf : H^n(\widetilde{R};\mathbb{Z}) \longrightarrow H^n(G;\mathbb{Z})\Big) = 0. \tag{3}$$

For the seven consecutive degrees beginning with degree four, the exponents $\big(\beta_{4+j}\big)_{0\le j\le 6}$ *are*

$$2, 4, 7, 11, 15, 20, 25. \tag{4}$$

If p *is odd and* $\dim_{\mathbb{F}_p} V = 2m$*, the exponents in these seven degrees can be replaced by*

$$\begin{array}{ll} m = 1 & (2,4,6,8,10,12,14),\\ m = 2 & (2,4,6,9,12,15,18),\\ m \ge 3 & (2,3,5,8,11,14,17). \end{array} \tag{5}$$

The coefficient bound underlying the last entry is

$$p^3 H^9(D;\mathbb{Z}) = 0 \qquad (p \text{ odd},\ m \ge 3). \tag{5a}$$

If $p = 3$ *and* $m \ge 3$*, let* $E_r^{s,t}$ *denote the integral LHS spectral sequence for* $A \to D \to V$*. Although the last entry of* (5) *requires a separate argument at this prime, the relevant middle filtration term in total degree nine vanishes:*

$$E_\infty^{5,4} = 0. \tag{5b}$$

More precisely, every class except the line represented on the fourth page by $q\omega u^2$ *is killed by the Kudo* d_5*, and an exact width-two filtration calculation followed by restriction shows that this last line also supports a nonzero integral* d_5*. There is also a width-uniform algebraic exclusion for the other middle filtration term. Put*

$$z(2) = P^1 q = \sum_i (X_i^3 y_i - x_i Y_i^3), \qquad w = \beta z(2) = \sum_i (X_i^3 Y_i - X_i Y_i^3).$$

For $m \ge 4$*, multiplication by* $z(2)$ *induces an injection*

$$\frac{\mathcal{A}^3}{\omega\mathcal{A}^1 + q\mathcal{A}^0} \hookrightarrow \frac{\mathcal{A}^{10}}{\omega\mathcal{A}^8 + q\mathcal{A}^7 + w\mathcal{A}^2}. \tag{5b$'$}$$

Thus the known d_2*-,* d_3*-, and* d_5*-transgression ideals cannot create a kernel for the possible* d_7 *from bidegree* $(3,6)$*. Any kernel on the actual seventh page must instead use a bottom-row boundary outside the displayed denominator. The unresolved possibilities are* d_4*- and* d_6*-images and any residual* d_5*-image not supplied by the Kudo family* $q\mathcal{A}^2u^2$*. This does not yet prove that* $E_\infty^{3,6}$ *vanishes. For widths* $m \ge 5$*, however, the bottom-row audit can be completed. If* $\overline{E}_r^{s,t}$ *denotes the mod-3 group spectral sequence, then the actual Kudo differential is injective:*

$$d_7 : \overline{E}_7^{3,6} \hookrightarrow \overline{E}_7^{10,0}, \qquad [au^3] \longmapsto [az(2)]. \tag{5b$''$}$$

Coefficient reduction therefore confines the integral survivor to the reduction-zero subspace:

$$\dim_{\mathbb{F}_3} E_\infty^{3,6} \le 2m, \tag{5b$'''$}$$

and every possible survivor in this bidegree is represented on the fourth page by a class in $\omega V^* u^3$*. For the first four widths at the prime three, a reproducible integral calculation gives the exact next coefficient:*

$$H^9(D;\mathbb{Z}) \cong \begin{cases} (C_3)^6, & m = 1,\\ (C_3)^{85}, & m = 2,\\ (C_3)^{679}, & m = 3,\\ (C_3)^{3198}, & m = 4. \end{cases} \tag{5c}$$

Consequently, at $p = 3$ the tuples for these four widths improve to

$$\begin{array}{ll} m = 1 & (2, 4, 6, 8, 10, 12, 13), \\ m = 2 & (2, 4, 6, 9, 12, 15, 16), \\ m = 3, 4 & (2, 3, 5, 8, 11, 14, 15). \end{array} \tag{5d}$$

At $p = 2$, write D_m^+ and D_m^- for the extraspecial groups of plus and minus type and width m, respectively. If $D = D_m^+$ occurs in the central descent, then $m \geq 3$, and the tuple in (4) *can be replaced by*

$$\begin{array}{ll} m = 3 & (2, 4, 5, 6, 7, 11, 12), \\ m \geq 4 & (2, 4, 5, 6, 7, 10, 11). \end{array} \tag{5e}$$

For an extraspecial 2-group P, put

$$\epsilon_j(P) = \min\{e \geq 0 : 2^e H^j(P; \mathbb{Z}) = 0\}.$$

An exact filtered-resolution calculation gives

P	$(\epsilon_2(P), \ldots, \epsilon_9(P))$
D_2^-	$(1, 1, 2, 1, 1, 1, 4, 1)$
D_3^+	$(1, 1, 2, 1, 1, 1, 4, 1)$
D_3^-	$(1, 1, 2, 1, 1, 1, 3, 1)$.

(5f)

Thus the edge tuples for D_2^- and D_3^- improve respectively to the first and second tuples in (5e). *These finite-width certificates do not by themselves give a uniform bound for all minus-type or almost-extraspecial kernels. For every $m \geq 3$, restriction and transfer through a plus-type central product give the componentwise uniform bound*

$$(\epsilon_2(D_m^-), \ldots, \epsilon_9(D_m^-)) \leq (1, 1, 2, 3, 3, 3, 5, 3). \tag{5g}$$

Consequently, for every minus-type width $m \geq 4$, the tuple in (4) *can be replaced by*

$$(2, 4, 7, 10, 13, 18, 21). \tag{5h}$$

The exact width-three row in (5f) *is sharper than this uniform comparison. Put $\mathsf{A}_m = D_m^- \circ C_4$, the almost-extraspecial group of width m. There is a normal index-two embedding*

$$\mathsf{A}_m \lhd D_{m+1}^+$$

whose quotient involution τ is trivial on $\mathsf{A}_m/Z(\mathsf{A}_m)$ and acts by inversion on $Z(\mathsf{A}_m) \cong C_4$. If $(a_2, \ldots, a_9)$ is

$$\begin{cases} (1, 1, 2, 1, 1, 1, 4, 1), & m = 2, \\ (1, 1, 2, 1, 1, 1, 3, 1), & m \geq 3, \end{cases}$$

then

$$2^{a_j}(1 + \tau)H^j(\mathsf{A}_m; \mathbb{Z}) = 0 \qquad (2 \leq j \leq 9). \tag{5i}$$

At the first permitted width, an exact calculation gives

$$(\epsilon_2(\mathsf{A}_2), \ldots, \epsilon_9(\mathsf{A}_2)) = (1, 1, 2, 1, 1, 1, 4, 1). \tag{5j}$$

In every remaining width one has the full coefficient bound

$$(\epsilon_2(\mathsf{A}_m), \ldots, \epsilon_9(\mathsf{A}_m)) \leq (1, 1, 2, 2, 2, 2, 4, 2) \qquad (m \geq 3). \tag{5k}$$

Consequently the corresponding seven-entry edge tuples for an almost-extraspecial kernel can be replaced by

$$\begin{array}{ll} m = 2 & (2,4,5,6,7,11,12), \\ m \geq 3 & (2,4,6,8,10,14,16). \end{array} \tag{5l}$$

Formula (5i) *by itself controls only the norm subgroup and does not see the τ-anti-invariant hidden extensions. The full bound* (5k) *instead comes from restriction and transfer through index-two extraspecial subgroups of A_m.*

There is also a sharper order bound in degree four. If $d_2^{3,0}$ is the homology differential

$$d_2^{3,0} : H_3(R;\mathbb{Z}) \longrightarrow H_1(R;V),$$

set

$$b = \dim_{\mathbb{F}_p} \operatorname{im} d_2^{3,0}, \qquad a = \dim_{\mathbb{F}_p} \bigl(\textstyle\bigwedge^2 V^*\bigr)^R.$$

The commutator form gives $a \geq 1$, and in every nonabelian elementary case

$$\left|\ker\Bigl(\inf : H^4(\widetilde{R};\mathbb{Z}) \longrightarrow H^4(G;\mathbb{Z})\Bigr)\right| \leq p^{a+b-1}. \tag{6}$$

More precisely, if $h = \dim_{\mathbb{F}_p} H^1(R;V^*)$, then $h = b$ when D is extraspecial and $h = b+1$ when D is almost extraspecial. In particular, the degree-four edge is injective if $b = 0$ and $a = 1$.

(iii) *If $\overline{N} = V$ and L is abelian, then*

$$p^{\,n-2} \ker\Bigl(\inf : H^n(\widetilde{R};\mathbb{Z}) \longrightarrow H^n(G;\mathbb{Z})\Bigr) = 0 \tag{7}$$

for every $n \geq 2$.

Proof. Proposition 9.13 identifies $\widetilde{R}$ with G/D. Formula (1) is therefore Proposition 5.5.

If $\overline{N} = S^r$, Proposition 9.15(i) makes D perfect, so $H^2(D;\mathbb{Z}) = 0$. In the inflation-edge filtration the d_3-source therefore vanishes, leaving at most $n-3$ successive quotients, for $4 \leq r \leq n$. Each is annihilated by $|D|$, proving (2); for $n = 3$ no source remains.

Suppose next that $\overline{N} = V$ and L is nonabelian. Proposition 9.15(ii) gives a central extension

$$1 \longrightarrow A \longrightarrow D \longrightarrow W = D/A \longrightarrow 1$$

with $A \cong C_p$ and W elementary abelian. In the integral cohomology spectral sequence for this extension, every nonzero positive-total-degree E_2-term is an elementary abelian p-group: this is true on the bottom row by the integral Kunneth theorem, and on the other rows because

$$H^b(C_p;\mathbb{Z}) = \begin{cases} C_p, & b > 0 \text{ even}, \\ 0, & b \text{ odd}. \end{cases}$$

There are at most $\lfloor j/2 \rfloor + 1$ nonzero filtration terms in total degree j. Hence

$$p^{\lfloor j/2 \rfloor + 1} H^j(D;\mathbb{Z}) = 0 \qquad (j \geq 5). \tag{8}$$

Moreover $H^2(D;\mathbb{Z})$ is dual to $D_{\mathrm{ab}} = D/A = W$, so it is elementary abelian, and Proposition 9.15(ii) shows that $H^3(D;\mathbb{Z})$ is elementary abelian as well.

We claim that the next coefficient always satisfies

$$p^2 H^4(D;\mathbb{Z}) = 0. \tag{9}$$

Suppose first that p is odd. Then D is extraspecial of exponent p, with $D/A = V$. In the spectral sequence for $A \to D \to V$, the transgression sends a generator $u \in H^2(A;\mathbb{Z})$ to the nonzero commutator class in $H^3(V;\mathbb{Z})$. The Leibniz rule gives

$$d_3(u^2) = 2u\, d_3(u) \neq 0;$$

the same nonzero L^2_ω differential is displayed in the proof of [29, Lemma 3.3]. Thus the $(0,4)$ term does not survive. Only the elementary abelian terms in bidegrees $(4,0)$ and $(2,2)$ can contribute to $H^4(D;\mathbb{Z})$, proving (9) in odd characteristic.

Now let $p = 2$. The squaring function

$$Q : W = D/A \longrightarrow A \cong \mathbb{F}_2, \qquad Q(xA) = x^2,$$

is a quadratic form whose polar form is commutation. The R-action preserves Q. If D is extraspecial, then $W = V$ has dimension $2m$ and Q is nondegenerate. The cases $m = 1$ cannot occur: their orthogonal groups are $O_2^+(2) \cong C_2$ and $O_2^-(2) \cong S_3$. Nor can the plus type occur when $m = 2$, since

$$O_4^+(2) \cong (S_3 \times S_3) \rtimes C_2$$

is solvable. Indeed, the image of the perfect group R in any of these solvable groups would be trivial, contrary to the nontrivial irreducible action on V. For $m \geq 3$, and for the minus type when $m = 2$, the calculation in [29, Section 3.3] shows directly that $H^4(D;\mathbb{Z})$ has exponent at most 4.

It remains to consider the almost-extraspecial case $Z_D \cong C_4$. The radical of the polar form on W is the line

$$T = Z_D/A,$$

and Q is nonzero on T. The quotient $W/T = V$ is symplectic of dimension $2m$. Here $m = 1$ is impossible because $\mathrm{Sp}_2(2) \cong S_3$ is solvable and the R-action on V is nontrivial. Thus $m \geq 2$. Choose a vector-space complement W_0 to T on which Q has minus type, and let E be its inverse image in D. Such a choice is always possible: replacing a complement by the graph of a linear form changes the quadratic refinement of the fixed nondegenerate polar form, and both Arf invariants occur. The group E is extraspecial of minus type and width m.

Compare the spectral sequences for $A \to E \to W_0$ and $A \to D \to W$. In the first sequence the differential

$$d_5 : E_5^{0,4} \longrightarrow E_5^{5,0}$$

is nonzero: this is the minus-type calculation for $m = 2$ and the uniform calculation for $m \geq 3$ in [29, Section 3.3]. Naturality forces the same differential for D to be nonzero. Hence the $(0,4)$ term again does not survive, and the total-degree-four filtration has only the two elementary abelian subquotients from bidegrees $(4,0)$ and $(2,2)$. This completes the proof of (9).

We can sharpen all the remaining low-degree coefficients uniformly for the plus-type extraspecial 2-groups. Let D_m^+ be the central product of m copies of D_8, let $V = D_m^+/Z(D_m^+)$, and let J_V be the ideal of $H^*(D_m^+;\mathbb{Z})$ generated by the positive-degree image of $H^*(V;\mathbb{Z})$. Harada and Kono's additive calculation [23], in the form restated in [40, Theorem 3.1], says that

$$C(m)^* = H^*(D_m^+;\mathbb{Z})/J_V$$

embeds additively in $H^*(D_m^+;\mathbb{Z})$. In the degrees needed here it gives

$$C(m)^k \cong C_{2^{\nu_2(k)}} \quad \text{if } \nu_2(k) \le m-1, \qquad C(3)^8 \cong C_{16}. \tag{9j}$$

Here C_1 denotes the zero group. Every positive-degree group $H^*(V;\mathbb{Z})$ is elementary abelian, so the ideal J_V is annihilated by 2. It follows from (9j) that

$$(\epsilon_2(D_m^+),\dots,\epsilon_9(D_m^+)) = \begin{cases} (1,1,2,1,1,1,4,1), & m=3, \\ (1,1,2,1,1,1,3,1), & m\ge 4. \end{cases} \tag{9k}$$

The exclusions preceding (9) show that a plus-type kernel in the central descent has $m \ge 3$. Taking partial sums of the coefficient exponents in (9k) proves (5e).

For comparison, the small-width calculation in (5f) is reproduced by

```
gap -q -c 'maxwidth:=3;; Read("scripts/check_extraspecial_2_primary.g");'
```

For each $P = D_m^{\pm}$, the script constructs `ResolutionNormalSeries([P,Center(P)],9)`, tensors the filtered resolution with the integers, and computes the Smith factors of $H_1(P;\mathbb{Z}),\dots,H_8(P;\mathbb{Z})$. Lemma 1.1 transfers the largest invariant factors to $H^2(P;\mathbb{Z}),\dots,H^9(P;\mathbb{Z})$ and gives exactly the three vectors in (5f). Their partial sums give the two stated edge tuples. The plus-type rows also independently check (9k) in the first computed widths. The calculation alone does not propagate the minus-type rows to unbounded width, and it does not cover the almost-extraspecial family.

There is nevertheless a uniform comparison for the minus type. Let z generate the common center of D_m^- and Q_8, and form the central product

$$P = D_m^- \circ Q_8 = \frac{D_m^- \times Q_8}{\langle (z,z^{-1}) \rangle}. \tag{9l}$$

The images of the two factors centralize one another, and the first factor embeds as a normal subgroup $D_m^- \lhd P$ of index four. The quadratic forms of D_m^- and Q_8 both have Arf invariant one, so the quadratic form of their central product is their orthogonal sum and has Arf invariant zero. Hence

$$P \cong D_{m+1}^+.$$

Moreover, conjugation by P on D_m^- is inner: the second central-product factor centralizes the first. The Mackey restriction–corestriction formula therefore gives, for every $x \in H^j(D_m^-;\mathbb{Z})$,

$$\operatorname{res}^P_{D_m^-} \operatorname{cor}^P_{D_m^-}(x) = 4x. \tag{9m}$$

For $m \ge 3$, equation (9k) applied to the plus group $P = D_{m+1}^+$ says that the coefficient exponents in degrees two through nine are bounded by

$$(1,1,2,1,1,1,3,1).$$

Equation (9m) adds at most two to each of these exponents. In degrees two, three, and four, the sharper bounds already proved above are $1,1,2$; in degrees five through nine, intersecting the comparison with the all-width estimate (8) gives

$$(\epsilon_2(D_m^-),\dots,\epsilon_9(D_m^-)) \le (1,1,2,3,3,3,5,3).$$

This is (5g), and partial sums give (5h).

The same plus-type overgroup also records a useful norm comparison for the almost-extraspecial case. Choose a cyclic subgroup $C_4 = \langle i \rangle$ in the Q_8 factor of (9l). The image

$$\mathsf{A}_m = D_m^- \circ \langle i \rangle$$

is normal of index two in $P = D_{m+1}^+$. If $j \in Q_8 \setminus \langle i \rangle$, conjugation by its image induces an involution τ on A_m. It centralizes the D_m^- factor and sends i to i^{-1}. Consequently, τ is the identity on $\mathsf{A}_m/Z(\mathsf{A}_m)$ and inversion on its cyclic center of order four. The Mackey formula is now

$$\operatorname{res}^P_{\mathsf{A}_m} \operatorname{cor}^P_{\mathsf{A}_m}(x) = (1+\tau)x. \tag{9n}$$

The plus-type coefficient bounds (9k) annihilate the corestriction on the right of (9n) and give (5i). Unlike (9m), this formula does not equal multiplication by the index: the possible τ-anti-invariant part is invisible to the norm. Although τ acts trivially on the associated graded of the central extension spectral sequence for

$$C_2 = [\mathsf{A}_m, \mathsf{A}_m] \longrightarrow \mathsf{A}_m \longrightarrow (C_2)^{2m+1},$$

this does not remove the obstruction, because inversion on a cyclic 2-power extension can be trivial on every elementary associated-graded quotient.

That obstruction belongs only to the overgroup norm comparison; it does not prevent a full coefficient bound. Indeed, the elementary rank-one central products satisfy

$$D_8 \circ C_4 \cong Q_8 \circ C_4. \tag{9o}$$

To see this directly, write

$$D_8 = \langle r, s : r^4 = s^2 = 1,\ srs = r^{-1} \rangle, \qquad C_4 = \langle c \rangle,$$

and identify $r^2 = c^2$ in the central product. Then r and sc both square to c^2, and $(sc)r(sc)^{-1} = r^{-1}$, so $\langle r, sc \rangle \cong Q_8$. Together with c this subgroup generates the whole central product, which proves (9o). Associativity of central products now gives

$$\mathsf{A}_m = D_m^- \circ C_4 \cong D_m^+ \circ C_4. \tag{9p}$$

Thus A_m contains an index-two copy of each of D_m^- and D_m^+. For either such subgroup $E \leq \mathsf{A}_m$, the usual restriction–corestriction identity is

$$\operatorname{cor}^{\mathsf{A}_m}_E \operatorname{res}^{\mathsf{A}_m}_E = 2\ \mathrm{id}_{H^j(\mathsf{A}_m;\mathbb{Z})}. \tag{9q}$$

If $m = 3$, choose $E = D_3^-$ and use its exact row in (5f); if $m \geq 4$, choose $E = D_m^+$ and use (9k). In either case

$$(\epsilon_2(E), \dots, \epsilon_9(E)) \leq (1,1,2,1,1,1,3,1).$$

For $x \in H^j(\mathsf{A}_m;\mathbb{Z})$, equation (9q) therefore gives

$$2^{\epsilon_j(E)+1} x = \operatorname{cor}^{\mathsf{A}_m}_E \big(2^{\epsilon_j(E)} \operatorname{res}^{\mathsf{A}_m}_E x\big) = 0.$$

Intersecting these bounds with the already proved elementary bounds in degrees two and three, the exponent-four bound in degree four, and (8) in the remaining degrees gives (5k). Taking partial sums gives the second row of (5l).

The first permitted width is checked exactly by

```
gap -q -c 'minwidth:=2;; maxwidth:=2;; \
  Read("scripts/check_almost_extraspecial_2_primary.g");'
```

The script constructs $\mathsf{A}_2 = (D_2^- \times C_4)/C_2$, uses the filtered normal-series resolution for its central derived subgroup, and computes the Smith factors of integral homology through degree eight. Duality gives (5j), whose partial sums give the first row of (5l).

We next record three odd-primary improvements needed below. Suppose p is odd, write $\dim_{\mathbb{F}_p} V = 2m$, and choose a symplectic basis

$$x_1, y_1, \dots, x_m, y_m$$

of $V^* = H^1(V;\mathbb{F}_p)$, with the commutator extension represented by $\omega = \sum_i x_i y_i$. Write $X_i = \beta x_i$ and $Y_i = \beta y_i$ in $H^2(V;\mathbb{F}_p)$. If u generates $H^2(A;\mathbb{Z})$, the reduction modulo p of $d_3(u)$ is

$$q = \beta\omega = \sum_i (X_i y_i - x_i Y_i) \in H^3(V;\mathbb{F}_p). \tag{9a}$$

The multiplication map

$$V^* \longrightarrow H^4(V;\mathbb{F}_p), \qquad f \longmapsto fq \tag{9b}$$

is injective. Indeed, after a symplectic change of basis a nonzero f is x_1, and the monomial $x_1 X_1 y_1$ occurs in fq and cannot cancel.

For $k \geq 2$, the multiplicative LHS spectral sequence gives

$$d_3(u^k) = kqu^{k-1}, \qquad d_3(fu^k) = (-1)^{|f|} kfqu^{k-1}. \tag{9c}$$

Because 2 and 4 are units modulo the odd prime p, equations (9a)–(9c) kill the top filtration term in total degrees five, eight, and nine. Every positive-total-degree term on the E_2-page is elementary abelian, so counting the remaining filtration pieces gives

$$p^2 H^5(D;\mathbb{Z}) = 0, \qquad p^4 H^8(D;\mathbb{Z}) = p^4 H^9(D;\mathbb{Z}) = 0. \tag{9d}$$

If $p \neq 3$, the same argument with $k = 3$ kills the top filtration terms in total degrees six and seven, and therefore

$$p^3 H^6(D;\mathbb{Z}) = p^3 H^7(D;\mathbb{Z}) = 0. \tag{9d'}$$

There is a further improvement in the next two degrees when $p \geq 5$ and $m \geq 2$:

$$p^3 H^8(D;\mathbb{Z}) = p^3 H^9(D;\mathbb{Z}) = 0. \tag{9d+}$$

To see this, put

$$\mathcal{A} = H^*(V;\mathbb{F}_p) = \mathbb{F}_p[X_1, Y_1, \dots, X_m, Y_m] \otimes \Lambda(x_1, y_1, \dots, x_m, y_m).$$

Multiplication by

$$q = \sum_i (X_i y_i - x_i Y_i)$$

makes $\mathcal{A}$ a Koszul complex for the regular sequence $X_1, Y_1, \dots, X_m, Y_m$. Its homology is one-dimensional, represented by $x_1 y_1 \cdots x_m y_m$ in total degree $2m$. In particular, for $m \geq 2$, multiplication by q is injective in degrees one and two, and its kernel in degree three is $q\mathcal{A}^0$.

In total degree eight, the d_3-differentials on the terms in bidegrees $(0,8)$ and $(2,6)$ are multiplication by $4q$ and $3q$, respectively. Both terms therefore vanish on the fourth page. Only the three

elementary abelian filtration terms in bidegrees $(8,0),(6,2),(4,4)$ can remain. In total degree nine, the term in bidegree $(1,8)$ is killed by multiplication by $4q$, while exactness of

$$\mathcal{A}^0u^4 \xrightarrow{\,\cdot 4q\,} \mathcal{A}^3u^3 \xrightarrow{\,\cdot -3q\,} \mathcal{A}^6u^2$$

shows that the term in bidegree $(3,6)$ also vanishes on the fourth page. Again only three elementary abelian filtration terms remain. This proves (9d+).

The missing degree-six assertion at $p=3$ follows from a coefficient comparison rather than from the Leibniz rule. Assume $p=3$ and $m\geq 2$, and write ${}^{\mathbb{Z}}E_r$ and ${}^{\mathbb{F}_3}E_r$ for the integral and mod-3 spectral sequences of $A\to D\to V$. Reduction of coefficients gives a morphism

$${}^{\mathbb{Z}}E_r \longrightarrow {}^{\mathbb{F}_3}E_r. \tag{9d"a}$$

On the E_2-page it sends the integral fiber generator u^3 to the mod-3 polynomial generator with the same name. The class $u^3\in {}^{\mathbb{Z}}E_2^{0,6}$ has no incoming differential and $d_3(u^3)=3qu^2=0$. The d_2, d_4, and d_6 targets vanish because the positive odd integral cohomology of A vanishes. Thus either an integral d_5 kills u^3, or it reaches ${}^{\mathbb{Z}}E_7$.

In the mod-3 spectral sequence for the width-two extraspecial group, the Cartan–Serre–Kudo calculation gives

$$d_7(u^3)=z(2)=P^1(\beta\omega)\neq 0;$$

see [43, equation (2.3), the paragraph following equation (3.5), and Section 4]. For larger m, restrict to the inverse image of a nondegenerate four-dimensional symplectic subspace; naturality gives the same nonzero differential. Consequently, if the integral class reaches page seven, then (9d"a) gives

$$\rho\big(d_7^{\mathbb{Z}}(u^3)\big)=d_7^{\mathbb{F}_3}\big(\rho(u^3)\big)=P^1(\beta\omega)\neq 0.$$

It is therefore killed by the integral d_7. In either alternative the $(0,6)$ term does not reach E_∞. The other three filtration terms in total degree six are elementary abelian, so

$$3^3H^6(D;\mathbb{Z})=0 \qquad (m\geq 2). \tag{9d"}$$

The same comparison, with one additional localization argument, gives the degree-seven bound

$$3^3H^7(D;\mathbb{Z})=0 \qquad (m\geq 2). \tag{9d"'}$$

We first prove the required assertion about the mod-3 spectral sequence. In width two, write $D=E_2$ in the presentation of [43, Section 3], and let

$$M=\langle c,a_1,a_3\rangle\cong C_3^3$$

be the maximal elementary abelian subgroup over the split isotropic plane. Let e be Yagita's product of the Bocksteins of the nonzero linear forms in $\langle x_1,x_3\rangle$. Its restriction to M is a nonzero polynomial. Yagita's formula for the actual group spectral sequence says that u^3 reaches page seven and gives

$$d_7(x_1u^3)=-x_1z(2), \qquad z(2)=P^1(\beta\omega). \tag{9d"'a}$$

Indeed, x_1 is permanent, and multiplicativity shows that x_1u^3 reaches page seven; it is nonzero there because no differential can enter bidegree $(1,6)$. This differential cannot vanish. Otherwise x_1u^3 would be a permanent cycle, since no differential of length at least eight can enter or leave bidegree $(1,6)$. It would remain nonzero after localizing at e: restriction to the split extension

$C_3 \to M \to C_3^2$ sends $e^k x_1 u^3$ to the nonzero class with the same name for every k. On the other hand, Yagita's [43, Corollary 3.11] identifies the localized cohomology as

$$[e^{-1}]H^*(E_2;\mathbb{F}_3) \cong [e^{-1}]S(2)[u^9] \otimes \Lambda(x_1,x_3) \otimes \Lambda(z). \tag{9d"'b}$$

This calculation is obtained from the spectral sequences (3.8) and (3.11) of that paper and retains the central fiber filtration: the displayed generators from $S(2)$ and x_1, x_3 have fiber degree zero, u^9 has fiber degree eighteen, and z has fiber degree one. Consequently its localized associated graded has no term of fiber degree six. This contradicts the permanent localized class represented by x_1u^3 and proves that (9d"'a) is nonzero.

The symplectic group $\mathrm{Sp}_4(3)$ acts on this spectral sequence, the class $z(2)$ is invariant, and the natural module V^* is irreducible. Hence

$$d_7 : V^*u^3 = E_7^{1,6} \longrightarrow E_7^{8,0}, \qquad fu^3 \longmapsto -fz(2), \tag{9d"'c}$$

is injective. For width $m > 2$, restriction to the inverse image of a nondegenerate four-dimensional symplectic subspace shows that this map is nonzero; irreducibility of the natural $\mathrm{Sp}_{2m}(3)$-module again makes it injective.

Return to the integral spectral sequence. Its term $E_2^{1,6} = V^*u^3$ has no incoming differential. The d_2, d_4, and d_6 targets vanish, while $d_3(fu^3) = -3fqu^2 = 0$. Thus a class is either killed by d_5 or reaches page seven. In the latter case its mod-3 reduction is a nonzero element of $E_7^{1,6}$, because that bidegree has no incoming mod-3 differential; injectivity in (9d"'c) and naturality of coefficient reduction force its integral d_7 to be nonzero. Hence $E_\infty^{1,6} = 0$. The other three possible filtration terms in total degree seven are elementary abelian, proving (9d"').

At $p = 3$ one can also sharpen the next even degree:

$$3^3H^8(D;\mathbb{Z}) = 0 \qquad (m \geq 2). \tag{9d3+}$$

The width-two case is already contained in (9g), so assume $m \geq 3$. In total degree eight, $d_3(u^4) = 4qu^3$ kills the $(0,8)$ term. The Koszul calculation above shows that the kernel of

$$d_3 : \mathcal{A}^4u^2 \longrightarrow \mathcal{A}^7u$$

is $q\mathcal{A}^1u^2$; there is no incoming d_3 because the coefficient on u^3 is three. We claim that d_5 is injective on this kernel.

Write

$$w = \beta P^1 q = \sum_i (X_i^3Y_i - Y_i^3X_i) \in \mathcal{A}^8.$$

The Cartan–Serre–Kudo formula in the mod-3 group spectral sequence gives

$$d_5(qfu^2) = fw \qquad (f \in V^*). \tag{9d3+a}$$

We verify that the right side is nonzero on the fifth page. Filter $\mathcal{A}$ by exterior degree. The d_2-boundaries on the bottom row are multiples of ω and have exterior degree at least two; the exterior-degree-one part of the d_3-boundaries is Sq. For a possible d_4-boundary, Leary's formula is a triple Massey product $\langle q, \chi, \omega\rangle$, where $\omega\chi = q\chi = 0$; see [33, Theorem 3]. The exterior-degree-one part of χ is zero because multiplication by the symplectic form $\omega : V^* \to \bigwedge^3 V^*$ is injective. Leary's factorwise description of the Massey products of an elementary abelian group at the prime three [33, Corollary 6, Proposition 7, and Corollary 8] then shows that this triple product has exterior degree at least three. Thus the exterior-degree-one quotient of the bottom row on page five is

$$(S \otimes V^*)/Sq, \qquad S = \mathbb{F}_3[X_1, Y_1, \ldots, X_m, Y_m]. \tag{9d3+b}$$

For nonzero $f \in V^*$, the vector wf is not a polynomial multiple of $q = \sum_i (X_i y_i - x_i Y_i)$: over the fraction field of S such an equality would make the vector of the algebraically independent coefficients X_i, Y_i proportional to the constant vector f. Hence fw is nonzero in (9d3+b).

The source qfu^2 is also nonzero through page five. Modulo the d_2-image this follows because the decomposable alternating form $q \wedge f$ cannot be a polynomial multiple of the nondegenerate form ω in dimension at least six. It survives d_3 since $q^2 = 0$, and it survives d_4 by multiplicativity and the fact that qu^2 supports the Kudo differential. Nor can it be an incoming d_4-boundary: the only possible source in bidegree $(0,7)$ has already been killed by $d_2(zu^3) = \omega u^3$. Thus (9d3+a) is injective on qV^*u^2. Reduction of coefficients now forces the corresponding integral d_5 to be injective on the integral d_3-kernel. Therefore $E_\infty^{4,4} = 0$. Together with the vanished $(0,8)$ term, this leaves only the three elementary abelian filtration terms in bidegrees $(8,0), (6,2), (2,6)$ and proves (9d3+).

We next isolate the middle total-degree-nine obstruction at $p = 3$ and $m \geq 3$. In the integral spectral sequence, the d_3-homology in bidegree $(5,4)$ is

$$E_4^{5,4} = q\mathcal{A}^2 u^2. \tag{9d3+c}$$

Indeed, d_3 on this term is multiplication by $2q$, the incoming coefficient on u^3 is three, and the Koszul calculation above is exact in degree five because its only homology has degree $2m \geq 6$.

We use two elementary symplectic-algebra identities. Filter $\mathcal{A} = S \otimes \Lambda(V^*)$ by exterior degree, and write $F_{\mathrm{ext}}^{\geq 3}\mathcal{A}$ for exterior degree at least three. Then

$$q\mathcal{A}^2 \cap \omega\mathcal{A}^3 = \langle q\omega \rangle, \tag{9d3+d}$$

$$\{a \in \mathcal{A}^2 : aw \in \omega\mathcal{A}^8 + q\mathcal{A}^7 + F_{\mathrm{ext}}^{\geq 3}\mathcal{A}^{10}\} = \langle \omega \rangle. \tag{9d3+e}$$

Here is a direct verification. Write $a = s + \eta$ with $s \in S^2$ and $\eta \in \bigwedge^2 V^*$. Exterior bidegrees first force $s = 0$ in either possible intersection. Regard the polynomial coefficients of q as the coordinates of a generic covector ℓ and pass to the fraction field of S. The first identity says that $\ell \wedge \eta$ lies in $\omega \wedge V^*$ for generic ℓ; the second says that η lies in $\langle \omega \rangle + \ell \wedge V^*$ for generic ℓ. In the second assertion one may invert the nonzero polynomial w. For a symplectic space of dimension at least six, the elementary coefficient identities

$$\{\eta : \ell \wedge \eta \in \omega \wedge V^* \text{ for every } \ell\} = \langle \omega \rangle, \qquad \bigcap_{\ell \neq 0} \big(\langle \omega \rangle + \ell \wedge V^*\big) = \langle \omega \rangle$$

follow by taking ℓ successively among the vectors of a symplectic basis and their pairwise sums. They prove (9d3+d) and (9d3+e); the reverse inclusions are immediate.

Reduce (9d3+c) to the mod-3 spectral sequence. Formula (9d3+d) shows that its kernel after the d_2-quotient is exactly the line $\langle q\omega u^2 \rangle$. There is no incoming d_4: its possible source is in bidegree $(1,7)$, and that term was already killed by multiplication $\omega : V^* \to \bigwedge^3 V^*$. Nor is there an outgoing d_4 on qau^2. In the notation of [33, Theorem 3], take $\chi = qa$; then $q\chi = q^2 a = 0$, so part (b) of that theorem gives zero. The Kudo differential is therefore

$$d_5(qau^2) = aw. \tag{9d3+f}$$

As in the proof of (9d3+a), every d_4-boundary on the bottom row has exterior degree at least three. Formula (9d3+e) now shows that (9d3+f) is injective away from the single class $q\omega u^2$. Naturality of coefficient reduction forces the same conclusion for the integral d_5. Thus $E_6^{5,4}$ is a quotient of that one-dimensional line. We shall kill this last line by the width-two filtered calculation below, thereby completing (5b).

We next prove the algebraic assertion (5b′). In the graded algebra $\mathcal{A}$ put

$$I_m = (\omega, q, w).$$

The assertion is equivalently the degree-three colon identity

$$(I_m : z(2))^3 = (\omega, q)^3. \tag{9d3+g}$$

The width-four calculation is exact and reproducible with the standard Python interpreter:

```
python3 scripts/check_extraspecial_d7_colon.py 4
```

The script enumerates the monomial bases of the polynomial–exterior algebra and performs sparse Gaussian elimination over $\mathbb{F}_3$. It returns

```
width=4
dim A^3=120
dim domain=111
dim A^10=19448
rank (omega,q,w)^10=7185
rank multiplication by z2=111
kernel dimension=0
```

Thus (9d3+g) holds in width four.

It remains to explain why width four detects every larger width in this degree. Write

$$W = \bigoplus_{i=1}^{m} W_i, \qquad W_i = \langle x_i, y_i \rangle, \qquad P = \bigoplus_{i=1}^{m} P_i, \qquad P_i = \langle X_i, Y_i \rangle.$$

For each four-element subset $J \subseteq \{1, \dots, m\}$, restriction to $\bigoplus_{j \in J} W_j$ induces a map

$$\frac{\mathcal{A}^3_m}{\omega_m W + \langle q_m \rangle} \longrightarrow \frac{\mathcal{A}^3_J}{\omega_J W_J + \langle q_J \rangle}. \tag{9d3+h}$$

The product of the maps (9d3+h) over all J is injective. Indeed,

$$\mathcal{A}^3_m = \bigwedge^3 W \oplus (P \otimes W),$$

and the two displayed relations lie in the two respective summands. Suppose first that $b \in P \otimes W$ restricts to a scalar multiple of q_J for every J. Its component in $P_i \otimes W_j$ vanishes for $i \neq j$, since a four-set can be chosen containing i and j and q_J has no such cross component. Its component in $P_i \otimes W_i$ is a scalar multiple of

$$q_i = X_i y_i - x_i Y_i.$$

Choosing a four-set containing any prescribed pair i, k shows that these scalars agree. Hence b is a scalar multiple of q_m.

Now suppose that $\eta \in \bigwedge^3 W$ restricts into $\omega_J W_J$ for every J. A form in $\omega_J W_J$ has no component using one vector from each of three distinct summands W_i, W_j, W_k; choosing J containing i, j, k shows that all such components of η vanish. Since $\dim W_i = 2$, there are then unique vectors $f_{i,r} \in W_r$ such that

$$\eta = \sum_{i \neq r} \omega_i f_{i,r}, \qquad \omega_i = x_i y_i.$$

On a four-set containing distinct i, k, r, comparison with $\omega_J f_J$ gives $f_{i,r} = f_{k,r}$. Thus, for fixed r, these vectors have a common value f_r, and

$$\eta = \sum_{i \neq r} \omega_i f_r = \omega_m \Big(\sum_r f_r \Big),$$

because $\omega_r f_r = 0$ in $\bigwedge^3 W_r$. This proves the claimed injectivity.

If $a \in \mathcal{A}^3_m$ satisfies $az(2)_m \in I_m$, its restriction to every coordinate width-four subalgebra satisfies the same relation. The verified width-four identity puts every restricted a in $(\omega_J, q_J)^3$, and the injectivity just proved puts a in $(\omega_m, q_m)^3$. This proves (9d3+g) for all $m \geq 4$.

In the mod-3 group spectral sequence the transgression targets $d_2(z) = \omega$, $d_3(u) = q$, and $d_5(qau^2) = aw$ account for subspaces of the three ideals in the target of the possible d_7 from bidegree $(3, 6)$. Quotienting by the full ideal I_m only enlarges those known boundary subspaces, so the injectivity just proved is the stronger test for this displayed family. A bottom-row boundary outside I_m may still arise from a d_4- or d_6-image, or from a residual d_5-source not represented by $q\mathcal{A}^2u^2$. This proves the final assertion following (5b′) and records precisely why it does not yet imply $E_\infty^{3,6} = 0$.

We now complete that bottom-row audit when $m \geq 5$. Let $\overline{E}_r$ denote the mod-3 group spectral sequence, and retain z for the exterior fiber generator and u for the polynomial fiber generator. After $d_2(z) = \omega$, its even and odd rows are

$$\overline{E}_3^{s,2k} = \big(\mathcal{A}^s / \omega \mathcal{A}^{s-2}\big) u^k, \tag{9d3+i}$$

$$\overline{E}_3^{s,2k+1} = \ker\big(\omega : \mathcal{A}^s \to \mathcal{A}^{s+2}\big) z u^k, \tag{9d3+j}$$

and d_3 is multiplication by the appropriate scalar multiple of q.

Consider first a class in bidegree $(5, 4)$, represented by au^2. The condition that it reach the fourth page is

$$qa = \omega b \qquad \text{for some } b \in \mathcal{A}^6. \tag{9d3+k}$$

Leary's formula, Theorem 3(b) of [33], gives

$$d_4(au^2) = 2zqb. \tag{9d3+l}$$

Here the target is

$$\overline{E}_4^{9,1} = \frac{\ker(\omega : \mathcal{A}^9 \to \mathcal{A}^{11}) z}{q \ker(\omega : \mathcal{A}^6 \to \mathcal{A}^8) z}. \tag{9d3+m}$$

Suppose that (9d3+l) is zero in this target. Then $qb = qc$ for some $c \in \ker(\omega : \mathcal{A}^6 \to \mathcal{A}^8)$. Multiplication by q has homology only in degree $2m$, so it is exact in degrees five and six when $m \geq 5$. Hence

$$b - c = qe \quad (e \in \mathcal{A}^3).$$

Using (9d3+k) and $\omega c = 0$ gives

$$q(a - \omega e) = 0,$$

and exactness in degree five gives $a - \omega e = qd$ for some $d \in \mathcal{A}^2$. Conversely, a representative of the form $qd + \omega e$ has zero d_4, by taking $b = qe$ in (9d3+k). There is no incoming d_4 from bidegree $(1, 7)$, because multiplication by ω is injective on $\mathcal{A}^1$. Together with (9d3+d), this proves

$$\overline{E}_5^{5,4} = \frac{q\mathcal{A}^2u^2}{\langle q\omega u^2 \rangle}. \tag{9d3+n}$$

Thus every d_5-image in bidegree $(10,0)$ is represented by aw for some $a \in \mathcal{A}^2$, by the Kudo formula (9d3+f); there is no additional d_5 family.

It remains to check the d_4- and d_6-sources of that bottom-row term. Representatives of the former, in $\overline{E}_4^{6,3}$, belong to the simultaneous kernel of multiplication by ω and q on $\mathcal{A}^6$; representatives of the latter, in $\overline{E}_6^{4,5}$, have already passed through the analogous simultaneous kernel on $\mathcal{A}^4$. We claim that, for $m \geq 5$,

$$\ker(\omega : \mathcal{A}^d \to \mathcal{A}^{d+2}) \cap \ker(q : \mathcal{A}^d \to \mathcal{A}^{d+3}) = 0 \qquad (d = 4, 6). \tag{9d3+o}$$

Give $\mathcal{A}$ the bidegree in which $S^j \otimes \bigwedge^e V^*$ has bidegree (j,e) and total degree $2j+e$. The two multiplications have bidegrees $(0,2)$ and $(1,1)$, so the assertion can be checked on each (j,e) separately. The Koszul complex for the regular sequence $X_1, Y_1, \ldots, X_m, Y_m$ says that, if $e < 2m$ and $qa = 0$ in bidegree (j,e), then

$$a = qb \quad \text{for some } b \text{ of bidegree } (j-1, e-1), \tag{9d3+o1}$$

where a bidegree with a negative entry is zero.

We shall also use that multiplication by ω is injective from $\bigwedge^1 V^*$ to $\bigwedge^3 V^*$ and from $\bigwedge^3 V^*$ to $\bigwedge^5 V^*$ when $m \geq 4$. This low-degree fact is valid in characteristic three by a direct incidence argument. Fixing the symplectic pairs from which a monomial uses exactly one generator decomposes the second map into the incidence maps from zero-subsets to one-subsets of $m-3$ available pairs and from one-subsets to two-subsets of $m-1$ available pairs. The first is injective when there is an available pair; the second is injective over $\mathbb{F}_3$ when there are at least three, since equations $c_i + c_j = 0$ for all $i \neq j$ force every c_i to vanish. The first exterior map is the same zero-subset incidence calculation.

In total degree four the bidegrees are $(0,4), (1,2), (2,0)$. Formula (9d3+o1) makes q injective on the first and last. In the middle bidegree, write $a = qb$ with b of bidegree $(0,1)$. If also $\omega a = 0$, then $q\omega b = 0$; another use of (9d3+o1), now with polynomial degree zero, gives $\omega b = 0$, and the exterior injectivity gives $b = 0$.

In total degree six the bidegrees are $(0,6), (1,4), (2,2), (3,0)$. The first and last again have no q-kernel. In bidegree $(1,4)$, write $a = qb$ with b of bidegree $(0,3)$. The equations $\omega a = 0$ and $q\omega b = 0$, followed by the polynomial-degree-zero case of (9d3+o1) and the injectivity on $\bigwedge^3 V^*$, give $b = 0$. Finally suppose that a has bidegree $(2,2)$. Write $a = qb$ with b of bidegree $(1,1)$. Exactness applied to $q\omega b = 0$ gives

$$\omega b = qc \qquad (c \in \bigwedge^2 V^*).$$

Identity (9d3+d) implies $qc = \lambda q\omega$ for some $\lambda \in \mathbb{F}_3$. Multiplication by q is injective on $\mathcal{A}^2$, so $c = \lambda\omega$. Hence $\omega(b - \lambda q) = 0$; injectivity on the exterior-degree-one factor gives $b = \lambda q$, and therefore $a = \lambda q^2 = 0$. This proves (9d3+o). Consequently $\overline{E}_4^{6,3} = 0$ and $\overline{E}_6^{4,5} = 0$, so there are no bottom-row d_4- or d_6-images.

It follows that every boundary in $\overline{E}_7^{10,0}$ lies in I_m^{10}. On the source side, $\overline{E}_7^{3,6}$ is a subspace of

$$\mathcal{A}^3/(\omega\mathcal{A}^1 + q\mathcal{A}^0);$$

there are no incoming differentials after the third page. The colon identity (9d3+g) now shows that a nonzero seventh-page class cannot have zero d_7. This proves (5b″).

Finally compare with the integral spectral sequence. On the fourth page, coefficient reduction in this bidegree is the quotient map

$$\frac{\mathcal{A}^3}{\langle q\rangle}u^3 \longrightarrow \frac{\mathcal{A}^3}{\omega\mathcal{A}^1+\langle q\rangle}u^3. \tag{9d3+p}$$

Its kernel is $\omega\mathcal{A}^1u^3$ and has dimension $2m$, since $\omega:\mathcal{A}^1\to\mathcal{A}^3$ is injective and the exterior bidegree separates its image from q. If an integral class surviving to $E_\infty^{3,6}$ had nonzero reduction in (9d3+p), then naturality and the absence of later incoming differentials would make its reduction a nonzero class on the mod-3 seventh page. The injective differential (5b″) would force its integral d_7 to have nonzero reduction, a contradiction. Therefore every integral survivor is represented by ωV^*u^3, proving (5b‴).

When $m=1$, choose a maximal elementary abelian subgroup $P\cong C_p^2$ of index p in D. The integral Kunneth theorem shows that positive-degree integral cohomology of P is annihilated by p, and $\operatorname{cor}_P^D\operatorname{res}_P^D=p$. Consequently

$$p^2H^j(D;\mathbb{Z})=0 \qquad (j>0) \tag{9e}$$

in width one. When $m=2$, choose a nondegenerate plane $V_0\le V$ and a line $T_0\le V_0^{\perp}$. The inverse image E of V_0 is extraspecial of width one. Choose $t\in D$ mapping to a generator of T_0. Because D has exponent p, one has $\langle t\rangle\cong C_p$; orthogonality gives $[t,E]=1$, and $t\notin E$. Thus the inverse image of $V_0\oplus T_0$ is the index-p subgroup

$$K=E\times\langle t\rangle\cong E\times C_p\le D. \tag{9f}$$

Equation (9e) annihilates the positive-degree integral cohomology of E by p^2. The split integral Kunneth sequence does the same for K, and restriction–corestriction gives

$$p^3H^j(D;\mathbb{Z})=0 \qquad (j>0) \tag{9g}$$

in width two.

We now verify the exact calculations in (5c). They are reproducible with GAP 4.16.0, HAP 1.75, and FLINT 3.4.0. From the root of this repository, run

```
mkdir -p tmp/extraspecial-h9
gap -q scripts/export_extraspecial_h9.g
cc -O3 scripts/check_extraspecial_h9_flint.c \
  -I/opt/homebrew/include -L/opt/homebrew/lib -lflint -lgmp \
  -o tmp/extraspecial-h9/check_h9
tmp/extraspecial-h9/check_h9 \
  tmp/extraspecial-h9/d8.gapmat tmp/extraspecial-h9/d9.gapmat
```

The GAP script computes the Smith forms directly in widths one and two and returns respectively six and eighty-five invariant factors, all equal to three. In width three it constructs the contracted integral chain complex through degree nine, whose dimensions are

$$1,7,28,84,210,462,924,1716,3003,5005,$$

and exports the two adjacent boundary matrices. The FLINT checker returns

```
rank(d8) over F_1000003 = 1107
rank(d9) over F_1000003 = 1896
rank(d9) over F_3 = 1217
log_3 |coker(d9) tensor Z/81| = 5107
Smith factors divisible by 3 = 679
sum min(v_3(s_i),4) = 679
certificate: H_8 is (Z/3)^679
```

Here is why these data are a certificate rather than a heuristic Smith calculation. Since D is finite, its positive-degree rational homology vanishes. The two ranks modulo the check prime sum to $1107 + 1896 = 3003 = \dim C_8$; because $d_8 d_9 = 0$, they therefore equal the rational ranks. Thus the free rank of $\operatorname{coker} d_9$ is $3003 - 1896 = 1107$. The rank drop modulo three says that exactly $1896 - 1217 = 679$ nonzero Smith factors s_i are divisible by three. Howell reduction modulo 81 gives

$$\log_3 |\operatorname{coker}(d_9) \otimes \mathbb{Z}/81| = 5107.$$

After subtracting the free contribution $4 \cdot 1107 = 4428$, one obtains

$$\sum_i \min\{v_3(s_i), 4\} = 679.$$

There are already 679 positive summands in this sum, so each has valuation exactly one. The torsion subgroup of $\operatorname{coker} d_9$ is $H_8(D;\mathbb{Z})$ because $\operatorname{im} d_8$ is free; it has no prime-to-three torsion because D is a 3-group. Lemma 1.1 gives the first three lines of (5c).

The width-four matrix has many more columns but remains sparse. The following calculation exports only its nonzero entries and replaces the full Howell reduction by an equivalent first-Bockstein rank certificate:

```
mkdir -p tmp/extraspecial-h9-width4
gap -q scripts/export_extraspecial_h9_width4_sparse.g
cc -O3 -Wall -Wextra \
  scripts/check_extraspecial_h9_bockstein_flint.c \
  -I/opt/homebrew/include -L/opt/homebrew/lib -lflint -lgmp \
  -o tmp/extraspecial-h9-width4/check_h9_bockstein
tmp/extraspecial-h9-width4/check_h9_bockstein \
  tmp/extraspecial-h9-width4/d8.sparse \
  tmp/extraspecial-h9-width4/d9.sparse
```

The contracted complex has dimensions

$$1, 9, 45, 165, 495, 1287, 3003, 6435, 12870, 24310.$$

The two exported matrices have respectively 44811 and 101888 nonzero entries. The checker returns

```
rank(d8) over F_5 = 4353
rank(d9) over F_5 = 8517
rank(d9) over F_3 = 5319
rank(first 3-Bockstein) = 3198
Smith factors divisible by 3 = 3198
Smith factors of exact 3-adic valuation one = 3198
certificate: H_8 is (Z/3)^3198
```

We spell out this second certificate. Put $A = d_9$. Since the group has order 3^9, its positive-degree homology over $\mathbb{F}_5$ vanishes; the two mod-5 ranks sum to $12870 = \dim C_8$. Together with rational exactness, this certifies the same two rational ranks. Hence A has rational rank 8517, and the rank drop modulo three shows that exactly $8517 - 5319 = 3198$ of its nonzero Smith factors are divisible by three.

Choose rows and columns giving an invertible 5319-square pivot block modulo three and write the resulting integral block decomposition as

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}.$$

The checker solves $BX = C$ modulo three and Hensel-lifts the solution to modulo nine. It then forms

$$\Delta = \frac{E - DB^{-1}C}{3} \pmod 3.$$

Block row and column operations invertible over $\mathbb{Z}_{(3)}$ show that $\operatorname{rank}_{\mathbb{F}_3} \Delta$ is exactly the number of Smith factors of A having 3-adic valuation one. The computed rank is 3198, equal to the total number of factors divisible by three. Thus every such factor has valuation exactly one. As above, the torsion subgroup of $\operatorname{coker} d_9$ is $H_8(D;\mathbb{Z})$ and has no prime-to-three torsion. Lemma 1.1 gives the fourth line of (5c).

We now use the width-two calculation to finish the integral bidegree-$(5,4)$ differential left open after (9d3+f). The following exact computation uses the filtered resolution underlying the integral homology LHS spectral sequence for $C_3 \to E_2 \to C_3^4$:

```
gap -q scripts/check_extraspecial_h9_filtration.g
```

HAP constructs a filtered complex $C_{\le s}$ whose total dimensions through degree nine are

$$1, 5, 15, 35, 70, 126, 210, 330, 495, 715.$$

The script first verifies $H_8(E_2;\mathbb{Z}) \cong (C_3)^{85}$ and $\operatorname{rank}_{\mathbb{F}_3}(d_9) = 209$. It then returns

s	0	1	2	3	4	5	6	7	8
$\dim_{\mathbb{F}_3} \operatorname{im}\big(H_8(C_{\le s};\mathbb{Z}) \to H_8(E_2;\mathbb{Z})\big)$	0	0	0	5	5	5	5	25	85.

(9h)

Here is an exact certificate for the displayed ranks. For each s, the script computes an integral lattice basis for $\ker(d_8|C_{\le s})$ using HAP's integer relation algorithm, embeds that basis in C_8, and reduces it modulo three. It adjoins these vectors to the mod-3 row space of d_9; the increase over rank 209 is the displayed image rank. This is legitimate because $H_8(E_2;\mathbb{Z}) \cong (C_3)^{85}$ makes coefficient reduction

$$H_8(E_2;\mathbb{Z}) \longrightarrow H_8(E_2;\mathbb{F}_3)$$

injective. Thus equality of the entries at $s = 4$ and $s = 5$ proves

$$E^{\infty}_{5,3} = 0$$

in the integral homology spectral sequence. Pontryagin duality on the finite filtered resolution identifies its dual with the cohomology term $E_{\infty}^{5,4}$, so (5b) holds in width two.

For any width $m \ge 3$, restrict to the inverse image of a coordinate nondegenerate four-space. The sole class not killed in the argument following (9d3+f) restricts as

$$q_m \omega_m u^2 \longmapsto q_2 \omega_2 u^2 \neq 0.$$

In width two the source in bidegree $(0,8)$ has already been killed by $d_3(u^4) = qu^3$, so no d_5 enters bidegree $(5,4)$; no later differential can enter or leave it. The vanishing just proved therefore says that $q_2\omega_2 u^2$ supports a nonzero integral d_5. Naturality forces the same for $q_m\omega_m u^2$. Combined with (9d3+f), this proves (5b) in every width $m \geq 3$.

In total degree nine the bidegree-$(1,8)$ term was already killed by d_3, and (5b) now removes bidegree $(5,4)$. The only remaining filtration quotients are in bidegrees $(9,0), (7,2), (3,6)$, and each is elementary abelian. Consequently

$$3^3 H^9(D;\mathbb{Z}) = 0 \qquad (p = 3,\ m \geq 3). \tag{9i}$$

Together with (9d+) this proves (5a) for every odd prime.

Corollary 5.4 gives $H^2(D;\mathbb{Z})^U = 0$, which removes the only incoming source in degree three. For $n \geq 4$, apply the inflation-edge filtration. Its $r = 3$ and $r = 4$ quotients are annihilated by p, its $r = 5$ quotient by p^2 using (9), and its $r \geq 6$ quotient by $p^{\lfloor (r-1)/2 \rfloor + 1}$ using (8). The product of these annihilators is p^{β_n}, proving (3); direct summation gives (4). If p is odd, write $\dim_{\mathbb{F}_p} V = 2m$. Equations (9d) and (8) bound the successive coefficient exponents in degrees two through nine by

$$1, 1, 2, 2, 4, 4, 4, 4.$$

For $m \geq 3$, [29, Lemma 3.3] makes $H^4(D;\mathbb{Z})$ elementary abelian; the resulting partial sums give the third line of (5), using (9d'), (9d"), (9d"'), (9d3+), and (5a) for the degree-six through degree-nine coefficients. Equation (9e) gives coefficient exponents $1, 1, 2, 2, 2, 2, 2, 2$ in width one and hence the first line. Equations (9d) and (9g) give coefficient exponents $1, 1, 2, 2, 3, 3, 3, 3$ in width two and hence the second line. Finally, (5c) replaces the degree-nine coefficient exponent by one at $p = 3$ and widths one through four. The corresponding partial sums give (5d).

Finally suppose that D is extraspecial and prove (6). One always has $H^2(D;\mathbb{Z}) \cong V^*$. For odd p and $m \geq 2$, the same cited lemma gives the R-module splitting

$$H^3(D;\mathbb{Z}) \cong \left(\bigwedge\nolimits^2 V^*\right)_\omega, \qquad \bigwedge\nolimits^2 V^* = \langle\omega\rangle \oplus \left(\bigwedge\nolimits^2 V^*\right)_\omega, \tag{10}$$

where ω is the commutator form. If p is odd and $m = 1$, the same spectral sequence instead gives $H^3(D;\mathbb{Z}) \cong V^*$, which has no R-invariants, while $(\bigwedge^2 V^*)^R = \langle\omega\rangle$. If $p = 2$, the exclusions above give $m \geq 2$, and [29, Section 3.3] gives an exact sequence of R-modules

$$0 \longrightarrow \langle B_Q \rangle \longrightarrow \bigwedge\nolimits^2 V^* \longrightarrow H^3(D;\mathbb{Z}) \longrightarrow 0. \tag{11}$$

Since R is perfect, $H^1(R;\mathbb{F}_p) = 0$. Taking invariants in (10) or (11), and using the R-invariance of the commutator form, gives in every extraspecial case

$$|H^3(D;\mathbb{Z})^R| = p^{a-1}. \tag{12}$$

The action of $U = \widetilde{R}$ on these coefficient modules factors through R. Since D is extraspecial, the kernel of $U \to R$ is C/A; it acts trivially on V^*, and

$$\mathrm{Hom}(C/A, V^*)^R = 0.$$

The cohomology five-term sequence therefore identifies

$$H^1(U; H^2(D;\mathbb{Z})) \cong H^1(R; V^*).$$

For $n = 4$, the inflation-edge filtration has only the $r = 3$ and $r = 4$ quotients. Their orders are bounded respectively by p^h and, by (12), by p^{a-1}. Thus in the extraspecial case the kernel has order at most p^{h+a-1}.

Now suppose that $p = 2$ and D is almost extraspecial. Put

$$W = D/A, \qquad T = Z_D/A.$$

Thus T is a trivial line, $W_R = 0$, and Proposition 9.15(ii) gives a nonsplit extension

$$0 \longrightarrow T \longrightarrow W \longrightarrow V \longrightarrow 0. \tag{13}$$

Since $D_{\mathrm{ab}} = W$, one has $H^2(D;\mathbb{Z}) \cong W^*$.

Let $Q \in H^2(W;\mathbb{F}_2) = \mathrm{Sym}^2(W^*)$ be the class of $A \to D \to W$. Its polar form B_Q is nonzero and has radical T, while Q is nonzero on T. In the integral LHS spectral sequence, [29, Section 3.3] identifies

$$d_3(u) = \mathrm{Sq}^1(Q), \qquad d_3(fu) = \mathrm{Sq}^1(fQ) = f\big(fQ + \mathrm{Sq}^1 Q\big) \tag{14}$$

for a generator $u \in H^2(A;\mathbb{Z})$ and $f \in W^*$; under the canonical identification $\mathrm{im}(\mathrm{Sq}^1 : \mathrm{Sym}^2 W^* \to \mathrm{Sym}^3 W^*) \cong \bigwedge^2 W^*$, the first value corresponds to B_Q. The second map is injective. Indeed, choose $z \in W^*$ which is nonzero on T and put $W_0 = \ker z$. Then

$$Q = z^2 + Q_0, \qquad \mathrm{Sq}^1 Q = \mathrm{Sq}^1 Q_0.$$

If $f \neq 0$ and $d_3(fu) = 0$, the polynomial ring $\mathrm{Sym}^\bullet(W^*)$ has no zero divisors, so $fQ = \mathrm{Sq}^1 Q$. Using $W = T \oplus W_0$, write $f = cz + \ell$ with $c \in \mathbb{F}_2$ and $\ell \in W_0^*$. Comparison of the z^3 coefficient gives $c = 0$, and comparison of the z^2 coefficient then gives $\ell = 0$, a contradiction because $\mathrm{Sq}^1 Q \neq 0$. Consequently the total-degree-three filtration gives an exact sequence of R-modules

$$0 \longrightarrow \langle B_Q \rangle \longrightarrow \bigwedge^2 W^* \longrightarrow H^3(D;\mathbb{Z}) \longrightarrow 0. \tag{15}$$

Since R is perfect, taking invariants yields

$$|H^3(D;\mathbb{Z})^R| = 2^{a-1}. \tag{16}$$

Indeed, exterior squaring the dual of (13) gives an exact sequence

$$0 \longrightarrow \bigwedge^2 V^* \longrightarrow \bigwedge^2 W^* \longrightarrow V^* \longrightarrow 0.$$

Since $(V^*)^R = 0$, it identifies $(\bigwedge^2 W^*)^R$ with $(\bigwedge^2 V^*)^R$, whose dimension is a.

It remains to bound the other degree-four filtration quotient. The kernel of $U \to R$ is $L/D = C/Z_D$ by Proposition 9.15, equation (5). It is central and acts trivially on W^*. Moreover $(W^*)^R = (W_R)^* = 0$, so restriction in the cohomology five-term sequence vanishes and

$$H^1(U;W^*) \cong H^1(R;W^*). \tag{17}$$

Dualizing (13) gives a nonsplit extension

$$0 \longrightarrow V^* \longrightarrow W^* \longrightarrow \mathbb{F}_2 \longrightarrow 0.$$

Its class is a nonzero element of $H^1(R;V^*)$. Since $(V^*)^R = (W^*)^R = 0$ and $H^1(R;\mathbb{F}_2) = 0$, the associated long exact sequence gives

$$\dim_{\mathbb{F}_2} H^1(R;W^*) = h - 1. \tag{18}$$

The two degree-four edge-filtration quotients therefore have orders at most 2^{h-1} and 2^{a-1} by (16)–(18). This gives the bound 2^{h+a-2} in the almost-extraspecial case; the nonzero extension class also shows $h \geq 1$.

It remains to express both bounds in terms of the intrinsic number b. Consider the homology spectral sequence for

$$1 \longrightarrow V \longrightarrow X = G/C \longrightarrow R \longrightarrow 1.$$

The proof of Proposition 9.13 shows that the image of

$$H_2(V;\mathbb{Z})_R \longrightarrow H_2(X;\mathbb{Z}) = C$$

is A. Hence the bottom piece of the total-degree-two filtration is $E^{\infty}_{0,2} = A$. Since $V_R = 0$, its top quotient is

$$E^{\infty}_{2,0} = H_2(R;\mathbb{Z}).$$

On the other hand, $U = G/D$ is the universal central extension of R; equation (5) of Proposition 9.15 identifies its kernel with $L/D = C/Z_D$. Therefore

$$H_2(R;\mathbb{Z}) \cong C/Z_D.$$

Since C is cyclic, its filtration subgroup above A and below the quotient C/Z_D must be Z_D. The middle filtration quotient is therefore

$$E^{\infty}_{1,1} \cong Z_D/A.$$

The only differential entering $E^2_{1,1} = H_1(R;V)$ is $d_2^{3,0}$, and no differential leaves it, so

$$H_1(R;V)/\operatorname{im} d_2^{3,0} \cong Z_D/A. \tag{19}$$

Duality on the finite bar complex gives $H^1(R;V^*) \cong H_1(R;V)^*$. Thus (19) says $h = b$ if $Z_D = A$, and $h = b+1$ if $Z_D \cong C_4$. Substitution into the two bounds above gives p^{a+b-1} in either case, proving (6).

In the abelian case Proposition 9.15(iii) makes D elementary abelian, and (7) is the elementary-abelian clause of Proposition 5.5. □

Corollary 9.17 (A scalar-endomorphism criterion in degree four)**.** *In the nonabelian elementary-layer case of Proposition 9.16, suppose that*

$$\operatorname{End}_{\mathbb{F}_p R}(V) = \mathbb{F}_p, \qquad H_3(R;\mathbb{Z})_{(p)} = 0.$$

Then inflation is injective in degree four:

$$H^4(\widetilde{R};\mathbb{Z}) \hookrightarrow H^4(G;\mathbb{Z}).$$

In particular, the endomorphism-ring hypothesis holds whenever V is absolutely irreducible over $\mathbb{F}_p$.

Proof. The nondegenerate R-invariant commutator form identifies V with V^*. Composition with this identification embeds the space of R-invariant alternating forms in $\operatorname{End}_{\mathbb{F}_p R}(V) = \mathbb{F}_p$; since it contains the commutator form, its dimension a is one. The target $H_1(R;V)$ of $d_2^{3,0}$ is an elementary abelian p-group, whereas the source has no p-primary part, so $d_2^{3,0} = 0$ and $b = 0$. Formula (6) of Proposition 9.16 makes the degree-four inflation kernel have order at most $p^{a+b-1} = 1$. □

Corollary 9.18 (A cyclic-Sylow persistence criterion)**.** *Retain the nonabelian elementary-layer hypotheses of Corollary 9.17, and suppose that a Sylow p-subgroup P of R is cyclic. Put*

$$e = |N_R(P) : C_R(P)|.$$

If

$$\mathrm{End}_{\mathbb{F}_p R}(V) = \mathbb{F}_p, \qquad e \nmid 2,$$

then inflation

$$H^4(\widetilde{R}; \mathbb{Z}) \hookrightarrow H^4(G; \mathbb{Z})$$

is injective.

Proof. Corollary 3.2, applied to R, gives $H_3(R; \mathbb{Z})_{(p)} = 0$. Corollary 9.17 now applies. □

Proposition 9.19 (Prime-to-layer alternating subledger)**.** *Retain either elementary-layer case of Proposition 9.16, suppose that R is simple, and suppose that $H_2(R; \mathbb{Z}) \neq 0$. If $R \cong A_n$, then*

$$H^4(G; \mathbb{Z})_{(3)} \neq 0.$$

Proof. The alternating multiplier is C_2 for $n = 5$ and $n \geq 8$, and is C_6 for $n = 6, 7$. Proposition 9.13 makes it a cyclic p-group, so the cases $n = 6, 7$ cannot occur and $p = 2$ in every remaining case. Proposition 4.2 gives a nonzero three-primary class on A_n. Its inflation to the universal central extension $\widetilde{R}$ is nonzero because the central kernel has order two. In either elementary-layer case the kernel $D = [G, L]$ of $G \twoheadrightarrow \widetilde{R}$ is a 2-group, so a second application of Lemma 5.1 proves the claim. □

Proposition 9.20 (Lie-type elementary-descent subledger)**.** *Retain either elementary-layer case of Proposition 9.16, suppose that R is simple, and suppose that $H_2(R; \mathbb{Z}) \neq 0$. If R is of Lie type, then*

$$H^i(G; \mathbb{Z}) \neq 0 \quad \textit{for some } i \leq 9.$$

Proof. Proposition 9.13 makes $H_2(R; \mathbb{Z})$ a cyclic p-group, and in either elementary-layer case the kernel $D = [G, L]$ of $G \twoheadrightarrow \widetilde{R}$ is a p-group. First suppose that $\widetilde{R}$ is the standard perfect simply connected fixed-point group over $\mathbb{F}_q$. If some prime $\ell \mid q^2 - 1$ differs from p, Proposition 4.5 gives $H^4(\widetilde{R}; \mathbb{Z})_{(\ell)} \neq 0$, and Lemma 5.1 makes this class survive on G.

It remains to consider the case in which every prime divisor of $q^2 - 1$ equals p. Since $q - 1$ and $q + 1$ differ by two, the elementary arithmetic alternatives are

$$(q, p) = (3, 2) \qquad \text{or} \qquad (q, p) = (2, 3). \tag{19a}$$

Suppose first that $(q, p) = (3, 2)$. The exact-order conclusion of Proposition 4.5, applied at $\ell = 2$, gives

$$H^4(\widetilde{R}; \mathbb{Z})_{(2)} \cong C_8. \tag{19b}$$

In the nonabelian elementary-layer case, formula (3) of Proposition 9.16 says that the degree-four inflation kernel is annihilated by $2^{\beta_4} = 2^2 = 4$. In the abelian elementary-layer case, formula (7) gives the same annihilator $2^{4-2} = 4$. An element of order eight cannot lie in either kernel, so it gives a nonzero class in $H^4(G; \mathbb{Z})$.

Now suppose that $(q, p) = (2, 3)$. The classification of the centers and full multipliers of simply connected groups leaves only unitary type ${}^2A_{n-1}$ in the standard-cover case [25, Theorem 6.1.4 and Table 6.1.3]; the exceptional full multiplier of ${}^2E_6(2)$ is mixed-prime and cannot occur. For ${}^2A_{n-1}$,

the signed fundamental degree $d = 4$ has $5 \mid 2^4 - 1$. The untwisting theorem, Proposition C.4, and Theorems C and D of Grodal–Lahtinen [27] inject the corresponding polynomial generator into

$$H^8(\widetilde{R}; \mathbb{F}_5) \neq 0.$$

The integral coefficient exact sequence makes $H^8(\widetilde{R}; \mathbb{Z})_{(5)}$ or $H^9(\widetilde{R}; \mathbb{Z})_{(5)}$ nonzero. Since $5 \neq p$, Lemma 5.1 carries the class to G.

Finally suppose that the universal cover is nonstandard. The complete exceptional-multiplier audit in the proof of Theorem 6.6, after the exceptional isomorphisms are removed, leaves exactly

$$\mathrm{PSU}_4(2), \quad \mathrm{PSp}_6(2), \quad G_2(3), \quad G_2(4), \quad F_4(2).$$

That proof gives a prime-to-p degree-four class on each universal cover, at the respective primes $3, 3, 2, 3, 3$. It survives across D. This completes the audit. □

Proposition 9.21 (Prime-to-layer sporadic subledger)**.** *Retain either elementary-layer case of Proposition 9.16, suppose that R is simple, and suppose that $H_2(R; \mathbb{Z}) \neq 0$. If R is sporadic, then*

$$H^i(G; \mathbb{Z}) \neq 0 \quad \textit{for some } i \leq 20.$$

Moreover, the putative cases $R = M_{22}, Suz, Fi_{22}$ cannot occur in this central descent.

Proof. Proposition 9.13 makes $H_2(R; \mathbb{Z})$ a cyclic p-group. The ATLAS multiplier list therefore excludes M_{22}, Suz, and Fi_{22}, whose multipliers are respectively C_{12}, C_6, C_6 [1]. The remaining sporadic groups with nontrivial prime-power multiplier are

$$M_{12}, HS, J_2, McL, O'N, J_3, \mathrm{Ru}, Co_1, \mathrm{Fi}'_{24}, B.$$

The smaller universal cover has the following prime-to-p classes, all in degree at most twenty:

R	p	ℓ	d	source of $H^d(\widetilde{R}; \mathbb{Z})_{(\ell)} \neq 0$
M_{12}	2	3	4	$H_3(2.M_{12}; \mathbb{Z}) \cong C_8 \oplus C_{24}$
HS	2	11	10	cyclic-Sylow class with automizer 5
J_2	2	3	4	$H_3(2.J_2; \mathbb{Z}) \cong C_{120}$
McL	3	7	6	cyclic Sylow 7 with automizer 3
$O'N$	3	2	4	$H_3(3.O'N; \mathbb{Z}) \cong C_8$
J_3	3	5	4	$H_3(3.J_3; \mathbb{Z}) \cong C_3 \oplus C_{15}$
Ru	2	5	8	$\mathrm{res}^{2.\mathrm{Ru}}_{C_5} c_4(\rho) = 4u^4 \neq 0$
Co_1	2	3	4	$H_3(2.Co_1; \mathbb{Z}) \cong C_{24}$
Fi'_{24}	3	2	4	$H^4(3.\mathrm{Fi}'_{24}; \mathbb{Z})_{(2)} \neq 0$
B	2	11	20	cyclic Sylow 11 with automizer 10.

(20)

The third-homology entries are from Johnson–Freyd and Treumann [29, table on p. 2]; the Ru row is Proposition 5.8, and the Fi'_{24} row is Proposition 5.9. For the McL and Baby Monster rows, Theorem 3.1 gives the displayed classes on R. It also gives the displayed degree-ten class in the HS row directly. In all three rows Lemma 5.1 lifts the class to $\widetilde{R}$. The class counts and automizers, including the unique class of order 11 in B, are reproduced from CTblLib by `scripts/check_sporadic_prime_to_layer.g` [8].

In either elementary-layer case the kernel $D = [G, L]$ of $G \twoheadrightarrow \widetilde{R}$ is a p-group. Since every prime ℓ in (20) differs from p, another application of Lemma 5.1 makes the indicated class survive on G. This proves every claimed row and completes the sporadic prime-power subledger in degree twenty. □

Corollary 9.22 (Simple nonzero-multiplier elementary descent)**.** *Retain either elementary-layer case of Proposition 9.16. If R is nonabelian simple and $H_2(R;\mathbb{Z}) \neq 0$, then*

$$H^i(G;\mathbb{Z}) \neq 0 \quad \textit{for some } i \leq 20.$$

Proof. By the classification of finite simple groups, R is alternating, of Lie type, or sporadic. Apply Propositions 9.19, 9.20, and 9.21, respectively. Their bounds are 4, 9, and 20. The Lie-type proposition closes every row through degree nine, and the sporadic table gives the common finite ceiling. □

Proposition 9.23 (A principal-block filter for abelian layers)**.** *Retain the abelian elementary-layer case of Proposition 9.13, and suppose that $C > A$, equivalently $H_2(R;\mathbb{Z}) = C/A \neq 0$. Then the extension*

$$1 \longrightarrow V \longrightarrow X = G/C \longrightarrow R \longrightarrow 1$$

is nonsplit. Consequently $H^2(R;V) \neq 0$, and the irreducible $\mathbb{F}_pR$-module V belongs to the principal p-block.

Proof. In the abelian case, Proposition 9.13 identifies the homology edge map with the quotient

$$H_2(X;\mathbb{Z}) = C \longrightarrow H_2(R;\mathbb{Z}) = C/A.$$

If $X \to R$ had a section, functoriality of homology would give a right inverse to this quotient. No such right inverse exists. Indeed, write $C \cong C_{p^k}$ with $k \geq 2$. Every homomorphism $C/A \cong C_{p^{k-1}} \to C$ has image in the subgroup pC, whose image in C/A is the proper subgroup $p(C/A)$. Thus the composite cannot be the identity.

The class in $H^2(R;V)$ classifying $X \to R$ is therefore nonzero. Since the trivial module lies in the principal block and Ext groups between distinct blocks vanish, a nonzero $\mathrm{Ext}^2_{\mathbb{F}_pR}(\mathbb{F}_p, V)$ forces V into that block. □

Lemma 9.24 (Quadratic kernel-type criterion)**.** *Retain the nonabelian elementary-layer case of Proposition 9.15, and suppose that $p = 2$. Let B be the commutator form on V. Then D is extraspecial if and only if V admits an R-invariant quadratic form whose polar form is B.*

Proof. Put $W = D/A$ and let

$$Q : W \longrightarrow A \cong \mathbb{F}_2, \qquad Q(xA) = x^2.$$

This is an R-invariant quadratic form with polar form induced by commutation. If D is extraspecial, then $W = V$, so Q itself is the required refinement.

Suppose instead that D is almost extraspecial. Then Proposition 9.15 gives a nonsplit sequence

$$0 \longrightarrow T = Z(D)/A \longrightarrow W \xrightarrow{\pi} V \longrightarrow 0, \tag{20a}$$

where T is the radical of the polar form and Q is nonzero on T. If an invariant refinement q of B existed on V, every fibre of π would contain a unique vector $s(v)$ satisfying

$$Q(s(v)) = q(v),$$

because adding the nonzero vector of T changes the value of Q. For $v, v' \in V$, equality of the two polar forms gives

$$Q(s(v) + s(v')) = q(v) + q(v') + B(v, v') = q(v + v').$$

Uniqueness therefore gives $s(v+v') = s(v) + s(v')$; the same uniqueness and the invariance of Q and q give $s(rv) = rs(v)$. Thus s is an R-linear section of (20a), contradicting nonsplitting. Hence no such refinement exists in the almost-extraspecial case, and the two alternatives exhaust the possibilities. □

Proposition 9.25 (Supplementary Rudvalis module subledger)**.** *Retain either elementary-layer case of Proposition 9.16 and suppose that* $R = \mathrm{Ru}$*. Proposition 9.21 already gives degree-eight persistence for every such layer. The finer two-primary degree-four audit is as follows. The irreducible* $\mathbb{F}_2 R$*-module dimensions are*

$$1,\ 28,\ 376,\ 1246,\ 7280,\ 16036,\ 16384,\ 102400. \tag{21}$$

In the nonabelian inverse-image case, degree-four persistence holds when

$$\dim_{\mathbb{F}_2} V \in \{28, 16384, 102400\}.$$

In the abelian inverse-image case none of these three dimensions can occur. Consequently, within the two-primary degree-four audit, the only modules not settled in either layer type have dimensions

$$376, \qquad 1246, \qquad 7280, \qquad 16036. \tag{22}$$

For the first two residual modules, the exact first cohomology is

$$\dim_{\mathbb{F}_2} H^1(\mathrm{Ru}; V) = \begin{cases} 3, & \dim V = 376, \\ 2, & \dim V = 1246. \end{cases} \tag{22a}$$

Moreover, a nonabelian 376*-dimensional layer can only have an almost-extraspecial canonical kernel, whereas a nonabelian* 1246*-dimensional layer can only have an extraspecial kernel. In both cases the kernel of degree-four inflation from* 2.Ru *has order at most four. In the abelian layer the corresponding bounds are eight and four, respectively.*

Proof. The characteristic-two Brauer table in CTblLib has absolutely irreducible degrees

$$1, 28, 376, 1246, 7280, 8192, 8192, 16036, 102400.$$

The 2-power Frobenius fixes every entry except the pair of degree 8192; finite-field descent therefore gives (21), with the pair producing one irreducible 16384-dimensional $\mathbb{F}_2$-module. Duality fixes every Brauer character. The Frobenius-fixed nontrivial modules have endomorphism ring $\mathbb{F}_2$ and a one-dimensional space of invariant alternating forms. Indeed, self-duality gives a nonzero invariant bilinear form; Schur's lemma makes it unique, its transpose equals itself, and its diagonal is an invariant linear functional, hence zero on a nontrivial irreducible module. The 16384-dimensional restriction of scalars has endomorphism ring $\mathbb{F}_4$ and a two-dimensional space of such forms: an invariant $\mathbb{F}_4$-valued alternating form, composed with the $\mathbb{F}_2$-linear functionals in $\mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_4, \mathbb{F}_2)$, gives all of them.

The principal block contains precisely the Brauer-character positions of degrees

$$1, 28, 376, 1246, 7280, 16036;$$

the two 8192 entries and the 102400 entry lie in a nonprincipal block of defect two. Hence $H^1(R; V) = 0$ for the 16384- and 102400-dimensional $\mathbb{F}_2$-modules. These table, Frobenius, duality, indicator, and block computations are reproduced by `scripts/check_ru_brauer_modules.g` [8].

For the 28-dimensional module the same vanishing is certified directly from the Online ATLAS standard generators. They satisfy

$$a^2 = b^4 = (ab)^{13} = (abb)^{14} = (ababb)^{29} = 1,$$

and the published class-representative words supply further valid word-order relations [6]. A derivation is determined by its values on a, b. Exact row reduction over $\mathbb{F}_2$ on the derivatives of these relations gives a solution space of dimension 28. The principal derivations already form a 28-dimensional subspace because V is nontrivial irreducible, so $H^1(R; V) = 0$. The identical calculation for the 376- and 1246-dimensional modules gives solution spaces of dimensions 379 and 1248, respectively. To prove that the resulting upper bounds are exact, start with the faithful Online ATLAS 4060-point permutation representation in the same standard generators. GAP constructs from it a certified presentation of Ru on a, b with 51 relators, of total length 11501. After quotienting the two solution spaces by their subspaces of principal derivations, evaluation of every full relator has rank zero on the residual spaces of dimensions three and two. Thus every residual vector is a genuine derivation for Ru, proving (22a). The matrix-format calculation for dimensions 28 and 376 is reproduced by `scripts/check_ru_semipresentation_h1.g`; the independent packed-bit implementation, including the Online ATLAS MeatAxe-format module of dimension 1246 and the full-presentation check, is `scripts/check_ru_meataxe_h1.py` [6].

There is a second exact matrix test. GAP's MeatAxe finds an invariant quadratic form with nondegenerate polar form on the 28- and 1246-dimensional modules, and proves that none exists on the 376-dimensional module. The calculation first verifies absolute irreducibility and then checks the defining equation on both published generators; it is reproduced by `scripts/check_ru_quadratic_forms.g`. Since the invariant alternating space has dimension one, any nonzero invariant quadratic form has polar form equal to the commutator form. Lemma 9.24 now forces the asserted kernel types in dimensions 376 and 1246.

Now suppose that the inverse image of V is nonabelian. In the notation of Proposition 9.16, the preceding calculation gives $(a, b) = (1, 0)$ for dimensions 28 and 102400, so formula (6) there makes degree-four inflation injective. For dimension 16384 it gives $(a, b) = (2, 0)$, so the inflation kernel has order at most two. Proposition 5.8 supplies a class on 2.Ru whose order is divisible by four, and that class cannot lie in a kernel of order at most two. Degree four therefore persists in all three cases.

Finally suppose that the inverse image is abelian. Proposition 9.23 excludes the two nonprincipal modules, while the nonsplit module extension in Proposition 9.12 requires $H^1(R; V^*) \neq 0$ and therefore excludes dimension 28. This leaves exactly the four dimensions in (22) in either layer type.

For dimension 376, equation (22a) gives $h = 3$. The forced almost-extraspecial type has $b = h - 1 = 2$, while $a = 1$, so formula (6) of Proposition 9.16 bounds the nonabelian degree-four kernel by $2^{a+b-1} = 4$. For dimension 1246 one has $h = 2$ and the forced extraspecial type has $b = h = 2$, giving the same bound. In the abelian case Proposition 9.27 gives 2^{h-1+a}, namely eight and four. The order-four class of Proposition 5.8 may still lie in a kernel of order four, so these sharper bounds do not close either module in degree four. The five-primary degree-eight class in the same proposition lies away from the layer prime and closes both, as well as the two larger modules in (22), by Proposition 9.21. □

A negative Baby Monster Chern audit. The prime-to-layer Chern search that closes Ru does not presently close B. The ordinary character table of $2.B$ gives the following exact negative certificate. For every complex representation ρ, every cyclic prime-power subgroup $C \leq 2.B$, and

$1 \le j \le 5$,

$$\operatorname{res}_C^{2.B} c_j(\rho) = 0. \tag{22b}$$

It is enough to check irreducible characters, since truncated total Chern classes multiply under direct sum. Applying Lemma 5.7 to every irreducible and every prime-power class gives zero in each case. This exhaustive calculation is part of `scripts/check_cover_chern.g` [8].

There is also no hidden Chern restriction in the audited low Chern degrees on a natural rank-four elementary abelian subgroup. The stored fusion chain

$$3^4{:}!A_6 < U_4(3) \leftarrow 2.U_4(3) < 2.(3^2{:}!D_8 \times U_4(3).2^2).2 < 2.B$$

identifies the three nonidentity orbits in the normal subgroup $E = C_3^4$ as

$$(20, 3B), \qquad (30, 3A), \qquad (30, 3A), \tag{22c}$$

where each pair records orbit size and fused $2.B$ class. Model the 20-element orbit as the nonzero singular cone of a minus-type quadratic form on $\mathbb{F}_3^4$. If χ is an irreducible character and $\lambda \in E^*$, the exact weight multiplicity is the Fourier coefficient

$$m_\lambda = \frac{1}{81} \sum_{v \in E} \chi(v) \zeta_3^{-\lambda(v)}. \tag{22d}$$

The two order-three classes are inverse-stable, so $m_\lambda = m_{-\lambda}$. Hence the restricted total Chern class is

$$\prod_{[\lambda] \in \mathbb{P}(E^*)} (1 - \lambda^2)^{m_\lambda}.$$

Exact evaluation of (22d) for every irreducible character gives zero homogeneous parts in Chern indices two and four; the opposite weight pairs make the odd indices one, three, and five zero. Thus

$$\operatorname{res}_E^{2.B} c_j(\rho) = 0 \qquad (1 \le j \le 5) \tag{22e}$$

for every complex representation ρ. The fusion checks, Fourier transform, and polynomial calculation are reproduced by `scripts/check_baby_elementary_chern.g`. Statements (22b) and (22e) are limits of this detection method, not vanishing assertions for $H^{2j}(2.B; \mathbb{Z})$ itself; a class not generated by these Chern restrictions may still exist.

Proposition 9.26 (Baby Monster Spin certificate and small-H^1 modules)**.** *The two-primary fourth cohomology of the Baby Monster cover satisfies*

$$\left|H^4(2.B; \mathbb{Z})_{(2)}\right| \ge 4. \tag{22f}$$

Retain either elementary-layer case of Proposition 9.16, suppose that $R = B$, and suppose that V is absolutely irreducible over $\mathbb{F}_2$ with

$$\dim_{\mathbb{F}_2} H^1(B; V^*) \le 1.$$

Then

$$H^4(G; \mathbb{Z}) \ne 0. \tag{22g}$$

In particular, this applies to the natural 4370*-dimensional $\mathbb{F}_2 B$-module.*

Proof. The universal central extension $2.B$ is superperfect. Hence every real representation of $2.B$ is oriented and Spin: its first two Stiefel–Whitney classes lie in $H^1(2.B;\mathbb{F}_2) = H^2(2.B;\mathbb{F}_2) = 0$. Let ρ be the real irreducible representation of degree 4371 and let

$$\kappa = q_1(\rho) = \frac{p_1}{2}(\rho) \in H^4(2.B;\mathbb{Z}).$$

If a Spin representation restricts to C_n as oriented real two-planes of weights a_i, then

$$\operatorname{res}_{C_n} q_1 = \frac{1}{2}\sum_i a_i^2 u^2 \quad \text{in} \quad H^4(C_n;\mathbb{Z}) = \mathbb{Z}/n\{u^2\}. \tag{22h}$$

For class $8A$ of $2.B$, the nonzero eigenvalue multiplicities of ρ are

$$[2:1106], \qquad [4:1080], \qquad [6:1106], \qquad [0:1079].$$

Thus the real two-plane square sum is

$$1106 \cdot 2^2 + 540 \cdot 4^2 = 13064,$$

and (22h) gives

$$\operatorname{res}_{\langle 8A\rangle} \kappa = 6532u^2 = 4u^2 \neq 0 \quad \text{in } \mathbb{Z}/8\{u^2\}. \tag{22i}$$

On class $4E$ the multiplicities are

$$[1:1024], \qquad [2:1172], \qquad [3:1024], \qquad [0:1151].$$

The square sum is $1024 + 586 \cdot 2^2 = 3368$, so

$$\operatorname{res}_{\langle 4E\rangle} \kappa = 1684u^2 = 0 \quad \text{in } \mathbb{Z}/4\{u^2\}. \tag{22j}$$

These Frobenius–Schur, eigenvalue, and fractional-Pontryagin calculations, including a validation on M_{11}, are reproduced by `scripts/check_baby_spin_q1.g` [8]. The same script scans all 229 irreducible real representation characters of $2.B$ on all 45 cyclic 2-power classes. Every restriction has order at most two, and the resulting binary restriction profiles have $\mathbb{F}_2$-rank one. The unique nonzero profile is supported on the classes $8A$, $8N$, and $8P$, while every profile vanishes on $4E$. Thus ordinary real representations provide exactly one cyclically detected q_1 direction; the Moonshine anomaly below supplies an independent second degree-four direction.

Let α be the two-primary component of the restricted Moonshine anomaly in Proposition 5.10. Its restriction to class $4E$ is nonzero, whereas (22j) says that the two-primary component of κ restricts to zero there; (22i) says that the latter component is nonzero. These are two distinct nonzero elements of the finite 2-group $H^4(2.B;\mathbb{Z})_{(2)}$, proving (22f).

It remains to apply the order bound. Put $h = \dim_{\mathbb{F}_2} H^1(B;V^*)$. In the nonabelian inverse-image case the commutator form identifies V with V^*. Absolute irreducibility gives

$$a = \dim_{\mathbb{F}_2}(\bigwedge^2 V^*)^B = 1,$$

and $b \le h \le 1$. Formula (6) of Proposition 9.16 therefore bounds the degree-four inflation kernel by $2^{a+b-1} \le 2$. In the abelian inverse-image case, the nonsplit module extension in Proposition 9.12 gives $h \ge 1$, hence $h = 1$. Absolute irreducibility gives

$$a = \dim_{\mathbb{F}_2}(\bigwedge^2 V^*)^B \le \dim_{\mathbb{F}_2} \operatorname{Hom}_B(V, V^*) \le 1.$$

Proposition 9.27 again bounds the kernel by $2^{h-1+a} \le 2$. In either case a homomorphism from the group in (22f) with kernel of order at most two has nonzero image, proving (22g).

Finally, the Online ATLAS certifies that the natural 4370-dimensional $\mathbb{F}_2 B$-module is absolutely irreducible, has one-dimensional first cohomology, and admits a nondegenerate invariant quadratic form of minus type [4]. Its polar form identifies it with its dual, so it satisfies the stated hypothesis. □

Supplementary M_{22} module certificate. Although the multiplier filter above means that M_{22} cannot occur in the present elementary central descent, the characteristic-two module audit gives a reusable certificate: every irreducible $\mathbb{F}_2 M_{22}$-module that admits an invariant alternating form has a one-dimensional space of such forms.

The Online ATLAS lists all faithful irreducible representations of M_{22} in characteristic two and supplies matrix generators for them [5]. Its filename convention records a finite field only when the matrices cannot be written over a proper subfield [7]. Since M_{22} is simple and V is nontrivial, every relevant irreducible is faithful. Finite-field Galois descent now turns each listed $\mathbb{F}_4$ Frobenius orbit into its 140-dimensional irreducible restriction of scalars over $\mathbb{F}_2$; the entries already over $\mathbb{F}_2$ stay as listed. Thus these entries give the complete list of irreducible $\mathbb{F}_2 M_{22}$-modules. Exact row reduction over $\mathbb{F}_2$ on the published generators gives the following certificate.

ATLAS module	$\dim_{\mathbb{F}_2} V$	$\dim \mathrm{End}_{\mathbb{F}_2 M_{22}}(V)$	$\dim \mathrm{Bil}_{M_{22}}(V)$	$\dim \mathrm{Alt}_{M_{22}}(V)$
$10a$	10	1	0	0
$10b$	10	1	0	0
34	34	1	1	1
$70a{\downarrow_{\mathbb{F}_2}}$	140	2	2	1
$70b{\downarrow_{\mathbb{F}_2}}$	140	2	2	1
98	98	1	1	1

(23)

For reproducibility, if A, B are the published generator matrices, the third column is obtained by solving

$$AX = XA, \qquad BX = XB,$$

and the fourth by solving

$$A^{\mathsf{T}} J A = J, \qquad B^{\mathsf{T}} J B = J.$$

Adding $J^{\mathsf{T}} = J$ and $J_{ii} = 0$ gives the last column. These are linear systems over $\mathbb{F}_2$; the exact calculation is implemented in `scripts/check_atlas_invariant_forms.py`. Each nonzero form in the last column has full rank, also directly from the row reduction (or from irreducibility).

The commutator form on V is a nondegenerate invariant alternating form, so the two 10-dimensional modules in (23) cannot occur. Every remaining possibility has

$$a = \dim_{\mathbb{F}_2}(\textstyle\bigwedge^2 V^*)^{M_{22}} = 1.$$

Supplementary characteristic-three module certificate. The multiplier filter likewise excludes Suz and Fi_{22} before their modules are considered. Independently, neither group has a nontrivial irreducible symplectic $\mathbb{F}_3$-module.

The complete characteristic-three Brauer tables in CTblLib 1.3.11 contain 20 absolutely irreducible characters for Suz and 27 for Fi_{22} [8]. In both tables the 3-power cyclotomic Galois operation fixes every irreducible Brauer character. Thus every Frobenius orbit has length one.

Finite-field Galois descent consequently makes every irreducible $\mathbb{F}_3R$-module absolutely irreducible. The modular Frobenius–Schur indicators computed from the same tables are

R	$\#\,\mathrm{IBr}_3(R)$	$\nu=+1$	$\nu=0$	$\nu=-1$
Suz	20	20	0	0
Fi_{22}	27	21	6	0.

(24)

In odd characteristic, indicator $+1$ means orthogonal type, indicator 0 means non-self-dual, and indicator -1 means symplectic type. Hence there is no irreducible symplectic $\mathbb{F}_3R$-module. The exact table calculation is reproduced by `scripts/check_sporadic_brauer_forms.g`.

Homology-jump context. These two quotients were singled out by the homology jumps

R	$H_3(R;\mathbb{Z})$	$H_3(6.R;\mathbb{Z})$
Suz	C_4	C_{24}
Fi_{22}	0	$C_3 \oplus A, \quad \lvert A\rvert \mid 4,$

in the table of Johnson–Freyd and Treumann [29, table on p. 2]. The scalar-endomorphism criterion would preserve the resulting 3-primary degree-four class in a broader extension problem. In the actual central descent, however, Proposition 9.21 excludes both quotients already at the multiplier stage.

Proposition 9.27 (Degree-four persistence through an abelian layer)**.** *Retain the notation of Proposition 9.13, suppose that $\overline{N} = V$ is elementary abelian and that its inverse image L in G is abelian, and put*

$$M = \Omega_1(L), \qquad U = G/M = \widetilde{R}.$$

Write $V^* = \mathrm{Hom}_{\mathbb{F}_p}(V, \mathbb{F}_p)$ and set

$$b = \dim_{\mathbb{F}_p} H^1(R; V^*) - 1, \qquad a = \dim_{\mathbb{F}_p} \bigl(\textstyle\bigwedge^2 V^*\bigr)^R.$$

If $d_2^{3,0}$ denotes the differential

$$d_2^{3,0} : H_3(R;\mathbb{Z}) \longrightarrow H_1(R;V)$$

in the homology Lyndon–Hochschild–Serre spectral sequence for $V \to X \to R$, then

$$b = \dim_{\mathbb{F}_p} \mathrm{im}\, d_2^{3,0}.$$

In particular b is nonnegative, and

$$\bigl|\ker\bigl(\mathrm{inf} : H^4(U;\mathbb{Z})_{(p)} \longrightarrow H^4(G;\mathbb{Z})_{(p)}\bigr)\bigr| \le p^{b+a}.$$

Consequently,

$$|H^4(U;\mathbb{Z})_{(p)}| > p^{b+a} \quad \Longrightarrow \quad H^4(G;\mathbb{Z})_{(p)} \neq 0.$$

If $b = a = 0$, inflation is injective on the p-primary part of degree four. More generally, if either $b = 0$ or $a = 0$, the kernel has exponent at most p; in that case every element of order at least p^2 in $H^4(U;\mathbb{Z})_{(p)}$ survives on G. The equality $a = 0$ holds, in particular, when V is not self-dual; it also holds when p is odd and V is absolutely irreducible of orthogonal type.

Proof. Consider first the homology spectral sequence for

$$1 \longrightarrow V \longrightarrow X \longrightarrow R \longrightarrow 1, \qquad X = G/C.$$

The composite

$$H_2(V;\mathbb{Z})_R \longrightarrow H_2(X;\mathbb{Z}) \xrightarrow{\cong} C$$

is the homology transgression of the restricted central extension $C \to L \to V$. It is zero because L is abelian: on $H_2(V;\mathbb{Z}) = \bigwedge^2 V$ this transgression is the commutator map. Thus $E^{\infty}_{0,2} = 0$. Moreover, $V_R = 0$, and Proposition 9.13 identifies the edge epimorphism

$$H_2(X;\mathbb{Z}) = C \longrightarrow H_2(R;\mathbb{Z}) = C/A$$

with the quotient map. Hence $E^{\infty}_{2,0} = H_2(R;\mathbb{Z})$ and the total-degree-two filtration gives $E^{\infty}_{1,1} \cong A$. The only differential entering $E^2_{1,1} = H_1(R;V)$ is $d_2^{3,0}$, and no differential leaves it. Therefore

$$H_1(R;V)/\operatorname{im} d_2^{3,0} \cong A. \tag{1}$$

Duality on the finite bar complex identifies $H^1(R;V^*)$ with $H_1(R;V)^*$, so equation (1) proves the asserted formula for b.

Let $M^\vee = \operatorname{Hom}(M, \mathbb{Q}/\mathbb{Z}) = H^2(M;\mathbb{Z})$. The action of U on M factors through R, and Proposition 9.12 gives $M_R = 0$. Since the central kernel C/A of $U \to R$ acts trivially, the cohomology five-term sequence gives an isomorphism

$$H^1(R;M^\vee) \xrightarrow{\cong} H^1(U;M^\vee).$$

The following restriction term is $\operatorname{Hom}(C/A, M^\vee)^R = 0$ because $(M^\vee)^R = 0$.

Dualizing the nonsplit module extension

$$0 \longrightarrow A \longrightarrow M \longrightarrow V \longrightarrow 0$$

gives

$$0 \longrightarrow V^* \longrightarrow M^\vee \longrightarrow A^* \longrightarrow 0.$$

Here $A^* \cong \mathbb{F}_p$ is trivial. Since R is perfect, $H^1(R;A^*) = 0$; since V is nontrivial irreducible and $M_R = 0$, both $(V^*)^R$ and $(M^\vee)^R$ vanish. The long exact sequence therefore identifies

$$H^1(R;M^\vee) \cong H^1(R;V^*)/\langle \xi \rangle, \tag{2}$$

where $\xi \neq 0$ is the class of the module extension. Equation (2) shows that

$$\dim_{\mathbb{F}_p} H^1(U;M^\vee) = b. \tag{3}$$

Also

$$H^3(M;\mathbb{Z}) \cong \operatorname{Hom}(\textstyle\bigwedge^2 M, \mathbb{Q}/\mathbb{Z}).$$

An R-invariant alternating form on M has A in its radical. Indeed, pairing with a fixed nonzero element of the trivial module A gives an R-map $M/A = V \to A^*$, and this map is zero because V is nontrivial irreducible. Thus every invariant form descends uniquely to V, and pullback gives the inverse construction. Hence

$$\dim_{\mathbb{F}_p} H^3(M;\mathbb{Z})^U = \dim_{\mathbb{F}_p} \big(\textstyle\bigwedge^2 V^*\big)^R = a. \tag{4}$$

If V is not self-dual, a nonzero invariant alternating form would give an isomorphism $V \cong V^*$, so $a = 0$. In the stated orthogonal case, Schur's lemma makes every invariant bilinear form a scalar multiple of the symmetric one; in odd characteristic it cannot also be alternating. Thus $a = 0$ in that case as well.

Finally use the integral Lyndon–Hochschild–Serre spectral sequence for $M \to G \to U$. The bottom-row term in total degree four is $E_2^{4,0} = H^4(U;\mathbb{Z})$. There is no outgoing differential, and the only possible incoming differentials are

$$d_2 : E_2^{2,1} \longrightarrow E_2^{4,0}, \qquad d_3 : E_3^{1,2} \longrightarrow E_3^{4,0}, \qquad d_4 : E_4^{0,3} \longrightarrow E_4^{4,0}.$$

The first source is zero because $H^1(M;\mathbb{Z}) = 0$. By (3) the second source has order at most p^b, and by (4) the third has order at most p^a. Therefore the kernel of the edge homomorphism, which is inflation, has order at most p^{b+a}. If one of the latter two sources vanishes, the kernel is the image of a single homomorphism from an elementary abelian p-group and consequently has exponent at most p. All of the stated consequences follow. □

The preceding results establish the following parts of the conjecture.

(1) Any universal bound must be at least six (Corollary 2.2).

(2) The conjecture holds for every finite simple group (Theorem 4.1).

(3) Cyclic-Sylow witnesses persist through arbitrary finite Frattini covers; in particular all finite Frattini covers of M_{23}, J_4, and Ly satisfy the conjecture (Corollary 6.4). The simple-quotient ledger records the exact scope of all current results for epimorphisms onto simple groups (Proposition 6.7); unconditionally it proves $\mathcal{Q}_2(C_p)$, while the nonabelian rows still impose split, prime-power-kernel, prime-to-kernel, universal-centralization order, central, elementary-kernel, or Frattini hypotheses. The universal-centralization row permits noncentral kernels and depends only on $[E, K]$. The central row uniformly includes every nonabelian simple group with cyclic prime-power Schur multiplier (Theorem 6.6).

(4) Nonabelian minimal normal subgroups whose simple factor has a split automorphism group reduce to a proper quotient. In the nonsplit cases, factor-stabilizer transfer reduces the problem to a restriction problem for almost-simple groups; the shared cyclic Sylow criterion solves this problem for A_6 in degree at most eight (Corollary 7.5), and the first-nonzero-row argument solves it for the Tits group in degree four (Corollary 7.7). The Evens-norm argument solves every symplectic factor stabilizer in mod-two degree four, hence in integral degree at most five (Theorem 7.28), and the same argument solves every factor stabilizer of type E_7 (Theorem 7.29). The Spin-cover version solves every factor stabilizer of type B_n (Theorem 7.30). The signed-square argument solves every untwisted E_6 factor stabilizer in degree eight (Theorem 7.19). The multiplier-power and iterated-Evens-norm argument gives every twisted E_6 factor stabilizer an integral witness in degree at most thirteen (Theorem 7.20). Proposition 7.21 proves that the automorphism-fixed degree-nine class $\beta(w)w^3$ is nonzero and isolates an optional lower-degree edge-lifting obstruction. Proposition 7.22 computes that obstruction: the lift is automatic when the odd outer Sylow three-subgroup is cyclic, and in rank two only one explicitly displayed scalar remains. The unitary model in Remark 7.23 has a nonzero analogous scalar, so the listed equivariance properties do not by themselves settle the twisted-E_6 coefficient. The fixed-point and degree-eight norm argument solves every even-orthogonal factor stabilizer, including genuine D_4 triality (Theorems 7.25 and 7.26). Theorem 7.34 also solves every linear and unitary factor stabilizer whose Schur multiplier is

supported at one prime. Its fixed-field reduction leaves only $\mathbb{F}_2$ and $\mathbb{F}_3$; degree-eight fixed-point classes handle the higher-rank cases, while an iterated norm on the triple-cover class handles $\mathrm{PSU}_3(2^m)$. Theorem 7.35 additionally closes every linear or unitary factor whose multiplier has prime support $\{2,3\}$, using a finite fixed-field list and fundamental degree-four classes. Theorem 7.36 then reduces the mixed-prime case to two even-field-image residues. Theorem 7.37 closes the unitary residue in degree at most twenty and uses a primitive cubic prime to sharpen the remaining linear residue to (9A). Theorem 7.38 gives degree at most $4n$ throughout that residue and therefore completes every bounded-rank part. Proposition 7.39 shows, without claiming a counterexample, that any prescribed finite list of the resulting primitive-divisor tests can be absorbed in a genuine nonsplit unbounded-rank family. Proposition 7.40 constructs a degree-four class on the inner-diagonal group at every absorbed prime satisfying its valuation test. Lemma 7.41 removes the later field–graph edge differentials once that class has an invariant preimage, and Theorem 7.42 proves this in the strict valuation range. The unresolved equality range is now the central-kernel correction formulated in Conjecture 11.8. The factor-stabilizer splitting criterion discards every split intermediate almost-simple group (Theorem 7.32). Theorem 7.48 packages all completed nonsplit families with common integral ceiling twenty, and Corollary 7.49 states their concrete exclusion from a least counterexample at a fixed finite threshold.

(5) A nonsplit elementary abelian extension is settled in odd characteristic when the irreducible kernel is orthogonal and the quotient has vanishing fourth mod-p cohomology (Theorem 8.2). Independently of self-duality or characteristic, it is also settled when the rank inequality in Theorem 8.1 holds. For a least counterexample, Proposition 5.12 further separates central universal covers from noncentral kernels with superperfect quotient. When the quotient by a central minimal kernel is simple, the quotient ledger sharpens this further to multiplier C_p and vanishing H^4 (Theorem 9.1). Proposition 5.16 supplies a degree-at-most-four witness on every standard Lie-type universal cover, and the exceptional-multiplier audit removes the remaining Lie-type cases. Hence such a central simple quotient must be sporadic (Corollary 9.3). The known third homology of sporadic Schur covers and a persistent degree-twenty class for 3.McL, together with the degree-four Chern class on 2.Ru and the degree-four classes on $3.\mathrm{Fi}'_{24}$ and $2.B$, eliminate the sporadic family as well (Corollary 9.4). Consequently, a central minimal kernel in a least counterexample cannot have a simple quotient (Corollary 9.5). The quotient has the same integral homology as $K(C_p, 2)$ through degree nine (Corollary 9.6), with all of those groups explicit in Proposition 9.8. If the central kernel is the unique minimal normal subgroup, its first quotient layer is constrained by the same-prime and Schur-multiplier coinvariant tests in Corollary 9.10. A nonabelian first layer, or an elementary abelian layer with nonabelian inverse image, leaves a nontrivial superperfect quotient. If the elementary inverse image is abelian, it is either a central cyclic group C_{p^2} or an elementary abelian nonsplit module extension whose quotient is again nontrivial and superperfect (Proposition 9.11). Passing from A to the full center absorbs the cyclic alternative: the center is cyclic, and an abelian elementary layer above the full central quotient produces a nontrivial superperfect group which is the universal central extension of the next quotient (Proposition 9.12). The full central quotient $X = G/Z(G)$ is centerless, $G \to X$ is universal, and every proper minimal-normal descent $X \to R$ makes $H_2(R;\mathbb{Z})$ a quotient of C/A and produces the smaller universal central extension as the quotient

$$\widetilde{R} \cong G/[G, L]$$

(Proposition 9.13). Within either elementary layer, the classification-family audit now closes the whole simple quotient subcase with nonzero multiplier: alternating quotients survive in

degree four, Lie-type quotients in degree at most nine, and sporadic quotients in degree at most twenty (Corollary 9.22). Only the Baby Monster needs the largest displayed ceiling in that subcase. This conclusion does not include a simple quotient with trivial multiplier, nor a nonsimple quotient. For a noncentral minimal kernel with simple quotient, that quotient has trivial multiplier and acts faithfully.

(6) Every least counterexample is confined to the two explicit configurations in Theorem 9.1.

Two families of obstructions remain.

(a) For a nonabelian minimal normal subgroup S^r, the remaining simple factors are precisely the linear groups $\mathrm{PSL}_n(q)$ whose automorphism extensions are nonsplit and whose Schur multipliers have at least two distinct prime divisors, one of them at least five, with field and diagonal data satisfying (9A) and the cyclotomic hierarchy following (9B). Every bounded-rank part is complete; Proposition 7.39 explains the remaining unbounded-rank mechanism. It is enough to construct a class on each possible nonsplit factor stabilizer whose restriction to S is nonzero in some uniformly bounded finite degree. By Lemma 7.2, a sufficient condition is a bounded-degree class on S with nonzero norm under the factor stabilizer. Proposition 7.8 also handles the stated outer quotients away from the primes in the Schur multiplier, while Corollary 7.11 handles every proper inner-diagonal factor stabilizer. Proposition 7.13 also handles the stated odd- characteristic graph and field mixtures by a quadratic scalar character. Proposition 7.15 further handles the stated mixed diagonal–field stabilizers in all three families and the unitary semilinear cases covered by its scalar-action criterion. Corollary 7.16 gives the complementary degree-four test and an explicit prime-support description of the remaining residue. Proposition 7.10 supplies the uniform central-cover coinvariant formulation for the remaining Lie types. Theorem 7.28 removes symplectic type altogether by applying the mod-two Evens norm from a pure-field subgroup of index two. Theorem 7.29 removes type E_7 by the same mechanism. Theorem 7.30 removes type B_n using the Spin double cover. Corollary 7.18 reduces the E_6 pair to small fixed fields. Theorem 7.19 then eliminates every untwisted E_6 stabilizer in degree eight. Theorem 7.20 closes every twisted stabilizer with integral bound thirteen. Hence neither member of the E_6 pair remains in the existence problem. The degree-nine calculation and its unitary model are retained only as diagnostics for the lifting method. Theorem 7.25 eliminates the full nontriality fixed-$\mathbb{F}_3$ residue in degree eight, and Theorem 7.26 eliminates the residual untwisted D_4 triality cases in the same degree. Thus no even-orthogonal factor remains. Theorem 7.34 eliminates every one-prime-support linear and unitary multiplier. Theorem 7.35 eliminates multiplier support $\{2, 3\}$. Theorem 7.36 reduces the mixed-prime case to two even-field-image residues. Theorem 7.37 eliminates the unitary residue and shows that an unresolved linear field order is divisible by twelve, while every prime divisor of $x^2 - 1$ and every odd prime divisor of $x^2 + 1$ already divides the diagonal multiplier. Theorem 7.38 further gives a restricting class by degree $4n$ and forces the primitive-divisor absorption conditions following (9B). The explicit family in Proposition 7.39 shows why this is still an unbounded-rank residue rather than a finite arithmetic checklist. This is the precise residue stated at the start of this item. Lemma 7.3 also converts a detected degree-four class on which the prime-to-ℓ outer quotient acts by signs into a restricting degree-eight class; applying it uniformly requires detection of the square.

(b) For an elementary abelian p-Frattini kernel, the cases not covered by Theorem 8.2 include characteristic two, non-self-dual modules, symplectic modules, and quotients with nonzero $H^4(-; \mathbb{F}_p)$. Theorem 8.1 covers any of these cases when its rank inequality holds, but iterated

p-Frattini covers still require control that is uniform along the entire tower. In the central case, Corollary 9.5 excludes a simple quotient, and Corollary 9.6 identifies its homology through degree nine with that of $K(C_p, 2)$. Corollary 9.10 shows that a unique-central-monolith case can descend only through elementary abelian layers of the same characteristic or through nonabelian layers whose factor-stabilizer coinvariants retain the kernel prime. Proposition 9.11 makes the next step explicit: every such nonabelian layer produces a nontrivial superperfect quotient and an exact multiplier-coinvariant sequence, while an elementary layer either does the same through a symplectic commutator form or has one of two exact abelian forms: a central C_{p^2}, or a nonsplit elementary module extension with superperfect quotient. The central cyclic thickening is not a separate iterated obstruction after one passes to the full center. Proposition 9.12 shows that the center is cyclic and that an abelian elementary layer above it instead yields a nontrivial superperfect group which universally covers the next quotient. Proposition 9.13 gives a strict induction invariant: every proper first layer lowers the exponent of the cyclic Schur multiplier, yields $|\widetilde{R}| < |G|$, and realizes $\widetilde{R}$ as $G/[G, L]$. Proposition 9.15 further identifies the canonical kernel $D = [G, L]$: it satisfies $D = [G, D]$; it is the perfect group $[L, L]$ in an S^r layer; in a nonabelian elementary layer it has $[D, D] = \Phi(D) = A$, elementary abelian D/A, and $|D| \le p^2|V|$, while $H_2(D; \mathbb{Z})$ is elementary abelian; and in an abelian elementary layer it is $\Omega_1(L)$. In odd characteristic the nonabelian kernel is more rigid: the p-power map and $D = [G, D]$ force D to be extraspecial of exponent p, so the almost-extraspecial alternative occurs only at $p = 2$. What remains is the persistence statement in Step 7 of Section 10: a witness on this smaller universal cover, within one common finite range, must force a witness on G within that range. Proposition 9.14 proves this away from the prime divisors of $[G, L]$; in particular the elementary cases are reduced to the single prime p. Proposition 9.16 treats every degree: the integral inflation kernel in degree n is annihilated by $|[G, L]|^{n-2}$. In an S^r layer the sharper exponent is $|[G, L]|^{n-3}$; in a nonabelian elementary layer it is p^{β_n} with

$$\big(\beta_{4+j}\big)_{0\le j\le 6} = (2, 4, 7, 11, 15, 20, 25);$$

when p is odd and $\dim V = 2m$, this improves further to

$$\begin{aligned} m &= 1 \quad (2, 4, 6, 8, 10, 12, 14),\\ m &= 2 \quad (2, 4, 6, 9, 12, 15, 18),\\ m &\ge 3 \quad (2, 3, 5, 8, 11, 14, 17). \end{aligned}$$

At $p = 3$, exact integral calculations in widths one through four replace the respective final entries by 13, 16, 15, and 15; the uniform final entry 17 applies from width three onward, while the exact computations retain the sharper width-three and width-four entry 15. At $p = 2$, the coefficient exponents through degree nine are now uniformly controlled for every possible nonabelian kernel. The corresponding seven-entry edge tuples are

$$\begin{aligned} D_m^+ \quad & m = 3 : (2, 4, 5, 6, 7, 11, 12),\\ & m \ge 4 : (2, 4, 5, 6, 7, 10, 11),\\ D_m^- \quad & m = 2 : (2, 4, 5, 6, 7, 11, 12),\\ & m = 3 : (2, 4, 5, 6, 7, 10, 11),\\ & m \ge 4 : (2, 4, 7, 10, 13, 18, 21),\\ \mathsf{A}_m \quad & m = 2 : (2, 4, 5, 6, 7, 11, 12),\\ & m \ge 3 : (2, 4, 6, 8, 10, 14, 16). \end{aligned}$$

In every nonabelian elementary case, put

$$b = \dim_{\mathbb{F}_p} \operatorname{im}\big(H_3(R; \mathbb{Z}) \to H_1(R; V)\big).$$

Then the degree-four kernel has order at most

$$p^{b+\dim_{\mathbb{F}_p}(\wedge^2 V^*)^R-1}.$$

It is therefore zero whenever $\mathrm{End}_{\mathbb{F}_p R}(V) = \mathbb{F}_p$ and $H_3(R;\mathbb{Z})_{(p)} = 0$; the latter condition follows, in particular, when a Sylow p-subgroup is cyclic with automizer order $e \nmid 2$. In that cyclic case, however, $H^4(\widetilde{R};\mathbb{Z})_{(p)} = 0$ as well, so the test is diagnostic rather than a new p-primary persistence case. In the abelian elementary case the all-degree annihilator is p^{n-2}, and Proposition 9.27 gives the sharper degree-four order bound

$$p^{\dim_{\mathbb{F}_p} H^1(R;V^*)-1+\dim_{\mathbb{F}_p}(\wedge^2 V^*)^R}.$$

Thus degree four persists whenever the smaller universal cover has more p-primary fourth cohomology than this explicit obstruction permits; in particular inflation is injective when both obstruction groups vanish.

10 A sufficient eight-step path to a universal bound

There is one endpoint: prove that some universal finite bound N_0 exists. No attempt should be made to optimize N_0 before this is achieved. The following eight steps are sufficient: completing the three partial proof obligations with any common finite bound N_0 exceeding the finite ceilings in the completed steps proves the theorem by Proposition 10.1 below. Bounds such as thirteen or twenty are therefore complete successes in their classification branches, not defects to be repaired. The steps are a proof plan, not merely a list of promising directions.

The status *complete* means that the full proof obligation stated in that step has been proved in the cited result; *partial* means that only the listed subcases are proved; *open* means that no uniform theorem of the stated scope is yet available; and *pending* means that the step is a formal assembly which depends on earlier open steps. Whenever a new result changes the scope of a step, both this list and the simple-quotient ledger in Section 6 must be updated.

Step 1. Complete: translate the problem to homology. For a finite group, identify $H^n(G;\mathbb{Z})$ for $n \geq 2$ with the Pontryagin dual of $H_{n-1}(G;\mathbb{Z})$. Thus a candidate bound N_0 is equivalent to finding nonzero integral homology in degrees one through $N_0 - 1$. Lemma 1.1 proves this, and Corollary 2.2 proves that no universal bound below six can work.

Step 2. Complete: reduce to a superperfect least counterexample. Assume a counterexample of least order, choose a minimal normal subgroup, and split into an elementary abelian kernel and a kernel S^r. Eliminate split extensions by inflation from the proper quotient. Proposition 5.11 and Theorem 9.1 carry out this reduction. The same theorem now also proves that the kernel of every maximal simple quotient of a least counterexample contains all three primes in the selected detecting set for that quotient; in particular such a kernel has at least three distinct prime divisors.

Step 3. Complete: close the simple and cyclic-central-simple base. The required state-

ment is:

$$\begin{gathered}\text{There is a finite } D_{\text{base}} \text{ such that}\\ H^i(S;\mathbb{Z}) \neq 0 \text{ for some } i \leq D_{\text{base}}\\ \text{for every nontrivial finite simple } S, \text{ and}\\ H^i(\widetilde{S};\mathbb{Z}) \neq 0 \text{ for some } i \leq D_{\text{base}}\\ \text{for every nonabelian simple } S \text{ for which}\\ H_2(S;\mathbb{Z}) \text{ is a cyclic prime-power group.}\end{gathered} \tag{S}$$

The first line is Theorem 4.1. The second line, including the complete exceptional-multiplier audit, is Theorem 6.6. Thus Step 3 is complete; the corresponding ledger row covers every central epimorphism onto such a simple group. The maximum of the finite ceilings in those two results is a valid value of D_{base}; optimizing that value is irrelevant to the existence argument.

Step 4. Partial: eliminate every nonabelian minimal normal kernel. It is sufficient to prove that for every

$$N = S^r \lhd E$$

which is minimal normal in a finite superperfect group E, some $H^i(E;\mathbb{Z})$ with uniformly bounded finite i is nonzero. By Theorem 7.1, it is enough, for every possible factor-stabilizer image

$$S \leq B \leq \mathrm{Aut}(S),$$

to construct a class of uniformly bounded finite degree on B whose restriction to S is nonzero. This is complete for the split cases, A_6, the Tits group, all symplectic types, E_7, B_n, and the other factor stabilizers listed as solved in Section 7. All even-orthogonal types, including genuine D_4 triality, and all untwisted E_6 towers are complete in degree eight. Every twisted E_6 stabilizer is complete in degree at most thirteen. Its residual degree-nine scalar is therefore an optional optimization and has been removed from the critical path. Every linear or unitary factor with prime-power Schur multiplier is now complete in integral degree at most thirteen, and every such factor with multiplier support $\{2,3\}$ is complete in integral degree at most nine. Theorem 7.37 now closes every unitary factor stabilizer with common integral ceiling twenty and sharpens the remaining mixed-prime linear residue to (9A). In particular its field-image order is divisible by twelve, and all primes of x^2-1 and all odd primes of x^2+1 are already absorbed by the diagonal multiplier. Theorem 7.38 now closes every bounded-rank part of this residue: it gives degree at most $4n$, and forces every primitive divisor of x^s-1 into the multiplier whenever an odd prime $s \leq n$ lies below one quarter of the asserted vanishing range. Proposition 7.39 constructs genuine nonsplit unbounded-rank examples absorbing any prescribed finite list of these primes. Proposition 7.40 supplies the projective degree-four class on the inner-diagonal part under the exact basic-level valuation test. Lemma 7.41 settles the subsequent field–graph edge differentials after an invariant preimage is obtained, and Theorem 7.42 obtains one in the strict valuation range. Proposition 7.43 puts the explicit absorption family exactly on the equality boundary and proves that its obstruction is already nonzero inside the projective cyclic line. Theorem 7.44 changes classes and closes that entire family in mod-three degree twelve. More generally its depth-s bound gives (9E): unless the full three-part of $q-1$ is absorbed, s must grow logarithmically with the asserted vanishing range. In defining characteristic three the same argument at the forced prime five gives (9F). The finite projective torsion class then proves (9G), forcing the field

image in that row to have order divisible by five, hence by sixty. The field-depth norm closes every subfamily of bounded field 3-depth or 5-depth and gives (9H). Thus any hypothetical sequence with increasing initial vanishing must have both unbounded rank and unbounded relevant field-primary depth. This gives the residual conditions (9D)–(9H). Conjecture 11.8 retains the off-line central-kernel H^1 correction as an optional projective-method question. The step remains partial for the narrower residue satisfying (9A)–(9H); the obligation is to find one degree bound independent of the rank, not merely to extend the finite cyclotomic list. Conjecture 11.10 isolates the exact non-toral semilinear permanence statement that would finish this step in integral degree forty-eight; Conjecture 11.9 is the toral alternative. Proposition 11.13 computes the full-depth field action on the two low multiplier rows: those cyclic modules are cohomologically trivial, and restriction from the semidirect product is onto in degrees two and three. This does not solve the permanence problem, because every extendable integral degree-three multiplier class vanishes on the non-toral detector. Proposition 11.12 shows that the standard restriction-of-scalars alternatives do not repair this: direct sum does not descend projectively, and tensor induction introduces an r-divisible scalar norm. Proposition 11.15 also proves that the natural point and hyperplane permutation representations miss the desired non-toral class at full depth. Proposition 11.16 replaces the isolated rank-two detector by an arbitrary tensor clock–shift subgroup C_r^{2b} and identifies the restriction as a nonzero symplectic invariant fixed pointwise by the field subgroup. Proposition 11.17 then rules out every ordinary complex Chern-class realization of that restriction once $b \geq 2$; the full symplectic Weyl group forces the first possible Chern restriction above the desired degree. Proposition 11.18 shows why the most immediate normalizer-transfer argument is incomplete: at full depth the rank-two detector has a conjugate inside its own centralizer.

Step 5. Partial: eliminate elementary kernels outside the unique central monolith. The required branch theorem is: if V is an elementary abelian minimal normal subgroup of a finite superperfect group E, and either $V \not\leq Z(E)$ or V is not the unique minimal normal subgroup of E, then $H^i(E;\mathbb{Z}) \neq 0$ for some uniformly bounded finite i. Theorems 8.1 and 8.2 prove this under their rank and orthogonal-module hypotheses, and Proposition 5.12 makes the noncentral quotient superperfect. Corollary 6.13 closes the simple-quotient subbranch for every alternating quotient and every standard ordinary or graph-twisted Lie-type quotient over $q \neq 2, 3$, with no condition on the p-group kernel or its module structure. Corollary 6.14 does the same for every sporadic quotient with bound 140 and for the Tits quotient with bound 24. Corollary 6.19 also closes every classical simple quotient, including all small-field families over $\mathbb{F}_2$ and $\mathbb{F}_3$, with bound 24 (and usually bound 8 in those two fields). Corollary 6.21 closes the remaining infinite Suzuki–Ree families with bound 24, and Corollary 6.22 closes the exceptional small-field residue with bound 60. Thus Theorem 6.29 closes the entire simple-quotient subbranch for arbitrary prime-power kernels with the explicit common bound 140. Proposition 6.8 gives an exact test at the first canonical p-Frattini layer over a base with $H^2(-;\mathbb{F}_p) = H^3(-;\mathbb{F}_p) = 0$: the new p-primary Schur multiplier is the kernel of one explicit differential on invariant alternating forms, or on invariant quadratic forms at $p = 2$. Corollary 6.9 also proves that prime-to-p and cyclic-Sylow witnesses persist up every fixed-p tower. These are new reductions, not a uniform survivor theorem. Characteristic two, non-self-dual and symplectic modules, failure of the rank inequality, the kernel of that canonical differential, iterated Frattini layers, and the multiple-central-minimal-subgroup case remain.

Step 6. Complete as a structural reduction: organize the unique central monolith. For the remaining central minimal subgroup $A = C_p$, prove that G/A is nonsimple and has the homology of $K(C_p, 2)$ through degree nine; classify its first minimal-normal layer; pass to the full center $C = Z(G)$; and exhibit a strictly decreasing multiplier invariant. Corollaries 9.5 and 9.9, Propositions 9.11, 9.12, and 9.13 prove these assertions, while Proposition 9.15 identifies the kernel of the canonical quotient. In particular, C is cyclic, $G \to X = G/C$ is the universal central extension of a centerless perfect group, and every proper minimal-normal descent $X \to R$ satisfies

$$C/A \twoheadrightarrow H_2(R;\mathbb{Z}), \qquad \widetilde{R} \cong G/[G,L], \qquad |\widetilde{R}| < |G|.$$

Step 7. Partial: prove one-layer persistence in the central descent. The exact missing statement is the following. In every configuration of Proposition 9.13 with X nonsimple, if the smaller universal central extension $\widetilde{R}$ has nonzero integral cohomology in some degree at most a common finite bound N_0, then G has nonzero cohomology in some degree at most the same global ceiling N_0. The class and its degree may change; only the ceiling must be preserved. This fixed-ceiling condition is essential for the least-counterexample induction, but N_0 may be chosen as large as necessary. It is enough to prove this separately for the three layer types

$$S^r, \qquad V \text{ with nonabelian inverse image}, \qquad V \text{ with abelian inverse image}. \tag{P}$$

A canonical inflation map is available uniformly: Propositions 5.3 and 9.13 identify $\widetilde{R}$ with $G/[G,L]$. Its possible kernel on cohomology is the remaining issue. Proposition 9.14 proves (P) for every primary component away from $|[G,L]|$. In particular, for either elementary layer it leaves only the p-primary part, while for an S^r layer it leaves only primes in $\{p\} \cup \pi(S)$. Proposition 9.16 also shows in every degree n that the integral inflation kernel is annihilated by $|[G,L]|^{n-2}$. In the three layer types this improves respectively to

$$\begin{array}{ll} |[G,L]|^{n-3} & S^r, \\ p^{\beta_n} & V \text{ with nonabelian inverse image}, \\ p^{n-2} & V \text{ with abelian inverse image}. \end{array}$$

where $\left(\beta_{4+j}\right)_{0\le j\le 6} = (2,4,7,11,15,20,25)$ and the first two inflation maps are injective in degree three. Thus only explicitly bounded-exponent primary classes remain. For an odd-prime nonabelian layer the kernel D is extraspecial of exponent p. If $\dim V = 2m$, the tuple improves further to

$$\begin{array}{ll} m = 1 & (2,4,6,8,10,12,14), \\ m = 2 & (2,4,6,9,12,15,18), \\ m \ge 3 & (2,3,5,8,11,14,17). \end{array}$$

At $p = 3$, the fourth through sixth entries use the coefficient-comparison bounds

$$3^3 H^6(D;\mathbb{Z}) = 3^3 H^7(D;\mathbb{Z}) = 3^3 H^8(D;\mathbb{Z}) = 0$$

proved in Proposition 9.16. Thus the width-at-least-three tuple through degree nine is uniform over all odd primes. The Koszul calculation gives $p^3 H^9(D;\mathbb{Z}) = 0$ for $p \ge 5$, while the filtered width-two calculation and restriction give the same bound at $p = 3$. At $p = 3$, the exact computations (5c) make $H^9(D;\mathbb{Z})$ elementary in widths one through

four; the corresponding sharper tuples are (5d). The filtration refinement (5b) now shows that $E_\infty^{5,4} = 0$: the Kudo d_5 kills every class away from the symplectic line $\langle q\omega u^2\rangle$, and the exact width-two filtration calculation proves that this line also has nonzero integral d_5; restriction propagates the result to every larger width. This removes one of the four remaining total-degree-nine filtration pieces and proves $3^3H^9(D;\mathbb{Z}) = 0$ uniformly. For the other middle term, the width-uniform colon calculation (5b′) proves that multiplication by $z(2)$ is already injective after quotienting by the full ideals generated by the known d_2-, d_3-, and d_5-transgression targets. The page audit (5b″) now proves that these are all the relevant bottom-row images in widths $m \geq 5$: Leary's d_4 formula reduces the surviving bidegree-$(5,4)$ source to the Kudo family, while the simultaneous (ω, q)-kernel calculation removes the bottom-row d_4- and d_6-sources. Thus the actual mod-3 d_7 from $(3,6)$ is injective. Coefficient reduction confines every integral survivor to $\omega V^* u^3$, giving the bound $\dim_{F_3} E_\infty^{3,6} \leq 2m$ in (5b‴). Determining the integral d_5 on this reduction-zero subspace remains the next stated odd-primary filtration target, although the coefficient exponent needed for the edge bound is now already sharp at three. At $p = 2$, the Harada–Kono additive theorem now settles the coefficient exponents through degree nine uniformly for every permitted plus-type extraspecial kernel. The corresponding edge tuples are

$$m = 3 : (2,4,5,6,7,11,12), \qquad m \geq 4 : (2,4,5,6,7,10,11).$$

The exact filtered-resolution certificate (5f) gives the first tuple for D_2^- and the second for D_3^-. Consequently the entire plus-type family, rather than merely a finite list of widths, now has a sharp low-degree coefficient bound. The normal index-four embedding $D_m^- \lhd D_{m+1}^+$ and restriction–transfer comparison give the uniform minus-type tuple

$$(2,4,7,10,13,18,21) \qquad (m \geq 4).$$

For an almost-extraspecial kernel A_m, the index-two embedding in a plus-type overgroup gives the norm relation (5i) for the involution that inverts $Z(\mathsf{A}_m) \cong C_4$. More decisively, $D_8 \circ C_4 \cong Q_8 \circ C_4$, so A_m contains both a plus- and a minus-type extraspecial subgroup of index two. Restriction and transfer through the sharper subgroup at each width give the full uniform coefficient bound (5k) and the edge tuples

$$m = 2 : (2,4,5,6,7,11,12), \qquad m \geq 3 : (2,4,6,8,10,14,16).$$

Thus uniform low-degree coefficient growth is now controlled for all three characteristic-two families. What remains is the actual persistence of a class from $\widetilde{R}$ across the inflation edge for the unbounded orthogonal-module actions. For every nonabelian elementary layer, set

$$b = \dim_{\mathbb{F}_p} \operatorname{im}\big(H_3(R;\mathbb{Z}) \to H_1(R;V)\big).$$

The degree-four calculation gives the uniform order bound

$$\Big|\ker\big(H^4(\widetilde{R};\mathbb{Z}) \to H^4(G;\mathbb{Z})\big)\Big| \leq p^{b+\dim_{\mathbb{F}_p}(\wedge^2 V^*)^R - 1};$$

in particular this edge is injective when $b = 0$ and the invariant alternating form is unique. This single formula includes the extraspecial cases in every characteristic and the almost-extraspecial case at $p = 2$. Corollary 9.17 makes these conditions automatic when V has scalar endomorphism ring and $H_3(R;\mathbb{Z})$ has no p-primary part, giving an explicit next subledger for the remaining finite and representation-theoretic audit. Corollary 9.18

replaces the homology condition by the concrete test that a Sylow p-subgroup of R is cyclic with automizer order not dividing two. The scope audit in Section 6 shows, however, that this specialization adds no p-primary case: the same cyclic-Sylow hypothesis makes the p-part of the multiplier vanish and forces $H^4(\widetilde{R};\mathbb{Z})_{(p)} = 0$. In the actual central descent, every nonzero multiplier must first be a cyclic p-group. Corollary 9.22 now closes the entire simple nonzero-multiplier subcase for the finite-bound goal: alternating groups give degree four, Lie-type groups degree at most nine, and sporadic groups degree at most twenty. The alternating argument uses the three-primary Pontryagin class. At the residual field $q = 3$, the Lie-type fixed-point calculation gives a degree-four class of exact order eight, while the inflation kernel is annihilated by four. At $q = 2$, the unitary signed fundamental degree four gives degree eight or nine. The sporadic argument excludes the mixed multipliers of M_{22}, Suz, and Fi_{22}, closes the remaining prime-power cases, and uses a cyclic Sylow 11-class to close the Baby Monster in degree twenty. Proposition 9.26 supplies a finer certificate: two independent degree-four detections give

$$|H^4(2.B;\mathbb{Z})_{(2)}| \geq 4.$$

This closes both elementary-layer types for every absolutely irreducible $\mathbb{F}_2B$-module with first-cohomology dimension at most one; the natural 4370-dimensional module is a certified example. Modules with larger or presently unknown first cohomology are not included in that certificate. If instead R is simple with trivial multiplier in either elementary-layer case, then $\widetilde{R} = R$ and $D = \ker(G \to R)$ is a p-group. Theorem 6.29 closes that case with the explicit bound 140. Hence every simple-R elementary-layer case is now complete for the finite-bound goal. Simple quotients arising after an S^r layer, and nonsimple quotients, remain part of the persistence problem. Proposition 9.25 retains the finer two-primary degree-four analysis as a supplementary certificate. In particular it computes exact first cohomology in dimensions 376 and 1246 and uses Lemma 9.24 to determine their possible nonabelian kernel types, but these module-by-module distinctions are no longer needed to close Ru. Proposition 9.27 also proves p-primary persistence in degree four whenever

$$|H^4(\widetilde{R};\mathbb{Z})_{(p)}| > p^{\dim_{\mathbb{F}_p} H^1(R;V^*)-1+\dim_{\mathbb{F}_p}(\wedge^2 V^*)^R};$$

it gives injectivity when $H^1(R;V^*)$ is generated by the nonsplit module-extension class and V has no invariant alternating form. Existing cyclic-Sylow, rank, and orthogonal-module criteria settle further special instances, so Step 7 is partial rather than open.

Step 8. Pending: assemble the induction. Once the incomplete parts of Steps 4 and 5 and the persistence statement (P) in Step 7 are proved with a common degree bound N_0 at least as large as the finite ceilings in the completed steps, take a least counterexample and apply Steps 2–7 to any minimal normal subgroup. Proposition 10.1 below verifies that the branches are exhaustive and that this assembly proves a finite bound.

Difficulty and prospects (subjective planning assessment). The following labels are not mathematical assertions. They estimate the kind of work still missing as of August 2026 and should be revised when a new uniform theorem changes the roadmap. In brief, Step 4 and the finite-family part of Step 7 are the pieces most likely to yield to further computation and classified case checking; Step 5 and the uniform part of Step 7 appear to require a genuinely deeper structural idea; Step 8 is formal once those inputs exist.

Steps 1–3: complete; no remaining research difficulty in their stated scope.
The homological translation, least-counterexample reduction, and simple base are proved. Further computation may sharpen degrees, but none is needed for these milestones.

Step 4: moderate difficulty; predominantly a classified finite and arithmetic audit.
The remaining nonsplit Lie-type factor stabilizers form a reduced classified list. The signed-square theorem closes every untwisted E_6 tower in degree eight, and the multiplier-power theorem closes every twisted E_6 tower in degree at most thirteen. Hence the residual degree-nine twisted scalar is not a main-goal task. Degree-eight theorems also close every even-orthogonal factor stabilizer, including genuine D_4 triality. The prime-power-multiplier theorem now also removes every linear and unitary factor with one-prime multiplier support, and the two-small-prime theorem removes support $\{2, 3\}$. What remains is the mixed-prime linear residue with a multiplier prime at least five, field order divisible by twelve, and all primes of $x^2 - 1$ and all odd primes of $x^2 + 1$ already absorbed by the diagonal multiplier. The unitary family is complete in degree at most twenty. The cyclotomic hierarchy now settles every bounded-rank portion and forces rank greater than one quarter of the vanishing threshold. Its explicit absorbing family shows that checking any fixed finite list of primitive divisors cannot be the whole argument; the multiplier-depth norm nevertheless closes that whole family in degree thirteen and forces $v_3(d_0) \neq 1$ in the true residue. At depth $s \geq 2$ it gives the finite factor bound $4 \cdot 3^s + 1$ whenever $v_3(q - 1) > s$. Thus an asserted vanishing range forces either full three-primary absorption or logarithmically growing depth. Character tables, regular embeddings, a fixed-degree replacement for the full-depth clock–shift detector, and restriction calculations should settle much of what remains. This is the part most plausibly finished by computation and classified case checking, although a short uniform operation or regular-embedding lemma is still needed at unbounded full depth. In defining characteristic three the forced prime five now gives the analogous depth dichotomy, so it is no longer an entirely separate mechanism. Degree lowering is not part of this task.

Step 5: very high difficulty; a deeper uniform idea is probably required.
These branches contain unbounded families of irreducible modules, Frattini towers, and the groups $H^1(R; V^*)$ and $(\bigwedge^2 V^*)^R$ with no present uniform bound. The entire simple-quotient subbranch has now been removed by Theorem 6.29, including the Suzuki–Ree families with bound 24 and the small-field exceptional families with bound 60. A finite catalogue of examples cannot close this scope. A structural theorem forcing a low-degree invariant, or a new functorial persistence mechanism insensitive to the module size, appears necessary.

Step 6: complete as a structural reduction; no missing classification or construction in its stated scope.
The multiplier descent and the quotient $G \twoheadrightarrow \widetilde{R}$ are available. The remaining work is not to construct the descent but to prove that a class survives it.

Step 7, elementary-layer simple slice: complete for the main goal.
For a simple R with nonzero multiplier in either elementary-layer case, the alternating, Lie-type, and sporadic families are now completely closed for the finite-bound goal in degrees 4, 9, and 20. The Baby Monster's degree-twenty witness is therefore sufficient. Its finer module audit is supplementary data, not unfinished work for the main goal. When the multiplier is trivial, Theorem 6.29 applies directly to $G \twoheadrightarrow R$. The remaining Step 7 problem lies in S^r layers, nonsimple quotients, and uniform layer persistence.

Step 7, uniform slice: high to very high difficulty; computation is diagnostic but a structural theorem is needed.
For S^r layers, simple quotients with trivial multiplier, and the odd-primary nonabelian elementary cases, computations can locate candidate classes and certify individual LHS differentials, but passing to every factor, width, and action still requires a uniform detection or transfer argument. The abelian elementary case and the remaining characteristic-two nonabelian case are harder: they involve unbounded module actions and quadratic refinements. The coefficient exponents are already controlled uniformly for the plus, minus, and almost-extraspecial families, so merely extending Smith-form calculations width by width should not be expected to finish the proof. A functorial persistence theorem insensitive to module dimension is the most likely decisive input.

Low difficulty once the inputs exist: Step 8.
Proposition 10.1 already proves that the final least-counterexample assembly is formal after Steps 4, 5, and 7.

On the present evidence, the planning estimate is an 86–96% chance that a recognizable extension of the program proves the existence of some finite N_0. This range expresses strategic uncertainty rather than a probabilistic model or a near-term delivery forecast. Conditional on a uniform structural theorem for the elementary-module branches, the remaining work looks finite and the estimate would rise substantially. Without such a theorem, further computation alone is likely to plateau.

The estimate reflects the fixed-point and prime-to-layer arguments that close the Lie-type residue, the common degree-twenty bound for every simple nonzero-multiplier elementary descent, and the degree-eight classes that close every even-orthogonal and untwisted-E_6 factor stabilizer. Every twisted-E_6 stabilizer is already closed in degree at most thirteen, so its unresolved degree-nine scalar does not affect the project estimate. The new type-A theorem also removes every linear and unitary factor with prime-power Schur multiplier. The subsequent two-small-prime theorem removes multiplier support $\{2, 3\}$, leaving only mixed support containing a prime at least five in that classification branch. The quadratic-prime escape theorem reduces this to two even-field residues; the subsequent unitary-completion theorem closes one of them and imposes the exact linear residue (9A). The cyclotomic-hierarchy theorem completes each bounded-rank part and forces rank greater than one quarter of any asserted vanishing range, while its absorbing family identifies the specific unbounded-rank obstruction to that method. The multiplier-depth norm then supplies that entire family with a degree-twelve mod-three class, and gives a depth-dependent bound beyond it. Thus the family is no longer a candidate counterexample family; the remaining linear row satisfies (9D)–(9H). The difficult remaining scope is the unbounded elementary-module and inflation-persistence problem. Thus the outlook is encouraging for the existence of some finite bound, but it is not yet close to a proof.

Proposition 10.1 (Sufficiency of the roadmap)**.** *Let $N_0 \geq D_{\mathrm{base}}$, with D_{base} as in* (S)*. Suppose the proof obligations in Steps 4, 5, and 7 are completed with every required witness and persistence conclusion in degree at most N_0. Then every nontrivial finite group has nonzero integral cohomology in some degree at most N_0.*

Proof. Suppose otherwise, and let G be a counterexample of least order. Steps 1 and 2 make G superperfect and reduce every minimal normal subgroup to the two cases in Proposition 5.11. If G is simple, Step 3 gives an immediate contradiction.

Let N be a proper minimal normal subgroup. If $N = S^r$, Step 4 gives nonzero cohomology of G in degree at most N_0. If N is elementary abelian and is noncentral or is not the unique minimal

normal subgroup, Step 5 gives the same contradiction. The only remaining case is therefore a unique central minimal subgroup $A = C_p$.

Apply Step 6 and write $C = Z(G)$ and $X = G/C$. If X is simple, then $G \to X$ is its universal central extension and $H_2(X;\mathbb{Z}) \cong C$ is a cyclic p-group, so the second line of (S) in Step 3 contradicts the choice of G. If X is not simple, Proposition 9.13 produces a nontrivial universal central extension $\widetilde{R}$ of order smaller than $|G|$. Minimality supplies a nonzero class on $\widetilde{R}$ in degree at most N_0, and Step 7 transfers the existence of such a class to G, again a contradiction. □

Proposition 10.2 (Prime-avoidance shortcut)**.** *Suppose there is a finite integer D_P such that, for every nontrivial finite perfect group Q and every prime p, there are a prime $\ell \neq p$ and an integer $i \leq D_P$ such that*

$$H^i(Q;\mathbb{Z})_{(\ell)} \neq 0. \tag{PA}$$

Then Conjecture 11.1 holds with

$$N_0 = \max\{2, D_P\}.$$

Moreover, in an elementary p-extension of a nontrivial perfect quotient, the class promised by (PA) *survives inflation without any module or differential calculation.*

Proof. Let G be nontrivial. If G is not perfect, then $H^2(G;\mathbb{Z})$, the Pontryagin dual of G_{ab}, is nonzero. If G is perfect, apply (PA), for example with $p = 2$, directly to G. This proves the displayed universal bound.

For the last assertion, let $E \twoheadrightarrow Q$ have elementary p-kernel and choose the prime-to-p class in (PA). Lemma 5.1 identifies its inflation with a nonzero class on E. This observation explains why (PA) is compatible with the elementary-kernel induction, although the first paragraph shows that the global perfect-group hypothesis already proves the main conjecture by itself. □

Stronger quotient-ledger shortcut. For any fixed N_0, there is a shorter sufficient route, but it asks for a stronger theorem: prove $\mathcal{Q}_{N_0}(\Gamma)$ for every finite simple group Γ. Indeed, every nontrivial finite group G has a maximal proper normal subgroup M, and G/M is simple; applying $\mathcal{Q}_{N_0}(G/M)$ to $G \twoheadrightarrow G/M$ proves the corresponding bound. The current ledger proves this unconditionally only for C_p; every nonabelian-simple row still has a hypothesis on the epimorphism. Theorem 6.28 nevertheless shows that every unresolved nonabelian-simple row has a kernel containing its entire selected three-prime detecting set. Thus the shortcut is open and does not replace the eight-step induction above.

11 Proof status, conjectures, and alternative approaches

This section is deliberately redundant with the detailed proofs. Its purpose is to mark the logical boundary of the project and to preserve useful approaches even if the main conjecture is not settled here.

11.1 The exact present boundary

The following statements are proved.

(P1) For finite groups, the integral-cohomology question is exactly the integral-homology question shifted by one degree (Lemma 1.1). Milgram's calculation for M_{23} proves that every universal bound is at least six (Corollary 2.2).

(P2) Every nontrivial finite simple group has nonzero integral cohomology in degree at most eight (Theorem 4.1). This does not by itself settle arbitrary groups, because a cohomology class on a simple quotient can die under inflation.

(P3) For any fixed candidate bound, a least counterexample is superperfect and has a minimal normal subgroup of one of two forms: an elementary abelian p-group in a nonsplit Frattini extension, or S^r for a nonabelian simple group S (Proposition 5.11). The finite-bound reduction in Theorem 9.1 further narrows both branches uniformly once N_0 exceeds the already-proved base ceilings. It also shows that every maximal-normal kernel contains the three-prime detecting set of its nonabelian simple quotient and therefore has at least three distinct prime divisors.

(P4) The S^r branch is closed for all factor stabilizers listed in Step 4 of Section 10. In particular, every untwisted E_6 and every even-orthogonal stabilizer is closed in degree at most eight. Every twisted-E_6 stabilizer is closed with integral bound thirteen. Every unitary stabilizer is now closed with integral bound twenty. Theorem 7.48 gives a common ceiling twenty for all completed nonsplit S^r families, and Corollary 7.49 proves that none of these factors can occur in a least counterexample once $N_0 \geq 20$. The multiplier-depth theorem additionally gives the finite factor ceiling $2r^s(r-1)+1$ whenever $s = v_r(\gcd(n, q-1))$ and $r^{s+1} \mid q-1$. Applied at the fixed residual primes three and five, it proves the quantitative alternatives (9E) and (9F). The finite projective torsion class proves (9G): in defining characteristic three the field-image order is divisible by five, and therefore by sixty. The field-depth norm then proves (9H) and closes every subfamily in which the relevant field-primary depth is bounded.

(P5) In the unique-central-monolith branch, the full-center quotient and the smaller universal central extension are canonically constructed (Propositions 9.12 and 9.13). Prime-to-layer persistence, explicit inflation-kernel annihilators, and several degree-four persistence criteria are proved. These results sharply constrain a counterexample, but do not prove that a low-degree class always survives the descent.

(P6) The eight steps in Section 10 are logically sufficient: if the stated obligations in Steps 4, 5, and 7 are supplied with a common bound, Proposition 10.1 proves the corresponding universal theorem. No further classification branch is hidden in the final assembly.

(P7) For the first canonical elementary abelian p-Frattini cover of a deeply vanishing perfect base, Proposition 6.8 identifies the new p-primary Schur multiplier exactly with the kernel of one LHS differential on invariant alternating or quadratic forms. Corollary 6.9 proves that prime-to-p classes and cyclic-Sylow witnesses cannot be removed anywhere in a fixed-p tower.

(P8) The two-prime criterion closes every epimorphism with prime-power kernel onto an alternating simple group, and onto every standard ordinary or graph-twisted simple group of Lie type over $q \neq 2, 3$, in degree at most four (Corollary 6.13). It also closes every sporadic quotient with bound 140 and the Tits quotient with bound 24 (Corollary 6.14), and every classical simple quotient with bound 24 (Corollary 6.19). The Suzuki–Ree bound 24 and the exceptional-small-field bound 60 complete the remaining families. Hence Theorem 6.29 closes the entire simple-quotient subbranch with common bound 140. These conclusions are independent of the dimension and structure of the kernel module. Theorem 6.28 goes beyond prime-power kernels: with a possibly larger but still finite common ceiling, it closes every epimorphism onto a simple group whose kernel order has at most two distinct prime divisors.

(P9) The witnesses used above occur at two distinct primes on every nonabelian simple group with explicit ceiling 140 (Theorem 6.25). More strongly, Theorem 6.26 proves a uniform, deliberately unoptimized three-prime ceiling for every nonabelian simple group. Corollary 6.27 shows more generally that an arbitrary kernel retains two witnesses whenever it contains at most one of the three detecting primes; in particular this holds for every prime-power kernel. For the ordinary existence question, only one witness is needed: Theorem 6.28 therefore settles an epimorphism whenever its kernel misses even one of the three detecting primes. Thus any unresolved epimorphism onto a nonabelian simple group has all three selected primes in its kernel. The prime-avoidance shortcut, Proposition 10.2, proves that the analogous uniform two-prime statement for all nontrivial perfect groups proves the universal conjecture immediately. Its prime-specific formulation also eliminates every elementary minimal-normal calculation at once. Thus the full simple-group intermediate target is now complete; extending it through arbitrarily many layers of a perfect group remains open.

The following statements are not proved.

(O1) It is not proved that there is any finite universal N_0. This is the main and only target of the project.

(O2) It is not proved that every nonsplit almost-simple factor stabilizer has a class of uniformly bounded finite degree restricting nontrivially to its socle. The unresolved classification residue is now precisely linear type with Schur multiplier divisible by at least two distinct primes, one of them at least five. All unitary type, prime-power-multiplier type A, $\{2,3\}$-support type A, both E_6 families, and all other completed families in Theorem 7.48 are outside this open scope. For an actual unresolved type-A stabilizer, the field image and diagonal multiplier must moreover satisfy (9A)–(9H). The cyclotomic hierarchy closes every bounded-rank subfamily and imposes the additional conditions following (9B). Proposition 7.39 shows that these conditions are not vacuous: any prescribed finite list can be realized inside the multiplier of a genuine nonsplit full automorphism extension. It does not show that those groups lack some different uniform restricting class. Conjecture 11.9 formulates a specific fixed-degree toral class that would bypass multiplier depth, while the calculation following it proves that the obvious projective-point permutation representation cannot produce that class once the depth is at least two. Theorem 7.45 proves the analogous non-toral class on the inner-diagonal group and whenever the field image is prime to the detecting prime. Theorem 11.11 more generally closes every bounded field-primary-depth subfamily and forces the quantitative growth condition (9H); Conjecture 11.10 states precisely the remaining LHS permanence problem for an r-divisible field image. Proposition 11.12 rules out the two most immediate semilinear representation constructions: the direct sum of Frobenius twists does not descend through PGL, while tensor induction does descend but multiplies the scalar-extension class by a norm divisible by r, so its mod-r detector vanishes. Proposition 11.13 proves that the low integral multiplier rows themselves have no positive-column cohomology, but that all extendable degree-three classes lose the detector. Proposition 11.15 rules out the natural projective-point and hyperplane Chern classes as a direct semilinear realization. Proposition 11.16 gives a larger field-fixed symplectic detector at every available tensor depth, and Proposition 11.17 proves that at tensor depth at least two every ordinary complex representation has only Dickson-type Chern restrictions, beginning strictly above the degree of z_b^{r-1}. Thus no ordinary Chern class, not just the natural permutation examples, can realize the non-toral survivor. Proposition 11.18 proves that the rank-two detector is not weakly closed, so essentiality alone cannot justify transfer from its normalizer.

(O3) It is not proved that every noncentral elementary abelian minimal normal subgroup forces low-degree cohomology uniformly in the dimension of the module. The existing rank, form, and first-cohomology criteria cover large subfamilies but not arbitrary irreducible modules or iterated Frattini layers. In the simple-quotient subbranch, the two-prime argument now closes every simple quotient with an arbitrary prime-power kernel in degree at most 140. The unresolved elementary-kernel branch therefore begins when the quotient is nonsimple, or before a simple quotient has been isolated compatibly with the chosen minimal kernel. For a direct epimorphism onto a simple quotient, Corollary 6.27 and Theorem 6.28 show that the remaining hard case for the ordinary existence ledger requires the kernel to contain all three primes in the chosen detecting set. If two surviving primes are demanded for the stronger prime-avoidance route, containing two of the three is the remaining obstruction.

(O4) It is not proved that a class on the smaller universal central extension survives one layer of the central descent at a uniformly bounded degree. The current estimates bound the possible kernel; they do not always make it zero.

(O5) No sequence of finite groups with unbounded initial homology vanishing has been constructed. Thus there is presently no counterexample to the existence of a finite universal bound either.

(O6) Even for a canonical universal p-Frattini tower, it is not known whether the differential in Proposition 6.8 always leaves a nonzero bounded-degree survivor, or whether selected layers can repeatedly make its kernel zero while suppressing all other low-degree rows.

(O7) It is not proved that nontrivial finite perfect groups have bounded-degree witnesses at two distinct primes. Theorem 6.26 and Corollary 6.27 prove it for simple groups and above a simple quotient whenever the kernel meets its detecting set in at most one prime. They do not control a tower using several different kernel characteristics. This is the exact remaining gap in the prime-avoidance shortcut.

The gap is therefore not a choice between a finished proof and a single missing calculation. Some remaining arithmetic cases are computational, but the elementary-module and persistence branches ask for a uniform structural theorem.

11.2 Positive conjectures and a rival conjecture

Conjecture 11.1 (Finite universal bound)**.** *There is an integer N_0 such that every nontrivial finite group G has*

$$H^i(G;\mathbb{Z}) \neq 0 \qquad \text{for some } 1 \leq i \leq N_0.$$

No numerical value is singled out in this conjecture. Every classification branch closed at any finite ceiling is complete for the main goal.

Conjecture 11.2 (Uniform factor-stabilizer detection)**.** *There is a finite d such that, for every nonabelian finite simple group S and every nonsplit intermediate group*

$$S \leq B \leq \mathrm{Aut}(S),$$

some prime p and some $x \in H^j(B;\mathbb{F}_p)$ with $j \leq d$ satisfy $\mathrm{res}^B_S(x) \neq 0$.

Theorem 7.1 explains why this is the exact local input needed for a minimal normal subgroup S^r. The conjecture is already proved for the completed rows of Step 4. By Theorem 7.34 and Theorem

7.35, its remaining classification scope is precisely linear type with mixed-prime Schur multiplier containing a prime at least five. Theorems 7.36 and 7.37 restrict the actual open stabilizers further to (9A) and close unitary type altogether. Theorem 7.38 proves this conjecture in every fixed rank and reduces a vanishing range of length N_0 to ranks greater than $N_0/4$, while Proposition 7.39 exhibits the unbounded-rank absorption mechanism that a uniform proof must overcome. It is stronger than the mere assertion that $H^*(B)$ is nonzero: classes inflated from B/S have zero restriction to S and are useless for factor transfer.

Conjecture 11.3 (Uniform elementary-kernel theorem)**.** *There is a finite N_E with the following property. If E is a finite superperfect group and $V \lhd E$ is a nontrivial elementary abelian minimal normal subgroup, then*

$$H^i(E;\mathbb{Z}) \neq 0 \qquad \text{for some } i \leq N_E.$$

This packages Step 5, including unbounded irreducible modules and nonsplit p-Frattini extensions. It is the most direct positive conjecture at the first genuinely uniform bottleneck.

Conjecture 11.4 (Uniform prime avoidance for perfect groups)**.** *There is a finite D_P such that, for every nontrivial finite perfect group Q and every prime p, some prime $\ell \neq p$ satisfies*

$$H^i(Q;\mathbb{Z})_{(\ell)} \neq 0 \qquad \text{for some } i \leq D_P. \tag{PA}$$

Equivalently, every nontrivial finite perfect group has bounded-degree integral witnesses at two distinct primes. Indeed, two witnesses imply (PA) immediately. Conversely, apply (PA) first with $p = 2$ to obtain a prime $\ell_1 \neq 2$, and then with $p = \ell_1$ to obtain a distinct prime ℓ_2. Theorem 6.25 proves this stronger conclusion for every nonabelian simple group with $D_P = 140$, while Theorem 6.26 and Corollary 6.27 leave two primes above any simple quotient whose kernel meets at most one detecting prime. Proposition 10.2 shows why the extension from simple to perfect groups would be decisive: it proves the main conjecture directly, and hence bypasses the entire minimal-normal and central-descent machinery. No such extension is proved here.

There is an equivalent quotient-ledger formulation which isolates the new hard case. Conjecture 11.4 holds if and only if there is a finite D such that, for every epimorphism

$$E \twoheadrightarrow S$$

from a nontrivial finite perfect group onto a nonabelian finite simple group, E has witnesses at two distinct primes in degrees at most D. The forward implication applies the conjecture to E. Conversely, every nontrivial finite perfect group has a maximal proper normal subgroup, and its simple quotient cannot be cyclic. Corollary 6.27 proves this equivalent ledger whenever the kernel contains at most one of the three chosen detecting primes. Thus the exact residual quotient case has a perfect source and a kernel divisible by at least two detecting primes.

Conjecture 11.5 (One-layer central-descent persistence)**.** *There is a finite N_P such that, in every configuration of Proposition 9.13, nonzero cohomology of the smaller universal central extension $\widetilde{R}$ in degree at most N_P forces nonzero cohomology of G in some degree at most N_P.*

This is exactly the uniform assertion missing from Step 7. It does not require the same class, prime, or degree to survive, only some class below one fixed finite ceiling. An uncontrolled increase of the ceiling at each descent layer would not suffice, because the number of layers is not known to be uniformly bounded.

There is also a precise rival to all of these positive statements.

Conjecture 11.6 (Unbounded evasion depth—the rival conjecture)**.** *For every $r \geq 1$ there is a nontrivial finite superperfect group G_r such that*

$$H_j(G_r; \mathbb{Z}) = 0 \qquad (1 \leq j \leq r).$$

By Lemma 1.1, Conjecture 11.6 is equivalent, up to the one-degree shift, to the negation of Conjecture 11.1. No such unbounded family is known in this note. The exact calculation for M_{23} supplies only the finite datum $r = 4$. Theorem 4.1 shows that an unbounded family cannot consist of simple groups; it must exploit extensions.

The most plausible place to search for the rival conjecture is a tower of nonsplit elementary kernels

$$1 \longrightarrow V_n \longrightarrow G_{n+1} \longrightarrow G_n \longrightarrow 1, \tag{R1}$$

where the G_n are superperfect, the V_n are irreducible $\mathbb{F}_{p_n}G_n$-modules, and successive LHS transgressions kill the first available classes. A tower of the form (R1) with vanishing depth tending to infinity would disprove every universal bound. Conversely, a theorem saying that some bounded-degree class must survive every such layer would prove the essential content of Conjectures 11.3 and 11.5. This makes Frattini towers a useful test of the truth of the original problem, not merely a technical nuisance in one proof.

11.3 The Frattini counterexample family under a microscope

There is a canonical version of (R1). Fix a finite group G_0 and a prime $p \mid |G_0|$, and define

$$\cdots \longrightarrow G_2 \longrightarrow G_1 \longrightarrow G_0 \tag{R1a}$$

to be the characteristic finite quotients of the universal p-Frattini cover. Each kernel

$$M_n = \ker(G_{n+1} \to G_n)$$

is elementary abelian. Gaschütz's theorem identifies it as

$$M_n \cong \Omega^2_{\mathbb{F}_p G_n}\mathbb{F}_p \tag{R1b}$$

[41, Theorem 2.4]. Thus (R1a) is not an ad hoc collection of extensions: the module and the unique nonsplit extension class are determined at every level. If G_0 is perfect, all G_n are perfect. Perfectness, however, is much weaker than the growing homology vanishing demanded by Conjecture 11.6.

The first unresolved calculation is already visible in total degree two. Suppose

$$H^2(G_n; \mathbb{F}_p) = H^3(G_n; \mathbb{F}_p) = 0.$$

Proposition 6.8 gives

$$H^2(G_{n+1}; \mathbb{F}_p) \cong \begin{cases} \ker\big((\Lambda^2 M_n^*)^{G_n} \to H^2(G_n; M_n^*)\big), & p \text{ odd}, \\ \ker\big((\mathrm{Sym}^2 M_n^*)^{G_n} \to H^2(G_n; M_n^*)\big), & p = 2, \end{cases} \tag{R1c}$$

where the arrows are $d_2^{0,2}$. Since G_{n+1} is perfect, the left side is nonzero exactly when $H^3(G_{n+1}; \mathbb{Z})_{(p)}$ is nonzero. This is a sharp, finite module-theoretic test for whether the next level immediately reacquires a Schur multiplier.

The direction of the dimension shift is important. From $M_n^* \simeq \Omega^{-2}\mathbb{F}_p$ stably one gets

$$H^1(G_n; M_n^*) \cong H^3(G_n; \mathbb{F}_p), \tag{R1d}$$

not a negative Tate group that is automatically nonzero. Consequently, when the base already vanishes deeply, the $(1,1)$ term supplies no cheap degree-two survivor. Every invariant form in (R1c) may in principle transgress. This correction makes the canonical family a more serious possible evasion mechanism than a superficial five-term argument would suggest. It does not prove that the kernel in (R1c) is zero in any example.

There are also strong obstacles to turning one fixed-prime tower into a counterexample.

(R1) A class at a prime $\ell \neq p$ survives every map in (R1a) by prime-to-kernel inflation. Thus a fixed-p tower cannot raise the vanishing depth past any low-degree prime-to-p class already on G_0.

(R2) If G_0 is nonabelian simple, the obstruction is stronger. Theorem 6.26 gives three bounded-degree primary witnesses on G_0, and Corollary 6.27 leaves at least two of them on every level of the fixed-p tower. Such a tower cannot even concentrate all bounded-degree cohomology at one prime. Any prime-support evasion strategy starting from a simple group must therefore use at least two kernel characteristics before it can reach the one-prime regime.

(R3) A cyclic-Sylow witness on G_0 survives every level, even when its prime equals p, by Theorem 6.3. This rules out many natural simple bases. In particular the M_{23} tower can never vanish through degree six: its degree-six seven-primary class persists.

(R4) Superperfectness is not automatic. A Frattini cover of a perfect group is perfect, but (R1c) shows that its Schur multiplier is a new calculation at each stage. Failure of superperfectness gives an immediate degree-three integral witness.

(R5) The kernels become enormous. If $d_n = \dim_{\mathbb{F}_p} M_n$, then

$$d_{n+1} = 1 + p^{d_n}(d_n - 1)$$

[41, Section 4]. The first 2-Frattini module of A_5 has dimension five, and the next has dimension 129. Hence even a two-level exact cohomology calculation can leave the range of present finite-group software.

These points narrow the counterexample search. A viable construction probably cannot be a raw fixed-p tower from an arbitrary simple group. It would have to vary the prime or select quotients of the canonical Frattini modules so as to attack the current first nonzero primary class. When a layer creates a new Schur multiplier, one can replace the group by its finite universal central extension, which is superperfect, but this central step can introduce new higher homology and therefore has to be audited again. The concrete counterexample-oriented program is:

(C1) start with a finite superperfect group and record the first nonzero integral homology group, including its primary decomposition;

(C2) choose a prime and an irreducible quotient of the canonical p-Frattini module whose extension transgression can hit that group;

(C3) compute the LHS maps, not only dimensions, and verify that every fiber class in all lower total degrees also dies;

(C4) if a new Schur multiplier appears, pass to the universal central extension and recompute the next low homology group;

(C5) repeat with a strictly increasing verified vanishing depth.

No instance of this five-stage cycle is known here to increase the record depth beyond the M_{23} datum. Steps (C2) and (C3) are finite computations for a fixed small module. Proving that the cycle can or cannot continue with unbounded depth requires the deeper uniform idea.

This suggests a more specific, and more vulnerable, version of the rival conjecture.

Conjecture 11.7 (Frattini–central evasion tower)**.** *For every r there is a finite chain beginning at a finite simple group and alternating selected elementary abelian Frattini extensions with universal central extensions, whose final nontrivial group X_r satisfies*

$$H_j(X_r;\mathbb{Z}) = 0 \qquad (1 \le j \le r).$$

This conjecture implies Conjecture 11.6, but is much more prescriptive and may well be false even if some other unbounded family exists. Its value is experimental: Proposition 6.8 specifies the first differential to compute, while Corollary 6.9 specifies two immediate reasons to discard a candidate. A single verified increase past depth four would be significant negative evidence for the finite-bound program; a uniform proof that the five-stage cycle always produces a bounded survivor would be significant positive evidence.

11.4 A sharp-bound diagnostic, not a main-goal obstruction: the twisted E_6 and unitary family

Every group in this family is already closed for the main goal by the degree-thirteen bound in Theorem 7.20. The discussion is retained because it is a useful warning about false lifting lemmas and about how an almost-simple obstruction could enter a larger counterexample, not because degree nine is still needed. The former sharp-bound arithmetic family has

$$S = {}^2E_6(2^m), \qquad 3 \mid m, \qquad m \text{ odd}, \tag{R2}$$

and an even outer image. Let B be its factor-stabilizer image and B_0 the odd diagonal–field subgroup. The three-part of the multiplier of S is generated by a class

$$w \in H^2(S;\mathbb{F}_3).$$

The local subgroup calculation in Proposition 7.21 proves that

$$c = \beta(w)w^3 \in H^9(S;\mathbb{F}_3) \tag{R3}$$

is nonzero and fixed by every automorphism. Thus the desired target class exists on the socle; the issue is whether it comes from the almost-simple group B.

The existing degree-six norm class on B_0 restricts to w^3. Therefore (R3) lifts and transfers to B if $\beta(w)$ lifts to B_0. Write

$$C_0 = B_0/S, \qquad \kappa = d_3(w) \in H^3(C_0;\mathbb{F}_3).$$

Proposition 7.22 proves that the edge obstruction is

$$d_4(\beta w) = [-\beta\kappa] \in H^4(C_0;\mathbb{F}_3)/(H^1(C_0;\mathbb{F}_3)\kappa). \tag{R4}$$

It vanishes whenever a Sylow three-subgroup of C_0 is cyclic. In the only remaining case,

$$P = D \times F \cong C_3 \times C_{3^b},$$

and graph–field anti-invariance forces

$$\operatorname{res}_P \kappa = Axv + Byu. \tag{R5}$$

For $b = 1$, the obstruction is the scalar $(A + B)uv$; for $b \geq 2$, it is Auv. This is an exact reduction, but the coefficient has not been computed for the groups in (R2).

The unitary control family explains why it would be unsafe to conjecture that the coefficient vanishes merely by equivariance. For $S_u = \mathrm{PSU}_3(8)$, the outer three-group is $C_3 \times C_3$. The exact Cohomolo calculation in Remark 7.23 gives

outer line	pure diagonal	pure field	mixed
$H_2(B_L;\mathbb{Z})_{(3)}$	C_3	C_3	0.

(R6)

The five-term sequence translates (R6) into vanishing of the unitary transgression on both pure lines and nonvanishing on the mixed line. In the coordinates (R5), its coefficient satisfies $A+B \neq 0$, so the corresponding Bockstein obstruction is nonzero.

This calculation has three different interpretations which must not be confused.

(a) It is a genuine counterexample to the tempting *lemma* that vanishing on the two pure lines plus graph anti-invariance forces $\beta\kappa = 0$ on the rank-two group.

(b) It is evidence that the particular class (R3) might fail to lift in some twisted-E_6 cases. If the twisted coefficient is nonzero, the present degree-nine construction fails and a different class or operation is needed for that optional lower-degree refinement.

(c) It is not a counterexample to Conjecture 11.1. The almost-simple groups B occurring here have a nontrivial solvable outer quotient, hence nontrivial abelianization, so they already have nonzero $H^2(B;\mathbb{Z})$. What fails is restriction of that easy quotient cohomology to S. Moreover, Theorem 7.20 gives every twisted-E_6 stabilizer a restricting integral class in degree at most thirteen.

Could this family nevertheless contribute to a genuine counterexample? Only indirectly. In the minimal-normal argument, B is the image of a factor stabilizer inside a larger superperfect group E with normal subgroup S^r. Classes inflated from B/S restrict trivially to S and therefore cannot be fed into the factor-transfer theorem. To turn the failure of (R3) into an actual counterexample, one would have to construct E so that the factor action uses the obstructed rank-two outer image and, simultaneously, every other low-degree class coming from the quotient, the permutation action on the factors, and the other LHS rows is killed. No such E is presently known. Thus this family identifies a real weakness in one proof route, while the elementary/Frattini towers in (R1) remain the more credible source of unbounded vanishing.

If one later chooses to optimize degrees, there are two concrete twisted-E_6 tasks rather than one vague one:

(E1) compute $A+B$ and A in (R5) from a regular embedding, root datum, or a certified finite-group calculation for the first fields in (R2);

(E2) independently search $H^j(S;\mathbb{F}_3)$, $j \leq 9$, for another automorphism-stable class whose LHS transgression is forced to vanish even when (R4) is nonzero.

The first task is mostly a difficult family-specific calculation and the second may require a new idea. Neither belongs to the critical path for the existence of a finite universal bound, so they are deferred.

11.5 Alternative approaches

The following routes are logically distinct enough that negative evidence for one need not damage the others.

A. Use a larger bound immediately. This is now the governing strategy, not merely an alternative. Prove Conjecture 11.1 without optimizing N_0. The twisted-E_6 branch is already closed by thirteen and the current sporadic central-descent audit by twenty; the Suzuki–Ree prime-power-kernel branch is closed by 24, and all simple prime-power-kernel quotients by 140. Higher-degree Evens norms, Steenrod operations, and characteristic classes have more room to avoid individual transgressions. This removes several sharp arithmetic obstacles, but does not remove the need for a uniform elementary-kernel or descent theorem. Thus extra computation and known operations can finish more classified rows, while all degree-only optimizations are deferred; a deeper persistence idea is still required.

B. Prove the strong simple-quotient ledger. For a finite simple group Γ, let $\mathcal{Q}_N(\Gamma)$ be the assertion that every epimorphism $E \twoheadrightarrow \Gamma$ with $E \neq 1$ has nonzero cohomology by degree N. If $\mathcal{Q}_N(\Gamma)$ holds for every simple Γ, every nontrivial finite group is settled by taking a maximal proper normal subgroup. This bypasses the minimal-normal induction entirely. Existing split, prime-to-kernel, central, Frattini, and universal-centralization rows are pieces of this strategy. Theorem 6.29 now proves its entire prime-power-kernel part with common bound 140. Its remaining difficulty is that an arbitrary kernel can contain all of the elementary-module and nonabelian-layer complications at once, so the shortcut asks for a stronger persistence theorem than the eight-step route.

For prime avoidance there is a weaker, more structured version: restrict to perfect sources and ask for two surviving primes rather than merely one class. As observed after Conjecture 11.4, this ledger is equivalent to that conjecture and hence already proves the main theorem. Corollary 6.27 closes every row in which the kernel contains at most one detecting prime. Its exact residue consists of perfect-source epimorphisms whose kernels contain at least two of the three primes selected on the simple quotient.

C. Change the class, not only the differential. The twisted-E_6 calculation illustrates a general spectral-sequence strategy. Instead of insisting that one multiplier class lift, form powers, Bocksteins, Steenrod operations, or Evens norms whose first possible differentials are incompatible with degree, parity, or outer action. The classes w^3, $\beta(w)w^3$, signed squares, and Spin refinements in this note are successful examples. Systematic computation of the low-degree module of stable classes under $\mathrm{Out}(S)$ can identify several targets at once; a proof then needs only one target to survive. This is well suited to the remaining finite arithmetic cases, although no general operation presently handles every elementary kernel.

D. Seek a uniform local-detection theorem. Cyclic Sylow detection, restrictions to elementary abelian subgroups, and characteristic classes all convert global nonvanishing into invariant local data. A strong enough theorem would say that every finite superperfect group has, in a bounded degree, a class detected on a controlled family of cyclic, elementary abelian, or extraspecial subgroups. Fusion stability would then turn a local monomial into a global class. Computation can map fusion and find candidates, but the bounded-degree assertion across all ranks and actions is the deep missing part.

E. Analyze universal p-Frattini towers directly. Rather than treating successive elementary kernels as errors in an induction, one can study the tower (R1) as the main object. Proposition 6.8 reduces the first new p-primary multiplier to $d_2^{0,2}$ on invariant alternating or quadratic forms. Compute this map, the later total-degree rows, and the maps on low integral homology from one stage to the next. A bounded survivor theorem would establish the positive persistence conjectures, while explicit stages with increasing vanishing depth would support Conjecture 11.6. The kernel dimension explosion makes even small stages demanding; either uniform outcome is a deeper problem.

F. Use the plus construction and homotopy fixed points more globally. For a perfect group, BG^+ is simply connected and has the same integral homology as G. Low homology vanishing makes BG^+ highly connected in the relevant range. In Lie type, completed homotopy fixed-point models already force low-degree classes from compact Lie groups. One alternative is to seek a uniform homotopy-theoretic obstruction to a finite group plus construction being too highly connected, possibly prime by prime. The present fixed-point theorems show the strength of this method for classification families; no theorem currently makes it uniform over arbitrary extensions.

G. Attack the module bounds with representation theory. The degree-four persistence formulas reduce part of the elementary branch to

$$\dim H^1(R; V^*), \qquad \dim(\Lambda^2 V^*)^R, \qquad \operatorname{im}\big(H_3(R;\mathbb{Z}) \to H_1(R;V)\big).$$

Uniform bounds or vanishing theorems for these three quantities would turn the existing edge estimates into a proof. Cross-characteristic and defining-characteristic modules may require different arguments. Character tables and MeatAxe calculations can close finite exceptional modules; unbounded rank and field families need structural representation theory.

H. Run a counterexample-oriented computation. A useful computation should record maps, not only dimensions. Starting from databases of perfect and superperfect groups, one can enumerate irreducible elementary kernels, build nonsplit extensions, and compute

$$H_j(G;\mathbb{Z}), \quad d_r \text{ in the LHS spectral sequence}, \quad \text{inflation and restriction maps}.$$

The search should maximize the initial vanishing depth and test whether it increases along Frattini layers. The unitary script in Remark 7.23 is a model of the required discipline: it certifies the relevant multiplier map and is interpreted only within its scope. Finite computations can find a counterexample to a proposed lemma or a fixed bound, but cannot by themselves prove a universal theorem over unbounded towers.

I. Prove prime avoidance instead of controlling each module. Conjecture 11.4 asks every nontrivial finite perfect group to have bounded-degree witnesses at two distinct primes. For an elementary p-kernel, one of those primes is automatically prime to the kernel, so inflation survives without examining the module, the extension class, or an LHS differential. Theorem 6.25 proves the simple base with bound 140, and Proposition 10.2 shows that the perfect-group extension by itself proves the main conjecture. This route is stronger than the elementary-kernel theorem but has a cleaner failure mode: a counterexample must be a family of perfect groups whose bounded-degree integral cohomology is supported at only one prime. Proposition 6.12 isolates a concrete intermediate target: a uniform three-prime theorem for nonabelian simple groups would prove prime

avoidance for every group mapping onto a simple group whose kernel meets at most one detecting prime. Three is the strongest possible uniform support count, since A_5 has only the group primes $2, 3, 5$. Theorem 6.26 now proves this target for every nonabelian simple group. Its infinite Lie-type part uses the exact exceptional-field list in Lemma 6.15, bounded projective Chern blocks for the remaining classical ranks, and a third homotopy-fixed-point prime beside the two sharp Suzuki–Ree tori. Corollary 6.27 is the resulting one-layer prime-avoidance theorem; it includes every prime-power kernel and many mixed-prime kernels. Any remaining hard simple quotient has a kernel divisible by at least two of its three detecting primes if two surviving primes are demanded, and by all three if only the one witness needed for the existence theorem is demanded. The next target is no longer a classification row: it is a persistence theorem through towers whose elementary kernels use more than one characteristic. No unbounded one-prime-support family of perfect groups is known.

J. Use the projective basic line at absorbed primes. The cyclotomic argument deliberately works at primes outside d_0, where the simply connected cover is a local equivalence. The absorbing family in Proposition 7.39 shows that a uniform proof must also exploit primes dividing the center. A natural replacement is the compact projective form

$$K_{n,d_0} = \mathrm{SU}(n)/C_{d_0}.$$

Its degree-four integral generator pulls back to a positive multiple of the basic generator on $B\mathrm{SU}(n)$; the multiplier is the basic level computed from the enlarged cocharacter lattice [42, Proposition 3.6.2]. At an odd prime r, its r-valuation is

$$\max\{0, 2v_r(d_0) - v_r(n)\}. \tag{A10}$$

This is at most $v_r(d_0)$. Thus whenever $v_r(q-1) > v_r(d_0)$, the projective basic level is still smaller than the r-part of $q^2 - 1$, exactly the numerical inequality one would expect to leave a nonzero degree-four fixed-point class. The nonsplitting certificate in the explicit family (19au) has this form at $r = 3$. Proposition 7.40 proves that this expected class does exist on $H = \mathrm{Inndiag}(S)$ and restricts nontrivially to S; it also computes the field and graph actions. Lemma 7.41 now proves that every later LHS edge differential vanishes once an invariant preimage has been found. Theorem 7.42 gives the exact valuation range in which the simply connected comparison supplies such a preimage. This proves the additional residual condition (9C).

The equality case is genuinely different. Proposition 7.43 shows that the explicit family (19au) lies exactly on the boundary: its invariant projective-line subgroup is invisible after pullback to $\mathrm{SL}_n(q)$. The cyclic universal-cover LHS calculation proves more: this subgroup also restricts trivially to S, and the resulting cocycle generates $H^1(C_f; K_{\mathcal{L}}) \cong C_3$ for the line kernel $K_{\mathcal{L}}$. Thus a projective correction, if one exists, must come from the full restriction kernel outside the cyclic line. Theorem 7.44 avoids that question by using a power of the multiplier class and an Evens norm; Corollary 7.46 closes the explicit family in mod-three degree twelve. The same theorem now treats arbitrary finite multiplier depth s when $v_r(q-1) > s$, in degree $2r^s(r-1)$. Its clock–shift calculation explains why the full-depth boundary $v_r(q-1) = s$ is not reached by the same polynomial detector.

Conjecture 11.8 (Kernel-corrected projective persistence)**.** *Retain the residual linear setting, and let r be an odd prime dividing d_0. Put $H = \mathrm{Inndiag}(S)$, let C be the cyclic field image of B/S, and put*

$$K_r = \ker\left(H^4(H; \mathbb{Z}_{(r)}) \longrightarrow H^4(S; \mathbb{Z}_{(r)})\right).$$

Suppose that

$$v_r(q^2-1) > \max\{0, 2v_r(d_0) - v_r(n)\}, \qquad p^{2a} \equiv 1 \pmod{r}. \tag{PBL}$$

Choose a multiple z of the projective basic class whose restriction x to S is nonzero of order r. Then x is C-fixed, and

$$\delta_z(c) = c^*z - z \in K_r \qquad (c \in C) \tag{PBL$'$}$$

defines a class in $H^1(C; K_r)$. The conjecture is that one may choose z so that $[\delta_z] = 0$. Equivalently, the fiber of x under $H^4(H; \mathbb{Z}_{(r)}) \to H^4(S; \mathbb{Z}_{(r)})$ contains a C-fixed element.

The cocycle in (PBL$'$) is independent of the chosen preimage up to a coboundary, so its class is the exact invariant-preimage obstruction. If it vanishes, Lemma 7.41, applied after adjoining graph, gives a degree-four class on B restricting nontrivially to S. Thus there are no further unidentified d_2 or d_3 obstructions. In the strict valuation range the class vanishes by Theorem 7.42. In the explicit equality family, Proposition 7.43 proves that the class is nonzero in the line kernel; whether its image vanishes in the full kernel remains a well-posed projective-method question, but is no longer a gap in excluding that family.

Conjecture 11.9 (Toral Dickson survivor)**.** *Let $S = \mathrm{PSL}_n(q)$ have its standard cyclic multiplier, let r be an odd prime, and put*

$$s = v_r(\gcd(n, q-1)) \geq 1.$$

Put $R = r^s$, let $T \cong C_R^2$ be the clock–shift subgroup constructed after Theorem 7.44, and let $E = \Omega_1(T) \cong C_r^2$. For polynomial generators $y_1, y_2 \in H^2(E; \mathbb{F}_r)$, put

$$\Delta_r = (y_1 y_2^r - y_1^r y_2)^{r-1} \in H^{2(r^2-1)}(E; \mathbb{F}_r). \tag{TD}$$

Then, for every $S \leq B \leq \mathrm{Aut}(S)$, some $z \in H^{2(r^2-1)}(B; \mathbb{F}_r)$ restricts to a nonzero multiple of Δ_r on E.

Theorem 7.45 proves a closely related but strictly different fixed-degree statement. Its detector $E_r \cong C_r^2$ is built directly from clock and shift blocks of size r, and is non-toral: its chosen lifts have scalar commutator of order r. The operation $\mathcal{B}_r\mathcal{P}^1(\bar{x})$ restricts there to the unsquared alternating determinant, so its $(r-1)$st power gives exactly the Dickson polynomial in (TD). It produces the required class on the inner-diagonal group, and on every factor stabilizer whose field image has order prime to r.

This does not prove the conjecture as stated. When $s \geq 2$, its subgroup $E = \Omega_1(T)$ is not the non-toral E_r above. Indeed, with $R = r^s$ and $[X, Y] = \xi I$ as in the full-depth construction,

$$[X^{R/r}, Y^{R/r}] = \xi^{(R/r)^2} I = I.$$

Thus E lifts to commuting matrices, the canonical degree-three projective class restricts trivially to it, and the finite projective torsion operation cannot detect Δ_r on this toral subgroup. In the residual semilinear problem the exact uncovered case for this new operation is an r-divisible field image. At the forced prime five in defining characteristic three, this yields the proved extra condition (9G) rather than a complete solution of the row. The depth-dependent replacement below additionally gives (9H), but still not one degree independent of the field depth.

Conjecture 11.10 (Semilinear projective torsion survivor)**.** *Retain the notation of Theorem 7.45. Let P be an r-subgroup of the pure-field outer group and put*

$$E_P = H \rtimes P.$$

Then the invariant class

$$\Theta_r = \theta_r^{r-1} \in H^{2(r^2-1)}(H;\mathbb{Z}_{(r)})^P$$

is in the image of restriction

$$H^{2(r^2-1)}(E_P;\mathbb{Z}_{(r)}) \longrightarrow H^{2(r^2-1)}(H;\mathbb{Z}_{(r)}). \tag{SPT}$$

Equivalently, Θ_r is a permanent cycle in the LHS spectral sequence for $H \lhd E_P$.

This is a second, non-toral route to completing Step 4. If it holds for $r = 3$ and $r = 5$, lift Θ_r over a Sylow-r field subgroup and corestrict exactly as in (19bp). Conditions (9D) and the proof of (9F) then close the two residual characteristic rows with a common integral ceiling of forty-eight.

There is concrete local evidence, but not yet a proof. The r-group P acts trivially on the order-r roots of unity, because its image in $\mathrm{Aut}(C_r)$ has order dividing both a power of r and $r-1$. It therefore centralizes the non-toral detector E_r pointwise, and the restriction of Θ_r extends over $E_r \times P$. Thus any differential obstructing (SPT) is invisible on the subgroup that proves nonvanishing. On the other hand, P can act nontrivially on the full scalar group C_{r^s}; the generator $x \in H^3(H;\mathbb{Z}_{(r)})$ used in (NT′) then need not extend. Naturality of the operation alone therefore does not prove (SPT). Computing those LHS differentials, or constructing the power directly on the semilinear group, is the exact remaining obligation for this route.

Proposition 11.14 below carries out the primary-operation part of that calculation at full field depth. It proves that the first two apparent differentials on the mod-r reduction of θ_r vanish, and reduces the question to a short, depth-independent list of genuinely later differentials. It does not prove permanence.

There is nevertheless an unconditional depth-dependent substitute. It is important here that the goal only asks for some finite common ceiling: the following result closes every subfamily in which the relevant field-primary depth is bounded, even though it does not prove the fixed-degree Conjecture 11.10.

Theorem 11.11 (The field-depth norm)**.** *Retain the notation of Theorem 7.45, and let*

$$S \le B \le \mathrm{Aut}(S), \qquad B^\sharp = BH.$$

If a Sylow r-subgroup of $B^\sharp/H$ (equivalently, of its field image) has order r^a, where $a \ge 0$, then there is a class

$$y \in H^{2r^a(r^2-1)}(B;\mathbb{F}_r)$$

whose restriction to S is nonzero. Consequently, if S^k is a minimal normal subgroup of a finite group G and B is its factor stabilizer image, then

$$H^i(G;\mathbb{Z}) \neq 0 \qquad \text{for some } i \le 2r^a(r^2-1)+1. \tag{FDN}$$

In particular, for fixed r every subfamily with bounded field r-depth a has a common finite bound.

Proof. Write $C = B^\sharp/H$, choose $P \in \mathrm{Syl}_r(C)$, and let B_P be its inverse image in $B^\sharp$. Since r is odd, P lies in the pure-field part. Put $D = 2(r^2-1)$ and let

$$\overline{\Theta}_r \in H^D(H;\mathbb{F}_r)$$

be the reduction of the class in Theorem 7.45. Apply the Evens norm:

$$Y_P = \mathcal{N}_H^{B_P}(\overline{\Theta}_r) \in H^{Dr^a}(B_P;\mathbb{F}_r).$$

The class $\overline{\Theta}_r$ is fixed by every automorphism of S, so the norm restriction formula gives

$$\operatorname{res}_H^{B_P}(Y_P) = \prod_{u\in P} u^*(\overline{\Theta}_r) = \overline{\Theta}_r^{r^a}. \tag{FDN$'$}$$

This class is nonzero: on the clock–shift subgroup $E_r \le S$ it restricts to

$$(y_1 y_2^r - y_1^r y_2)^{(r-1)r^a},$$

a nonzero polynomial.

Now corestrict Y_P to $B^\sharp$. Since H is normal, the Mackey formula and (FDN$'$) give

$$\operatorname{res}_H^{B^\sharp} \operatorname{cor}_{B_P}^{B^\sharp}(Y_P) = \sum_{cP\in C/P} c^*(\overline{\Theta}_r^{r^a}) = [C:P]\overline{\Theta}_r^{r^a} \neq 0, \tag{FDN$''$}$$

because $[C:P]$ is prime to r. Restricting this class from $B^\sharp$ to B preserves its nonzero restriction to S, proving the first assertion. Theorem 7.1 supplies a nonzero mod-r class of the same degree on G, and the mod-r universal coefficient sequence places a nonzero integral class in that degree or the next one. This proves (FDN). □

The dependence on r^a is not removed by transferring a fixed-degree class one layer at a time. If

$$H \rtimes P_b \le H \rtimes P_a, \qquad [P_a : P_b] = r^{a-b} > 1,$$

and $c \in H^*(H \rtimes P_b; \mathbb{F}_r)$ restricts to a class z on H which is fixed by P_a, then the Mackey formula gives

$$\operatorname{res}_H^{H\rtimes P_a} \operatorname{cor}_{H\rtimes P_b}^{H\rtimes P_a}(c) = r^{a-b} z = 0. \tag{FDN$'''$}$$

The Evens norm avoids this zero only by replacing z with $z^{r^{a-b}}$ and multiplying the degree by the same index. Thus a uniform solution cannot come from ordinary transfer up the cyclic field tower; it must use the LHS edge, a fixed-degree stable-elements class, or a genuinely different characteristic class.

The most natural representation-theoretic attempt has a parallel, exactly computable defect.

Proposition 11.12 (The tensor-induction scalar norm)**.** *Retain the standard cyclic-multiplier hypotheses of Theorem 7.45. Let $q = q_0^{r^a}$ with $a \ge 1$, let φ be the q_0-power field automorphism, and put*

$$K = \mathrm{PGL}_n(q) \rtimes \langle\varphi\rangle, \qquad |\varphi| = r^a.$$

For $V = \mathbb{F}_q^n$, form the tensor induction

$$T(V) = \bigotimes_{j=0}^{r^a-1} V^{(q_0^j)}. \tag{TI}$$

Cyclic permutation of the factors extends the projective action to a homomorphism

$$\tau : K \longrightarrow \mathrm{PGL}(T(V)).$$

Let $\xi_\tau \in H^3(K; \mathbb{Z}_{(r)})$ be the r-primary class of the pulled-back scalar extension, and let $x \in H^3(\mathrm{PGL}_n(q); \mathbb{Z}_{(r)})$ be the canonical projective generator in Theorem 7.45. Up to a unit,

$$\operatorname{res}^K_{\mathrm{PGL}_n(q)}(\xi_\tau) = \frac{q-1}{q_0-1}\, x, \qquad v_r\left(\frac{q-1}{q_0-1}\right) = a. \tag{TI$'$}$$

Consequently its mod-r reduction restricts to zero on $\mathrm{PGL}_n(q)$*, and applying the projective torsion operation to* ξ_τ *cannot recover* θ_r*. By contrast, direct sum of the Frobenius twists in* (TI) *does not even descend through* $\mathrm{PGL}_n(q)$*.*

Proof. For $g \in \mathrm{GL}_n(q)$ put

$$\widetilde{\tau}(g) = \bigotimes_{j=0}^{r^a-1} g^{(q_0^j)}.$$

A cyclic permutation matrix on the tensor factors conjugates $\widetilde{\tau}(g)$ to $\widetilde{\tau}(g^{(q_0)})$, after choosing the orientation of the cycle appropriately. If $g = \lambda I$, then

$$\widetilde{\tau}(\lambda I) = \lambda^{1+q_0+\cdots+q_0^{r^a-1}} I = \lambda^{(q-1)/(q_0-1)} I. \tag{TI$''$}$$

Thus the action descends projectively and extends over φ. Equation (TI$''$) also shows that the induced map on scalar central extensions is the power map with exponent $(q-1)/(q_0-1)$. Pairing its r-part with a generator of $\mathrm{Hom}(C_{r^s}, \mathbb{Q}/\mathbb{Z})$ gives (TI$'$), up to the choice of a generator. Since the residue of q_0 in $\mathbb{F}_r^\times$ has r-power order, it is one, and the odd-prime lifting-the-exponent formula gives

$$v_r(q-1) - v_r(q_0-1) = a.$$

The class in (TI$'$) is therefore divisible by r, so its reduction modulo r is zero. Finally, on the direct sum $\bigoplus_j V^{(q_0^j)}$ the scalar λI acts with the generally distinct scalars $\lambda^{q_0^j}$ on the different summands; hence that linear construction does not factor through the projective quotient. □

The low multiplier rows can nevertheless be computed completely. The answer both removes a possible spectral-sequence obstruction and explains why simply extending a divisible multiple of the degree-three class cannot prove Conjecture 11.10.

Proposition 11.13 (The full-depth field module)**.** *Let* $P = C_{r^a}$*,* $a \geq 1$*, be a pure-field subgroup in the setting of Conjecture 11.10, and write*

$$q = q_0^{r^a}.$$

Assume the full-depth equality

$$s = v_r(\gcd(n, q-1)) = v_r(q-1).$$

Then

$$v_r(q_0 - 1) = s - a, \qquad 1 \leq a < s. \tag{FD1}$$

For either of the two field modules

$$M = H^2(H; \mathbb{Z}_{(r)}) \cong C_{r^s}, \qquad M = H^3(H; \mathbb{Z}_{(r)}) \cong C_{r^s}, \tag{FD2}$$

one has

$$H^j(P; M) = 0 \qquad (j > 0), \tag{FD3}$$

and $M^P = r^a M$*. Consequently restriction is onto in degrees two and three:*

$$H^j(H \rtimes P; \mathbb{Z}_{(r)}) \longrightarrow H^j(H; \mathbb{Z}_{(r)})^P \quad \text{is surjective for } j = 2, 3. \tag{FD4}$$

Nevertheless, every class in the image of (FD4) *for* $j = 3$ *restricts trivially to the non-toral detector* E_r *of Theorem 7.45.*

Proof. The residue of q_0 in $\mathbb{F}_r^\times$ has r-power order, hence is one. The odd-prime lifting-the-exponent formula gives

$$v_r(q-1) = v_r(q_0^{r^a}-1) = v_r(q_0-1)+a,$$

which proves (FD1). In particular $s \geq 2$.

Equation (19ba) gives (FD2). On either cyclic group the field generator acts by multiplication by q_0 or q_0^{-1}; replacing q_0 by its inverse does not change the valuations below. Put $c = q_0$ and

$$\mathcal{N} = 1 + c + \cdots + c^{r^a-1}.$$

On C_{r^s} one has

$$v_r(c-1) = s-a, \qquad v_r(\mathcal{N}) = a.$$

The standard cyclic-group formulas now give

$$H^{2i}(P;M) = M^P/\mathcal{N}M = 0 \quad (i \geq 1), \qquad H^{2i+1}(P;M) = \ker(\mathcal{N})/(c-1)M = 0 \quad (i \geq 0),$$

because

$$M^P = \mathcal{N}M = r^a M, \qquad \ker(\mathcal{N}) = (c-1)M = r^{s-a}M.$$

This proves (FD3).

In the integral LHS spectral sequence for $H \lhd H \rtimes P$, the only possible differential from the degree-two invariant edge has zero target; the same holds in degree three by (FD3) and $H^1(H;\mathbb{Z}_{(r)}) = 0$. A possible differential into the bottom row also vanishes because the extension is split, so inflation from P is injective. This proves (FD4).

Finally, if x generates the second group in (FD2), its invariant subgroup is generated by $r^a x$. The restriction of x to E_r has order r by Theorem 7.45; since $a \geq 1$, $r^a x$ restricts to zero. Thus the low-row edge is surjective, but none of its integral degree-three classes retains the required detector. □

Proposition 11.14 (Primary-operation audit at full field depth)**.** *Under the hypotheses of Proposition 11.13, consider the mod-r LHS spectral sequence*

$$E_2^{i,j} = H^i\big(P; H^j(H;\mathbb{F}_r)\big) \Longrightarrow H^{i+j}(H \rtimes P;\mathbb{F}_r). \tag{OA1}$$

Let

$$\xi = \bar{x} \in E_2^{0,3}, \qquad \eta = \beta\mathcal{P}^1(\xi) in E_2^{0,,2r+2}, \tag{OA2}$$

where x and θ_r are as in Theorem 7.45. Thus η is the mod-r reduction of θ_r. Then:

(i) *$d_2(\eta) = d_3(\eta) = 0$, without any assumption on $d_2(\xi)$;*

(ii) *if $d_2(\xi) = 0$, then*

$$d_j(\eta) = 0 \qquad (2 \leq j \leq r+3); \tag{OA3}$$

(iii) *if $d_2(\xi) = d_3(\xi) = 0$, then ξ is a permanent cycle and Conjecture 11.10 holds for this semidirect product.*

Consequently the primary transgression formulas leave only the following possible outgoing differentials on η:

r	$d_2(\xi) \neq 0$	$d_2(\xi) = 0,\ d_3(\xi) \neq 0$
3	d_4, d_5, d_6, d_7, d_8	d_7, d_8
5	$d_4, d_5, d_6, d_7, d_8, d_9, d_{10}, d_{11}, d_{12}$	$d_9, d_{10}, d_{11}, d_{12}$

(OA4)

The displayed list is independent of the field depth a. Its nonemptiness is the exact gap: this proposition does not prove that η, still less merely its power η^{r-1}, is permanent in the two remaining cases.

Proof. First record the coefficient actions needed to apply the spectral-sequence operations. The r-primary part of H_{ab} is C_{r^s} and a field generator acts on it by $q_0^{\pm 1}$. Since $q_0 \equiv 1 \pmod r$, P acts trivially on $H^1(H;\mathbb{F}_r)$. The coefficient sequence gives

$$0 \longrightarrow H^2(H;\mathbb{Z}_{(r)})/r \longrightarrow H^2(H;\mathbb{F}_r) \xrightarrow{\mathcal{B}_r} H^3(H;\mathbb{Z}_{(r)})[r] \longrightarrow 0. \tag{OA5}$$

Both end terms are fixed. Moreover, the right-hand generator has a P-fixed lift: it is represented by the field-equivariant scalar extension

$$1 \longrightarrow C_r \longrightarrow \mathrm{GL}_n(q)/C_{(q-1)/r} \longrightarrow \mathrm{PGL}_n(q) \longrightarrow 1, \tag{OA6}$$

where $C_{(q-1)/r}$ denotes the unique scalar subgroup of that order. The field action fixes the kernel because $q_0 \equiv 1 \pmod r$. Hence P acts trivially on $H^2(H;\mathbb{F}_r)$ as well.

Apply Araki's front and back reduced-power operations to (OA1) [32, Summary (L) and Corollary 4.1]. If $d_2(\xi) = \delta_2$, the page-shift formula gives

$$d_2(\eta) = 0, \qquad d_3(\eta) = \pm {}_B\beta\mathcal{P}^1(\delta_2). \tag{OA7}$$

Because the two low coefficient actions are trivial,

$$H^*(P;\mathbb{F}_r) = \mathbb{F}_r[v] \otimes \Lambda(t), \qquad |t| = 1, \quad |v| = 2, \quad \beta(v) = 0,$$

and $\delta_2 = v \otimes m$ for some $m \in H^2(H;\mathbb{F}_r)$. Araki's product formula therefore gives

$${}_B\beta\mathcal{P}^1(v \otimes m) = \pm\beta(v) \otimes m^r = 0. \tag{OA8}$$

This proves (i).

Now suppose $d_2(\xi) = 0$ and write $d_3(\xi) = \delta_3$. For an operation $\beta\mathcal{P}^1$ on a class of vertical degree three, Araki's shifted page is

$$\mu' = r(t-1) + (2-3)(r-1) + 2 = r(t-2) + 3. \tag{OA9}$$

Taking $t = 3$ shows that η survives to E_{r+3} and

$$d_{r+3}(\eta) = \pm {}_B\beta\mathcal{P}^1(\delta_3). \tag{OA10}$$

Here $\delta_3 = tv \otimes h$ for some $h \in H^1(H;\mathbb{F}_r)$. The same product formula makes (OA10) zero because the vertical factor h has odd degree. This proves (ii).

If also $d_3(\xi) = 0$, the only remaining outgoing differential on ξ is d_4 into the bottom row. It vanishes because the extension is split, so inflation from P is injective. Thus ξ is the restriction of a class $\widetilde{\xi} \in H^3(H \rtimes P;\mathbb{F}_r)$. Naturality now shows that

$$\mathcal{B}_r\mathcal{P}^1(\widetilde{\xi}) \quad \text{restricts to} \quad \theta_r;$$

raising this integral class to the $(r-1)$st power extends Θ_r and proves (iii).

Finally, a differential from $E_j^{0,,2r+2}$ reaches the bottom row only at $j = 2r+3$, and that differential again vanishes by splitness. Removing this last page, together with the vanishing ranges just proved, gives exactly (OA4). □

There is also a tempting representation-theoretic construction which can now be ruled out at the same boundary.

Proposition 11.15 (Projective points miss the non-toral class)**.** *Under the hypotheses of Proposition 11.13, let ρ_{pt} be the complex permutation representation of $H \rtimes P$ on the points of $\mathbb{P}^{n-1}(\mathbb{F}_q)$. Then*

$$\mathrm{res}^{H\rtimes P}_{E_r}\big(\overline{c_{r^2-1}(\rho_{\mathrm{pt}})}\big) = 0 \quad \text{in } H^{2(r^2-1)}(E_r;\mathbb{F}_r). \tag{FD5}$$

The same conclusion holds for the hyperplane permutation representation and for any direct sum of copies of the point and hyperplane representations. Hence these natural semilinear representations do not realize the detected restriction of Θ_r.

Proof. Write

$$Q_m = \frac{q^m - 1}{q - 1}.$$

The clock–shift action of E_r has no fixed projective point. For every line $L \le E_r$, $|L| = r$, a nonidentity element of L has r eigenspaces of dimension n/r. Thus the numbers of point orbits with stabilizer L and with trivial stabilizer are respectively

$$a_L = Q_{n/r}, \qquad a_0 = \frac{Q_n - (r+1)rQ_{n/r}}{r^2}. \tag{FD6}$$

By (FD1), full depth and $a \ge 1$ imply $r^2 \mid (q-1)$ and $r^2 \mid n$. Hence

$$a_L \equiv n/r \equiv 0 \pmod{r}.$$

Moreover, expanding the geometric sums modulo r^3 gives

$$Q_n \equiv n \pmod{r^3}, \qquad Q_{n/r} \equiv n/r \pmod{r^3}.$$

The numerator in (FD6) is therefore congruent to $-rn$ modulo r^3, and $a_0 \equiv 0 \pmod{r}$ as well.

Let y_1, y_2 generate the polynomial part of $H^*(E_r;\mathbb{F}_r)$. For an orbit E_r/L, its permutation representation has total Chern class

$$\prod_{b\in\mathbb{F}_r}(1 + b\ell) = 1 - \ell^{r-1}$$

for a linear form ℓ annihilating L. For a free orbit the total Chern class is the product over all linear forms in y_1, y_2; all of its nonconstant homogeneous degrees are multiples of $r-1$. Every orbit multiplicity in (FD6) is divisible by r, so Frobenius shows that every nonconstant homogeneous term of the total Chern class of $\rho_{\mathrm{pt}}|_{E_r}$ has polynomial degree divisible by $r(r-1)$. But

$$r^2 - 1 = (r+1)(r-1)$$

is not divisible by $r(r-1)$. This proves (FD5). The dual clock–shift action gives the identical hyperplane calculation, and taking direct sums only multiplies total Chern classes with the same degree divisibility. □

Proposition 11.16 (The symplectic tensor detector)**.** *Retain the hypotheses and notation of Theorem 7.45. If $r^b \mid n$, then H contains an elementary abelian subgroup*

$$A_b \cong C_r^{2b}$$

whose inverse image generated by the chosen lifts is extraspecial of order r^{1+2b} and exponent r. There are exterior generators $a_1, b_1, \dots, a_b, b_b$ and polynomial generators

$$u_j = \beta(a_j), \qquad v_j = \beta(b_j)$$

for which

$$\overline{\mathrm{res}^{H}_{A_b}(\theta_r)} = z_b := \sum_{j=1}^{b} (u_j v_j^r - u_j^r v_j), \tag{FD7}$$

and hence

$$\overline{\mathrm{res}^{H}_{A_b}(\Theta_r)} = z_b^{r-1} \neq 0. \tag{FD8}$$

The polynomial z_b is invariant under $\mathrm{Sp}_{2b}(\mathbb{F}_r)$. If a pure-field r-subgroup P is present, then P centralizes A_b pointwise, so the class in (FD8) *extends over $A_b \times P$.*

Proof. Let $W = \mathbb{F}_q^r$, and write the natural module as

$$\mathbb{F}_q^n \cong W^{\otimes b} \otimes \mathbb{F}_q^{n/r^b}.$$

On the jth copy of W, place the size-r clock and shift matrices X_j, Y_j used in Theorem 7.45, acting as the identity on all other tensor factors. Matrices belonging to different tensor factors commute, while

$$[X_j, Y_j] = \omega^{\pm 1} I.$$

After changing the sign of b_j if necessary, their projective images generate A_b, and the scalar commutator pairing is the standard symplectic form

$$\sum_{j=1}^{b} a_j b_j.$$

The lift group is therefore the central product of b Heisenberg groups of order r^3, hence is extraspecial of the stated order and exponent.

The homology map from $H_2(A_b; \mathbb{Z})$ to the cyclic group in (19ba) is precisely this scalar commutator pairing. The calculation in the proof of Theorem 7.45, now applied to every symplectic pair, gives

$$\overline{\mathrm{res}^{H}_{A_b}(x)} = \sum_{j=1}^{b} (a_j v_j - b_j u_j)$$

up to one common nonzero scalar. Reduced power and Bockstein are additive, so

$$\mathcal{B}_r \mathcal{P}^1 \left(\sum_{j=1}^{b} (a_j v_j - b_j u_j) \right) = \sum_{j=1}^{b} (u_j v_j^r - u_j^r v_j),$$

again up to a nonzero scalar. This proves (FD7) and (FD8).

If $U = (u_1, v_1, \ldots, u_b, v_b)^{\mathsf{T}}$ and J is the standard symplectic matrix, then z_b is, up to sign,

$$U^{\mathsf{T}} J\, U^{[r]},$$

where $U^{[r]}$ is obtained by taking rth powers. For a matrix $g \in \mathrm{Sp}_{2b}(\mathbb{F}_r)$ one has $g^{[r]} = g$ and $g^{\mathsf{T}} J g = J$, proving invariance. Finally a pure-field r-subgroup fixes ω: its action on $\langle \omega \rangle$ has order dividing both a power of r and $r - 1$. It therefore fixes all the displayed clock and shift matrices, and the last assertion follows from the projection $A_b \times P \to A_b$. □

This is the first detector in the present argument which retains the full symplectic commutator geometry at arbitrary multiplier depth. The polynomials z_b are the standard symplectic invariants occurring in the cohomology of extraspecial groups; the relation with Chern classes and transfer is studied in [18, 19]. More precisely, z_b is the invariant denoted $H_{2b,1}$ by Gu, whose restriction theorem identifies it as the restriction of a torsion class for the compact projective unitary group [22, Proposition 5.1]. That compact-group realization confirms that the detector is canonical; it does not supply the still missing extension across the finite pure-field subgroup.

There is also a uniform reason why this torsion class cannot be replaced by an ordinary Chern class once two clock–shift factors are available.

Proposition 11.17 (Ordinary Chern classes miss the symplectic detector)**.** *Retain the hypotheses and notation of Proposition 11.16, and suppose $b \geq 2$. If ρ is any finite-dimensional complex representation of H—including the restriction of a representation of any finite overgroup containing H—then*

$$\operatorname{res}^{H}_{A_b}\big(\overline{c_i(\rho)}\big) = 0 \qquad \Big(0 < i < r^{2b} - r^{2b-1}\Big). \tag{FD8a}$$

The same vanishing range holds for the positive-degree part of the subring generated by Chern classes of complex representations. In particular, since

$$r + 1 < r^2 - 1 < r^{2b} - r^{2b-1},$$

neither z_b nor z_b^{r-1} can be the restriction of an ordinary Chern class, or of a polynomial in such classes, in its displayed degree. Consequently no ordinary complex representation of the semilinear group can realize θ_r or Θ_r on this detector.

Proof. First note that the image of $N_H(A_b)$ in $\mathrm{Aut}(A_b)$ contains $\mathrm{Sp}_{2b}(\mathbb{F}_r)$. Indeed, the inverse image of A_b generated by the clock and shift matrices is the extraspecial Heisenberg group $\widetilde{A}_b$. Its standard presentation shows directly that every symplectic change of the generators lifts to an automorphism of $\widetilde{A}_b$ fixing the scalar center: the symplectic condition is exactly preservation of the displayed commutator relations, and the exponent-r relations are unchanged. Those matrices generate the full matrix algebra $M_{r^b}(\mathbb{F}_q)$, so the lifted automorphism induces an automorphism of this matrix algebra. By Skolem–Noether it is conjugation by an element of $\mathrm{GL}_{r^b}(q)$. Tensoring that element with the identity on the multiplicity space gives the required element of H.

Decompose the restriction of ρ to A_b into one-dimensional characters:

$$\rho|_{A_b} = \bigoplus_{\lambda \in A_b^*} m_\lambda L_\lambda.$$

Conjugation by $N_H(A_b)$ preserves this representation. The symplectic group is transitive on the nonzero vectors of A_b^*, so $m_\lambda = m$ is independent of nonzero λ. Trivial summands do not affect Chern classes, and therefore

$$c(\rho|_{A_b}) = \left(\prod_{0 \neq \lambda \in A_b^*} (1 + \ell_\lambda) \right)^m, \tag{FD8b}$$

where $\ell_\lambda = c_1(L_\lambda)$ is the corresponding linear form in the polynomial generators of $H^*(A_b; \mathbb{F}_r)$.

The Dickson identity in $2b$ variables is

$$\prod_{\lambda \in A_b^*} (T + \ell_\lambda) = T^{r^{2b}} + \sum_{j=0}^{2b-1} (-1)^{2b-j} D_j T^{r^j}.$$

After setting $T = 1$, every nonconstant homogeneous term on the right of (FD8b) has polynomial degree at least $r^{2b} - r^{2b-1}$. This proves (FD8a). Products and inverses of total Chern classes preserve the same first possible positive degree, so the assertion about the Chern subring and virtual representations follows as well. Finally, z_b and z_b^{r-1} have polynomial degrees $r + 1$ and $r^2 - 1$, respectively, which proves the last assertions. □

Thus the compact projective-unitary calculation points specifically to a torsion operation outside the ordinary representation ring. It does not yet give a class on the whole finite semilinear group. In fact the small detector has a concrete fusion defect at full depth.

Proposition 11.18 (Failure of weak closure)**.** *If $r^2 \mid n$, the subgroup $E_r = A_1$ in Theorem 7.45 is not weakly closed in $N_H(E_r)$ with respect to H. Consequently, essentiality of z_1^{r-1} on E_r does not by itself prove that corestriction from $N_H(E_r)$ to H is nonzero.*

Proof. Use the construction of Proposition 11.16 with $b = 2$, and write

$$A_2 = E_1 \times E_2$$

for the two clock–shift tensor factors. They centralize one another, so $E_2 \le C_H(E_1) \le N_H(E_1)$. The linear map interchanging the two copies of W conjugates E_1 to E_2 in H. On the other hand, E_1 is normal in its normalizer, so no element of $N_H(E_1)$ conjugates E_1 to the distinct subgroup E_2. This is exactly the failure of weak closure. In the Mackey formula for corestriction from $N_H(E_1)$, the conjugate E_2 therefore produces an additional full-intersection double coset; essentiality only kills terms coming from proper intersections and does not determine this term. □

The first exact Mackey diagnostic also cancels. In a four-dimensional symplectic space, let the sum run over all nondegenerate two-planes W. Use symplectic projection to regard the rank-two polynomial z_W as a polynomial on the full space, and put

$$\Sigma_r = \sum_{\substack{W \le \mathbb{F}_r^4 \\ \dim W = 2,\ W \text{ nondegenerate}}} z_W^{r-1}. \tag{FD9}$$

The exact polynomial calculation

```
python3 scripts/check_symplectic_plane_sum.py
```

gives

```
r=3 planes=130 nondegenerate=90 nonzero_terms=0
r=5 planes=806 nondegenerate=650 nonzero_terms=0
```

and hence $\Sigma_3 = \Sigma_5 = 0$. The script enumerates every two-plane by its unique reduced-row-echelon matrix and performs polynomial arithmetic over the prime field, so this is an identity rather than a random evaluation. It is not a proof that the actual group-theoretic Mackey sum vanishes: stabilizer weights and normalizer orbits still have to be matched. It does prove that the most symmetric unweighted plane average cannot provide the missing transfer class at either residual prime.

The alternating determinant in (TD) changes by the determinant under $\mathrm{GL}_2(\mathbb{F}_r)$, so its $(r-1)$st power is independent of the chosen basis and is nonzero. Thus this is a concrete fixed-degree replacement for the nilpotent multiplier detector, not merely a reformulation of uniform factor-stabilizer

detection. At the two fixed residual primes it predicts mod-three degree sixteen and mod-five degree forty-eight, hence integral ceilings seventeen and forty-nine. No construction of the class z is presently proved; plausible sources are a more refined projective Chern class, a regular-embedding characteristic class, or a stable-elements calculation on the clock–shift normalizer. Conditions (9A) and (9D) make these two instances sufficient for the whole remaining factor-stabilizer row: outside defining characteristic three, $r = 3$ divides d_0, while in defining characteristic three, the proof of (9F) gives $5 \mid d_0$. Thus Conjecture 11.9 for $r = 3, 5$ would complete Step 4 with common integral ceiling forty-nine.

The most obvious Chern realization does not work. Assume $s \geq 2$ and let B act on the projective points of the natural module (adjoin the dual action on hyperplanes if graph automorphisms are present). Write

$$Q_j = \frac{q^j - 1}{q - 1}.$$

For the subgroup E above, every line subgroup has r eigenspaces of dimension n/r, while E has r^2 simultaneous eigenspaces of dimension n/r^2. Hence the number of size-r orbits with any fixed stabilizer line, and the number of free orbits, are respectively

$$a_L = Q_{n/r} - rQ_{n/r^2}, \qquad a_0 = \frac{Q_n - (r+1)rQ_{n/r} + r^3 Q_{n/r^2}}{r^2}. \tag{TD$'$}$$

Since $v_r(q-1) \geq s$ and $r^s \mid n$, the expansion $Q_j = j + \binom{j}{2}(q-1) + O((q-1)^2)$ gives

$$r^{s+1} \mid a_L, \qquad r \mid a_0. \tag{TD$''$}$$

For a size-r orbit the mod-r total Chern class is $1 - \ell^{r-1}$, while the first positive term for a free E-orbit has polynomial degree $r(r-1)$. Frobenius and (TD$''$) therefore make the total Chern class of the projective-point permutation representation equal to one through polynomial degree $r^2 - 1$, exactly the degree of Δ_r. The hyperplane summand has the same defect. Thus Conjecture 11.9, if true, requires a subtler representation or a non-Chern stable-elements construction.

11.6 Work classification and present assessment

The remaining work can now be classified more sharply.

Primarily more computation and case checking.

Finish the remaining mixed-prime linear almost-simple row, whose multiplier support contains a prime at least five and whose field data satisfy (9A)–(9H) and the hierarchy after (9B), using whatever finite degree is available. The depth-dependent norm is now proved; the next targeted calculation is a fixed-degree replacement at the full-depth boundaries: $v_3(q-1) = v_3(d_0)$ outside characteristic three, or $v_5(q-1) = v_5(d_0)$ in characteristic three. Do not append another fixed finite cyclotomic list. The toral Dickson class in Conjecture 11.9 is a concrete degree-sixteen target at three and degree-forty-eight target at five; those two cases alone would complete the remaining factor-stabilizer row. The non-toral alternative in Conjecture 11.10 asks only for LHS permanence of the already-constructed integral class and would give common integral ceiling forty-eight. The field-depth norm already supplies the unconditional ceiling $2r^a(r^2-1)+1$, so only unbounded a remains in this route. The primary-operation audit in Proposition 11.14 now proves that d_2 and d_3 on the mod-r class $\eta = \beta\mathcal{P}^1(\bar{x})$ vanish universally; if $d_2(\bar{x}) = 0$, it also removes every differential through d_{r+3}. Thus this route has been reduced to the finite, depth-independent lists in (OA4). Those later

differentials are genuinely unresolved, and permanence of the power η^{r-1} could still hold even if η itself dies. The full-depth field-module calculation shows that the low multiplier rows are cohomologically trivial but their extendable integral classes vanish on the detector; the point and hyperplane permutation representations also miss the target Chern degree. Restriction of scalars offers no shortcut here: direct sum of Frobenius twists fails to descend through the projective quotient, while tensor induction descends only after applying the scalar norm $(q-1)/(q_0-1)$, whose r-adic valuation is the field depth a. The resulting mod-r projective class is therefore already zero on the inner-diagonal subgroup. The tensor clock–shift construction now identifies its restriction on C_r^{2b} as a standard nonzero symplectic invariant, fixed pointwise by the field subgroup. The full symplectic Weyl action further shows that every ordinary complex representation restricts as trivial characters plus equal copies of all nontrivial characters. Its Chern classes are therefore Dickson invariants and begin above the degree of the desired detector once $b \geq 2$. This rules out the entire ordinary-Chern-class route, rather than only the point and hyperplane representations. A non-Chern normalizer-transfer construction would therefore be natural, but the rank-two subgroup is not weakly closed once $r^2 \mid n$, so the full Mackey sum rather than essentiality alone must be computed. The exact rank-four calculation (FD9) shows that the unweighted sum over all nondegenerate planes already cancels at both residual primes; the actual stabilizer weights and normalizer orbits are therefore essential data. Thus the next calculation must address the higher operation itself or that symplectic Mackey sum, not only its degree-three input or the natural permutation representations; replacing them by another ordinary complex representation cannot help. The off-line kernel-correction class (PBL′) remains a useful diagnostic. Also compute $d_2^{0,2}$ in (R1c) for the first feasible Frattini modules; and search small perfect-group and Frattini–central extension databases for increasing vanishing depth. The twisted-E_6 degree-nine scalar and finer Baby Monster module audit are optional optimizations and are not included in this main-goal work list.

Likely to require a substantially deeper idea.

Prove Conjecture 11.3; prove a uniform form of Conjecture 11.5; prove the alternative prime-avoidance Conjecture 11.4; control arbitrary irreducible modules and iterated Frattini layers; or rule out the evasion tower (R1). Extending Smith-form calculations one width at a time will not settle these statements.

Formal after the inputs exist.

The least-counterexample assembly in Proposition 10.1, and the bookkeeping that propagates a common degree bound through Steps 1–8.

The planning estimate in Section 10 is now 86–96% for eventual success in proving some finite universal bound. It is not a near-term completion forecast or an objective probability that the conjecture is true. Near-term completion is materially less likely because the uniform elementary and persistence theorems do not yet have proofs. The modest upward revision reflects closure of the entire simple-quotient prime-power-kernel branch; it is deliberately small because the two uniform bottlenecks are unchanged. The prime-avoidance shortcut is a genuine alternative to those bottlenecks, but its perfect-group hypothesis is presently a conjecture rather than a new uniform theorem, so it does not by itself justify a further numerical increase. The main evidence that could genuinely reverse the overall outlook would be a computed sequence of Frattini–central extensions whose initial homology vanishing depth grows.

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