PROFINITE RIGIDITY OF FINITE-VOLUME HYPERBOLIC 3–MANIFOLDS

Abstract. We prove that every orientable finite-volume hyperbolic 3–manifold is determined, among compact orientable 3–manifolds with empty or toral boundary, by the profinite completion of its fundamental group. In particular, two orientable finite-volume hyperbolic 3–manifolds with isomorphic profinite completions are isometric, possibly after reversing orientation. The proof combines a finite-quotient marked reconstruction of primitive closed geodesics with a cohomological reconstruction of splittings of completed surface groups along simple curves.

WARNING: this was created by OpenAI under the limited direction and supervision of Frank Calegari.

Contents

1. Introduction

For a finitely generated group Γ, its profinite completion

$$\widehat{\Gamma} = \varprojlim_{\Gamma \twoheadrightarrow F} F$$

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records all finite quotients of Γ and the maps between them. A group, or a manifold through its fundamental group, is *profinitely rigid in a class* if no other member of that class has an isomorphic profinite completion. The question considered here asks whether the topology of a finite-volume hyperbolic 3–manifold can be recovered from this apparently finite algebraic information.

The problem combines two established rigidity themes. Grothendieck rigidity asks when a finitely generated subgroup is determined by the map that it induces on profinite completions; Long–Reid proved the relevant result for finite-volume hyperbolic 3–manifold groups [30]. At the other end, Mostow–Prasad rigidity says that an isomorphism of the discrete lattices determines the hyperbolic manifold [36, 38]. The possibility of extracting complete 3–dimensional information from finite or quantum data also appears in the work of Calegari–Freedman–Walker [11], and Reid formulated the lattice version of the present problem explicitly in his survey [40, Question 9]. A profinite isomorphism, however, need not identify either discrete lattice inside the two completions. Recovering discrete elements, and then their incidence, is the central difficulty.

Our main result is the following stronger form of profinite rigidity.

Theorem 1.1 (Profinite rigidity among 3–manifolds)**.** *Let M be an orientable finite-volume hyperbolic 3–manifold and let N be a compact connected orientable 3–manifold with empty or toral boundary. If*

$$\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)},$$

then N is homeomorphic to a compact core of M.

Theorem 1.2 (Hyperbolic profinite rigidity)**.** *Let M and N be orientable finite-volume hyperbolic 3–manifolds. If*

$$\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)},$$

then M and N are homeomorphic. Consequently they are isometric, possibly after reversing orientation.

Theorem 1.2 is the immediate hyperbolic case of Theorem 1.1; the last assertion is Mostow–Prasad rigidity [36, 38].

Corollary 1.3 (Geometric and arithmetic invariants)**.** *The profinite completion of the fundamental group of an orientable finite-volume hyperbolic 3–manifold determines its hyperbolic volume (or, equivalently, the covolume of its lattice), its arithmeticity, its invariant trace field up to field isomorphism, and its invariant quaternion algebra up to the induced field isomorphism.*

Proof. Theorem 1.2 and Mostow–Prasad rigidity recover the manifold up to isometry and orientation reversal. The listed quantities are isometry invariants; orientation reversal only applies complex conjugation to the standard algebraic realization. In particular, the first assertion displays explicitly the volume conclusion that is also obtained internally in Sections 3–4. □

Previous work. Several individual hyperbolic 3–manifolds and lattices were already known to be profinitely rigid: these include the figure-eight knot complement [9], the Weeks manifold and the nonuniform lattice of minimum covolume [10], and further closed examples, including the four-fold cyclic branched cover of the figure-eight knot [8]. More generally, Liu proved that a finite-volume hyperbolic 3–manifold is determined up to finitely many possibilities and constructed a correspondence between its Zariski-dense algebraic PSL_2–representations [27, 28]. His work also detects fibered classes and completed fiber subgroups. The recent regularity, normalizer, filling, and simple-curve theorems of Hanany, Belolipetsky–Cheetham-West, Xu, and Wang–Xu [18, 3, 52, 53, 46] leave a final comparison problem for the mapping-torus structures that arise in the closed fibered case. The first step below strengthens Liu's volume comparison unconditionally by using the finite-field Chern classes of Calegari–Garoufalidis–Zagier [12].

Overview of the proof. A profinite isomorphism identifies all finite quotients, but there is no a priori reason for it to carry the original discrete lattice inside one completion onto the other lattice. The first task is therefore to recover actual discrete elements from profinite data. Primitive elements corresponding to closed geodesics can be studied through higher Reidemeister torsions of the manifold and its finite covers. Untwisted cover data naturally remembers only the cyclic subgroup generated by the image of a geodesic in a finite quotient, leaving a profinite power ambiguity.

We remove that ambiguity by twisting the canonical local systems by every complex representation of a finite quotient. Liu's full-group ambiguity is a quadratic character [28, Lemma 8.10], and therefore disappears in every even symmetric power. Proposition 8.1 proves the needed based Shapiro–Artin formula: its Shapiro part identifies the based torsion complex of an induced representation with that of the corresponding, possibly nonnormal, cover, while its Artin part matches the Euler factors cycle by cycle. The higher-torsion identities then recover the character-marked prime-geodesic data. Lemma 8.3 uses Adams operations to record the character of each finite-quotient power and Möbius inversion to subtract all proper-power contributions. Applying the result simultaneously to finite products of quotients shows that a fixed primitive element is conjugate, in every finite quotient, to one of finitely many geodesics of the other manifold. Compactness produces one profinite conjugator to one fixed discrete geodesic. This gives the first actual discrete element on the target side from which the later incidence reconstruction can be based.

For a fiber element represented by a simple closed curve, one anchor is not enough: conjugators for neighboring curves may contain unrelated profinite powers of the mapping-torus generator. Cutting the fiber along the anchored curve gives a one-edge splitting of its surface group. We associate to the completed splitting a cohomology class with coefficients in the completed edge permutation module. A completed finite-free Poincaré-duality calculation makes the relevant cohomology group one-dimensional, while profinite malnormality shows that the only ambiguity in a based cocycle is switching the two endpoints of the chosen edge. The cocycle therefore reconstructs the completed splitting itself.

This reconstruction forces every disjoint neighboring curve into the appropriate completed subsurface group. A transporter theorem then shows that the remaining mapping-torus time is an ordinary integer. Propagating through a connected finite configuration of curves produces a simplicial map on a finite rigid set in the curve complex. It is induced by a homeomorphism of the fibers; including the monodromy image of a finite marking forces the two monodromies to be conjugate up to inversion. This proves the fibered closed case. Normalizer and Dehn-filling reductions give the general theorem.

Relation to the ingredients. The scalar spectral inverse mechanism is due to Menal–Ferrer–Porti [31], and the line-twisted Ruelle identities come from Fried, Müller, Wotzke, and Bénard–Dubois–Heusener–Porti [17, 35, 47, 4]. The surface descent is a profinite version of the relation between first cohomology, relative ends, and splittings developed by Houghton, Kropholler–Roller, Sageev, and Scott–Swarup [20, 25, 41, 42]. The compact coefficient formalism is from Boggi–Corob Cook and Wilkes [6, 49]; profinite malnormality is from Wilton–Zalesskii [50]. Our new points are the finite-character marked inverse theorem, exact ambient descent, the profinite cut class and fixed-edge reconstruction, and their synchronization.

Section 2 gives the geometric reductions. Sections 3–4 prove representation volume and recognize holonomy. Sections 5–9 give the analytic marked descent. Sections 10– 13 construct and recognize the cut class, and Section 14 completes the proof.

2. The geometric reduction

We first isolate exactly what is supplied by the global 3–manifold literature.

Proposition 2.1 (Reduction to hyperbolic competitors)**.** *Let M be finite-volume hyperbolic and let N be as in Theorem 1.1. If their fundamental groups have isomorphic profinite completions, then the interior of N is finite-volume hyperbolic. Moreover M and N are either both closed or both cusped.*

Proof. This is Xu's precise summary of hyperbolicity and cusp detection [52, Proposition 2.3]: a compact orientable 3–manifold with the same profinite completion as a finite-volume hyperbolic 3–manifold is itself a finite-volume hyperbolic compact core and has the same number of cusps. The underlying geometry- and decomposition-detection theorems are due to Wilton–Zalesskii [50, 51]. This formulation directly applies to the given groups and requires no choice of a vertex subgroup. □

Proposition 2.2 (Closed reduction)**.** *If profinite rigidity holds for closed fibered hyperbolic 3–manifolds, then it holds for all closed orientable hyperbolic 3–manifolds.*

Proof. This is precisely the implication (2) ⇒ (1) in Belolipetsky–Cheetham-West [3, Corollary 1.2]. Their proof combines Agol's virtual fibering theorem [1], their normalizer theorem [3, Theorem 3.3], and the normalizer-inheritance theorem of Bridson–Reid [8, Theorem 4.4]. In particular, the subgroup is chosen so that its full lattice normalizer is the original lattice; the conclusion therefore returns to the original groups and is not merely a finite-index statement. □

Lemma 2.3 (Drilling corresponding fillings)**.** *If profinite rigidity holds for closed hyperbolic 3–manifolds, then it holds for cusped finite-volume hyperbolic 3–manifolds.*

Proof. Let M, N be cusped finite-volume hyperbolic manifolds with isomorphic profinite completions. Xu's peripheral regularity theorem [53, Theorem 1.4], together with his Dehn-filling theorem [52, Theorem A], gives a cusp bijection and integral boundary isomorphisms carrying every full multislope s on M to a multislope s' on N, with

$$\pi_1\widehat{(M(s))} \cong \pi_1\widehat{(N(s'))}.$$

The boundary isomorphisms are integral, so a sequence of full multislopes tending to infinity is carried to another such sequence.

Apply the short-core form of hyperbolic Dehn surgery [26, Theorem 2.1] to the two corresponding sequences. It supplies a positive uniform lower bound for every geodesic which is not a power of a filling core, while the core lengths tend to zero. Choose $\delta > 0$ below that lower bound and below the systoles of both unfilled manifolds. For all sufficiently long full multislopes the fillings are closed hyperbolic and all filling cores have length below δ. The primitive cores are therefore exactly the primitive closed geodesics below δ. Closed profinite rigidity and Mostow–Prasad rigidity [36, 38] give an isometry between the two fillings. It must carry the entire short core link to the short core link, allowing a permutation of components and reversal of orientation. Drilling that link recovers M and N, which are therefore homeomorphic. This proves the lemma. □

By Propositions 2.1, 2.2, and Lemma 2.3, it remains to prove rigidity for closed pseudo-Anosov mapping tori.

3. Odd-primary Bloch classes and specialization

We now prove the representation-volume theorem used to recognize holonomy. For a field F, let $B(F)$ be the Bloch group in the convention of Calegari–Garoufalidis–Zagier [12]. It differs from Suslin's Bloch group by a group of exponent two [12, Lemma 2.2]; the distinction is therefore invisible with odd coefficients and to the Borel regulator. Put $\mathrm{SL}(F) = \varinjlim_n \mathrm{SL}_n(F)$. The stable Hurewicz and Dupont–Sah–Suslin maps give a natural homomorphism

$$\partial_F : H_3(\mathrm{SL}(F), \mathbf{Z}) \longrightarrow B(F)$$

[15, 45]. Write st for stabilization from $\mathrm{SL}_2(F)$.

For an infinite field, the stable Hurewicz map identifies

$$K_3(F)/(\{-1\} \cdot K_2(F)) \cong H_3(\mathrm{SL}(F), \mathbf{Z})$$

[22, Proposition 2.4]; its kernel in $K_3(F)$ is killed by two. For odd primes l it therefore induces

$$K_3(F)/l \cong H_3(\mathrm{SL}(F), \mathbf{Z})/l.$$

If F is a number field and l is sufficiently large, Suslin's exact sequence identifies these groups with $B(F)/lB(F)$ [22, Corollary 2.9]. If $k = \mathbf{F}_q$, l is odd, and $q \equiv -1 \pmod{l}$, the finite-coefficient Hurewicz isomorphism [22, Lemma 4.3] and the finite-field Bloch map [12, Section 4.2] give an isomorphism

$$\partial_{q,l} : H_3(\mathrm{SL}(\mathbf{F}_q), \mathbf{Z}/l) \xrightarrow{\sim} B(\mathbf{F}_q) \otimes \mathbf{Z}/l.$$

We first dispose of the two ambiguities in Liu's representation correspondence.

Lemma 3.1 (Stable ambiguities)**.** *Let E be a field, Δ a group, and l an odd prime.*

1. $\mathrm{GL}_2(E)$*–conjugate representations $\Delta \to \mathrm{SL}_2(E)$ induce the same map to stable* SL*–homology with any coefficients.*
2. *If $\rho : \Delta \to \mathrm{SL}_2(E)$ and $\xi : \Delta \to \{1, -1\}$, then ρ and $\xi\rho$ induce the same map to stable* SL*–homology with $\mathbf{Z}/l$–coefficients.*

Proof. If $T \in \mathrm{GL}_2(E)$, stabilization turns conjugation by T into conjugation by the element

$$\mathrm{diag}(T, \det(T)^{-1}) \in \mathrm{SL}_3(E),$$

hence into an inner automorphism. For the second assertion let $D = \{1, -1\}$, and compare the projection $D \times \mathrm{SL}_2(E) \to \mathrm{SL}_2(E)$ with multiplication $(d, g) \mapsto dg$. Because $H_i(D; \mathbf{Z}/l) = 0$ for $i > 0$, Künneth says that projection is a homology isomorphism. Both maps agree after the inclusion $g \mapsto (1, g)$, so they induce the same map. Apply this to $g \mapsto (\xi(g), \rho(g))$. □

For a finite place $\mathfrak{q}$ of a number field F, let $k_{\mathfrak{q}}$ be its residue field and let $\mathrm{sp}_{\mathfrak{q}}$ denote the Bloch specialization of [12, Section 4.3].

Lemma 3.2 (Specialization compatibility)**.** *Let F be a number field, $c \in H_3(\Delta, \mathbf{Z})$, and $\rho : \Delta \to \mathrm{SL}_2(F)$. Set*

$$\beta(\rho, c) = \partial_F(\mathrm{st}(\rho_* c)) \in B(F).$$

Fix a finite bar cycle representing c. For every sufficiently large odd prime l, and for all but finitely many places $\mathfrak{q}$ with $k_\mathfrak{q} = \mathbf{F}_q$ and $q \equiv -1 \pmod{l}$, an integral $\mathrm{GL}_2(F)$–conjugate $\widetilde{\rho}$ satisfies

$$\mathrm{sp}_\mathfrak{q}(\beta(\rho, c) \bmod l) = \partial_{q,l}\big(\mathrm{st}((\overline{\rho}_\mathfrak{q})_*(c \bmod l))\big).$$

Proof. Compute the Dupont–Sah map on the chosen bar cycle in the generic configuration complex of $\mathbf{P}^1$. Exclude the finitely many places at which an entry is nonintegral or two of the finitely many projective points coalesce. For all remaining sufficiently large residue fields, the auxiliary points can be chosen integrally with generic reduction. The cross-ratio formula then commutes term by term with reduction. The asserted equality is exactly the compatibility in [12, Lemma 4.4], combined with [22, Lemma 4.3]. Lemma 3.1 removes the conjugacy ambiguity, and the difference between the two Bloch conventions is 2–primary. □

For a number field F let $\mathcal{Q}_l(F)$ consist of the primes $\mathfrak{q}$ above rational primes q which split completely in F and satisfy $q \equiv -1 \pmod{l}$.

Proposition 3.3 (Finite-field detection)**.** *For every sufficiently large odd prime l, the product map*

$$B(F)/lB(F) \longrightarrow \prod_{\mathfrak{q} \in \mathcal{Q}_l(F)} B(\mathbf{F}_q) \otimes \mathbf{Z}/l$$

is injective, and remains injective after deleting finitely many factors.

Proof. This is [12, Proposition 4.2 and Remark 4.3], transported from $K_3(F)/l$ to the Bloch group through the odd-primary comparison above; [12, Lemma 4.4] identifies the local Chern maps with the displayed specializations. □

Lemma 3.4 (One common sign)**.** *Let $x, y \in B(F)$. Suppose that for every sufficiently large odd l there are a sign ϵ_l and a finite set $S_l \subset \mathcal{Q}_l(F)$ such that*

$$\mathrm{sp}_\mathfrak{q}(x) = \epsilon_l \, \mathrm{sp}_\mathfrak{q}(y) \quad \textit{in } B(\mathbf{F}_q) \otimes \mathbf{Z}/l$$

for $\mathfrak{q} \in \mathcal{Q}_l(F) \setminus S_l$. Then $x - \epsilon y$ is torsion for one sign $\epsilon \in \{1, -1\}$.

Proof. Proposition 3.3 gives $x \equiv \epsilon_l y \pmod{lB(F)}$. One sign occurs for infinitely many l. Since the Bloch group of a number field is finitely generated, the image of the corresponding difference in its free quotient is divisible by infinitely many primes and hence vanishes. □

4. Representation volume and holonomy

We isolate the unconditional local part of Liu's construction. Here $\mathrm{Fratt}_2(\Delta) = \ker(\Delta \to H_1(\Delta; \mathbf{F}_2))$.

Lemma 4.1 (Liu's residual comparison)**.** *Let (M_A, M_B, Ψ) be a profinite isomorphism setting of pseudo-Anosov mapping tori, and let ρ_A, ρ_B be corresponding Zariski-dense SL_2–representations defined over a number field K. Outside a finite set of rational primes q splitting completely in K, there are integral $\mathrm{GL}_2(K)$–conjugates whose reductions are identified, up to $\mathrm{GL}_2(\mathbf{F}_q)$–conjugacy, on the completed 2–Frattini subgroups.*

Proof. This is the reduction modulo $\mathfrak{q}$ of diagrams (10.1) and (10.2) in the proof of [28, Lemma 10.1]. Their construction uses only the boundedness and character comparison of [28, Lemma 9.1]; the regulator hypothesis in that paper enters only after these diagrams. □

Proposition 4.2 (Fibered representation volumes)**.** *In the preceding mapping-torus setting, corresponding Zariski-dense* $\mathrm{SL}_2(\overline{\mathbf{Q}})$*–representations satisfy*

$$\mathrm{Vol}(M_A, \sigma\rho_A) = \delta\,\mathrm{Vol}(M_B, \sigma\rho_B)$$

for every complex embedding σ *and one sign* $\delta \in \{1, -1\}$ *independent of* σ*.*

Proof. Choose a common field of definition K. Put $\Delta_X = \mathrm{Fratt}_2(\pi_1 M_X)$ and let $M_X^\natural$ be the corresponding regular cover. The two covers have the same degree. For their fundamental classes $c_X = [M_X^\natural]$, define

$$\beta_X = \partial_K\big(\mathrm{st}((\rho_X|_{\Delta_X})_* c_X)\big) \in B(K).$$

These classes are unaffected by the conjugacies in Lemma 4.1, by Lemma 3.1.

The fundamental-class comparison in the proof of [28, Lemma 10.1], obtained from [28, Lemmas 5.3 and 7.1], says that for every odd l there is a sign δ_l such that

$$\Psi_*[M_A^\natural]_l = \delta_l [M_B^\natural]_l$$

in continuous H_3 of the completed 2–Frattini groups. Goodness gives the same equality for the discrete mod-l fundamental classes. For $\mathfrak{q} \in \mathcal{Q}_l(K)$ outside the finite exceptional sets, Lemma 4.1 lets us push the equality through the residual representations. Lemma 3.2 then gives

$$\mathrm{sp}_{\mathfrak{q}}(\beta_A \bmod l) = \delta_l\,\mathrm{sp}_{\mathfrak{q}}(\beta_B \bmod l).$$

Lemma 3.4 supplies one sign δ for which $\beta_A - \delta\beta_B$ is torsion.

The degree-three Borel regulator kills torsion. Its component at each complex embedding is a fixed nonzero multiple of signed representation volume, compatibly with the Dupont–Sah–Suslin map [28, (2.3)]; real components vanish. The displayed volume identity follows first for the two covers. Representation volume multiplies by covering degree, so division by their common degree proves the result. □

Theorem 4.3 (Representation-volume comparison)**.** *Let* M_A, M_B *be closed orientable hyperbolic* 3*–manifolds and* $\Psi : \widehat{\pi_1 M_A} \to \widehat{\pi_1 M_B}$ *an isomorphism. If* ρ_A, ρ_B *are corresponding Zariski-dense algebraic* PSL_2*–representations in Liu's correspondence, then one sign* δ *satisfies*

$$\mathrm{Vol}(M_A, \sigma\rho_A) = \delta\,\mathrm{Vol}(M_B, \sigma\rho_B)$$

for every embedding $\sigma : \overline{\mathbf{Q}} \to \mathbf{C}$*.*

Proof. Use the reduction in [28, Theorem 11.1]. There are corresponding finite-index subgroups inside the 2–Frattini subgroups whose covers are pseudo-Anosov mapping tori, and the two indices agree. The restricted projective representations lift to corresponding $\mathrm{SL}_2(\overline{\mathbf{Q}})$–representations. Their residual ambiguity is a quadratic scalar character; it is invisible to odd-primary stable homology by Lemma 3.1, and projective representation volume does not depend on the lift. Proposition 4.2 applies. Divide its identity by the common covering degree. □

Corollary 4.4 (Volume and holonomy recognition)**.** *If M_A, M_B are as in Theorem 4.3, then*

$$\mathrm{Vol}(M_A) = \mathrm{Vol}(M_B),$$

and Liu's correspondence takes holonomy to holonomy or antiholonomy. The statement is compatible with restriction of Ψ and the representations to corresponding finite covers.

Proof. Take ρ_B to be holonomy and ρ_A its correspondent. The theorem and volume rigidity give

$$\mathrm{Vol}(M_B) = |\,\mathrm{Vol}(M_A, \rho_A)| \leq \mathrm{Vol}(M_A),$$

with equality only for a discrete faithful representation [28, Example 2.1 and Remark 2.2]. Apply the same argument to Ψ^{-1} for the reverse inequality. Thus equality holds and ρ_A is holonomy or its complex conjugate. Restriction to corresponding subgroups multiplies both representation volumes by their common index, and all fundamental-class comparisons above are induced by the restricted isomorphism. Hence the same conclusion holds simultaneously in corresponding covers. □

5. The fibered setting

Write

$$M_X = M_{\varphi_X}, \qquad \Gamma_X = \Pi_X \rtimes_{\varphi_X} \langle t_X \rangle, \qquad \Pi_X = \pi_1(S_X),$$

where S_X is closed of genus at least two and φ_X is pseudo-Anosov. Let

$$\Psi : \widehat{\Gamma}_A \xrightarrow{\sim} \widehat{\Gamma}_B.$$

Hanany's regularity theorem [18, Theorem 1.1] and Liu's detection of fibered classes and fiber subgroups [27, Theorem 1.3 and Corollary 6.2] allow us to choose the fibered classes so that, initially,

$$\Psi(\widehat{\Pi}_A) = \widehat{\Pi}_B, \qquad \nu_B \Psi = \varepsilon_{\mathrm{fib}} \nu_A, \qquad \varepsilon_{\mathrm{fib}} \in \{1, -1\},$$

where $\nu_X : \widehat{\Gamma}_X \to \widehat{\mathbf{Z}}$ is the completed fibration. If $\varepsilon_{\mathrm{fib}} = -1$, replace the target mapping-torus presentation by the reversed fibration

$$(\nu_B, t_B, \varphi_B) \longmapsto (-\nu_B, t_B^{-1}, \varphi_B^{-1}).$$

This changes neither M_B nor Γ_B; it only reverses the chosen orientation of the base circle. From now on the symbols (ν_B, t_B, φ_B) denote these normalized choices. Thus throughout the proof

$$\Psi(\widehat{\Pi}_A) = \widehat{\Pi}_B, \qquad \nu_B \Psi = \nu_A. \tag{5.0}$$

At the end we undo the possible reversal, which is exactly the inversion allowed in the conclusion. Put $\alpha = \Psi|_{\widehat{\Pi}_A}$.

Lemma 5.1 (Exact completed fiber sequence)**.** *The natural sequence*

$$1 \longrightarrow \widehat{\Pi}_X \longrightarrow \widehat{\Gamma}_X \xrightarrow{\nu_X} \widehat{\mathbf{Z}} \longrightarrow 1$$

is exact.

Proof. Given a finite-index subgroup of Π_X, intersect the finite orbit under φ_X of its normal core. The resulting finite-index normal subgroup is φ_X-invariant, and the induced automorphism of the finite quotient has finite order. Thus every finite quotient of Π_X extends across a suitable finite cyclic quotient of Γ_X. The topology induced on Π_X is its full profinite topology and completion preserves the kernel. □

Apply Corollary 4.4 to the fixed isomorphism Ψ. Choose compatible $\mathrm{SL}_2(\overline{\mathbf{Q}})$ lifts ρ_X of the recognized holonomy representations. In the antiholonomy case the target representation is replaced by its algebraic complex conjugate; the local scalar transport is specified below.

6. Full-group comparison of canonical local systems

The point of this section is that no passage to a characteristic Frattini cover is needed. Write $\mathrm{Fratt}_2(H)$ for the kernel of $H \to H_1(H; \mathbf{F}_2)$.

Lemma 6.1 (Quadratic ambiguity disappears)**.** *Let K be a characteristic-zero field, let H be finitely generated, and let $\rho, \rho' : H \to \mathrm{SL}_2(K)$ be Zariski dense. If their characters agree on $\mathrm{Fratt}_2(H)$, then, over a splitting field, there are a character $\xi : H \to \{1, -1\}$ and $A \in \mathrm{GL}_2$ such that*

$$\rho'(h) = A\,\xi(h)\rho(h)A^{-1}.$$

Consequently $\mathrm{Sym}^{2r} \rho'$ and $\mathrm{Sym}^{2r} \rho$ are conjugate on all of H for every $r \geq 1$.

Proof. The first assertion is Liu's quadratic-ambiguity lemma [28, Lemma 8.10]. Applying Sym^{2r} kills the scalar $\xi(h)$ because $\xi(h)^{2r} = 1$. □

Fix $r \geq 1$, a finite quotient q occurring below, and a representation η of that quotient. Choose, for these data, a number field E which is stable under complex conjugation and contains the entries of the two lifts and the character values and a splitting field for η. Let $\mathfrak{p}$ be a place of the common invariant trace field at which the two restricted representations in [28, Lemma 9.1] are bounded, and let v be a place of E above $\mathfrak{p}$. These are exactly the places to which Liu's local character comparison applies; all but finitely many places are available for any fixed finite collection of algebraic data. Put

$$V_{X,r} = \mathrm{Sym}^{2r} \rho_X.$$

In the antiholonomy case, complex conjugation carries v to a place $\bar{v}$ and gives an isomorphism $E_v \to E_{\bar{v}}$. We write $V^{\dagger}_{B,r}$ for $V_{B,r}$ in the holonomy case and for the scalar transport of the conjugate representation from $E_{\bar{v}}$ to E_v in the antiholonomy case. Thus no automorphism of one fixed completion is being assumed. A bounded E_v–representation is conjugate into $\mathrm{GL}_n(\mathcal{O}_v)$; its reductions modulo $\mathfrak{m}_v^j$ have finite image and therefore extend uniquely to the profinite completion. Their inverse limit is the associated continuous E_v–representation of the completion. We denote this extension of $V_{X,r}$ by $\widehat{V}_{X,r}$, and use the same convention for $V^{\dagger}_{B,r}$ after scalar extension.

Proposition 6.2 (Full-group local-system comparison)**.** *For every v just specified, after a finite splitting extension of E_v, there is a conjugacy of continuous representations*

$$\widehat{V}_{A,r} \cong \widehat{V}^{\dagger}_{B,r} \circ \Psi \quad \text{on } \widehat{\Gamma}_A.$$

If $q_B : \widehat{\Gamma}_B \to Q$ is finite, $q_A = q_B \Psi$, and $\eta : Q \to \mathrm{GL}(W)$ is realized over E, tensoring gives

$$\widehat{V}_{A,r} \otimes (\eta q_A) \cong (\widehat{V}^{\dagger}_{B,r} \otimes (\eta q_B)) \circ \Psi.$$

More generally, on any pair of corresponding open subgroups, q_B may be an arbitrary finite quotient of the target open subgroup and $q_A = q_B\Psi$ on the source. For fixed (r, q_B, η, v) one conjugacy restricts to all further corresponding open subgroups.

Proof. Liu's local comparison gives equality of the continuous local characters on the completed 2–Frattini subgroups [28, Lemma 9.1], after the scalar transport just described. Their images are Zariski dense, so the restrictions are absolutely irreducible. For absolutely irreducible representations in characteristic zero, equality of characters implies conjugacy: Burnside's theorem makes the image span the full matrix algebra, and the nondegenerate trace pairing constructs the intertwiner (equivalently, use [14, Proposition 1.5.2]).

Apply Lemma 6.1 on the discrete full group. It says that the only ambiguity extending this conjugacy from the 2–Frattini subgroup is a quadratic character. That character has finite image and extends continuously to the completion. Every Sym^{2r} kills it, so the conjugacy holds on the full completed group. Tensoring with the literally corresponding finite representation gives the second display. Restriction proves the open-subgroup and further-cover statements. If one starts with a finite quotient of an open subgroup instead, the same argument is applied there after choosing its own field E and place v. □

Lemma 6.3 (Continuous inner automorphisms)**.** *Let P be a profinite group, let L be a finite continuous P–module, and let $p \in P$. The pair consisting of $\mathrm{Int}(p)$ on P and the coefficient transport $x \mapsto px$ on L induces the identity on $H^*_{\mathrm{cts}}(P; L)$. The identity is natural in L. Consequently it remains true for an inverse limit of finite modules whenever continuous cohomology is computed by that inverse limit.*

Proof. Use the continuous homogeneous bar resolution. The group automorphism and coefficient transport give the map induced on the nerve of the one-object topological category P by the inner automorphism functor. Multiplication by p is a natural transformation from that functor to the identity. The standard nerve homotopy inserts this natural-transformation arrow in each of the finitely many positions of an n–simplex, with alternating signs. It is the usual bar homotopy proving that inner automorphisms act trivially in group cohomology. Every map in the formula is a finite composition of multiplication and coordinate insertion in P, hence is continuous. The formula is functorial in L and therefore commutes with all transition maps; passing to inverse limits proves the last assertion. □

Choose invariant $\mathcal{O}_v$–lattices $\mathcal{V}_{X,r}$, taking the target lattice to be the image of the source lattice under Proposition 6.2. After enlarging E, finite-image representations are unitary and integral.

Proposition 6.4 (Cohomology and monodromy comparison)**.** *Let Γ'_A, Γ'_B be corresponding finite-index mapping-torus subgroups and $\Pi'_X = \Gamma'_X \cap \Pi_X$. For fixed (r, q_B, η, v) as above and every i, there is an isomorphism*

$$H^i(\Pi'_A; V_{A,r} \otimes \eta q_A) \cong H^i(\Pi'_B; V^\dagger_{B,r} \otimes \eta q_B)$$

of finite-dimensional E_v–spaces. It intertwines the maps F_A, F_B induced by the normalized primitive generators of the cyclic quotients. It is compatible with corresponding further covers. An arbitrary finite quotient of either open subgroup is allowed after making the field-and-place choice for that quotient.

Proof. Let $\mathcal{L}_X$ be the indicated integral tensor-product lattice and set $L_{X,n} = \mathcal{L}_X/\mathfrak{m}_v^n\mathcal{L}_X$. These are finite modules. Surface groups are good (apply the 3–manifold goodness cited in the proof of [49, Theorem 6.21] to $S \times I$), so naturally in the group and module,

$$H^i(\Pi'_X; L_{X,n}) \cong H^i_{\mathrm{cts}}(\widehat{\Pi}'_X; L_{X,n}). \tag{5.1}$$

Proposition 6.2 identifies the two right sides.

Choose primitive stable letters t'_X. The completed fibration relation gives

$$\Psi(t'_A) = pt'_B, \qquad p \in \widehat{\Pi}'_B.$$

The two actions on the right side of (5.1) therefore differ by $\mathrm{Int}(p)$ together with its coefficient transport. Lemma 6.3 removes that factor and proves, for every n, the claimed intertwining with F_B. All maps are natural in n and under further restriction.

It remains only to pass from finite coefficients to E_v. Let $C^\bullet_X$ be the cellular cochain complex of the finite surface $K(\Pi'_X, 1)$ with coefficients in $\mathcal{L}_X$. It is bounded and its terms are finite free $\mathcal{O}_v$–modules, with $C^\bullet_X/\mathfrak{m}_v^n = C^\bullet(\Pi'_X; L_{X,n})$. The Milnor sequence is

$$0 \to \varprojlim{}^1 H^{i-1}(C^\bullet_X/\mathfrak{m}_v^n) \to H^i(C^\bullet_X) \to \varprojlim H^i(C^\bullet_X/\mathfrak{m}_v^n) \to 0.$$

For each target level n, the images of all later cohomology groups form a descending chain of subgroups of the finite group $H^{i-1}(C^\bullet_X/\mathfrak{m}_v^n)$, so they stabilize. Thus the system is Mittag–Leffler and the $\varprojlim^1$ term vanishes. Taking inverse limits in the compatible finite-level isomorphisms and then tensoring with E_v proves the proposition. Because the original differentials and monodromy maps are defined over E, equality of their characteristic polynomials after the embedding in E_v is an equality over E (or its algebraic conjugate), which is the comparison used for complex torsion below. □

7. Mapping-cone torsion in all finite covers

We use cellular cochains. Cell lifts and bases of the coefficient module give geometric bases. A change of coefficient basis contributes $|\det A|^{\chi(M)} = 1$. Changing cell lifts contributes determinants of group elements; these have modulus one because the canonical representations are unimodular and finite-image representations are unitary. Thus the absolute torsion is independent of these choices.

For a cochain endomorphism f of $C^\bullet$, our shifted cone is $T^i = C^i \oplus C^{i-1}$ with

$$D(x, y) = (dx, fx - dy).$$

We use the cohomological torsion convention in which a two-term acyclic complex $0 \to C^0 \xrightarrow{d} C^1 \to 0$ has torsion $\det(d)^{-1}$. This is the cohomological parity convention of [4, Appendix A.4]; consequently no reciprocal is inserted when the Ruelle–torsion formula from that paper is used below.

Lemma 7.1 (Mapping-cone determinant)**.** *Let $C^\bullet$ be a finite based cochain complex over a characteristic-zero field and $F : C^\bullet \to C^\bullet$ a based cochain equivalence. If every $1 - H^i(F)$ is invertible, then* $\mathrm{Cone}(1 - F)$ *is acyclic and*

$$\tau(\mathrm{Cone}(1 - F)) = \prod_i \det(1 - H^i(F))^{(-1)^{i+1}}$$

up to the conventional sign.

Proof. Split each cochain group into boundaries, cohomology representatives, and a complement mapping isomorphically to the next boundary group. Boundary and complement blocks cancel in adjacent degrees. The uncancelled blocks are $1 - H^i(F)$ with the displayed parity. □

For later basis control, choose a cellular representative of the monodromy after a common subdivision. Give its mapping torus the cells $e \times \{0\}$ and the open prisms $e \times (0, 1)$, one of each type for every cell e of the fiber. With compatible lifts, the horizontal and prism cochains in degree i identify respectively with $C^i(S;V)$ and $C^{i-1}(S;V)$, and the cellular differential is, up to the displayed cone sign convention,

$$(x,y) \longmapsto (dx, (1-F)x - dy).$$

Thus the identification with $\mathrm{Cone}(1-F)$ is based: its only basis changes are permutations, signs, and changes of cell lifts. The last contribute coefficient matrices of determinant of modulus one. A different cellular representative is related by the simple cellular moves for a mapping cylinder, so gives the same absolute torsion. This proves, with geometric bases, the mapping-torus cone assertion used below.

Lemma 7.2 (Acyclicity of finite twists)**.** *Let $q : \Gamma \to Q$ be finite and let $\eta \in \mathrm{Irr}(Q)$. For every $r \geq 1$, $V_r \otimes \eta q$ is acyclic on a closed hyperbolic 3–manifold.*

Proof. The odd-dimensional canonical systems $V_r = \mathrm{Sym}^{2r} \rho$ are acyclic on every closed hyperbolic 3–manifold and every finite cover by the Raghunathan vanishing theorem; see [33, Corollary 0.2]. The regular representation is

$$\mathbf{C}[Q] = \bigoplus_{\eta \in \mathrm{Irr}(Q)} (\dim \eta)\eta.$$

Shapiro identifies cohomology with coefficients $V_r \otimes \mathbf{C}[Q]$ with canonical cohomology in the cover associated to $\ker q$, which vanishes. Every irreducible direct summand is therefore acyclic. □

Theorem 7.3 (Twisted canonical torsion comparison)**.** *For every corresponding pair of finite covers with groups Γ'_X, every $r \geq 1$, every corresponding finite quotient $q_X : \Gamma'_X \to Q$, and every $\eta \in \mathrm{Irr}(Q)$,*

$$\left|\tau(M'_A; V_{A,r} \otimes \eta q_A)\right| = \left|\tau(M'_B; V_{B,r} \otimes \eta q_B)\right|.$$

Antiholonomy complex-conjugates one torsion. The assertion includes the target-fibration normalization above and is compatible with further corresponding covers.

Proof. Lemma 6.4 identifies the fiber cohomology and intertwines F_A with F_B. Lemma 7.2 and the mapping-torus cone sequence show that every $1-H^i(F_X)$ is invertible. Lemma 7.1, together with the algebraic characteristic-polynomial comparison at the end of Proposition 6.4, gives equality in a common coefficient field for compatible bases. On returning to geometric bases, the only extra factors are signs and determinants of finite-image coefficient matrices, hence roots of unity. Their complex absolute values are one, which gives the display. Acyclicity is Lemma 7.2; the preceding geometric-basis convention removes all choice dependence in absolute value. Antiholonomy changes every algebraic value by complex conjugation. All constructions came from one full-group conjugacy, hence remain compatible in further covers. □

8. Finite-character marked spectral inversion

We adopt exactly the prime convention of [4, Section 5]. Thus $\mathcal{P}(M) = \mathrm{PC}(M)$ is the set of primitive conjugacy classes in $\Gamma = \pi_1 M$, equivalently prime *oriented* closed geodesics. Both $[\gamma]$ and $[\gamma^{-1}]$ occur; reversing orientation is an involution of this indexing set. We use the complex-length convention

$$\lambda(c) = \ell(c) + i\theta(c), \qquad \ell(c) > 0, \qquad z(c) = e^{-\lambda(c)}.$$

The rotation angle is taken modulo 2π with the oriented axis, and

$$z(c^{-1}) = \overline{z(c)}.$$

For a finite group Q, let $\mathrm{Conj}(Q)$ denote its finite set of conjugacy classes. The involution relevant to absolute torsion is

$$\iota(z, [g]) = (\bar{z}, [g^{-1}]).$$

For a finite quotient $q : \Gamma \to Q$, define the primitive marked measure

$$\mu_q = \sum_{c \in \mathcal{P}(M)} \delta_{(z(c), [q(c)])}$$

on the punctured unit disk times $\mathrm{Conj}(Q)$. Orientation reversal is a bijection of $\mathcal{P}(M)$, so

$$\iota_* \mu_q = \mu_q. \tag{6.1}$$

The measure is locally finite and has discrete support. For $N > 2$ define the explicit complex weighted measure $\mu_q^{(N)}$ by

$$d\mu_q^{(N)}(z, [g]) = z^N \, d\mu_q(z, [g]).$$

The prime-geodesic counting estimate $\#\{c \in \mathcal{P}(M) : \ell(c) \leq L\} = O(e^{2L}/L)$, in the form recorded in [4, (20)], implies by summation over unit length intervals that

$$\sum_{c \in \mathcal{P}(M)} |z(c)|^N = \sum_{c \in \mathcal{P}(M)} e^{-N\ell(c)} < \infty \qquad (N > 2).$$

Thus $\mu_q^{(N)}$ has finite total variation.

If $\eta : Q \to U(W)$ is an actual representation, define

$$R_{\eta,k}(M, q) = \prod_{c \in \mathcal{P}(M)} \det(1 - \eta(q(c)) z(c)^k), \qquad k > 2. \tag{6.2}$$

It depends only on the character of η and is multiplicative under direct sums. Since every factor is nonzero, for a virtual character $\chi = [\eta_+] - [\eta_-]$ we may therefore define unambiguously

$$\log |R_{\chi,k}| = \log |R_{\eta_+,k}| - \log |R_{\eta_-,k}|.$$

Proposition 8.1 (Based Shapiro–Artin formalism)**.** *Let $H \leq Q$, let $\xi : H \to \mathbf{C}^\times$ be a one-dimensional unitary character, put $\Gamma_H = q^{-1}(H)$, define $q_H = q|_{\Gamma_H} : \Gamma_H \to H$, and let $M_H \to M$ be the corresponding possibly nonnormal cover. With geometric cellular bases,*

$$\Big|\tau\big(M; V_r \otimes (\mathrm{Ind}_H^Q \xi) q\big)\Big| = \big|\tau\big(M_H; V_r|_{\Gamma_H} \otimes \xi q_H\big)\big|, \tag{6.3}$$

$$R_{\mathrm{Ind}_H^Q \xi, k}(M, q) = R_{\xi,k}(M_H, q_H). \tag{6.4}$$

Both identities are multiplicative for direct sums and hence additive after taking logarithms for virtual differences.

Proof. Choose right coset representatives for $H \backslash Q$ and compatible lifts of every cell of M to M_H and to the universal cover. The usual Shapiro map is then an isomorphism of based cellular complexes

$$C^\bullet\big(M; V_r \otimes \mathrm{Ind}_H^Q \xi\big) \longrightarrow C^\bullet\big(M_H; V_r|_{\Gamma_H} \otimes \xi\big).$$

Changing coset representatives only permutes basis vectors and multiplies them by matrices from the canonical unimodular system and the unitary line system. Its determinant therefore has modulus one. This proves the based torsion identity (6.3), not merely cohomological Shapiro.

For (6.4), fix a primitive $\gamma \in \Gamma$. The permutation of $H\backslash Q$ induced by $q(\gamma)$ decomposes into cycles. If a cycle has length f and is represented by Hx, minimality of f gives $h = xq(\gamma)^f x^{-1} \in H$, and the corresponding cycle block $T_{\gamma,Hx}$ of the induced matrix satisfies

$$\det(1 - uT_{\gamma,Hx}) = 1 - \xi(h)u^f.$$

We spell out why these are exactly the primitive lifts, since the cover need not be normal. Lift x to $g \in \Gamma$. The cycle gives

$$\delta = g\gamma^f g^{-1} \in \Gamma_H.$$

The centralizer of a nontrivial element of the torsion-free closed hyperbolic 3–manifold group Γ is its maximal cyclic subgroup. Indeed, a nontrivial element has a unique invariant geodesic axis. Its centralizer preserves that axis; the kernel of the signed-translation homomorphism is a discrete subgroup of the compact rotation group about the axis, hence is finite and therefore trivial, while the image is a discrete subgroup of $\mathbf{R}$. Thus the centralizer is infinite cyclic. Since γ is primitive, if $\delta = \beta^m$ in Γ_H with $m > 1$, then $\beta = (g\gamma g^{-1})^a$ and $f = am$. But $\beta \in \Gamma_H$ would make the coset Hx return after $a < f$ steps, contradicting the minimality of f. Thus the lift is primitive. Conversely, every primitive oriented geodesic in M_H projects to an iterate of a unique primitive oriented geodesic in M. Writing its element as $g\gamma^f g^{-1}$, primitivity in Γ_H forces f to be the least positive return time of Hx under $q(\gamma)$; otherwise a smaller returning power would be a proper root in Γ_H. Hence the cycles are in bijection with the primitive lifts, and their complex-length parameters are $z(\gamma)^f$. Multiplying over cycles and then over all primitive conjugacy classes proves (6.4). Direct-sum multiplicativity is immediate from block determinants and from torsion multiplicativity. □

Proposition 8.2 (Finite-character Ruelle values)**.** *For every actual representation η of Q, of degree d, the absolute torsions*

$$\{|\tau(M; V_r \otimes \zeta q)| : r \geq 1,\ \zeta \in \mathrm{Irr}(Q)\}$$

and the volume determine, for all sufficiently large k, the number

$$\log |R_{\eta,k}|.$$

The determination is additive and hence extends to every virtual character.

Proof. For a finite-order unitary line character ξ, the precise closed formula is [4, Theorem 7.1]: for $m \geq 3$,

$$\log \left| \frac{\tau(M; V_m \otimes \xi)}{\tau(M; V_2 \otimes \xi)} \right| = \sum_{k=3}^{m} \log |R_{\xi,k}| - \frac{\mathrm{Vol}(M)}{\pi}(m-2)(m+3). \tag{R}$$

Here the paper's ρ_{2m+1} is our $V_m = \mathrm{Sym}^{2m} \rho$, and its

$$R_{\xi,-2k}(k) = \prod_{c \in \mathrm{PC}(M)} (1 - \xi(c)e^{-k\lambda(c)})$$

is exactly (6.2). Finite-order characters are "rational" in the sense of [4, Definition 3.1], so the stated hypothesis applies. This also verifies that both orientations are already present in the product.

Now let η have degree d. Brauer induction writes its character in the representation ring as an integral combination

$$[\eta] = \sum_j n_j [\mathrm{Ind}_{H_j}^{Q} \xi_j]$$

of finite-order line characters of subgroups [44, Chapter 10]. Apply the line formula in each cover M_{H_j} and use Proposition 8.1. Since $\mathrm{Vol}(M_{H_j}) = [Q : H_j]\,\mathrm{Vol}(M)$ and $\dim \mathrm{Ind}_{H_j}^{Q} \xi_j = [Q : H_j]$, additivity gives the exact general formula

$$\log\left|\frac{\tau(M; V_m \otimes \eta q)}{\tau(M; V_2 \otimes \eta q)}\right| = \sum_{k=3}^{m} \log|R_{\eta,k}| - \frac{d\,\mathrm{Vol}(M)}{\pi}(m-2)(m+3). \qquad (R_\eta)$$

Thus the character degree in the volume term is explicit. No root is chosen: torsion moduli are positive and Brauer coefficients are integers.

Subtracting (R_η) for $m-1$ from that for m gives

$$\log|R_{\eta,m}| = \log\left|\frac{\tau(M; V_m \otimes \eta q)}{\tau(M; V_{m-1} \otimes \eta q)}\right| + \frac{2md}{\pi}\,\mathrm{Vol}(M),$$

so all $m \geq 3$ values are determined. This is the line-twisted counterpart of the unmarked formula [31, Theorem 7.9]. □

For a virtual character χ, write $(\psi_m\chi)(g) = \chi(g^m)$ for its mth Adams operation. The standard Adams operation preserves the integral representation ring (equivalently, apply Brauer's generalized-character criterion on cyclic subgroups), so this is again a virtual character [44, Chapter 10]. Put

$$\nu_q = \mu_q + \iota_*\mu_q = 2\mu_q, \qquad (6.5)$$

where the last equality is the oriented-prime convention (6.1). For $n > 2$ define the character-valued moments

$$M_n(\chi) = \int z^n \chi(g)\, d\nu_q(z, [g]).$$

Lemma 8.3 (Adams–Möbius removal of proper powers)**.** *Put $L_n(\chi) = -2\log|R_{\chi,n}|$. Then*

$$L_n(\chi) = \sum_{m\geq 1} \frac{1}{m} M_{nm}(\psi_m\chi), \qquad (*)$$

and

$$M_n(\chi) = \sum_{m\geq 1} \frac{\mu(m)}{m} L_{nm}(\psi_m\chi), \qquad (**)$$

where μ is the number-theoretic Möbius function. Both series converge absolutely for $n > 2$.

Proof. Expanding $\log\det(1 - Az^n)$ gives

$$\log\det(1 - Az^n) = -\sum_{m\geq 1} \frac{\mathrm{tr}(A^m) z^{nm}}{m}.$$

For a unitary finite-group representation, complex conjugation replaces $\mathrm{tr}(\eta(g)^m)z^{nm}$ by $\mathrm{tr}(\eta(g^{-m}))\bar{z}^{nm}$. Adding this conjugate is integration against $\mu_q + \iota_*\mu_q = \nu_q$. Since $\log|w| = \frac{1}{2}(\log w + \overline{\log w})$, the factor is exactly -2 in $L_n = -2\log|R_{\chi,n}|$. This proves $(*)$ and verifies the orientation normalization against the product of [4, Section 5].

Substitute $(*)$ into the right side of $(**)$. The coefficient of $M_{nr}(\psi_r\chi)$ is

$$\sum_{m|r} \frac{\mu(m)}{m}\frac{1}{r/m} = \frac{1}{r}\sum_{m|r}\mu(m),$$

which is 1 for $r = 1$ and 0 otherwise. The prime-geodesic bound and $|z(c)| < 1$ justify rearranging the absolutely convergent series. $\square$

Theorem 8.4 (Finite-character marked inverse theorem)**.** *For a finite quotient $q : \Gamma \to Q$, the family of absolute canonical torsions for all even symmetric powers and all irreducible complex representations of Q determines the symmetrized measure*

$$\nu_q = \mu_q + \iota_*\mu_q.$$

Equivalently, it determines with multiplicity the measure on the orbit space for the single simultaneous involution

$$(z, [g]) \longleftrightarrow (\bar{z}, [g^{-1}]).$$

With the oriented-prime convention, $\nu_q = 2\mu_q$. The determination is functorial when Q is replaced by a quotient, a product of finite quotients, or the quotient induced in a finite cover.

Proof. The volume is itself recovered from the quadratic asymptotics of the untwisted canonical torsions [31, Theorem 7.1]; in the comparison application it is already equal by Corollary 4.4. Proposition 8.2 supplies $L_n(\chi)$ for every virtual character and all $n \geq N$, for some $N > 2$. Lemma 8.3 therefore supplies $M_n(\chi)$ for every $n \geq N$. Irreducible complex characters form a basis of the class functions on Q, so these are all moments of every conjugacy-class coordinate of ν_q.

Suppose two marked measures give the same data. For each conjugacy class $C \subset Q$, let

$$\nu_{q,C}(A) = \nu_q(A \times \{C\})$$

be the corresponding complex-plane coordinate. The indicator of C is a complex linear combination of irreducible characters, so all moments $\int z^n \, d\nu_{q,C}$ for $n \geq N$ are known. If σ_C is the difference of the two coordinates, define a finite complex measure ω_C by

$$d\omega_C(z) = z^N \, d\sigma_C(z),$$

and give it no atom at 0. Then for every $j \geq 0$,

$$\int z^j \, d\omega_C(z) = \int z^{N+j} \, d\sigma_C(z) = 0.$$

The closure of the support together with 0 is compact: only finitely many prime geodesics have length below a fixed bound, so 0 is the only possible new accumulation point. It is countable, hence has planar measure zero. Its complement is connected: by the classical planar separator theorem, a compact set that separates the plane has a nondegenerate continuum component, whereas a connected metric space containing two points is uncountable (apply the distance from one of the points). A countable compact set therefore has no such component; see also [48]. The Cauchy-transform uniqueness theorem [31, Propositions 8.4–8.6], applied to the complex Radon measure ω_C, gives $\omega_C = 0$. Multiplication by z^N is nonzero on the punctured disk and σ_C has no atom at zero, so $\sigma_C = 0$. This holds for every C, proving equality of the symmetrized marked measures.

Every operation in the proof is natural in the finite quotient. Applying the theorem to $Q_1 \times \cdots \times Q_s$ gives the simultaneous marking needed for any finite collection of quotients. On passing to a finite cover, the group is replaced by the corresponding open subgroup and

q may be any finite quotient of that subgroup, as in Proposition 6.2. Corresponding open subgroups have the same index, so the volume term in (R) is compatible as well. □

Remark 8.5. The theorem refines Liu's rational-character and table-of-marks framework: rational permutation characters naturally retain the cyclic subgroup generated by $q(c)$, while all complex irreducible characters retain the element conjugacy class. The inverse/conjugation pairing cannot be removed from absolute torsion data.

9. Exact ambient descent

Theorem 9.1 (Exact ambient descent)**.** *For every primitive $a \in \Gamma_A$ there are a primitive $b \in \Gamma_B$, a sign $\sigma \in \{1, -1\}$, and $u \in \widehat{\Gamma}_B$ such that*

$$\Psi(a) = ub^{\sigma}u^{-1}.$$

If $a \in \Pi_A$, then $b \in \Pi_B$. The assertion is compatible with corresponding finite covers and also holds for Ψ^{-1}.

Proof. By Theorems 7.3 and 8.4, every finite quotient q of Γ_B marks the image of the primitive geodesic represented by a. The set D of primitive geodesics in M_B having its complex length, allowing complex conjugation, is finite because the length spectrum is locally finite. Thus for every q there are $b_q \in D$ and $\sigma_q \in \{1, -1\}$ such that $q(\Psi(a))$ is conjugate to $q(b_q^{\sigma_q})$.

Apply the marked theorem to the product of any finite collection of quotients. The corresponding nonempty subsets of the finite set $D \times \{1, -1\}$ have the finite-intersection property. Fix one pair (b, σ) that works in every finite quotient. For each open normal $U \triangleleft \widehat{\Gamma}_B$, let

$$C_U = \{u \in \widehat{\Gamma}_B : \Psi(a)^{-1}ub^{\sigma}u^{-1} \in U\}.$$

These are nonempty closed subsets of the compact group $\widehat{\Gamma}_B$. Product quotients again give the finite-intersection property, so their total intersection is nonempty. An element of that intersection is the required conjugator.

If $a \in \Pi_A$, apply the completed fibration ν_B. Conjugacy does not change its value and $\nu_B\Psi(a) = 0$. Since b is discrete, $\nu_B(b)$ is an ordinary integer whose image in $\widehat{\mathbf{Z}}$ is zero; hence it is zero and $b \in \Pi_B$. All analytic, product-quotient, and compactness arguments commute with restriction to corresponding open subgroups. The same proof applies to Ψ^{-1}. □

10. A quasiconvex transporter theorem

For $a \in G$ and $H \leq G$, put

$$\mathrm{Tran}_G(a, H) = \{x \in G : xax^{-1} \in H\}.$$

Theorem 10.1 (Quasiconvex transporter)**.** *If G is torsion-free, hyperbolic, and virtually compact special, $H \leq G$ is quasiconvex, and $a \in G$ has infinite order, then*

$$\mathrm{Tran}_{\widehat{G}}(a, \overline{H}) = \overline{\mathrm{Tran}_G(a, H)}.$$

Proof. Suppose $xax^{-1} \in \overline{H}$. Quasiconvex subgroup-into-conjugacy separability, in the form of [5, Corollary 6.4], first gives $g \in G$ with $gag^{-1} \in H$. Indeed, if no conjugate of $\langle a \rangle$ lay in H, that theorem would separate the two subgroups in a finite quotient, contrary to $x\overline{\langle a \rangle}x^{-1} \leq \overline{H}$. After replacing a by this conjugate and x accordingly, assume $a \in H$.

The canonical completion and retraction construction in the proof of [21, Theorem 7.3] gives a finite-index subgroup $U \leq G$ containing H and a retraction $r : U \to H$. Write $x = uk$, where $u \in \widehat{U}$ and k belongs to a fixed finite set of right-coset representatives. Put $b = kak^{-1}$. Since $ubu^{-1} \in \overline{H} \subset \widehat{U}$, we have $b \in U$.

Applying the completed retraction to that equality gives

$$ubu^{-1} = r(u)r(b)r(u)^{-1}.$$

Thus $c = r(u)^{-1}u$ conjugates b to $r(b)$. The group U is hereditarily conjugacy separable by [37, Theorem 1.1]; hence there is $y \in U$ with $yby^{-1} = r(b)$. It follows that

$$y^{-1}r(u)^{-1}u \in C_{\widehat{U}}(b).$$

Minasyan's profinite centralizer criterion [34, Corollary 12.3] gives $C_{\widehat{U}}(b) = \overline{C_U(b)}$. Therefore

$$u \in \overline{H\, y\, C_U(b)}.$$

Every element $hyzk$, with $h \in H$ and $z \in C_U(b)$, transports a into H. Hence $x = uk$ is a limit of discrete transporters. The reverse inclusion follows from continuity. □

Let d be simple in a closed fiber S, and let H be the group of a component of $S \setminus d$. It is a proper finitely generated subgroup of the fiber group. It is the group of a compact surface with boundary and is therefore free, so it cannot be commensurable with the closed surface fiber. The tameness and covering theorems imply that it is geometrically finite rather than a virtual fiber; hence it is quasiconvex in the mapping-torus group [13, Theorem 2.1]. If a simple $b \in \Pi$ is discretely transported into H by an element of time n, then $\varphi^n(b)$ can be represented in that component and

$$d_{\mathcal{C}(S)}(d, \varphi^n(b)) \leq 1.$$

The pseudo-Anosov action on the curve graph is loxodromic [32, Section 4], so only finitely many integers n have this property.

Corollary 10.2 (Integral time)**.** *If $u \in \widehat{\Gamma}$, $b \in \Pi$ is simple, and $ubu^{-1} \in \overline{H}$, then the image of u in $\widehat{\mathbf{Z}}$ is an ordinary integer.*

Proof. Theorem 10.1 says that u lies in the closure of the discrete transporter. The image of that transporter in $\mathbf{Z}$ is the finite set just described. A finite subset of $\mathbf{Z}$ is closed in $\widehat{\mathbf{Z}}$. □

11. The profinite cut class

Let S be a closed oriented surface, $\Pi = \pi_1(S)$, and $G = \widehat{\Pi}$. Let c be a primitive simple curve and $C = \overline{\langle c\rangle} \leq G$. The surface splitting along c is efficient: the surface group is residually finite, and the cyclic edge group and the compact-subsurface vertex groups are virtual retracts by Scott's almost-geometric theorem [43], hence closed and endowed with the full induced profinite topology. Therefore [39, Exercise 9.2.7] identifies the completion with the profinite fundamental group of the completed graph of groups and identifies its standard profinite Bass–Serre tree T_c.

Let T be a minimal profinite G–tree, without inversion, with one edge orbit and edge stabilizer C. Work over $k = \mathbf{F}_2$. Its augmented profinite chain complex is exact:

$$0 \longrightarrow k[[E(T)]] \xrightarrow{\partial} k[[V(T)]] \xrightarrow{\varepsilon} k \longrightarrow 0.$$

Identify $P_C = k[[E(T)]]$ with $k[[G/C]]$ after choosing a distinguished edge.

Fix $v \in V(T)$. For $g \in G$, define $z_{T,v}(g)$ by

$$\partial z_{T,v}(g) = gv - v.$$

Lemma 11.1 (Cut class)**.** *The map $z_{T,v} : G \to P_C$ is a continuous 1–cocycle. Its class*

$$\kappa(T) = [z_{T,v}] \in H^1_{\mathrm{cts}}(G, k[[G/C]])$$

is independent of v and is natural under equivariant isomorphisms and isomorphisms of the pair (G, C).

Proof. The two chains $z(gh)$ and $z(g) + gz(h)$ have the same boundary, so injectivity of ∂ makes them equal. The map $g \mapsto gv - v$ is continuous. The boundary is a continuous injection between compact Hausdorff profinite modules, hence a homeomorphism onto its image; its inverse proves continuity of z. If w is another vertex and $\partial m = w - v$, then

$$z_{T,w}(g) - z_{T,v}(g) = gm - m.$$

This proves basepoint independence and naturality. □

We spell out the coefficient category. Let $\Lambda = k[[G]]$. We work in the category $\mathsf{PMod}(\Lambda)$ of compact Hausdorff profinite left Λ-modules and continuous maps. Completed tensor products are denoted $\widehat{\otimes}$; continuous Hom is used throughout. Because the resolution below is finite free, continuous cohomology is computed by its continuous-Hom complex; this is the projective-resolution description in [6, Proposition 6.7 and Sections 7–8]. Right modules are converted to left modules by the involution $g \mapsto g^{-1}$.

Proposition 11.2 (One-dimensionality)**.** *There is a natural isomorphism*

$$H^1_{\mathrm{cts}}(G, k[[G/C]]) \cong k.$$

Proof. Choose a one-vertex cell structure on S. The cellular complex of its universal cover is a length-two finite free resolution over $k\Pi$. Surface groups are good [19, Proposition 3.4] and of type FP_∞; the absolute case of [49, Proposition 6.6] therefore says that completion gives the exact finite-free resolution $P_\bullet = \Lambda\widehat{\otimes}_{k\Pi}C_\bullet$ of k.

Before completion, cellular cap product with the mod-2 fundamental class, including the side-changing involution, gives a chain homotopy equivalence

$$\mathrm{Hom}_{k\Pi}(C_\bullet, k\Pi) \simeq C_{2-\bullet}.$$

This is the cellular proof of Poincaré duality: dual cells give the chain map, and cellular subdivision and its inverse give the homotopies. Every module, map, and homotopy is finite free. Completing the maps and both homotopies therefore gives the chain homotopy equivalence

$$\mathrm{Hom}^{\mathrm{cts}}_{\Lambda}(P_\bullet, \Lambda) \simeq P_{2-\bullet}$$

actually needed here; no chain-level conclusion is being inferred from an abstract profinite duality theorem.

For a compact left Λ-module M, finite freeness gives, degree by degree and compatibly with differentials,

$$\mathrm{Hom}^{\mathrm{cts}}_{\Lambda}(P_\bullet, M) \cong \mathrm{Hom}^{\mathrm{cts}}_{\Lambda}(P_\bullet, \Lambda)\widehat{\otimes}_{\Lambda}M \simeq P_{2-\bullet}\widehat{\otimes}_{\Lambda}M.$$

Here $\mathrm{Hom}^{\mathrm{cts}}_{\Lambda}(P_\bullet, \Lambda)$ is a right Λ–complex by postmultiplication, with the involution $g \mapsto g^{-1}$ supplying the side change in the duality equivalence, and $\widehat{\otimes}_{\Lambda}$ is balanced between this right action and the given left action on M. Thus $H^i(G; M) \cong H_{2-i}(G; M)$. With

$$k[[G/C]] = \Lambda\widehat{\otimes}_{k[[C]]}k,$$

the completed homological Shapiro lemma [6, Theorem 7.11 and Corollary 7.12] gives

$$\begin{aligned} H^1_{\mathrm{cts}}(G, k[[G/C]]) &\cong H_1(G, k[[G/C]]) \\ &\cong H_1(C; k) \\ &\cong k. \end{aligned}$$

The last isomorphism uses $C \cong \widehat{\mathbf{Z}}$ and $C^{\mathrm{ab}} \widehat{\otimes}_{\widehat{\mathbf{Z}}} \mathbf{F}_2 \cong \mathbf{F}_2$. □

12. Fixed chains in the edge permutation module

If a profinite P–space X has a fixed basepoint x_0, let $k[[(X, x_0)]]$ be the quotient of $k[[X]]$ in which x_0 is zero.

Lemma 12.1 (Free-orbit fixed points)**.** *Let $P \cong \mathbf{Z}_2$ act continuously on a profinite space X. Suppose x_0 is its unique fixed point and P acts freely on $X \setminus \{x_0\}$. Then*

$$k[[(X, x_0)]]^P = 0, \qquad k[[X]]^P = kx_0.$$

Proof. Finite pointed P–equivariant quotients are cofinal among the finite quotients defining $k[[(X, x_0)]]$. At a finite level, an invariant vector is a sum of orbit sums. Fix a non-basepoint orbit O and let $K \leq P$ be the stabilizer of one of its points. The preimage Y of O is compact and omits x_0. Let a topologically generate the open copy $K \cong \mathbf{Z}_2$. Freeness makes the graph $\{(ay, y) : y \in Y\}$ disjoint from the diagonal. A finite clopen quotient separates these compact sets; refining by the finitely many translates of the partition makes it a finite P–quotient.

In that refinement a fixes no point over O. Every stabilizer above O is therefore a proper open subgroup of K, so every refined orbit has even index over O. Its orbit sum maps to zero over $\mathbf{F}_2$. Compatibility forces the coefficient of O in an invariant inverse-limit vector to be zero. This holds for every non-basepoint orbit at every level, proving the first assertion. Taking invariants in

$$0 \longrightarrow kx_0 \longrightarrow k[[X]] \longrightarrow k[[(X, x_0)]] \longrightarrow 0$$

gives the second. □

Lemma 12.2 (The only invariant edge chain)**.** *If $e = C \in G/C$, then*

$$k[[G/C]]^C = ke.$$

Proof. The subgroup $\langle c \rangle$ is maximal cyclic, quasiconvex, and malnormal in the surface group. It is separable and carries its full profinite topology, so $C \cong \widehat{\mathbf{Z}}$. Wilton–Zalesskii's profinite malnormality theorem [50, Theorem 3.3] says

$$C \cap gCg^{-1} \neq 1 \quad \Longrightarrow \quad g \in C.$$

Let $P \cong \mathbf{Z}_2$ be the pro-2 Sylow subgroup of C. Its stabilizer at gC is $P \cap gCg^{-1}$, which is P for $gC = C$ and trivial otherwise. Lemma 12.1 gives $k[[G/C]]^P = ke$. The C-invariants are contained in the P-invariants and e is fixed by C. □

Corollary 12.3 (Nonvanishing of the standard cut class)**.** *The cut class of the standard tree T_c is nonzero.*

Proof. Choose v to be one endpoint of the distinguished edge e. Suppose $z = z_{T_c,v}$ were a coboundary, $z(g) = gm - m$. Since z vanishes on C, Lemma 12.2 gives $m = 0$ or $m = e$. In the first case $z = 0$, so $gv = v$ for all g. In the second case, changing the basepoint to the other endpoint w gives $z_{T_c,w}(g) = z(g) + ge - e = 0$. Either endpoint would be globally fixed, contrary to minimality of the nontrivial standard tree. □

13. Fixed-edge reconstruction

Theorem 13.1 (Fixed-edge reconstruction)**.** *Let T, T' be reduced minimal profinite G–trees, acting without inversion, with one edge orbit and edge stabilizer C. Suppose their augmented k-chain complexes are exact and their cut classes are nonzero. Then T and T' are G–equivariantly isomorphic.*

Proof. Choose distinguished edges fixed by C and identify both edge spaces with G/C, taking the distinguished edge to $e = C$. Choose endpoints v, v' and write $z = z_{T,v}$ and $z' = z_{T',v'}$. Both cocycles vanish on C. Proposition 11.2 makes their nonzero classes equal, so

$$z'(g) - z(g) = gm - m$$

for some $m \in k[[G/C]]$. Restriction to C and Lemma 12.2 give $m = 0$ or e. If $m = e$, switch v to the other endpoint w; then $z_{T,w}(g) = z_{T,v}(g) + ge - e$. After this possible switch, $z' = z$.

We reconstruct the whole tree from the actual cocycle. Minimality and the absence of isolated vertices imply that the vertex space is the union of the two endpoint orbits of e. Map these orbits to the edge module by

$$gv \longmapsto z(g), \qquad gw \longmapsto z(g) + ge. \tag{1}$$

The maps are well defined and injective: applying the injective boundary map turns equality in the first or second coordinate into equality of the corresponding vertices. An equality between coordinates of different types is likewise equivalent to equality of those vertices. This also describes precisely the overlap in the HNN case, where the endpoint orbits coincide. Each map is continuous and descends from a compact coset space, so it is a homeomorphism onto its image.

In the coordinates (1), the edge gC joins

$$z(g) \quad \text{to} \quad z(g) + ge.$$

The vertex coordinates have affine action

$$a * x = z(a) + ax,$$

while G/C has its usual left action. Thus z and e reconstruct the vertex space, the edge space, both incidence maps, and the action. The reconstructions for T and T' coincide. The resulting continuous bijection is a homeomorphism because its source is compact and its target Hausdorff, and it is G–equivariant by the displayed action. □

Theorem 13.2 (Standard-edge rigidity, SER)**.** *Let $\beta : \widehat{\pi_1(S_A)} \to \widehat{\pi_1(S_B)}$ be an isomorphism. If, after an inner adjustment and choice of orientation, $\beta(c) = d$ for discrete primitive simple curves, then the transported standard tree $T_c \cdot \beta$ is equivariantly isomorphic to T_d.*

Proof. The isomorphism induces the semilinear identification $\mathbf{F}_2[[G_A/C_c]] \cong \mathbf{F}_2[[G_B/C_d]]$. The transported and target trees have edge stabilizer C_d; their chain complexes are exact and their cut classes are nonzero by Corollary 12.3 and naturality in Lemma 11.1. Apply Theorem 13.1. □

14. Propagation and synchronization

Lemma 14.1 (Propagation across one edge)**.** *Let c_0, c_1 be distinct disjoint simple curves on S_A. If $\alpha(c_0)$ is conjugate in $\widehat{\Pi}_B$ to a discrete simple curve d_0, then $\alpha(c_1)$ is conjugate in $\widehat{\Pi}_B$ to a discrete simple curve of the same topological type as c_1.*

Proof. Make a temporary inner adjustment so that $\alpha(c_0) = d_0$, and base c_1 in a component of $S_A \setminus c_0$. Standard-edge rigidity identifies the transported standard tree with T_{d_0}, so $\alpha(c_1)$ lies in the closure of the group H of the corresponding component of $S_B \setminus d_0$.

Theorem 9.1 gives

$$\alpha(c_1) = ub^\sigma u^{-1}, \qquad b \in \Pi_B, \quad u \in \widehat{\Gamma}_B.$$

Since $\widehat{\Pi}_B$ is normal, $\mathrm{Int}(u^{-1})\alpha$ remains an isomorphism of completed surface groups and sends the discrete simple curve c_1 to the discrete element b^σ. After identifying the equal genera, Wang–Xu's [46, Theorem 1.3] says that b is a simple curve of the same topological type. To apply the theorem in precisely its automorphism form, choose any discrete isomorphism between the equal-genus surface groups and compose its completed map with $\mathrm{Int}(u^{-1})\alpha$. Wang–Xu's finite-quotient invariant EpiIm is preserved in both directions, so the conclusion transfers back to b.

Corollary 10.2 gives $\nu_B(u) = n \in \mathbf{Z}$. Write $u = qt_B^n$ with $q \in \widehat{\Pi}_B$. Then

$$\alpha(c_1) = q\varphi_B^n(b^\sigma)q^{-1},$$

whose unoriented target is a discrete simple curve. Undo the temporary inner adjustment. □

Proof of Theorem 1.1. We first prove the closed fibered case. The completed first homology of a closed genus-g surface group is $\widehat{\mathbf{Z}}^{2g}$; hence S_A, S_B have the same genus.

Choose an Alexander-method marking $\mathcal{R}_0$ whose pointwise stabilizer in the extended mapping class group is the kernel of the action on the curve complex; such finite markings are supplied by the Alexander method [16, Proposition 2.8]. Add nonseparating curves representing a symplectic basis $e_1, \ldots, e_{2g}$ and primitive classes $e_1 + e_j$ for $2 \leq j \leq 2g$. Also add $\varphi_A(\mathcal{R}_0)$. By the finite rigid exhaustion of Aramayona–Leininger [2, Theorem 1.1], a finite rigid set $\mathcal{X}$ contains all these curves. Adjoin finitely many edge paths in the curve graph so that the disjointness graph of the resulting finite set $\mathcal{R}$ is connected. Rigidity will be used on $\mathcal{X} \subset \mathcal{R}$.

Theorem 9.1, followed by one ambient inner adjustment, gives an exact discrete anchor for one vertex of $\mathcal{R}$; the restriction to the normal completed fiber remains an isomorphism and preserves the outer monodromy relation. Wang–Xu recognizes the anchor as simple. Repeatedly apply Lemma 14.1 along the connected disjointness graph. For every $c \in \mathcal{R}$ there is a discrete simple unoriented curve $f(c)$ with

$$\alpha(c) \sim_{\widehat{\Pi}_B} f(c)^{\pm 1}.$$

The map f is injective. Closed surface groups are conjugacy separable, for example by [37, Theorem 1.1] since they are hyperbolic and virtually compact special. If two targets represented the same unoriented curve, their group representatives would be conjugate or inverse-conjugate; the same would hold for the two source curves in the completion. Conjugacy separability then makes the source unoriented curves equal. The profinite Reidemeister pairing is natural under α, and Boggi–Zalesskii's zero criterion [7, Theorem 4.3] says that its vanishing is equivalent to zero geometric intersection. Thus targets of disjoint sources are disjoint, and $f|_{\mathcal{X}}$ is a locally injective simplicial map. The rigidity assertion for the stages in that exhaustion [2, Theorem 1.1] gives a homeomorphism $h : S_A \to S_B$ inducing f on $\mathcal{X}$, unique up to the pointwise stabilizer.

The outer equivariance of α gives, for $c \in \mathcal{R}_0$,

$$h\varphi_A(c) = \varphi_B h(c)$$

as unoriented curves. Indeed both sides are discrete and conjugate in the completion, so surface-group conjugacy separability gives ordinary conjugacy. Consequently

$$k_0 = \varphi_B^{-1} \circ h \circ \varphi_A \circ h^{-1}$$

fixes $h(\mathcal{R}_0)$ pointwise. Notice that the marking here is $h(\mathcal{R}_0)$, not literally $\mathcal{R}_0$.

It remains to remove the genus-two kernel and all orientation signs. On completed first homology,

$$\alpha_*(e_i) = s_i h_*(e_i), \qquad \alpha_*(e_1 + e_j) = s_{1j} h_*(e_1 + e_j), \qquad s_i, s_{1j} \in \{1, -1\}.$$

Linearity and independence force all signs to equal one common sign s; hence $\alpha_* = sh_*$. The scalar cancels from the outer monodromy relation, so

$$h_* \varphi_{A*} h_*^{-1} = \varphi_{B*}.$$

Thus k_0 acts trivially on first homology. For genus at least three the curve-complex action is faithful. In genus two its kernel is generated by the hyperelliptic involution, which acts as -1 on first homology [23, 24, 29]. Therefore $k_0 = 1$ in every genus, and the normalized monodromies are conjugate. Undoing the possible reversal of the target fibration made in (5.0), the original monodromies are conjugate up to inversion. Their mapping tori are homeomorphic.

Proposition 2.2 now proves the closed case for all hyperbolic manifolds. Lemma 2.3 proves the cusped case. Finally Proposition 2.1 shows that the arbitrary compact competitor N in the theorem is one of these hyperbolic compact cores. □

Acknowledgements and attribution

The proof builds on Yi Liu's program relating profinite isomorphisms of hyperbolic 3–manifold groups to fibered classes, periodic trajectories, twisted torsions, and algebraic representations [27, 28]. The marked spectral reconstruction here is a complex-character refinement of that framework and uses the higher-torsion inverse theorem of Menal–Ferrer and Porti [31] and twisted Ruelle–torsion identities [17, 35, 47, 4]. The reduction to fibered lattices uses work of Agol [1], Hanany [18], Belolipetsky–Cheetham-West [3], Bridson–Reid [8], and Xu [52, 53]. The surface argument uses Wang–Xu's recognition of simple curves [46], the profinite Reidemeister pairing of Boggi–Zalesskii [7], and finite rigid sets of Aramayona–Leininger [2]. The cut-class construction is informed by the classical theory of ends, relative ends, almost-invariant sets, duality groups, splittings, and cubings developed by Houghton, Kropholler–Roller, Sageev, Scott–Swarup, and others [20, 25, 41, 42], together with the profinite cohomology and tree methods of Boggi–Corob Cook [6], Wilkes [49], and Wilton–Zalesskii [50].

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