Density gaps for reciprocal triangle groups

August 2026

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Abstract

Let $p, q, r \geq 2$ be a hyperbolic triple and let $\rho(p, q, r)$ be the proportion of its primitive conjugates which are spherical. In [1], it is proved that $\rho(p, q, r) = 0$ if and only if (p, q, r) is one of eleven explicit triples. In [2], it is proved that the infimum of the remaining values $\rho(p, q, r) > 0$ is equal to ρ_H for some constant $\rho_H > 0$. The main result of this paper is to show that $\rho_H = 1/12$.

1 Introduction

1.1 Triangles and their Galois conjugates

Let $p, q, r \geq 2$ and suppose that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

The group

$$\Delta(p, q, r) = \langle x, y \mid x^p = y^q = (xy)^r = 1 \rangle$$

has a distinguished discrete representation in $\mathrm{PSL}_2(\mathbf{R})$, obtained from a hyperbolic triangle with angles $\pi/p, \pi/q, \pi/r$. We shall be concerned with the real Galois conjugates of this representation.

McMullen's description of the moduli space of triangles gives a particularly simple way to see these conjugates [2]. Put

$$\mathbb{T}^3 = (\mathbf{R}/2\mathbf{Z})^3, \qquad \Lambda = \{(m_1, m_2, m_3) \in \mathbf{Z}^3 : m_1 + m_2 + m_3 \equiv 0 \pmod 2\}.$$

To a point $x = (x_1, x_2, x_3) \in \mathbb{T}^3$, McMullen associates an ordered, oriented triangle $T(x)$ whose angles, with the appropriate orientations, are $\pi x_1, \pi x_2, \pi x_3$. His geometric criterion is

$$T(x) \text{ is spherical} \quad \Longleftrightarrow \quad \mathrm{dist}_1(x, \Lambda) > 1.$$

Here dist_1 is distance for the norm $|x_1| + |x_2| + |x_3|$. Thus the word *spherical* refers throughout the paper to the spherical triangle $T(x)$, or equivalently to the displayed distance inequality.

The criterion can also be written without any reference to the lattice. Fold each coordinate of x into the interval $[0, 1]$, and denote the result by $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$. Then $T(x)$ is spherical if and only if

$$\begin{aligned} \bar{x}_1 + \bar{x}_2 + \bar{x}_3 &> 1, \\ \bar{x}_1 + \bar{x}_2 - \bar{x}_3 &< 1, \\ \bar{x}_1 - \bar{x}_2 + \bar{x}_3 &< 1, \\ -\bar{x}_1 + \bar{x}_2 + \bar{x}_3 &< 1. \end{aligned}$$

In the folded cube $[0,1]^3$, these inequalities cut out the interior of the tetrahedron with vertices $(1,0,0),(0,1,0),(0,0,1),(1,1,1)$.

Let

$$L = \operatorname{lcm}(p,q,r), \qquad N = 2L.$$

The primitive Galois conjugates of the original triangle are represented by

$$x_k = \left(\frac{k}{p}, \frac{k}{q}, \frac{k}{r}\right), \qquad k \in (\mathbf{Z}/N\mathbf{Z})^\times.$$

Consequently their spherical proportion is

$$\rho(p,q,r) = \frac{1}{\varphi(N)} \# \left\{ k \in (\mathbf{Z}/N\mathbf{Z})^\times : \operatorname{dist}_1(x_k, \Lambda) > 1 \right\}.$$

This density has a direct arithmetic meaning. The triangle group has an associated quaternion algebra over its trace field. At a real place of the trace field this algebra is ramified precisely when the corresponding triangle is spherical. Hence $\rho(p,q,r)$ is the proportion of real places at which the quaternion algebra is ramified.

It was in this setting that McMullen introduced the ramification spectrum and proved that zero is an isolated point in its closure [2, Theorem 1.1 and Corollary 1.2]. In particular,

$$\rho_H = \inf\{\rho(p,q,r) : \rho(p,q,r) > 0\}$$

is positive, although that argument does not determine its value. Calegari–Chen subsequently proved that $\rho(p,q,r) = 0$ exactly for the eleven triples in the Hilbert series [1]. The purpose of the present paper is to determine McMullen's constant ρ_H.

Theorem 1.1 (Density gap)**.** *Let (p,q,r) be a hyperbolic triple. Then*

$$\rho(p,q,r) = 0 \qquad \text{or} \qquad \rho(p,q,r) \geq \frac{1}{12}.$$

The zero-density triples are exactly the eleven triples

$$(2,4,6),(2,6,6),(2,6,10),(3,4,4),(3,6,6),(3,10,10),$$
$$(4,6,12),(5,6,6),(6,9,18),(6,10,15),(14,21,42),$$

up to permutation. If $\min(p,q,r) \leq 100$, equality holds exactly for

$$(6,30,45), \quad (6,45,90), \quad (9,15,45), \quad (14,30,105),$$

again up to permutation.

1.2 From the real line to primitive points

The starting point, both here and in Calegari–Chen, is the closed rational line

$$\ell_{p,q,r}(t) = \left(\frac{t}{p}, \frac{t}{q}, \frac{t}{r}\right) \quad \text{in } \mathbb{T}^3.$$

The primitive Galois conjugates are obtained by restricting t to the integers prime to N. Before imposing this arithmetic restriction, Calegari–Chen use Fourier analysis to prove the stronger geometric assertion that there is a real number t_0 such that

$$\operatorname{dist}_1(\ell_{p,q,r}(t_0), \Lambda) \geq 1 + \frac{1}{5}.$$

This is Theorem I of [1]. In particular, the line does not merely touch the spherical region. Since the left side of this inequality changes by at most

$$|t-t_0|\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right),$$

an interval of real parameters around t_0 lies in that region. The interval may be taken to be

$$|t-t_0|<\frac{1}{5\,(p^{-1}+q^{-1}+r^{-1})}.$$

The subsequent twisting argument in [1, §§4–6] modifies a suitable real parameter so as to obtain an integer k with $(k,N)=1$, without losing the inequality $\operatorname{dist}_1(\ell_{p,q,r}(k),\Lambda)>1$.

This is exactly what is required to prove that $\rho(p,q,r)>0$, but it does not give a lower bound for $\rho(p,q,r)$. There are two separate reasons. First, the length of the interval furnished by the preceding inequality, as a proportion of the full period N, need not be bounded below independently of p,q,r. Second, the Jacobsthal estimates used in the twisting step say that a sufficiently long interval contains *one* integer prime to a given modulus. They do not count a positive proportion of all such integers. A density theorem therefore requires a counting version of the passage from the real line to its primitive integral points.

1.3 Overview of the proof

We first explain this counting argument when p is fixed. For $x\in\mathbf{R}/2\mathbf{Z}$, write

$$\|x\|=\operatorname{dist}(x,\mathbf{Z})\in[0,1/2].$$

If the first two coordinates x,y are held fixed, the set of third coordinates z for which (x,y,z) is spherical is a union of two intervals of normalized total length

$$2\min(\|x\|,\|y\|).$$

This elementary identity suggests averaging the quantity on the right over the primitive values of the first two coordinates. We therefore define

$$A(p,q)=\frac{1}{\varphi(2\operatorname{lcm}(p,q))}\sum_{\substack{k \bmod 2\operatorname{lcm}(p,q)\\ (k,2\operatorname{lcm}(p,q))=1}}\min\left(\left\|\frac{k}{p}\right\|,\left\|\frac{k}{q}\right\|\right).$$

We call $A(p,q)$ the *pair depth*. Thus this term means the explicit average just displayed, and the fiber-length formula says that $2A(p,q)$ is the spherical proportion one would obtain if the last coordinate were continuously and uniformly distributed.

The arithmetic error in making this last comparison is measured as follows. For $n\geq 1$, let

$$R_n=\{u/n\in\mathbf{R}/\mathbf{Z}:(u,n)=1\},$$

and define its circular discrepancy by

$$\delta(n)=\sup_I\left|\frac{\#(R_n\cap I)}{\varphi(n)}-|I|\right|,$$

where I ranges over circular intervals and $|I|$ is normalized length. Thus circular discrepancy, rather than a bound for the largest gap between reduced residues, records the loss in the *number* of primitive points in every interval. Applying this definition to the two intervals in the fiber-length formula gives a lower bound for $\rho(p,q,r)$ in terms of $A(p,q)$ and an explicit discrepancy. Applying the same argument with q and r interchanged severely restricts the arithmetic of any triple with $\rho(p,q,r) \le 1/12$.

To use the common arithmetic of q and r, write

$$q = ad, \qquad r = bd, \qquad (a,b) = 1, \qquad a \le b.$$

For a primitive residue u mod $2p$, the last two coordinates now move on the rational circle

$$v \longmapsto (2bv, 2av), \qquad v \in \mathbf{R}/\mathbf{Z}.$$

We introduce its average spherical length by the formula

$$W_p(a,b) = \frac{1}{\varphi(2p)} \sum_{u \in (\mathbf{Z}/2p\mathbf{Z})^\times} \left| \left\{ v \in \mathbf{R}/\mathbf{Z} : \left(\frac{u}{p}, 2bv, 2av \right) \text{ is spherical} \right\} \right|.$$

The vertical bars in this formula denote normalized length on $\mathbf{R}/\mathbf{Z}$. The quantity $W_p(a,b)$ depends on p and on the coprime direction (a,b), but not on the scale d.

Here is the finiteness mechanism in concrete terms. Put

$$m_a = \frac{\operatorname{lcm}(p, ad, bd)}{\operatorname{lcm}(p, bd)},$$
$$m_b = \frac{\operatorname{lcm}(p, ad, bd)}{\operatorname{lcm}(p, ad)}.$$

The pair-depth bounds show that a triple of density at most $1/12$ can have only finitely many values of m_a and m_b. On the other hand,

$$\frac{a}{(a,p)} \mid m_a, \qquad \frac{b}{(b,p)} \mid m_b,$$

so only finitely many coprime directions (a,b) remain. For each of these directions, the average $W_p(a,b)$ can be evaluated exactly. The primitive points on the corresponding circle have multiplier

$$M = \frac{\operatorname{lcm}(p, ad, bd)}{p} = \frac{abd}{(p, abd)}.$$

If $W_p(a,b) > 1/12$, a counterexample is possible only when the discrepancy of this finite set of points is large enough to account for the difference $W_p(a,b) - 1/12$. The exact formula for that discrepancy leaves only finitely many values of M, and hence finitely many values of d. The density of every remaining triple is then obtained by direct inclusion–exclusion on the intervals defining $W_p(a,b)$. This proves the gap in the range where the smallest denominator is bounded.

For large denominators we return to Fourier analysis, but with a different test function from the one used for the existence theorem of Calegari–Chen. We construct a trigonometric polynomial P satisfying

$$P(x) \le \mathbf{1}_{\mathcal{S}}(x) \quad \text{for every } x \in \mathbb{T}^3, \qquad \int_{\mathbb{T}^3} P > \frac{1}{12},$$

where $\mathcal{S}$ is the spherical region. Averaging this inequality over the primitive Galois conjugates gives a lower bound for the density itself, rather than merely producing one spherical point.

If this average were smaller than $1/12$, a nonconstant Fourier term of P would have to survive the primitive average. The possible terms form a fixed finite set. For large p, q, r, the survival of one of them forces an exact reciprocal relation

$$\frac{c_1}{p} + \frac{c_2}{q} + \frac{c_3}{r} = 0$$

with fixed, explicitly bounded integers c_i, not all zero. Restricting the argument to the plane defined by this equation either proves the desired density bound or forces a second independent relation of the same kind. Two independent relations determine the ratio $1/p : 1/q : 1/r$, so only a common integral scale remains, and that scale is handled by an exact one-dimensional count.

In the intermediate range, the same argument can instead produce an affine relation

$$t\left(\frac{c_1}{p} + \frac{c_2}{q} + \frac{c_3}{r}\right) = 2s$$

for explicitly bounded integers t, s, c_i. Once one denominator is fixed, this relation factors into a divisor equation in the other two denominators. Its nondegenerate solutions are finite, while its degenerate solutions are exactly the rational circles described above. Thus the Fourier argument reduces the three-dimensional problem to planes and then to rational circles; the discrepancy argument counts the primitive points on those circles. The remaining computations evaluate finite sets supplied by these reductions, using exact integer and rational arithmetic.

1.4 Exact formulation

Definition 1.2 (Basic arithmetic data)**.** Let $p, q, r \geq 2$. Define

$$L = \operatorname{lcm}(p, q, r), \qquad N = 2L,$$

and

$$\Lambda = \{(a, b, c) \in \mathbf{Z}^3 : a + b + c \equiv 0 \pmod 2\}.$$

The spherical region in $(\mathbf{R}/2\mathbf{Z})^3$ is

$$\mathcal{S} = \{x : \operatorname{dist}_1(x, \Lambda) > 1\}.$$

The triple is *hyperbolic* if

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

Definition 1.3 (Folded residues)**.** For $n \geq 2$ and $k \in \mathbf{Z}$, let $\bar{k}_n$ be the least nonnegative residue of k modulo $2n$, and put

$$s_n(k) = \min(\bar{k}_n, 2n - \bar{k}_n).$$

If $n \mid L$, define the integer

$$x_{n,L}(k) = \frac{L}{n}\, s_n(k).$$

Lemma 1.4 (Integer spherical criterion)**.** *Put*

$$x = x_{p,L}(k), \qquad y = x_{q,L}(k), \qquad z = x_{r,L}(k).$$

Then

$$(k/p, k/q, k/r) \in \mathcal{S}$$

if and only if

$$\begin{aligned} x+y+z &> L, \\ x+y-z &< L, \\ x-y+z &< L, \\ -x+y+z &< L. \end{aligned} \tag{1.1}$$

Proof. Fold each coordinate into $[0,1]$ using sign changes and translations by $2\mathbf{Z}$. In the resulting cube, the relevant points of Λ are

$$(0,0,0), \quad (1,1,0), \quad (1,0,1), \quad (0,1,1).$$

The condition that the ℓ^1-distance from each of them be greater than one is precisely the four strict real triangle inequalities. Multiplication by L gives (1.1). $\square$

Definition 1.5 (Density)**.** Let

$$\mathcal{U}_N = \{k \in \{0, \ldots, N-1\} : (k,N) = 1\}.$$

Define

$$g(p,q,r) = \#\{k \in \mathcal{U}_N : k \text{ satisfies } \mathit{(1.1)}\}$$

and

$$\rho(p,q,r) = \frac{g(p,q,r)}{\varphi(N)}. \tag{1.2}$$

Remark 1.6. The four inequalities in (1.1) remove all boundary and endpoint ambiguity from the density. They also make it clear that g and ρ are symmetric under permutation of p,q,r.

2 Circular discrepancy in primitive lifting fibers

The points R_n below are the primitive points of a rational circle. Their discrepancy is the sole arithmetic loss when the length of a spherical slice is replaced by a primitive point count. The full function $\Phi_n(m)$, rather than only the first interval length containing a reduced residue, is useful because a density problem must count every primitive point in an interval.

Definition 2.1 (Circular discrepancy)**.** For $n \geq 1$, let

$$R_n = \{u/n \in \mathbf{R}/\mathbf{Z} : 0 \leq u < n,\ (u,n) = 1\}.$$

For a circular interval $I \subset \mathbf{R}/\mathbf{Z}$, put

$$\operatorname{err}_n(I) = \frac{\#(R_n \cap I)}{\varphi(n)} - |I|.$$

Define

$$\delta(n) = \sup_I |\operatorname{err}_n(I)|, \qquad \delta(1) = 1. \tag{2.1}$$

Lemma 2.2 (Finite evaluation of discrepancy)**.** *For $n > 1$, the supremum in (2.1) is a maximum. It is attained by a circular interval whose endpoints lie in R_n. Equivalently, if the elements of R_n are put in increasing order, $\delta(n)$ is the range of the left and right limits of their empirical distribution function minus the identity function.*

Proof. Represent a positively oriented circular interval by endpoints $a, b \in [0,1]$, allowing $b < a$ for a wrapping interval. If the interval contains a point of R_n, move its initial endpoint forward to the first contained point and its terminal endpoint backward to the last contained point, in the circular order. The count does not change and the length can only decrease, so the positive error can only increase. If the interval is empty, its positive error is nonpositive. Hence the largest positive error is attained by a closed interval whose endpoints lie in R_n.

It remains to control negative error without hiding an endpoint convention. For distinct a, b, the two oppositely oriented closed arcs from a to b and from b to a have lengths summing to 1, and together cover R_n. Their counts therefore sum to at least $\varphi(n)$. Consequently

$$-\operatorname{err}_n([a,b]) \le \operatorname{err}_n([b,a]).$$

When $a = b$, the closed interval has length zero and nonnegative error. Thus every absolute error is bounded by a positive error of a closed interval. There are only finitely many ordered endpoint pairs in R_n, so the supremum is a maximum. This is equivalently the usual statement that between successive points of R_n the empirical distribution function is constant and its difference from the identity is affine. $\square$

Definition 2.3 (Minimal reduced-residue count)**.** For $n \ge 1$ and $0 \le m \le n$, define

$$\Phi_n(m) = \min_{b \in \mathbf{Z}} \#\{j \in \{1, \ldots, m\} : (b+j, n) = 1\}. \tag{2.2}$$

Thus $\Phi_n(m)$ is the least number of reduced residues in any block of m consecutive integers.

Lemma 2.4 (Discrepancy from minimal interval counts)**.** *For $0 \le m < n$, the largest number of reduced residues in a block of $m+1$ consecutive residue classes is*

$$\varphi(n) - \Phi_n(n-m-1). \tag{2.3}$$

Consequently

$$\varphi(n)\delta(n) = \max_{0 \le m < n} \left\{ \frac{(m+1)\varphi(n)}{n} - \Phi_n(m) \right\}. \tag{2.4}$$

Proof. A block of $m+1$ consecutive residue classes and its complementary block of $n-m-1$ consecutive residue classes partition $\mathbf{Z}/n\mathbf{Z}$. The first claim follows by taking the minimum over the complementary block.

By Lemma 2.2, circular discrepancy is the largest positive error of a closed circular interval whose endpoints are reduced residues; the complementary open interval has the opposite error. A closed interval of grid span m/n contains a block of $m+1$ consecutive residue classes. Conversely, if a maximizing block has a nonreduced endpoint, trimming that endpoint does not change its count and only increases its error. Thus

$$\delta(n) = \max_{0 \le m < n} \left\{ \frac{\varphi(n) - \Phi_n(n-m-1)}{\varphi(n)} - \frac{m}{n} \right\}.$$

Replacing $n-m-1$ by m gives (2.4). $\square$

Corollary 2.5 (Relation with the Jacobsthal function)**.** *If $J(n)$ is the least positive integer such that every block of $J(n)$ consecutive integers contains an integer prime to n, then*

$$J(n) = \min\{m \ge 1 : \Phi_n(m) \ge 1\}. \tag{2.5}$$

Lemma 2.6 (A finite inclusion–exclusion lower bound)**.** *Let $p_1, \dots, p_R$ be the distinct primes dividing n. For a nonempty subset $S \subseteq \{1, \dots, R\}$, put*

$$d_S = \prod_{i \in S} p_i.$$

Then

$$\begin{aligned}\Phi_n(m) \geq \max\Bigg(0,\ m - &\sum_{\substack{\varnothing \neq S \subseteq \{1,\dots,R\} \\ |S| \text{ odd}}} \left\lceil \frac{m}{d_S} \right\rceil \\ &+ \sum_{\substack{\varnothing \neq S \subseteq \{1,\dots,R\} \\ |S| \text{ even}}} \left\lfloor \frac{m}{d_S} \right\rfloor \Bigg).\end{aligned} \tag{2.6}$$

Proof. Apply inclusion–exclusion to the multiples of the primes p_i in an arbitrary block of m consecutive integers. An intersection indexed by S is one residue class modulo d_S, hence has either $\lfloor m/d_S \rfloor$ or $\lceil m/d_S \rceil$ members. Use the upper choice for a term with negative sign and the lower choice for a term with positive sign, then take the minimum over the initial point of the block. □

Remark 2.7 (Self-improvement of the discrepancy bound). The elementary estimate $\delta(n) \leq 2^{\omega(n)}/\varphi(n)$ is uniform in n, but the argument below only requires a bound on the moduli compatible with a putative counterexample. Suppose that a first application of the multiplicative inequalities places the prime-power part $n = M_0$ of a lifting modulus in a finite set $\mathcal{N}$. On this set define

$$\mathfrak{D}(\mathcal{N}) = \max_{n \in \mathcal{N}} \varphi(n)\delta(n). \tag{2.7}$$

If only a size and prime-factor bound is known, one may instead use

$$\mathfrak{D}_R(X) = \max_{\substack{n \leq X \\ \omega(n) \leq R}} \varphi(n)\delta(n). \tag{2.8}$$

Lemma 2.4 determines either maximum from the finite collection of values $\Phi_n(m)$.

This gives a useful self-improvement. A coarse discrepancy estimate first restricts the possible moduli. The exact discrepancies on that restricted set then strengthen the pair-depth and slice inequalities, which may restrict the moduli further. Repeating this process produces a descending chain of finite sets and hence eventually stabilizes. The point is mathematical: the discrepancy need only be uniform on the arithmetic family which the preceding part of the proof permits, rather than on all integers. For example, when $\omega(n) \leq 15$, the inclusion–exclusion bound (2.6) involves at most 2^{15} terms for each prime set.

Definition 2.8 (The D-decomposition)**.** Fix $D, M \geq 1$. Write

$$M = M_0 M_1$$

where, for every prime ℓ, the full power $\ell^{v_\ell(M)}$ belongs to M_0 if $\ell \nmid D$, and belongs to M_1 if $\ell \mid D$. Define

$$\Delta_D(M) = \frac{\delta(M_0)}{M_1}. \tag{2.9}$$

Lemma 2.9 (Exact primitive-fiber discrepancy)**.** *Fix $u \in (\mathbf{Z}/D\mathbf{Z})^\times$. Among the lifts*

$$u + Dt \pmod{DM}, \qquad t \in \mathbf{Z}/M\mathbf{Z},$$

which are primitive modulo DM, an affine circular coordinate of exact order M has circular discrepancy exactly $\Delta_D(M)$.

Proof. Put $n = M_0$ and $k = M_1$, so that $M = nk$ and $(n,k) = 1$. Every prime divisor of k already divides D. Since $(u,D) = 1$, the numbers $u + Dt$ are automatically coprime to the full k-part of M. Consequently

$$(u + Dt, DM) = 1 \quad \Longleftrightarrow \quad (u + Dt, n) = 1. \tag{2.9a}$$

Moreover $(D,n) = 1$, so multiplication by D is invertible modulo n. There is therefore a fixed $s \bmod n$ such that, simultaneously for every integer t,

$$(u + Dt, n) = 1 \quad \Longleftrightarrow \quad (t + s, n) = 1. \tag{2.9b}$$

Thus every consecutive block of parameter values has exactly the corresponding reduced-residue block count, with one fixed cyclic shift.

First use the coordinate t/M. Ignoring the harmless cyclic shift in (2.9b), its primitive empirical measure is

$$\mu_{n,k} = \frac{1}{k}\sum_{j=0}^{k-1}(S_j)_*\mu_n, \qquad S_j(x) = \frac{j+x}{k}, \tag{2.9c}$$

where μ_n is the uniform measure on R_n. Each cell $[j/k,(j+1)/k]$ has both $\mu_{n,k}$-mass and Lebesgue mass $1/k$. Hence a complete cell contributes zero to the signed error $\mu_{n,k} - \lambda$. At common cell boundaries use half-open cells with one consistent convention; the corresponding one-sided limits give the same supremum.

Let I be any circular interval. Delete from I all complete cells. At most two boundary pieces remain. After dilating those pieces by k and joining them across 0 when necessary, they form a circular interval J in the base circle. Formula (2.9c) gives

$$\mathrm{err}_{n,k}(I) = \frac{1}{k}\,\mathrm{err}_n(J). \tag{2.9d}$$

It follows that the discrepancy is at most $\delta(n)/k$. Conversely, start with an interval J attaining $\delta(n)$, scale its nonwrapping pieces into one cell, and, if J crosses 0, include the intervening complete cells. Their signed error is zero, so (2.9d) is an equality. This proves that the discrepancy is exactly $\delta(n)/k$.

Finally, an affine coordinate of exact order M writes the coordinate index as $at + c \bmod M$ with $(a,M) = 1$. Reparametrizing by this index changes (2.9a) to another affine coprimality condition modulo n, whose slope is still a unit modulo n. The same fixed-shift argument (2.9b) and the preceding replication calculation apply unchanged. Therefore the exact discrepancy is

$$\frac{\delta(n)}{k} = \frac{\delta(M_0)}{M_1} = \Delta_D(M).$$

□

Definition 2.10 (Multiplicative discrepancy cost)**.** Put

$$F(n) = \frac{\varphi(n)}{2^{\omega(n)}}.$$

When D is even, define

$$G_D(M) = M_1 F(M_0). \tag{2.10}$$

Lemma 2.11 (Elementary discrepancy bound)**.** *For every $n \geq 1$,*

$$\delta(n) \leq \frac{2^{\omega(n)}}{\varphi(n)} = \frac{1}{F(n)}. \tag{2.11}$$

Consequently

$$\Delta_D(M) \leq \frac{1}{G_D(M)}. \tag{2.12}$$

Proof. For an interval, count integers coprime to n by inclusion–exclusion over the squarefree divisors of n. Replacing the length of each integer interval by its real length introduces an error of absolute value at most one for each squarefree divisor. Division by $\varphi(n)$ gives (2.11), and (2.12) follows from Lemma 2.9. $\square$

Lemma 2.12 (Prime factorization of G_D)**.** *If D is even, then*

$$G_D(M) = \prod_{\substack{\ell^e \| M \\ \ell | D}} \ell^e \prod_{\substack{\ell^e \| M \\ \ell \nmid D}} \frac{\ell^{e-1}(\ell - 1)}{2}. \tag{2.13}$$

Corollary 2.13 (Finiteness below a rational cost)**.** *For every positive rational B, the set*

$$\{M \geq 1 : G_D(M) \leq B\} \tag{2.14}$$

is finite.

Proof. Every local factor in (2.13) is positive and increases without bound with the exponent. If $\ell \nmid D$, the first local factor $(\ell - 1)/2$ also tends to infinity with ℓ. Hence only finitely many primes and exponents can occur. Induction over the ordered prime divisors therefore gives the entire set; equivalently, (2.13) gives an effective proof of finiteness. $\square$

Proposition 2.14 (Exact finite envelope at the 13/120 threshold)**.** *For every even D, let $M = M_0 M_1$ be the D-decomposition. If*

$$\Delta_D(M) = \frac{\delta(M_0)}{M_1} \geq \frac{13}{120}, \tag{2.15}$$

then

$$M \in \mathcal{E}_{13/120} := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 24, 26, 28, 30, 33, 34, 36, 42, 60, 66\}. \tag{2.16}$$

Thus the coarse multiplicative bound loses no information at this threshold once discrepancy is evaluated on the finite set which it determines.

Proof. The elementary estimate first gives

$$F(M_0) \leq \frac{120}{13}.$$

The prime-power factorization in Lemma 2.12 leaves exactly 91 possibilities for M_0, the largest being 630. Evaluate the finite formula of Lemma 2.2 for each one. On this set the optimal value of $\varphi(M_0)\delta(M_0)$ is

$$\frac{328}{77},$$

attained at $M_0 = 462$. For each of the 91 values, consider the positive integers M_1, coprime to M_0, for which $\delta(M_0)/M_1 \geq 13/120$. Their products $M_0 M_1$ are exactly the 31 integers in (2.16). $\square$

3 The pair-depth bootstrap

We first use discrepancy without choosing a rational direction for the whole triple. If two coordinates are fixed, the length of the spherical fiber in the third coordinate is twice the smaller of their folded distances from an integer. Averaging this minimum gives the pair depth $A(p,q)$. A lower bound for $A(p,q)$, followed by discrepancy in the remaining coordinate, already places strong restrictions on the relative least common multiples of a counterexample.

Definition 3.1 (Pair depth)**.** For $n \geq 2$, write

$$\left\| \frac{k}{n} \right\| = \frac{1}{n} \min(k \bmod n, n - (k \bmod n)).$$

Define

$$A(p,q) = \frac{1}{\varphi(2\operatorname{lcm}(p,q))} \sum_{\substack{k \bmod 2\operatorname{lcm}(p,q) \\ (k, 2\operatorname{lcm}(p,q))=1}} \min\left(\left\| \frac{k}{p} \right\|, \left\| \frac{k}{q} \right\| \right). \tag{3.1}$$

For $u \in (\mathbf{Z}/2p\mathbf{Z})^\times$, put

$$\alpha_u = \left\| \frac{u}{p} \right\|,$$

$$H_p = \frac{1}{\varphi(2p)} \sum_u (\alpha_u - \alpha_u^2), \qquad K_p = \frac{1}{\varphi(2p)} \sum_u \alpha_u. \tag{3.2}$$

Proposition 3.2 (Uniform pair-depth estimate)**.** *Let*

$$M = \frac{\operatorname{lcm}(p,q)}{p}.$$

Then

$$A(p,q) \geq H_p - 2K_p \Delta_{2p}(M). \tag{3.3}$$

Proof. Fix $u \in (\mathbf{Z}/2p\mathbf{Z})^\times$. The continuous mean over $y \in \mathbf{R}/\mathbf{Z}$ of

$$\min(\alpha_u, \|y\|)$$

is $\alpha_u - \alpha_u^2$. Its layer-cake superlevel sets have at most two circular interval components, and the integral of the component-count error is at most $2\alpha_u \Delta_{2p}(M)$. Average over u. $\square$

Proposition 3.3 (One-coordinate density estimate)**.** *Let*

$$B = \operatorname{lcm}(p,q), \qquad D = 2B, \qquad M = \frac{\operatorname{lcm}(p,q,r)}{B}.$$

Then

$$\rho(p,q,r) \geq 2A(p,q) - 2\Delta_D(M). \tag{3.4}$$

Proof. For a fixed primitive residue modulo D, the length of the spherical third-coordinate fiber is twice

$$\min\left(\left\| \frac{k}{p} \right\|, \left\| \frac{k}{q} \right\| \right).$$

The fiber is the union of at most two circular intervals. Apply Lemma 2.9 in each primitive lifting fiber and average over the primitive base residues. $\square$

Corollary 3.4 (Finite verification of a constant c_p)**.** *Let $c_p < H_p$. To prove*

$$A(p,q) \geq c_p \qquad \textit{for every } q \geq 2, \tag{3.5}$$

it suffices to evaluate (3.1) for the finitely many q satisfying

$$\frac{\operatorname{lcm}(p,q)}{p} \in \left\{ M : \Delta_{2p}(M) > \frac{H_p - c_p}{2K_p} \right\}. \tag{3.6}$$

Corollary 3.5 (Relative multiplier restriction)**.** *Assume (3.5). If*

$$\rho(p,q,r) \leq \frac{1}{12},$$

then, with D, M as in Proposition 3.3,

$$\Delta_D(M) \geq c_p - \frac{1}{24}. \tag{3.7}$$

Proof. Substitute (3.5) into (3.4) and rearrange. ☐

Theorem 3.6 (Pair-depth constants for $p \leq 72$)**.** *For $2 \leq p \leq 72$, (3.5) holds with the following constants.*

p	2	3	4	5	6	7	8	9	10	11
c_p	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{10}{63}$	$\frac{7}{48}$	$\frac{4}{27}$	$\frac{2}{15}$	$\frac{5}{33}$

p	12	13	14	15	16	17	18	19	20	21
c_p	$\frac{1}{8}$	$\frac{35}{234}$	$\frac{17}{126}$	$\frac{2}{15}$	$\frac{9}{64}$	$\frac{5}{34}$	$\frac{7}{54}$	$\frac{25}{171}$	$\frac{2}{15}$	$\frac{17}{126}$

p	22	23	24	25	26	27	28	29	30
c_p	$\frac{3}{22}$	$\frac{10}{69}$	$\frac{13}{96}$	$\frac{7}{50}$	$\frac{16}{117}$	$\frac{34}{243}$	$\frac{17}{126}$	$\frac{25}{174}$	$\frac{2}{15}$.

p	31	32	33	34	35	36	37	38	39	40
c_p	$\frac{40}{279}$	$\frac{107}{768}$	$\frac{3}{22}$	$\frac{7}{51}$	$\frac{17}{126}$	$\frac{5}{36}$	$\frac{95}{666}$	$\frac{47}{342}$	$\frac{16}{117}$	$\frac{11}{80}$.

p	41	42	43	44	45	46	47	48	49	50
c_p	$\frac{35}{246}$	$\frac{17}{126}$	$\frac{55}{387}$	$\frac{3}{22}$	$\frac{37}{270}$	$\frac{19}{138}$	$\frac{20}{141}$	$\frac{53}{384}$	$\frac{430}{3087}$	$\frac{52}{375}$

p	51	52	53	54	55	56	57	58	59	60
c_p	$\frac{7}{51}$	$\frac{16}{117}$	$\frac{15}{106}$	$\frac{67}{486}$	$\frac{3}{22}$	$\frac{139}{1008}$	$\frac{47}{342}$	$\frac{4}{29}$	$\frac{25}{177}$	$\frac{17}{120}$.

p	61	62	63	64	65	66	67
c_p	$\frac{155}{1098}$	$\frac{77}{558}$	$\frac{26}{189}$	$\frac{427}{3072}$	$\frac{16}{117}$	$\frac{3}{22}$	$\frac{85}{603}$.

p	68	69	70	71	72
c_p	$\frac{7}{51}$	$\frac{19}{138}$	$\frac{44}{315}$	$\frac{10}{71}$	$\frac{5}{36}$.

Theorem 3.7 (Ordered pair depth throughout the Fourier middle range)**.** *If*

$$41 \leq p \leq 4788, \qquad q \geq p,$$

then

$$A(p,q) \geq \frac{3}{20}. \tag{3.8}$$

Consequently, if p is the smallest denominator and $\rho(p,q,r) \leq 1/12$, then

$$\frac{\operatorname{lcm}(p,q,r)}{\operatorname{lcm}(p,q)} \in \mathcal{E}_{13/120}, \qquad \frac{\operatorname{lcm}(p,q,r)}{\operatorname{lcm}(p,r)} \in \mathcal{E}_{13/120}. \tag{3.9}$$

Proof. For each p, Corollary 3.4 with $c_p = 3/20$ reduces (3.8) to finitely many values of $q \geq p$. At most 1096 values occur for any p in the stated range. Möbius inversion rewrites the numerator of (3.1) as a signed sum of complete-grid averages of

$$t \longmapsto \min\left(\left\|\frac{2\operatorname{lcm}(p,q)}{p}\,t\right\|, \left\|\frac{2\operatorname{lcm}(p,q)}{q}\,t\right\|\right).$$

This function is affine between the rational breakpoints supplied by the two frequencies, their sum, and their difference. Summing each affine piece on the relevant rational grid evaluates every average by integer arithmetic, with work controlled by the relative frequencies rather than by $\operatorname{lcm}(p,q)$. Now $3/20 - 1/24 = 13/120$. Apply Corollary 3.5 and Proposition 2.14, first to $(p,q); r$ and then to $(p,r); q$, to obtain (3.9). □

4 Coprime rational slices

The pair-depth estimate treats q and r separately. To exploit their common arithmetic, write $q = ad$ and $r = bd$, with $(a,b) = 1$. The coprime pair (a,b) determines the rational direction of the last two coordinates, while d determines how the primitive orbit samples that direction. The next lemmas compute the continuous spherical length along this rational line before any discrepancy loss is imposed.

Definition 4.1 (Coprime slice coordinates)**.** Write

$$q = ad, \qquad r = bd, \qquad d = (q,r), \qquad (a,b) = 1, \qquad a \leq b. \tag{4.1}$$

For $u \in (\mathbf{Z}/2p\mathbf{Z})^\times$, put

$$X_u = \frac{\min(u, 2p-u)}{p}.$$

For $n \geq 1$, define

$$E_n(X) = \left\{v \in \mathbf{R}/\mathbf{Z} : \left\|nv - \frac{1}{2}\right\| < \frac{X}{2}\right\}, \qquad E_0(X) = \varnothing. \tag{4.2}$$

Lemma 4.2 (Exact spherical slice)**.** *For fixed u, a, b, the spherical part of the circle*

$$v \longmapsto \left(\frac{u}{p}, 2bv, 2av\right)$$

is

$$E_{a+b}(X_u) \mathbin{\triangle} E_{|a-b|}(X_u), \tag{4.3}$$

with every boundary point removed. It has at most $2b$ open components.

Proof. Apply the four inequalities of Lemma 1.4 in their real form. The two relevant parity alternatives are exactly membership in one, but not both, of the two sets in (4.3). The boundary points are zeros of one of the four strict inequalities. The first set has $a+b$ intervals and the second has $|a-b|$ intervals; their symmetric difference has at most

$$(a+b) + |a-b| = 2b$$

components. □

Definition 4.3 (Bernoulli slice mean)**.** Let

$$B_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}.$$

If $a = b = 1$, set $W_X(a,b) = X$. Otherwise put

$$g = (a+b, |a-b|), \quad m = (a+b)/g, \quad n = |a-b|/g, \quad \epsilon \equiv m+n \pmod 2,$$

and define

$$W_X(a,b) = 2X(1-X) - \frac{2}{mn}\left[B_2\left(\frac{(m-n)X+\epsilon}{2}\right) - B_2\left(\frac{(m+n)X+\epsilon}{2}\right)\right]. \tag{4.4}$$

Finally set

$$W_p(a,b) = \frac{1}{\varphi(2p)} \sum_{u \in (\mathbf{Z}/2p\mathbf{Z})^\times} W_{X_u}(a,b). \tag{4.5}$$

Lemma 4.4 (Exact slice measure)**.** *The normalized measure of (4.3) is $W_{X_u}(a,b)$. In particular, $W_p(a,b)$ is the exact primitive average of the spherical slice lengths.*

Proof. The measure of a symmetric difference is the sum of the two measures minus twice their overlap. The overlap of the two periodic interval sets is an elementary second Bernoulli sum. Reducing the two frequencies by their gcd gives (4.4). Every argument of B_2 is rational. ☐

Lemma 4.5 (Relative factors carried by a and b)**.** *Let*

$$L = \operatorname{lcm}(p, ad, bd),$$

$$m_a = \frac{L}{\operatorname{lcm}(p, bd)}, \qquad m_b = \frac{L}{\operatorname{lcm}(p, ad)}.$$

Then

$$\frac{a}{(a,p)} \mid m_a, \qquad \frac{b}{(b,p)} \mid m_b, \qquad (m_a, m_b) = 1. \tag{4.6}$$

Proof. Compare valuations prime by prime. Since $(a,b) = 1$, a prime can contribute an exponent beyond $\operatorname{lcm}(p, bd)$ only through a, and can contribute an exponent beyond $\operatorname{lcm}(p, ad)$ only through b. The excesses cannot both be positive. ☐

Corollary 4.6 (Finiteness of the coprime pair list)**.** *Fix p and a constant $c_p > 1/24$ satisfying (3.5). If $\rho(p, ad, bd) \le 1/12$, then both m_a and m_b lie in the finite inclusive discrepancy set forced by (3.7). Together with (4.6), this leaves only finitely many coprime pairs (a,b).*

5 The remaining scale

Once p, a, b are fixed, the continuous slice mean is fixed as well. Only the scale d remains, and its effect is entirely arithmetic. The following estimate compares the primitive density with the continuous mean and then reconstructs the possible scales prime by prime.

Definition 5.1 (Remaining multiplier)**.** For a coprime pair (a,b), put

$$M = \frac{\operatorname{lcm}(p, ad, bd)}{p} = \frac{abd}{(p, abd)}. \tag{5.1}$$

Proposition 5.2 (Slice discrepancy estimate)**.** *For every p, a, b, d as above,*

$$\rho(p, ad, bd) \geq W_p(a, b) - 2b\,\Delta_{2p}(M). \tag{5.2}$$

Proof. For each primitive $u \bmod 2p$, apply Lemma 2.9 to every open component in Lemma 4.2. There are at most $2b$ components. Average the result over u and use Lemma 4.4. □

Corollary 5.3 (Finite scale inequality)**.** *Assume $W_p(a, b) > 1/12$. If*

$$\rho(p, ad, bd) \leq \frac{1}{12},$$

then

$$G_{2p}(M) \leq \frac{2b}{W_p(a, b) - 1/12}. \tag{5.3}$$

Proof. Combine (5.2) with $\Delta_{2p}(M) \leq 1/G_{2p}(M)$ and rearrange. □

Lemma 5.4 (Prime-by-prime reconstruction of d)**.** *Fix p, a, b, M, and put*

$$\pi = v_\ell(p), \qquad \alpha = v_\ell(ab), \qquad \mu = v_\ell(M), \qquad x = v_\ell(d).$$

Equation (5.1) is equivalent, for every prime ℓ, to

$$\mu = \max(0, \alpha + x - \pi). \tag{5.4}$$

Consequently:

1. *if $\mu > 0$, then necessarily*
$$x = \pi + \mu - \alpha \geq 0;$$
2. *if $\mu = 0$, solutions exist precisely when $\alpha \leq \pi$, and then*
$$0 \leq x \leq \pi - \alpha.$$

Thus every d satisfying (5.1) is obtained from a finite Cartesian product of explicit exponent choices.

Proof. Take v_ℓ of (5.1):

$$\mu = \alpha + x - \min(\pi, \alpha + x) = \max(0, \alpha + x - \pi).$$

The two cases follow immediately. □

Theorem 5.5 (Finite fixed-parameter reduction)**.** *Fix $p \geq 2$, and suppose $c_p > 1/24$ satisfies (3.5) and that every pair left by Corollary 4.6 satisfies $W_p(a, b) > 1/12$. Then there is an explicit finite set $\mathcal{C}_p$ such that every hyperbolic triple containing p and having density at most $1/12$ belongs, up to permutation, to $\mathcal{C}_p$.*

The set $\mathcal{C}_p$ is obtained by:

1. *the finite inclusive discrepancy sets in (3.7);*
2. *the coprime pairs forced by (4.6);*
3. *every multiplier satisfying (5.3), enumerated through (2.13);*
4. *every scale d supplied by Lemma 5.4.*

No independent upper bound on q, r, d, M, or L is used.

6 A second pair-depth estimate

Before counting any retained triple exactly, one can apply pair depth in either order to the other two coordinates. Taking the better of the two lower bounds removes a large part of the finite set while preserving the borderline case $\rho = 1/12$.

Proposition 6.1 (Two-stage lower bound)**.** *Let x, y be q, r in either order, and define*

$$B = \mathrm{lcm}(p, x), \qquad M_x = B/p, \qquad M_y = \mathrm{lcm}(B, y)/B.$$

Then

$$\rho(p, x, y) \geq 2H_p - \frac{4K_p}{G_{2p}(M_x)} - \frac{2}{G_{2B}(M_y)}. \tag{6.1}$$

Proof. By Proposition 3.2 and (2.12),

$$A(p, x) \geq H_p - \frac{2K_p}{G_{2p}(M_x)}.$$

Insert this in Proposition 3.3 and apply (2.12) to the second lifting fiber. □

Corollary 6.2 (A strict exclusion criterion)**.** *Let $\mathcal{L}_{x,y}$ denote the right side of (6.1). If*

$$\max(\mathcal{L}_{q,r}, \mathcal{L}_{r,q}) > \frac{1}{12},$$

then (p, q, r) cannot have density zero or $1/12$. Thus a triple may be excluded from an inclusive classification under this strict inequality, but not under equality.

7 Exact counting on a retained rational slice

The reductions so far use inequalities. Once only finitely many rational slices remain, their densities are recovered exactly. The essential point is that open endpoints matter: the spherical region is defined by strict triangle inequalities. The following grid count keeps those conventions visible and reduces the final calculation to inclusion–exclusion.

Definition 7.1 (Open rational grid count)**.** For $Q \geq 1$, $s \in \mathbf{Q}/\mathbf{Z}$, and an open circular interval $I \subset \mathbf{R}/\mathbf{Z}$ with rational endpoints, define

$$\mathrm{Grid}(Q, s, I) = \#\left\{0 \leq j < Q : s + \frac{j}{Q} \in I \pmod 1\right\}. \tag{7.1}$$

Lemma 7.2 (Integer evaluation of an open grid count)**.** *The value in (7.1) is computable using integer floor and ceiling operations. For a non-wrapping interval $I = (A, B) \subset [0, 1)$, after replacing $A - s$ and $B - s$ by compatible rational lifts,*

$$\mathrm{Grid}(Q, s, I) = \max\left(0,\ \lceil Q(B - s)\rceil - \lfloor Q(A - s)\rfloor - 1\right). \tag{7.2}$$

A wrapping interval is the disjoint union of its two non-wrapping parts, with the point 0 included exactly when it is interior to the original circular interval.

Proof. For a non-wrapping interval, (7.1) asks for the integers j satisfying

$$Q(A-s) < j < Q(B-s).$$

The first possible integer is the left floor plus one, and the last is the right ceiling minus one. The wrapping statement follows by splitting at zero. $\square$

Proposition 7.3 (Inclusion–exclusion formula for a primitive fiber). *Let $D = 2p$, fix $u \in (\mathbf{Z}/D\mathbf{Z})^\times$, and let M be as in (5.1). Let*

$$\mathcal{P} = \{\ell : \ell \mid M,\ \ell \nmid D\}.$$

For a squarefree product e of primes in $\mathcal{P}$, let

$$c_e \equiv -uD^{-1} \pmod{e}, \qquad 0 \le c_e < e,$$

with $c_1 = 0$, and put $Q_e = M/e$. Then the number of primitive lifts in the spherical slice above u is an alternating sum

$$\sum_{e \mid \prod_{\ell \in \mathcal{P}} \ell} \mu(e) \sum_{I \in \mathcal{A}_{u;a,b}} \operatorname{Grid}(Q_e, s_{u,e}, I), \tag{7.3}$$

where $\mathcal{A}_{u;a,b}$ is the finite list of open arcs in (4.3), and $s_{u,e}$ is the rational shift induced by

$$v = \frac{u + 2pt}{2abd}, \qquad t \equiv c_e \pmod{e}. \tag{7.4}$$

Proof. A lift $u + Dt$ fails to be primitive precisely when

$$t \equiv -uD^{-1} \pmod{\ell}$$

for some $\ell \in \mathcal{P}$. Apply inclusion–exclusion. Under the congruence modulo e, write $t = c_e + ej$. As j varies modulo Q_e, (7.4) is an affine rational grid of exact order Q_e, so each open arc is counted by (7.1). $\square$

Corollary 7.4 (Exact density of a rational slice). *Summing (7.3) over $u \in (\mathbf{Z}/2p\mathbf{Z})^\times$ gives $g(p, ad, bd)$. The denominator in (1.2) is*

$$\varphi(2pM).$$

Thus the comparisons

$$\rho = 0, \qquad \rho = \frac{1}{12}, \qquad 0 < \rho < \frac{1}{12}$$

are decidable by integer arithmetic.

8 The finite classification through denominator 100

Definition 8.1 (Exceptional sets). Let $\mathcal{Z}$ be

$$\begin{gathered}\{(2,4,6),(2,6,6),(2,6,10),(3,4,4),(3,6,6),\\ (3,10,10),(4,6,12),(5,6,6),(6,9,18),(6,10,15),(14,21,42)\},\end{gathered} \tag{8.1}$$

and let $\mathcal{H}$ be

$$\{(6,30,45),(6,45,90),(9,15,45),(14,30,105)\}. \tag{8.2}$$

Both sets are understood up to permutation.

Theorem 8.2 (Exact finite classification for each fixed denominator)**.** *Let $2 \le P \le 100$, and let (p, q, r) be a hyperbolic triple containing P. Then:*

1. *$\rho(p, q, r) = 0$ if and only if the sorted triple belongs to $\mathcal{Z}$;*
2. *if the sorted triple does not belong to $\mathcal{Z}$, then $\rho(p, q, r) \ge 1/12$;*
3. *$\rho(p, q, r) = 1/12$ if and only if the sorted triple belongs to $\mathcal{H}$.*

Proof. For $P \le 72$, use the constants in Theorem 3.6. For $73 \le P \le 100$, use Theorem 3.7, which supplies the uniform constant $c_P = 3/20$. In either case Corollary 4.6 and Theorem 5.5 give an explicit finite inclusive set containing every possible zero or equality case. Corollary 6.2 removes only triples whose density is strictly greater than 1/12. Formula (7.3), summed as in Corollary 7.4, evaluates every retained triple.

Over $2 \le P \le 100$, the first reduction produces 68736243 candidates. The symmetric pair-depth test removes 37276467 of them, leaving 31459776 terminal exact counts. There are 282940 small-modulus cases in which the optimized count is also compared directly with enumeration of the primitive residue classes. The resulting zero triples are precisely the members of $\mathcal{Z}$, the equality triples are precisely the members of $\mathcal{H}$, and no retained triple has positive density below 1/12. If an exceptional triple contains more than one denominator in $[2, 100]$, its occurrence is checked in every corresponding fixed-denominator calculation. Finally, direct enumeration of primitive residue classes independently confirms the density of each member of $\mathcal{Z} \cup \mathcal{H}$. $\square$

Theorem 8.3 (Smallest denominator at most 100)**.** *Let (p, q, r) be hyperbolic and suppose*

$$\min(p, q, r) \le 100.$$

Then

$$\rho(p, q, r) = 0 \qquad \text{or} \qquad \rho(p, q, r) \ge \frac{1}{12}. \tag{8.3}$$

The zero-density triples are exactly $\mathcal{Z}$, and the triples of density 1/12 are exactly $\mathcal{H}$.

Proof. Permute the triple so that $P = \min(p, q, r)$. Then $2 \le P \le 100$, and Theorem 8.2 applies. $\square$

9 Fourier descent for large denominators

We now turn to the opposite regime, where all three denominators are large. The guiding idea comes from the Fourier descent of Calegari–Chen [1, §2]: a small primitive average can occur only if some low-frequency mode is arithmetically resonant. An existence argument may use a test function which is large somewhere. A density argument needs the stronger pointwise inequality

$$\mathcal{P} \le \mathbf{1}_{\mathcal{S}},$$

so that the primitive average of $\mathcal{P}$ is itself a lower bound for the spherical density. Once such a minorant has mean greater than 1/12, Ramanujan sums measure exactly how much its nonconstant modes can lower that mean.

Put

$$\Delta(x) = 4 - 4\sum_{i=1}^{3} \cos^2(\pi x_i) - 8\prod_{i=1}^{3} \cos(\pi x_i).$$

Then $x \in \mathcal{S}$ if and only if $\Delta(x) > 0$, and $-16 \le \Delta(x) \le 4$. There is an explicit polynomial $R \in \mathbf{Q}[u]$ of degree 18 such that

$$\mathcal{P}(x) = R\left(\frac{\Delta(x)+6}{10}\right) \le \mathbf{1}_{\mathcal{S}}(x). \tag{9.1}$$

The pointwise inequalities follow from Sturm's theorem. The finite Fourier expansion

$$\mathcal{P}(x) = \sum_{\nu \in \mathbf{Z}^3} c_\nu e^{\pi i \nu \cdot x}$$

has 25345 nonzero rational coefficients, $\|\nu\|_1 \le 108$ on its support,

$$c_0 = \frac{2116121513335867589}{15258789062500000000}, \qquad \sum_{\nu \ne 0} |c_\nu| = \frac{12729933427286634256\,29}{190734863281250000000}. \tag{9.2}$$

The coefficients of R, the pointwise comparison, and the Fourier identities in (9.2) are all rational.

For $L = \operatorname{lcm}(p, q, r)$, put

$$(A, B, C) = (L/p, L/q, L/r), \qquad N = 2L,$$

and, for a Fourier mode ν, put

$$q_\nu = \frac{N}{(N, \nu \cdot (A, B, C))}.$$

Primitive averaging of (9.1) and the exact Ramanujan-sum formula give

$$\rho(p, q, r) \ge c_0 + \sum_{\nu \ne 0} c_\nu \frac{\mu(q_\nu)}{\varphi(q_\nu)}. \tag{9.3}$$

In particular, if every nonconstant mode has $\varphi(q_\nu) \ge 121$, then (9.2) gives $\rho > 1/12$.

Lemma 9.1 (Small quotient forces an exact Fourier relation)**.** *Let $p_0 = \min(p, q, r)$. If $q_\nu \le Q$, $p_0 > 54Q$, and ν lies in the support of $\mathcal{P}$, then*

$$\nu \cdot (A, B, C) = 0.$$

Proof. If the dot product is nonzero, its absolute value is at least $N/Q = 2L/Q$. On the other hand, $A, B, C \le L/p_0$ and $\|\nu\|_1 \le 108$, so it is at most $108L/p_0 < 2L/Q$. $\square$

Proposition 9.2 (Finite Fourier descent data)**.** *The polynomial $\mathcal{P}$ has the following arithmetic properties.*

1. *There are 126 squarefree s with $\varphi(s) \le 120$, and their maximum is 462. There are 306 squarefree s with $\varphi(s) \le 308$, and their maximum is 1110.*
2. *For every primitive Fourier line h orthogonal to a positive direction, the mean of the modes parallel to h, including the constant term, is at least*

$$\frac{8009338009785455891}{76293945312500000000} > 0.10497.$$

3. *The 7623 relevant Fourier lines determine 3841327 positive primitive directions. Exactly 1536 ordered directions, or 256 classes under coordinate permutation, have polynomial mean below 21/200.*

4. *For those* 256 *classes, the multiplicative scale inequality has* 3336 *solutions, with largest scale* 715 *and largest modulus* 143220*. Exact reduced-residue interval counts find density below* 1/12 *only for*

$$(4,6,12), \quad (6,9,18), \quad (14,21,42), \quad (6,10,15),$$

and each of these densities is zero.

Theorem 9.3 (Smallest denominator greater than 59940)**.** *If* (p,q,r) *is hyperbolic and*

$$\min(p,q,r) > 59940,$$

then

$$\rho(p,q,r) \geq \frac{1}{12}. \tag{9.4}$$

Proof. Suppose instead that $\rho < 1/12$. Equation (9.3) first supplies a mode with $\varphi(q_\nu) \leq 120$. Only squarefree quotients contribute, so Proposition 9.2 gives $q_\nu \leq 462$, and Lemma 9.1 makes this an exact Fourier relation.

After one exact relation, (9.2) and the lower mean in Proposition 9.2 show that all nonparallel modes having $\varphi(q_\nu) \geq 309$ would give

$$\rho \geq 0.10497 - \frac{6.6742}{309} > \frac{1}{12}.$$

Thus there is a nonparallel contributing mode with $\varphi(q_\nu) \leq 308$. Its squarefree quotient is at most 1110, and the inequality $59940 = 54 \cdot 1110$, together with the strict hypothesis, makes the second relation exact.

The reciprocal direction is therefore one of the directions in Proposition 9.2. If its polynomial mean is at least 21/200, the nonexact terms in (9.3) have total absolute value less than 6.6742/309, again giving density above 1/12. Otherwise it is one of the 256 residual classes. The exact scale computation in the proposition shows that every scale there with density below 1/12 has smallest denominator at most 14, a contradiction. □

10 Coefficient-sensitive Fourier descent

The constant 59940 comes from applying the largest possible Fourier frequency and one common totient cutoff to every mode. Those two worst cases need not occur simultaneously. We now retain the coefficient and the geometry of each mode separately. This gives a substantially smaller rigorous endpoint.

Pair the nonzero modes in (9.1) under $\nu \leftrightarrow -\nu$. Choose one representative from each pair and write

$$C_\nu = c_\nu + c_{-\nu} = 2c_\nu. \tag{10.1}$$

There are 12672 paired modes. If

$$t_\nu = \frac{N}{(N, \nu \cdot (A,B,C))},$$

then the combined contribution of the pair is exactly

$$C_\nu \frac{\mu(t_\nu)}{\varphi(t_\nu)}. \tag{10.2}$$

It is zero unless t_ν is squarefree. For a nonzero rational number C, put

$$\epsilon(C) = \begin{cases} -1, & C > 0, \\ 1, & C < 0. \end{cases}$$

Thus (10.2) is negative only when $\mu(t_\nu) = \epsilon(C_\nu)$.

Definition 10.1 (Signed inverse-totient profile)**.** For $\epsilon \in \{-1, 1\}$ and a nonnegative rational number X, define

$$\Psi_\epsilon(X) = \min\{\varphi(t) : t \text{ squarefree}, \ \mu(t) = \epsilon, \ t > X\}. \tag{10.3}$$

The minimum in (10.3) is determined by a finite set, despite the absence of an upper bound on t. Indeed, choose a squarefree $t_0 > X$ of the required Möbius sign and put $B = \varphi(t_0)$. A minimizer must satisfy

$$\varphi(t) = \prod_{\ell | t} (\ell - 1) \le B.$$

There are only finitely many squarefree integers with this property: every prime divisor is at most $B + 1$, and the product of the local factors is bounded. Considering this finite set gives $\Psi_\epsilon(X)$ exactly. This observation justifies every use of the profile below without imposing an extraneous bound on the order.

Permute the denominators so that

$$p_0 = \min(p, q, r), \qquad x = \frac{p_0}{p}, \quad y = \frac{p_0}{q}, \quad z = \frac{p_0}{r},$$

and then $1 = x \ge y \ge z > 0$. Fourier symmetry permits the same permutation of every mode. For $\nu = (a, b, c)$, define its ordered reciprocal width

$$w(\nu) = \max\{|a|, \ |a + b|, \ |a + b + c|\}. \tag{10.4}$$

These are precisely the absolute values at the three vertices of

$$\{(1, y, z) : 1 \ge y \ge z \ge 0\}.$$

Lemma 10.2 (Modewise order separation)**.** *Let $P \ge 0$, suppose $p_0 > P$, and suppose $\nu \cdot (A, B, C) \ne 0$. Then*

$$t_\nu > \frac{2P}{w(\nu)}. \tag{10.5}$$

Consequently the contribution of this paired mode is at least

$$-\frac{|C_\nu|}{\Psi_{\epsilon(C_\nu)}(2P/w(\nu))}. \tag{10.6}$$

Proof. The definition of t_ν gives

$$|\nu \cdot (A, B, C)| \ge \frac{N}{t_\nu} = \frac{2L}{t_\nu}.$$

On the other hand,

$$(A, B, C) = \frac{L}{p_0}(1, y, z),$$

so (10.4) gives

$$|\nu \cdot (A, B, C)| \le \frac{L}{p_0} w(\nu).$$

Hence $t_\nu w(\nu) \ge 2p_0 > 2P$, proving (10.5). Only the sign $\epsilon(C_\nu)$ can make (10.2) negative, and (10.3) then gives (10.6). ∎

Define the exact rational residual

$$\mathcal{R}(P) = \sum_{\{\nu,-\nu\}} \frac{|C_\nu|}{\Psi_{\epsilon(C_\nu)}(2P/w(\nu))}. \tag{10.7}$$

If none of the negatively contributing modes is homogeneous, (9.3) and Lemma 10.2 give

$$\rho(p,q,r) \geq c_0 - \mathcal{R}(P). \tag{10.8}$$

Proposition 10.3 (First coefficient-sensitive relation)**.** *If $p_0 > 1752$ and $\rho(p,q,r) < 1/12$, then there is a paired mode with*

$$C_\nu < 0, \qquad \nu \cdot (A,B,C) = 0. \tag{10.9}$$

Proof. Exact evaluation of (10.7) gives

$$c_0 - \mathcal{R}(1752) - \frac{1}{12} = \frac{247319013262580881893247507 9}{263863758850097656250000000000000} > 0. \tag{10.10}$$

At $P = 1751$ the corresponding residual comparison is not positive, so 1752 is the first integer detected by this particular residual. If (10.9) failed, (10.8) and (10.10) would contradict $\rho < 1/12$. □

We next use the geometry left after the first relation. For a primitive mixed-sign normal h, put

$$\mathcal{K}_h = \{x \in [0,1]^3 : \max_i x_i = 1,\ h \cdot x = 0\}. \tag{10.11}$$

For a mode ν not parallel to h, define

$$w_h(\nu) = \max_{x \in \mathcal{K}_h} |\nu \cdot x|. \tag{10.12}$$

The vertices in (10.11) are obtained by fixing two coordinates to 0 or 1 and solving the remaining linear equation. Thus (10.12) is an exact maximum of finitely many rational numbers. The use of the full cube in (10.11), rather than a single ordered simplex, makes the calculation simultaneously valid for every coordinate orientation.

Let

$$m_h = c_0 + \sum_{\nu \in \mathbf{Q}h,\ \nu \neq 0} c_\nu \tag{10.13}$$

be the exact mean of all modes made homogeneous by h. If no independent homogeneous relation holds, the analogue of (10.8) is

$$\rho(p,q,r) \geq m_h - \sum_{\nu \notin \mathbf{Q}h} \frac{|C_\nu|}{\Psi_{\epsilon(C_\nu)}(2P/w_h(\nu))}. \tag{10.14}$$

Proposition 10.4 (Second coefficient-sensitive relation)**.** *If $p_0 > 4510$ and $\rho(p,q,r) < 1/12$, then the reciprocal direction (A,B,C) satisfies two independent homogeneous Fourier relations.*

Proof. Proposition 10.3 supplies a first relation whose line contains a negative paired coefficient. Up to coordinate permutation and overall sign, there are 406 such primitive line orbits. For every one of them, (10.11)–(10.14) are evaluated exactly.

Across all line–mode pairs the smallest nonzero transverse width is

$$\min w_h(\nu) = \frac{2}{21}.$$

For the values $P \leq 59940$ which occur here, the largest strict order query in (10.3) is therefore 1258740. The finite argument following (10.3) reduces all the required profiles to the 262230 squarefree orders of totient at most 262144; each Möbius sign has a witness beyond every query.

Every one of the 406 exact inequalities (10.14) is strictly greater than $1/12$ at $P = 4510$. The last one to become positive is represented by

$$h = (-3, -1, 3), \qquad m_h = \frac{8009338009785455891}{76293945312500000000}.$$

For this line the surplus at 4510 is

$$\frac{108761250901048225078880553420685435 27}{65064116295926061767578125000000000000000} > 0,$$

whereas the same residual comparison at 4509 is negative. Thus, if no second relation existed, (10.14) would contradict $\rho < 1/12$. $\square$

Proposition 10.5 (High-mean directions)**.** *Suppose that the polynomial mean $M_{\mathcal{P}}(A, B, C)$ is at least $21/200$. If $p_0 > 4788$, then $\rho(p, q, r) > 1/12$.*

Proof. All nonhomogeneous modes satisfy Lemma 10.2. Subtracting the residual (10.7), including harmless extra terms for modes which may already be homogeneous, gives

$$\rho(p, q, r) \geq \frac{21}{200} - \mathcal{R}(4788).$$

The exact surplus is

$$\frac{21}{200} - \mathcal{R}(4788) - \frac{1}{12} = \frac{1573813888070309002065790275148634 3}{11559393147238137817382812500000000000000} > 0. \tag{10.15}$$

At 4787, the same rational comparison is negative. $\square$

Theorem 10.6 (Improved large-denominator theorem)**.** *If (p, q, r) is hyperbolic and*

$$\min(p, q, r) > 4788,$$

then

$$\rho(p, q, r) \geq \frac{1}{12}. \tag{10.16}$$

Proof. Suppose that $\rho < 1/12$. Since $4788 > 4510$, Proposition 10.4 forces two independent homogeneous relations. Hence (A, B, C) is one of the 3841327 exact positive directions in Proposition 9.2.

If its polynomial mean is at least $21/200$, Proposition 10.5 is a contradiction. Otherwise it is one of the 256 permutation classes in the residual direction calculation. The exact scale calculation in Proposition 9.2 shows that every density below $1/12$ in those classes has smallest denominator at most 14, again a contradiction. $\square$

11 The upper middle range

Between 1753 and 4788, the first homogeneous relation is already forced, but a second relation need not follow from one uniform estimate. The geometry of the plane cut out by the first relation contains the missing information. Each possible line has its own threshold $T(h)$: below that threshold its integral points can be described directly, while above it a second relation is forced. This gives an exhaustive division into one-relation and two-relation cases.

For a primitive negative Fourier line $h = (a, b, c)$, let $T(h)$ be the smallest-denominator threshold obtained by applying the exact transverse residual (10.14) on $\mathcal{K}_h$. Thus

$$\begin{gathered} p_0 > T(h), \qquad h \cdot (A, B, C) = 0, \qquad \rho(p, q, r) < \frac{1}{12} \\ \Longrightarrow (A, B, C) \text{ has a second independent relation.} \end{gathered} \tag{11.1}$$

There are 2274 oriented primitive negative lines, forming the 406 orbits used in Proposition 10.4; their thresholds are exact integers and satisfy

$$\max_h T(h) = 4510. \tag{11.2}$$

Lemma 11.1 (Signed-divisor parametrization on one line)**.** *Fix p, and suppose a, b, c are nonzero and*

$$\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 0. \tag{11.3}$$

Put

$$g = (a, p), \qquad U = -\frac{a}{g}, \qquad V = \frac{p}{g}.$$

Then

$$(Uq - Vb)(Ur - Vc) = V^2 bc. \tag{11.4}$$

Consequently all ordered positive solutions $p \le q \le r$ of (11.3) are obtained by running through the signed divisors x of $V^2 bc$, putting

$$y = \frac{V^2 bc}{x}, \qquad q = \frac{x + Vb}{U}, \qquad r = \frac{y + Vc}{U}, \tag{11.5}$$

and retaining exactly the integral positive ordered values.

Proof. After multiplication by pqr/g, (11.3) becomes

$$Uqr = V(br + cq).$$

Expanding the left side of (11.4) and substituting this identity gives $V^2 bc$. Conversely, expansion of (11.4) recovers (11.3). The two factors in (11.4) are signed divisors of their product, so (11.5) is exhaustive. ☐

The cases in which one coefficient of h vanishes are one-dimensional: they fix either one denominator or the ratio of the two remaining denominators. In the range in which such a line is not already routed by (11.1), the only resulting families are, up to coordinate permutation,

$$q = 3p, \qquad r = 3p, \qquad r = 3q. \tag{11.6}$$

For the first two families, the exact pair depth of $(p, 3p)$, followed by the finite discrepancy envelope, gives all possibilities for the free denominator. For $r = 3q$, the exact slice mean $W_p(1, 3)$, (5.3), and Lemma 5.4 give all possible scales.

Lemma 11.2 (Direction and scale under two relations)**.** *If two independent Fourier lines vanish on (A, B, C), their cross product determines a positive primitive direction $d = (d_1, d_2, d_3)$. Put $\lambda = \operatorname{lcm}(d_1, d_2, d_3)$. Every denominator triple in this direction has the form*

$$(p, q, r) = \left(\frac{s\lambda}{d_1}, \frac{s\lambda}{d_2}, \frac{s\lambda}{d_3}\right), \qquad s \in \mathbf{N}. \tag{11.7}$$

Moreover

$$\frac{L}{\operatorname{lcm}(p, q)} = (d_1, d_2), \qquad \frac{L}{\operatorname{lcm}(p, r)} = (d_1, d_3). \tag{11.8}$$

Proof. The cross product gives the rational line containing (A, B, C); primitivity selects d. Since (A, B, C) is primitive when $L = \operatorname{lcm}(p, q, r)$, it equals d, and allowing the common denominator scale gives (11.7). The identities in (11.8) follow prime by prime from

$$\operatorname{lcm}(L/d_i, L/d_j) = L/(d_i, d_j).$$

□

For a positive direction d, define its routing threshold

$$T(d) = \max\{T(h) : h \cdot d = 0,\ h \text{ is a negative Fourier line}\}, \tag{11.9}$$

with maximum zero if the set is empty. If $p_0 \leq T(d)$, at least one negative line through d is retained in the one-line branch. If $p_0 > T(d)$, (11.1) puts the triple in the two-line branch. Thus the two branches are exhaustive without duplicating a direction unnecessarily.

Proposition 11.3 (Finite reduction in the upper middle range)**.** *Assume*

$$1753 \leq p = \min(p, q, r) \leq 4788, \qquad \rho(p, q, r) < \frac{1}{12}. \tag{11.10}$$

Then, after ordering $p \leq q \leq r$, the triple belongs to one of the following three exact finite sets.

1. *The signed-divisor calculation (11.5), restricted by $p \leq T(h)$ and by both conditions (3.9), contains* 250722 *generic triples.*
2. *The degenerate families (11.6), which need occur only for* $1753 \leq p \leq 1785$*, contain* 6252 *triples after their exact pair-depth and slice reductions.*
3. *The* 3841327 *positive Fourier directions are restricted by $p > T(d)$, by (11.8), and by $(d_1, d_2), (d_1, d_3) \in \mathcal{E}_{13/120}$. This leaves* 248593 *direction orbits and* 9784295 *scales in (11.7).*

Proof. Proposition 10.3 supplies a negative homogeneous line. If $p \leq T(h)$, Lemma 11.1 gives the complete generic set; a zero coefficient is covered by (11.6). Theorem 3.7 supplies both exact multiplier restrictions (3.9).

If every negative line through the reciprocal direction has threshold below p, (11.1) supplies a second independent line. Lemma 11.2 then gives the complete direction and scale parametrization. The routing rule (11.9), the two gcd tests (11.8), the hyperbolic inequality, and the stated inclusive range for p are all finite integer conditions. Solving them gives the displayed cardinalities. □

It remains to compare the minorant mean with 1/12 on these finite sets. Scale all coefficients in the paired Fourier minorant by their common positive denominator D. Write the scaled constant as C_0 and the 12672 paired scaled coefficients as C_ν. For a retained triple put

$$t_\nu = \frac{N}{(N, \nu \cdot (A, B, C))}.$$

The exact primitive mean of the minorant is

$$\overline{\mathcal{P}} = \frac{1}{D}\left(C_0 + \sum_{\{\nu,-\nu\}} C_\nu \frac{\mu(t_\nu)}{\varphi(t_\nu)}\right). \tag{11.11}$$

Since $t_\nu \mid N$, one can compare (11.11) with 1/12 using the single integer

$$C_0\varphi(N) + \sum_{\{\nu,-\nu\}} C_\nu \mu(t_\nu)\frac{\varphi(N)}{\varphi(t_\nu)}. \tag{11.12}$$

Thus the comparison involves no numerical approximation.

Theorem 11.4 (The upper middle range)**.** *If (p, q, r) is hyperbolic and*

$$1753 \le \min(p, q, r) \le 4788,$$

then

$$\rho(p, q, r) \ge \frac{1}{12}. \tag{11.13}$$

Proof. Apply the exact comparison (11.12) to Proposition 11.3. In the generic one-line branch, 248300 of the 250722 triples have minorant mean at least 1/12. Direct evaluation of (1.1) for the remaining 2422 puts every one strictly above 1/12.

In the degenerate branch, (11.12) settles 6232 of the 6252 triples; direct evaluation puts all remaining 20 strictly above 1/12.

In the routed two-line branch, (11.12) settles 9784051 of the 9784295 triples. Direct evaluation puts all remaining 244 strictly above 1/12. These three exhaustive alternatives contradict (11.10). □

12 Affine Fourier relations below the homogeneous threshold

For smaller p_0, a Fourier mode of small order need not be homogeneous. It nevertheless gives an exact Diophantine relation, and that relation is enough to make the other two denominators finite. This is the natural continuation of the homogeneous descent: one keeps the small order instead of discarding it, and the lost homogeneity reappears as the integer translation on the right side of (12.1).

Lemma 12.1 (A small order is an affine reciprocal relation)**.** *Let $\nu = (a, b, c)$, and let t_ν be its order in (10.2). There is an integer s, with $(s, t_\nu) = 1$, such that*

$$t_\nu\left(\frac{a}{p} + \frac{b}{q} + \frac{c}{r}\right) = 2s. \tag{12.1}$$

If $p = \min(p, q, r)$ and $p \le q \le r$, then necessarily

$$\left\lceil \frac{t_\nu \min(a, a + b, a + b + c)}{2p} \right\rceil \le s \le \left\lfloor \frac{t_\nu \max(a, a + b, a + b + c)}{2p} \right\rfloor. \tag{12.2}$$

Proof. By the definition of t_ν,

$$\nu \cdot (A, B, C) = \frac{N}{t_\nu} s$$

for an integer s coprime to t_ν. Divide by $L = N/2$ to get (12.1). The linear form

$$a + b\frac{p}{q} + c\frac{p}{r}$$

on $1 \geq p/q \geq p/r \geq 0$ has its minimum and maximum at the three vertices, where its values are $a, a+b, a+b+c$. This gives (12.2). $\square$

Fix P. For each paired mode, put

$$Q_\nu(P) = \left\lfloor \frac{2P}{w(\nu)} \right\rfloor.$$

Let $\mathcal{T}_\nu$ be a finite set of squarefree orders of sign $\epsilon(C_\nu)$, all greater than $Q_\nu(P)$, and define

$$d_\nu(P) = \min\{\varphi(t) : t \text{ squarefree},\ \mu(t) = \epsilon(C_\nu),\ t > Q_\nu(P),\ t \notin \mathcal{T}_\nu\}. \tag{12.3}$$

The finite argument following (10.3) determines the minimum and proves that the chosen set $\mathcal{T}_\nu$ contains every omitted order which could improve it.

Proposition 12.2 (Affine relation cover)**.** *Suppose*

$$c_0 - \sum_{\{\nu,-\nu\}} \frac{|C_\nu|}{d_\nu(P)} > \frac{1}{12}. \tag{12.4}$$

If $p_0 > P$ and $\rho(p, q, r) < 1/12$, then either a negative-coefficient mode is homogeneous, or for some paired mode ν and some $t \in \mathcal{T}_\nu$, equation (12.1) holds with $(s, t) = 1$.

Proof. If a negatively contributing mode is not homogeneous, Lemma 10.2 gives $t_\nu > Q_\nu(P)$. If its order does not belong to $\mathcal{T}_\nu$, its contribution is at least $-|C_\nu|/d_\nu(P)$. Summing over the paired modes and using (12.4) contradicts $\rho < 1/12$. A retained order gives (12.1) by Lemma 12.1. $\square$

There is no canonical choice of the sets $\mathcal{T}_\nu$, nor is one needed. Starting from (10.7), include signed orders in increasing totient levels until the strict inequality (12.4) holds. Any finite family obtained in this way proves the proposition; only the displayed rational inequality, and not the manner in which the orders were chosen, enters the argument.

For $P = 1500$, the exact cover retains 1122 mode–order pairs, involving 709 paired modes. Its residual and strict surplus are

$$\mathcal{R}_{\text{aff}}(1500) = \frac{5616202609442974118264625052501}{10147038757324218750000000000000}.$$

$$c_0 - \frac{1}{12} - \mathcal{R}_{\text{aff}}(1500) = \frac{622037999205639622104 27499}{10147038757324218750000000000000} > 0. \tag{12.5}$$

At $p = 1501$, (12.2) leaves only 1107 feasible affine mode–order–shift instances.

Once p is fixed, (12.1) becomes

$$(ta - 2sp)qr + tbpr + tcpq = 0. \tag{12.6}$$

If its three coefficients are nonzero, Lemma 11.1 applies verbatim. A zero coefficient fixes one denominator or fixes a reduced ratio $q : r = x : y$. In the latter case the exact slice estimate is

$$\rho(p, xd, yd) \geq W_p(x, y) - 2y\,\Delta_{2p}\left(\frac{xyd}{(p, xyd)}\right). \tag{12.7}$$

The coarse G_{2p}-inequality first gives a finite modulus set; evaluating the exact $\Delta_{2p}(M) = \delta(M_0)/M_1$ on that set removes every modulus which fails the exact threshold in (12.7). In this way the exact discrepancy on the restricted arithmetic family sharpens the possible scales in the affine relation.

Proposition 12.3 (Finite affine reduction from 1501 to 1752)**.** *Every possible counterexample with*

$$1501 \leq p = \min(p, q, r) \leq 1752$$

belongs to a generic set of 4529273 *triples or a degenerate set of* 843271 *triples, each defined by exact integer conditions.*

Proof. Proposition 12.2, (12.2), and (12.6) give every relation line. Include also all 2274 negative homogeneous lines. Lemma 11.1 gives the generic solutions, and (3.9) removes only triples incompatible with density at most 1/12. The zero-coefficient lines are reduced by exact pair depth or by (12.7). All comparisons are inclusive. Solving the resulting finite divisibility conditions gives the two stated cardinalities. □

Theorem 12.4 (Affine closure of the next band)**.** *If* (p, q, r) *is hyperbolic and*

$$1501 \leq \min(p, q, r) \leq 1752,$$

then

$$\rho(p, q, r) \geq \frac{1}{12}. \tag{12.8}$$

Proof. Apply the integer Fourier comparison (11.12) to the two sets in Proposition 12.3. In each set exactly 220 triples are not settled by the minorant comparison. Direct evaluation of (1.1) puts every remaining triple strictly above 1/12. □

There is one further useful refinement at the lower end. Exactly 18 paired modes depend only on the fixed coordinate. When the smallest denominator is the fixed integer p, their orders are known exactly:

$$t_{(a,0,0)} = \frac{2p}{(2p, a)}. \tag{12.9}$$

Their exact Ramanujan contributions can therefore be added to c_0 before constructing the affine cover, rather than being charged to a uniform absolute-value residual.

Proposition 12.5 (The affine range through 300)**.** *Every hyperbolic triple whose smallest denominator lies in*

$$101 \leq \min(p, q, r) \leq 300$$

has density at least 1/12*.*

Proof. For each value of p, use (12.9) for the 18 fixed-coordinate modes. At $p = 101$, the resulting exact affine cover gives 759759 distinct relation lines. The signed-divisor parametrization and both relative-multiplier restrictions give 135160 generic triples. The exact discrepancy values in (12.7) leave 80632 degenerate triples. The integer comparison (11.12) is at least $1/12$ for every triple in both sets.

Over the complete block $101 \le p \le 190$, the aggregate counts are 29805856 generic triples and 13583157 degenerate triples. There are 77 occurrences in each branch not settled by the Fourier comparison. Direct evaluation of (1.1) puts every one strictly above $1/12$.

For $191 \le p \le 300$, the same argument gives 31300979 generic triples and 13252371 degenerate triples. There are 94 occurrences in each branch not settled by (11.12), and exact evaluation of (1.1) again puts every one strictly above $1/12$. ☐

Proposition 12.6 (The cases 301 and 302)**.** *Every hyperbolic triple whose smallest denominator is* 301 *or* 302 *has density at least* $1/12$*.*

Proof. Use the uniform affine relation family with boundary $P = 300$. Formula (12.2) and the signed-divisor parametrization give 285283 generic triples. Exact pair-depth and discrepancy-filtered slice reduction give 141112 degenerate triples. The integer Fourier comparison (11.12) is at least $1/12$ for every triple in both sets.

For $p = 302$, the same cover gives 153564 generic triples and 89759 degenerate triples. One triple in each branch is not settled by the Fourier comparison; exact evaluation of (1.1) puts both strictly above $1/12$. ☐

Theorem 12.7 (Uniform affine closure through 1500)**.** *Let* (p, q, r) *be hyperbolic and suppose*

$$303 \le \min(p, q, r) \le 1500.$$

Then

$$\rho(p, q, r) \ge \frac{1}{12}. \tag{12.10}$$

Proof. The five rows in the table below use Proposition 12.2 with thresholds

$$P = 300,\ 500,\ 750,\ 1000,\ 1250,$$

respectively. In each row the selected mode–order pairs satisfy the strict rational inequality (12.4). Hence a triple of density below $1/12$ whose smallest denominator lies in the indicated interval satisfies either a retained affine relation (12.6) or a retained negative homogeneous relation.

For a relation having three nonzero coefficients, Lemma 11.1 and its converse enumerate every positive integral solution $p \le q \le r$. A zero coefficient either fixes one denominator or fixes a reduced ratio $q : r = x : y$. The former branch is a finite pair-depth calculation; in the latter, (12.7) and the exact primitive-fiber discrepancy leave a finite set of scales. Thus the generic and degenerate columns below are exhaustive sets, not samples.

The exact integer Fourier comparison (11.12) proves the required lower bound for every entry except the entries in the last column. Formula (7.3) evaluates each of those remaining entries exactly, and every result is strictly greater than $1/12$.

smallest denominator	generic triples	degenerate triples	terminal exact counts
302–500	65216583	26238433	344
501–750	51408631	18008815	432
751–1000	28480477	8451468	438
1001–1250	15172189	3499032	438
1251–1500	8167018	1446173	440

The first row includes the already separated case $p = 302$; removing that duplication gives precisely the interval in the theorem. The five range records check all 1199 values $302 \le p \le 1500$, and their endpoints agree. This proves (12.10). ☐

Corollary 12.8 (The complete finite range)**.** *If* (p, q, r) *is hyperbolic and*

$$2 \leq \min(p, q, r) \leq 1500,$$

then

$$\rho(p, q, r) = 0 \qquad \textit{or} \qquad \rho(p, q, r) \geq \frac{1}{12}.$$

Proof. Use Theorem 8.3 through 100, Proposition 12.5 from 101 through 300, Proposition 12.6 for $301, 302$, and Theorem 12.7 from 303 through 1500. ☐

13 Completion of the argument

We conclude by recording how the two main reductions meet. For a fixed small denominator, Sections 2–5 compare the continuous length of a spherical slice with its primitive point count. Pair depth first restricts the rational direction, and the slice mean then restricts the remaining scale. The valuation identities (4.6) and (5.4) are what make both restrictions exhaustive. Only after this finiteness has been proved does the open-arc formula (7.3) enter. It yields the classification in Theorem 8.3.

For large denominators, the minorant (9.1) and the Ramanujan identity (9.3) turn a deficit in density into a reciprocal relation. The coefficient-sensitive estimates of Section 10 force a second relation when the smallest denominator is large enough. The first relation is retained separately in Section 11, where its plane geometry supplies the line-dependent threshold needed to cover the interval down to 1753. Once two relations are present, their cross product fixes the reciprocal direction and only the scale remains.

Section 12 supplies the overlap between these arguments. A low-order mode which is not homogeneous gives an affine reciprocal relation. Its signed-divisor factorization reduces the generic case to finitely many denominator pairs, while its degenerate cases are exactly the rational slices governed by the discrepancy estimates of Sections 2–5. This closes the interval from 101 to 1752, using separate exact covers on the successive finite bands.

Proof of Theorem 1.1. Put $p_0 = \min(p, q, r)$. If $p_0 \leq 1500$, apply Corollary 12.8. If $1501 \leq p_0 \leq 1752$, apply Theorem 12.4; if $1753 \leq p_0 \leq 4788$, apply Theorem 11.4; and if $p_0 > 4788$, apply Theorem 10.6. These four intervals exhaust all possibilities and prove the gap.

Calegari–Chen prove that the primitive orbit meets the spherical region unless the sorted triple belongs to $\mathcal{Z}$. The direct calculation in Theorem 8.2 verifies that every member of $\mathcal{Z}$ has density zero. This gives the asserted zero classification. The equality classification for $p_0 \leq 100$ is the corresponding assertion of Theorem 8.3. ☐

All finite evaluations used in these final steps take place only after a mathematical finiteness statement has supplied the set in question. The minorant coefficients, discrepancy values, interval endpoints, Ramanujan sums, and final density comparisons are rational or integral. Thus no floating-point inequality or unspecified cutoff enters any of the stated theorems.

References

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