Two routes to profinite rigidity in dimension three: an expository and referee-oriented comparison

A comparison of Xu, arXiv:2608.07350v1,
and the experimental AI-assisted draft `Profinite.tex`

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WARNING: this was created by OpenAI.

Abstract

This report compares Xiaoyu Xu's paper *Profinite rigidity in lattices of* $\mathrm{PSL}_2(\mathbf{C})$, arXiv:2608.07350v1, with an experimental draft entitled *Profinite rigidity of finite-volume hyperbolic* 3*–manifolds*. The second document was produced as an experiment in AI-assisted mathematical research, largely by ChatGPT/Codex under human guidance; it is not being presented here as a competing claim of authorship or priority. The purpose of the comparison is mathematical: to explain the two proof architectures to a reader who is not already an expert, to identify their genuine overlap, to make clear where their central mechanisms differ, and to provide a section-by-section checklist that may be useful when reading or refereeing Xu's paper.

The two arguments share a local geometric bottleneck. Both reduce to a surface-by-cyclic group, obtain a profinitely distinguished simple curve, and recover the splitting of a completed surface group obtained by cutting along that curve. They are nevertheless substantially different proofs. The experimental draft uses representation volume, higher torsion with all finite-group characters, exact descent of individual primitive elements, a cohomological cut class, a transporter theorem, and finite rigidity in the curve complex. Xu uses virtual crossfibering, a profinite cut-and-glue theorem, an internal proof of regularity, drilled orbifolds, semiauthenticity, and algebraic automorphisms of surface character varieties. Xu's conclusion is broader: it treats all lattices, including orbifold lattices, proves that the given profinite isomorphism is genuine, and computes the profinite outer automorphism group. The experimental draft proves several independent arithmetic, spectral, and cohomological intermediate theorems not present in Xu.

Contents

1 Purpose, provenance, and how to use this report

There are two rather different reasons to compare these documents.

First, Xu's paper proves a major theorem and introduces a long chain of new constructions. A reader or referee needs a map of the argument: which parts are standard reductions, which parts are genuinely new, which implications are load-bearing, and where a local mistake would propagate into the main theorem.

Second, the experimental draft arrived at the same headline conclusion by a very different route. Even if that draft is never submitted, it provides useful mathematical contrasts. In particular, it isolates an exact element-marking problem, gives a short cohomological reconstruction of a one-edge surface splitting, and proves a transporter statement that cleanly separates a profinite conjugator from its ordinary mapping-torus time. These ideas can serve as independent checks on several stages of Xu's proof.

Throughout this report:

- X denotes Xu's paper, arXiv:2608.07350v1, submitted on 7 August 2026;
- E denotes the experimental AI-assisted draft `Profinite.tex`;
- references such as "Theorem 9.2" without an additional label refer to Xu's numbering;
- theorem names such as "Exact ambient descent" or "Fixed-edge reconstruction" refer to the named results in E.

A non-expert can read Sections 2–8 as an introduction to the two proofs. A specialist or referee can proceed directly to Sections 9–12, which compare the load-bearing theorems and give a detailed audit of Xu's argument.

2 The problem in plain language

2.1 What a profinite completion remembers

Let Γ be a finitely generated group. Every homomorphism from Γ to a finite group records a finite shadow of Γ. The profinite completion

$$\widehat{\Gamma} = \varprojlim_{N \triangleleft_f \Gamma} \Gamma/N$$

packages all finite quotients, together with all maps between them, into one compact topological group. Two finitely generated groups have isomorphic profinite completions precisely when they have the same finite quotients in the appropriate compatible sense.

The basic rigidity question is:

> Can one recover a hyperbolic 3–manifold, or at least its fundamental group, from these finite shadows?

For a finite-volume orientable hyperbolic 3–manifold M, the group $\pi_1(M)$ is a torsion-free lattice in $\mathrm{PSL}_2(\mathbf{C})$. If orbifolds are allowed, one obtains arbitrary lattices in $\mathrm{PSL}_2(\mathbf{C})$, possibly with torsion. Mostow–Prasad rigidity says that an isomorphism of the discrete lattices recovers the hyperbolic geometry. The difficulty is that an isomorphism

$$\Phi : \widehat{\Gamma_1} \longrightarrow \widehat{\Gamma_2}$$

does not visibly carry the dense copy of Γ_1 onto the dense copy of Γ_2. It matches all finite quotients, but it may send a specific discrete element to an apparently non-discrete profinite element.

2.2 Three levels of possible conclusion

It is useful to distinguish three increasingly strong statements.

1. *Group rigidity:* $\widehat{\Gamma_1} \cong \widehat{\Gamma_2}$ implies $\Gamma_1 \cong \Gamma_2$.
2. *Manifold rigidity:* if $\Gamma_i = \pi_1(M_i)$, the manifolds are homeomorphic, hence isometric up to orientation reversal.
3. *Genuineness of the map:* the particular profinite isomorphism Φ, after composing with an inner automorphism of the target completion, is the completion of a discrete isomorphism $\Gamma_1 \to \Gamma_2$.

The third conclusion is much stronger than the first. It implies, for example, that every outer automorphism of the profinite completion comes from an outer automorphism of the discrete lattice.

2.3 Why fibered manifolds are the natural testing ground

A closed hyperbolic 3–manifold is *fibered* if it is a mapping torus of a pseudo-Anosov homeomorphism $\varphi : S \to S$ of a closed surface:

$$M_\varphi = S \times [0,1]/(x,1) \sim (\varphi(x),0).$$

Its fundamental group has the form

$$\Gamma_\varphi = \Pi \rtimes_\varphi \mathbf{Z}, \qquad \Pi = \pi_1(S).$$

The normal subgroup Π is the fiber group, and the quotient $\mathbf{Z}$ records how many times one travels around the base circle. In the profinite completion the quotient becomes $\widehat{\mathbf{Z}}$. This creates a central difficulty: an automorphism of $\widehat{\mathbf{Z}}$ can multiply by any unit in $\widehat{\mathbf{Z}}^\times$, not merely by ± 1. Thus a profinite isomorphism can appear to move a periodic orbit by a mysterious "profinite power."

A second difficulty appears inside the fiber. A simple closed curve $c \subset S$ determines a splitting of $\pi_1(S)$ over the cyclic group $\langle c \rangle$. For a nonseparating curve this is an HNN splitting; for a separating curve it is an amalgam. Recognizing that splitting after profinite completion is the local geometric bottleneck shared by both proofs.

2.4 A small dictionary

Term	**Meaning in this comparison**
Fibered class	A homomorphism $\pi_1(M) \to \mathbf{Z}$ coming from a fibration $M \to S^1$. Its kernel is a surface group.
Regularity	The induced map on the completed cyclic quotient is multiplication by ± 1, rather than by an arbitrary unit of $\widehat{\mathbf{Z}}$.
Genuine isomorphism	A profinite isomorphism induced, up to inner automorphism, by an isomorphism of the discrete groups.
Periodic trajectory	A closed orbit of the pseudo-Anosov suspension flow of a fibered hyperbolic 3–manifold.
Cut-and-glue / fixed-edge reconstruction	Recovery, from profinite data, of the surface splitting obtained by cutting along a simple curve.
Semiauthenticity	Xu's condition that a group-generating semigroup maps, element by element, to profinite conjugates of discrete target elements.
Marked spectral data	In E, the complex length of a primitive geodesic together with the conjugacy class of its image in a chosen finite quotient.
Character of a finite group	The trace of a finite-dimensional representation of a finite quotient; used in E to separate conjugacy classes.
Character variety	An algebraic parameter space of SL_n representations of an infinite group; used in X to realize a profinite surface automorphism.

3 What the two principal theorems say

3.1 The experimental draft

The main theorem of E is formulated for orientable finite-volume hyperbolic 3–manifolds and compact orientable 3–manifold competitors with empty or toral boundary. It concludes that the competitor is homeomorphic to a compact core of the hyperbolic manifold. In particular, two finite-volume hyperbolic 3–manifolds with isomorphic profinite completions are isometric up to orientation reversal.

The draft is written for torsion-free manifold groups. It does not claim that the specified profinite isomorphism is genuine. Its strongest intermediate conclusion of this type is elementwise: every primitive element on one side is profinitely conjugate to a discrete primitive element on the other side, up to inversion.

3.2 Xu's paper

Xu proves that every lattice in $\mathrm{PSL}_2(\mathbf{C})$ is profinitely rigid among all such lattices. This includes orbifold lattices with torsion. He proves more:

$$\mathrm{Out}(\Gamma) \xrightarrow{\sim} \mathrm{Out}(\widehat{\Gamma})$$

for every lattice $\Gamma \leq \mathrm{PSL}_2(\mathbf{C})$. The operative statement in the proof is that every profinite isomorphism between lattices is genuine. Corollary 1.8 then treats finitely generated groups virtually isomorphic to lattices among finitely generated virtually 3–manifold groups.

Thus Xu's final theorem is broader both in objects and in control of the map. If Xu's theorem is correct, it subsumes the headline manifold-rigidity conclusion of E. It does not subsume the independent representation-volume, marked-spectrum, transporter, or cut-class theorems proved en route in E.

	Experimental draft E	**Xu's paper X**
Source objects	Orientable finite-volume hyperbolic 3–manifolds; torsion-free lattices.	All lattices in $\mathrm{PSL}_2(\mathbf{C})$, including orbifold lattices with torsion.
Target class	Compact connected orientable 3–manifolds with empty or toral boundary.	Lattices for Theorem 1.2; finite extensions and virtually 3–manifold groups in Corollary 1.8.
Primary conclusion	Homeomorphism of compact cores; isometry in the hyperbolic case.	Abstract lattice isomorphism and genuineness of the specified profinite isomorphism.
Outer automorphisms	No theorem identifying $\mathrm{Out}(\widehat{\Gamma})$.	Proves $\mathrm{Out}(\widehat{\Gamma}) \cong \mathrm{Out}(\Gamma)$.
Volume and holonomy	Proved internally, including Galois-conjugate representation volumes.	Not used and not proved.
Element marking	Every primitive element descends exactly, up to conjugacy and inversion.	A generating semigroup descends elementwise; the whole map is then shown genuine.

	Experimental draft E	Xu's paper X
Surface realization	A finite rigid configuration of curves determines the monodromy.	Character varieties and characteristic covers realize the whole surface-group isomorphism.
Length	Roughly twenty-five compiled pages in the supplied draft.	Ninety-six pages in v1.

4 The two proof architectures at a glance

The two proofs solve the same problem by attacking different intermediate questions.

4.1 The experimental route

$$\begin{aligned}
&\text{finite-field Bloch classes and representation volume}\\
&\quad\Longrightarrow \text{holonomy or antiholonomy recognition}\\
&\quad\Longrightarrow \text{comparison of all finite-character canonical torsions}\\
&\quad\Longrightarrow \text{finite-quotient marked primitive length spectrum}\\
&\quad\Longrightarrow \text{exact profinite conjugacy to discrete elements}\\
&\quad\Longrightarrow \text{cohomological reconstruction of a cut surface}\\
&\quad\Longrightarrow \text{integral-time propagation through a rigid curve set}\\
&\quad\Longrightarrow \text{conjugate monodromies and profinite rigidity.}
\end{aligned}$$

The key philosophy is to make enough individual elements discrete and then synchronize them.

4.2 Xu's route

$$\begin{aligned}
&\text{virtual crossfibering}\\
&\quad\Longrightarrow \text{a periodic orbit lying in a second fiber}\\
&\quad\Longrightarrow \text{profinite cut-and-glue and internal regularity}\\
&\quad\Longrightarrow \text{drilled orbifolds with retractions to the fibers}\\
&\quad\Longrightarrow \text{virtually hereditary bi-semiauthenticity}\\
&\quad\Longrightarrow \text{algebraic maps of surface character varieties}\\
&\quad\Longrightarrow \text{a genuine surface-group isomorphism}\\
&\quad\Longrightarrow \text{a genuine lattice isomorphism, uniformly and cusped.}
\end{aligned}$$

The key philosophy is to obtain enough elementwise discreteness to build an algebraic automorphism of a character variety, then recover the entire map at once.

4.3 The shared bottleneck

Both proofs eventually reach the following local situation. There is an isomorphism between completed closed surface groups and a simple closed curve c whose procyclic subgroup is carried to the corresponding procyclic subgroup on the other side, perhaps with a unit power. One must recover the completed surface obtained by cutting along c, together with the way the two boundary components are glued.

The proofs of this local statement are related in spirit but different in form:

- E attaches a canonical cohomology class to the edge and proves that this class reconstructs the whole profinite tree.
- X studies fixed vertices and stabilizer intersections in the profinite Bass–Serre tree, using a profinite PD^2 fixed-point theorem.

This is the most important similarity between the two arguments. It is not, however, enough to make the proofs "essentially the same."

5 A guided tour of the experimental draft

This section explains the proof in E without assuming familiarity with Bloch groups, analytic torsion, or profinite cohomology. The technical details matter, but the role of each step is more important for comparison.

5.1 Step 1: reduce to closed fibered manifolds

The draft uses existing 3–manifold results to isolate the genuinely new case. Profinite almost-rigidity and geometry detection reduce an arbitrary compact 3–manifold competitor to a finite-volume hyperbolic one. A recent normalizer theorem reduces closed lattices to suitably chosen closed fibered finite-index lattices. Peripheral regularity and corresponding Dehn fillings reduce the cusped case to the closed case.

At the new core of the proof, therefore, one has two pseudo-Anosov mapping-torus groups

$$\Gamma_X = \Pi_X \rtimes_{\varphi_X} \mathbf{Z}, \qquad \Pi_X = \pi_1(S_X),$$

and an isomorphism of their profinite completions that identifies the completed fiber groups. After reversing the target fibration if necessary, the map on the completed cyclic quotient is the identity.

5.2 Step 2: recognize the correct holonomy local system

Higher Reidemeister torsion is built from linear representations of the fundamental group. The useful representations here are the even symmetric powers of the hyperbolic holonomy representation. A profinite isomorphism, however, does not automatically tell us which algebraic representation on the other side is holonomy.

The draft solves this through algebraic K–theory. A hyperbolic representation determines a Bloch class, and the Borel regulator of that class records signed representation volume. Liu's representation correspondence compares residual representations at many finite places, while finite-field Chern classes detect the global Bloch class modulo large odd primes. The conclusion is that corresponding representations have the same Galois-conjugate volumes up to one common sign. Volume rigidity then forces the representation corresponding to holonomy to be holonomy or antiholonomy.

For the rest of the proof, this step has a simple meaning: the two mapping tori now carry genuinely corresponding canonical local systems.

5.3 Step 3: compare all finite-character twisted torsions

Liu's local comparison initially has a quadratic ambiguity on the full group. If ρ' differs from ρ by a quadratic character ξ, then

$$\mathrm{Sym}^{2r}(\xi\rho) = \mathrm{Sym}^{2r}(\rho).$$

Thus even symmetric powers remove the ambiguity. After tensoring by any representation of any finite quotient, the resulting local systems agree on the full completed mapping-torus groups and on every corresponding finite cover.

The cochain complex of a mapping torus is a cone of $1 - F$, where F is the monodromy action on the fiber cohomology. Goodness of surface groups and an inverse-limit argument compare these cohomology groups and the operator F. Consequently, all absolute canonical torsions with finite-group twists agree.

The point is not merely to produce many numerical invariants. The finite-group twists will mark each closed geodesic by its image in a finite quotient.

5.4 Step 4: recover a finite-quotient marked length spectrum

A primitive closed geodesic c has a complex length $\lambda(c) = \ell(c) + i\theta(c)$ and, for a finite quotient $q : \Gamma \to Q$, a label $[q(c)]$ in the set of conjugacy classes of Q. The draft considers Ruelle products twisted by every irreducible character of Q.

Why do all characters matter? Permutation characters naturally see the cyclic subgroup generated by $q(c)$. Irreducible complex characters span all class functions, so they distinguish the conjugacy class of the element $q(c)$ itself. This is precisely what is needed to remove a profinite unit-power ambiguity.

There are two technical obstructions.

1. The available torsion formula is first stated for finite-order line characters. Based Shapiro–Artin formalism and Brauer induction extend it to arbitrary finite-group characters, including nonnormal covers.
2. The logarithm of an Euler product contains all powers c^m. The label on that term is $q(c)^m$, so ordinary scalar Möbius inversion is not enough. The draft uses the Adams operation $\psi_m\chi(g) = \chi(g^m)$ and a compatible Möbius inversion to remove proper powers while retaining their finite-group labels.

A Cauchy-transform uniqueness theorem then recovers the marked primitive measure

$$\sum_{c \text{ primitive}} \delta_{(e^{-\lambda(c)},[q(c)])}$$

with multiplicity, modulo the unavoidable simultaneous operation of complex conjugation and inversion.

5.5 Step 5: compactness gives exact ambient descent

Fix a primitive element a on the source side. Only finitely many primitive geodesics on the target side have the same complex length. For each finite quotient, marked spectral reconstruction shows that the image of a is conjugate to the image of one of these finitely many target geodesics, possibly inverted.

Applying the theorem to products of finite quotients makes these choices compatible. A finite-intersection argument selects one fixed target geodesic b that works in every finite quotient. A second compactness argument selects one profinite conjugator u such that

$$\Phi(a) = ub^{\pm 1}u^{-1}$$

in the full profinite completion. If a lies in the fiber, then b also lies in the fiber because its ordinary integer coordinate maps to zero in $\widehat{\mathbf{Z}}$.

This is the theorem called *Exact ambient descent.* It is stronger than what the final manifold theorem alone requires: it applies to every primitive element.

5.6 Step 6: reconstruct the cut surface from a cohomology class

Let $G = \widehat{\pi_1(S)}$, let c be a primitive simple curve, and let $C = \overline{\langle c \rangle}$. The standard profinite Bass–Serre tree T_c has edge set G/C. Choosing a vertex v gives a cocycle

$$z_{T,v} : G \longrightarrow \mathbf{F}_2[[G/C]]$$

characterized by the boundary equation

$$\partial z_{T,v}(g) = gv - v.$$

For a finite tree this cocycle is simply the mod-2 sum of edges in the path from v to gv. The profinite version is a completed “which side of the cut” record.

The proof has two decisive calculations.

1. Completed finite-free Poincaré duality and Shapiro give

$$H^1_{\mathrm{cts}}(G, \mathbf{F}_2[[G/C]]) \cong \mathbf{F}_2.$$

   Thus every nonzero cut class is the same cohomology class.

2. Profinite malnormality implies

$$\mathbf{F}_2[[G/C]]^C = \mathbf{F}_2 e,$$

   where e is the distinguished edge. Hence two based cut cocycles differ only by 0 or by the coboundary of e. The second possibility merely switches the two endpoints of the distinguished edge.

After this possible endpoint switch, the actual cocycles agree. The two vertex orbits are embedded in the edge module by

$$gv \longmapsto z(g), \qquad gw \longmapsto z(g) + ge.$$

These coordinates reconstruct the vertex space, edge space, incidence maps, and the full G–action. This is the *Fixed-edge reconstruction* theorem.

5.7 Step 7: turn profinite times into integers

Exact descent gives a discrete target curve but only after conjugation by an element of the ambient mapping-torus completion. That conjugator may have a coordinate in $\widehat{\mathbf{Z}}$ that is not visibly an integer.

The transporter theorem says that if G is hyperbolic and virtually compact special and H is quasiconvex, then

$$\mathrm{Tran}_{\widehat{G}}(a, \overline{H}) = \overline{\mathrm{Tran}_G(a, H)}.$$

For a complementary subsurface subgroup H, a discrete transporter of time n can exist only when the corresponding monodromy iterate lies at curve-graph distance at most one from the cutting curve. A pseudo-Anosov acts loxodromically, so only finitely many integers n occur. The closure of this finite set in $\widehat{\mathbf{Z}}$ is the same set. Therefore the profinite time is an ordinary integer.

5.8 Step 8: propagate one anchor and recover the monodromy

Fixed-edge reconstruction places every curve disjoint from the anchored curve inside the correct completed complementary-subsurface group. Exact descent and the transporter theorem turn it into an ordinary simple curve. Repeating this along a connected finite graph of disjointness propagates discreteness through a finite curve configuration.

Wang–Xu recognizes the discrete targets as simple curves. The Boggi–Zalesskii profinite intersection pairing shows that disjoint curves remain disjoint. An Aramayona–Leininger finite rigid set is therefore realized by a homeomorphism of the fibers. Including a finite Alexander marking and its monodromy image forces the two monodromies to be conjugate up to inversion.

5.9 Independent outputs of the experimental route

Even if one ignores the final rigidity theorem, the route produces several stand-alone statements:

- a Galois-equivariant representation-volume comparison;
- unconditional holonomy/antiholonomy recognition;
- finite-character reconstruction of a marked primitive length spectrum;
- exact ambient descent for every primitive element;
- a quasiconvex transporter theorem;

- a cohomological uniqueness theorem for one-edge profinite surface splittings.

None of these is a theorem of Xu's paper in the same form.

6 A guided tour of Xu's proof

Xu's proof is longer because it proves a stronger conclusion and develops an independent surface-realization machine. The central innovation is not exact spectral marking of every element. It is the passage from partial elementwise discreteness to a genuine isomorphism of the whole surface group.

6.1 Step 1: construct a virtual crossfibered cover

A crossfibered closed hyperbolic 3–manifold carries two distinct fibered classes. One is called axial and one facial. A periodic orbit of the axial pseudo-Anosov flow is freely homotopic into the facial fiber.

This arrangement connects two kinds of profinite information already present in Liu's work:

- fibered classes and completed fiber subgroups are profinitely detectable;
- periodic trajectories of the pseudo-Anosov flow are profinitely aligned.

Theorem 3.2 proves that every closed hyperbolic 3–manifold has a finite crossfibered cover. The proof uses virtual fibering, large first Betti number, Agol's RFRS technology, the polyhedral geometry of fibered cones, and Fried's description of periodic-orbit homology directions.

The practical output is a periodic orbit whose conjugacy class lies inside a second surface subgroup. This gives a distinguished procyclic subgroup in a profinite surface group.

6.2 Step 2: cut and glue in a profinite surface group

Xu's Theorem 9.2 treats an isomorphism between completed closed surface groups that sends the procyclic subgroup generated by a nonseparating simple curve to a unit power of the corresponding subgroup on the target. The theorem recovers an isomorphism of the completed complementary-surface groups and controls the HNN stable letter.

The proof uses the standard profinite Bass–Serre trees. The action is 1–acylindrical because distinct conjugates of a simple-curve procyclic subgroup have trivial intersection. The main task is to show that the completed complementary-surface group fixes a vertex in the target tree.

Xu passes to a finite cover of the surface with boundary, kills all peripheral subgroups, and obtains the completion of a closed surface group acting on a quotient tree with trivial edge stabilizers. A fixed-point theorem for profinite PD^2 groups forces a vertex fixed point. Applying the same argument in the opposite direction identifies the full vertex stabilizers. Finally, in a 1–acylindrical tree, two vertices are adjacent exactly when their stabilizers intersect nontrivially. This reconstructs the tree and then the HNN data.

Theorem 9.2 is the closest analogue of Fixed-edge reconstruction in E. Xu's proof is action-theoretic; the experimental proof is cohomological.

6.3 Step 3: use the cut theorem to prove regularity

Liu's trajectory alignment naturally permits a unit $\mu \in \widehat{\mathbf{Z}}^{\times}$. Xu's cut-and-glue theorem records how this unit scales the fundamental class of the facial surface. Liu's results also give a scaling on the mapping-torus fundamental class and on the fibration class. Naturality of cap product relates the three scalars.

Schematically, if the H_3 class scales by λ_3, the H_2 surface class by λ_2, and the fibration class by μ, then

$$\lambda_3 = \lambda_2 \mu.$$

The external trajectory comparison gives $\lambda_3 = \pm\mu^3$, while the cut calculation gives $\lambda_2 = \pm\mu$. Therefore $\mu = \pm 1$. This proves regularity in Theorem 10.2.

This step has no analogue inside E, which takes regularity from Hanany and normalizes the sign by reversing the target fibration.

6.4 Step 4: drill two curves in a product and retain retractions

Xu next considers two simple curves on a surface, drills them at different levels from $\Sigma \times S^1$, and studies the resulting link exterior. The cut-and-glue theorem is used twice to construct an isomorphism of profinite completions of these link-exterior groups, compatible with natural retractions onto the surface groups and with the peripheral meridians.

When the curves fill the surface, the link exterior is hyperbolic. Orbifold Dehn fillings along large meridian powers produce uniform lattices while preserving retractions to the surface groups. These retractions are the bridge that transfers information about periodic trajectories in 3–manifold groups back to the profinite surface-group isomorphism.

This drilled-orbifold construction is a major component of Xu's proof and has no counterpart in E.

6.5 Step 5: semiauthenticity

A homomorphism between profinite completions is semiauthentic if there is a semigroup that generates the source group and whose elements map, one by one, to profinite conjugates of discrete target elements. "Bi-semiauthentic" means the condition holds in both directions. "Virtually hereditarily" means it continues to hold after passing to a suitable finite-index subgroup and all further finite-index subgroups.

Why use a semigroup rather than all elements? A Markov partition for a pseudo-Anosov flow supplies a transition graph whose directed cycles represent periodic trajectories and generate the mapping-torus group as a group. Liu's trajectory correspondence, combined with regularity, makes those cycle elements discrete up to profinite conjugacy. The drilled-orbifold retractions transfer this property to the facial surface group.

Lemma 12.16 concludes that the relevant profinite surface-group isomorphism is virtually hereditarily bi-semiauthentic.

This is weaker than exact ambient descent for every primitive element, but it is tailored to the next algebraic step.

6.6 Step 6: semiauthentic maps act on character varieties

For a finitely generated group Γ, its SL_n character variety $X_n(\Gamma)$ parametrizes characters of semisimple SL_n representations. Theorem 13.5 says that a semiauthentic profinite homomorphism canonically induces an algebraic map of character varieties.

The construction is p-adic. Start with an algebraic representation of the target group. At a suitable finite place it becomes integral and extends continuously to the profinite completion. Pull it back through the profinite homomorphism. On each element of the generating semigroup, semiauthenticity identifies the trace with the trace of a discrete target element. Since the trace functions on a generating semigroup generate the character algebra, the pulled-back character is algebraic and the construction defines a regular map.

Functoriality is proved for compositions, and a bi-semiauthentic isomorphism therefore gives an algebraic isomorphism of character varieties.

This is entirely different from the finite-group character theory in E. There, characters label geodesics in finite quotients. Here, characters are coordinates on an algebraic moduli space of surface-group representations.

6.7 Step 7: realize the surface map

For a closed surface of genus at least three, Marché–Simon determine the algebraic automorphism group of the $\mathrm{SL}_2(\mathbf{C})$ character variety. It is generated by the extended mapping class group and central sign twists from $H^1(S; \mathbf{Z}/2)$. The automorphism induced by a profinite isomorphism fixes the trivial character, so the sign-twist component vanishes. The resulting character-variety automorphism therefore comes from a mapping class.

Xu then passes through a cofinal family of standard characteristic covers. Functoriality of the character-variety construction forces compatibility with deck transformations. Centerless extension arguments descend the mapping classes from the covers to the base surface. Agreement on all characteristic finite quotients finally shows that the original profinite surface isomorphism is genuine. This is Theorem 14.1.

The use of finite covers also avoids the genus-two hyperelliptic kernel.

6.8 Step 8: recover all lattices and the specified map

For a crossfibered closed manifold, the preceding steps make the facial surface map genuine. A centerless extension criterion then makes the whole mapping-torus map genuine. Virtual crossfibering and a finite-index genuineness theorem treat all uniform lattices, including orbifold lattices.

For nonuniform lattices, Xu uses peripheral regularity and factorial orbifold Dehn fillings. Every fixed finite quotient of the cusped group eventually factors through these fillings. The uniform theorem makes the induced maps on the filled lattices genuine. Compatibility through all characteristic quotients then makes the original cusped profinite isomorphism genuine.

This final result is stronger than recovering an isometry of the manifolds: it identifies the given profinite map, up to inner automorphism, with a discrete isomorphism.

7 Where the proofs genuinely overlap

7.1 A common reduction to a surface-by-cyclic problem

Both arguments exploit the fact that fibered classes and completed fiber subgroups can be detected profinitely. Both eventually study an isomorphism of completed groups with a normal completed surface subgroup and a completed cyclic quotient. Both need the quotient map to be regular, and both use pseudo-Anosov dynamics to obtain distinguished conjugacy classes.

This shared setup comes largely from the same background literature, especially Liu's work. It should not be mistaken for a shared proof of the new theorem.

7.2 A simple-curve anchor

Both arguments need one element that is not merely discrete but geometric: it must represent a simple closed curve in a fiber.

- In E, exact ambient descent makes a primitive fiber element discrete, and Wang–Xu recognizes it as a simple curve.
- In X, crossfibering places a periodic orbit inside the facial fiber; a characteristic-cover lemma makes all its relevant elevations nonseparating simple curves.

Thus the two proofs arrive at the same local input by different routes.

7.3 Reconstructing the completed cut surface

Let c be the anchored curve and T_c the standard profinite Bass–Serre tree. This is the sharpest comparison.

	Experimental fixed-edge theorem	**Xu Theorem 9.2**
Basic invariant	A cocycle in $H^1(G, \mathbf{F}_2[[G/C]])$.	The action of the completed complementary-surface group on the target profinite tree.
Uniqueness input	The cohomology group is one-dimensional.	A profinite PD^2 group acting with trivial edge stabilizers fixes a vertex.
Malnormality input	Computes the C–fixed completed edge chains as $\mathbf{F}_2 e$.	Gives 1–acylindricity and characterizes intersections of vertex stabilizers.
How vertices are recovered	Explicit affine coordinates $gv \mapsto z(g)$ and $gw \mapsto z(g) + ge$.	Identify full vertex stabilizers in both directions; recover adjacency from nontrivial intersections.
Power ambiguity	Removed before the theorem is applied.	A unit power is allowed and its effect on H_2 is recorded.
Range	One theorem handles HNN and amalgam patterns once the tree hypotheses are satisfied.	Written for the nonseparating HNN case used later; remarks indicate extensions.

	Experimental fixed-edge theorem	Xu Theorem 9.2
Extra output	Canonical cut class and abstract uniqueness of one-edge actions.	Precise control of the HNN stable letter and homological scaling.

Both proofs are modern profinite versions of the classical relation between relative ends, almost-invariant sets, first cohomology, and splittings. One is a cohomological reconstruction; the other is a fixed-point and stabilizer reconstruction.

7.4 Finite covers and malnormality

Both arguments use characteristic covers to control simple curves, the fact that surface subgroups and cyclic curve subgroups carry their full profinite topology, and profinite malnormality of simple-curve cyclic subgroups. These are shared technical tools, not evidence that the main engines coincide.

7.5 Dehn filling

Both proofs use Dehn filling in the cusped case, but with different objectives. The experimental draft needs only to recover the unfilled manifolds: match long closed fillings, use closed rigidity, identify the short core link, and drill. Xu must prove the specified profinite map genuine. He therefore uses factorial orbifold fillings so that every finite quotient is eventually seen by a filling, and tracks genuine maps through characteristic quotients.

8 Where the proofs fundamentally differ

8.1 Two solutions to the profinite power problem

Liu's trajectory correspondence naturally leaves a unit $\mu \in \widehat{\mathbf{Z}}^{\times}$.

- E removes μ analytically. All irreducible characters of every finite quotient determine the conjugacy class of the finite image of the trajectory, so only the element or its inverse survives.
- X removes μ homologically. Crossfibering places the trajectory in a second surface, the cut theorem calculates its action on the surface fundamental class, and cap-product scaling forces $\mu = \pm 1$.

The two mechanisms solve the same obstruction but do not share a proof.

8.2 Local exact descent versus global genuineness

The experimental theorem

$$\Phi(a) = ub^{\pm 1}u^{-1}$$

for every primitive a is a strong local statement. The conjugator u may depend on a. The rest of the proof selects finitely many elements and synchronizes their conjugators geometrically.

Xu initially asks for much less: only a group-generating semigroup must map elementwise to discrete conjugacy classes. He then uses this partial information to build a character-variety automorphism and proves that the entire map is genuine. Thus Xu's intermediate elementwise theorem is weaker, but his final map-level theorem is stronger.

8.3 The word "character" refers to different mathematics

In E, a character is an irreducible character of a finite group Q. The finite character is inserted into a Ruelle product to recover the finite label $q(c)$ of a geodesic.

In X, a character is the trace function of an SL_n representation of a surface group. These characters form an algebraic variety, and automorphisms of that variety are used to recover a mapping class.

The only common feature is that traces turn conjugacy-sensitive data into functions. The representation theories, inputs, and conclusions are otherwise different.

8.4 Volume and holonomy

Volume is load-bearing in E. It identifies holonomy, supplies the canonical local systems, and appears in the torsion-to-Ruelle formula. The draft therefore proves a substantial representation-volume theorem.

Xu's proof avoids representation volume. It never needs to identify the holonomy representation selected by Liu's representation correspondence. The only character variety used is the surface-group character variety, and its role is realization of a mapping class, not hyperbolic geometry.

8.5 Regularity

The experimental route imports regularity from Hanany. Xu proves regularity inside the paper. This is one reason Xu's argument is longer, but it also makes his theorem less dependent on the recent normalizer/regularity package used in E.

8.6 The final surface theorem

The experimental route reconstructs only a finite configuration of simple curves. Wang–Xu gives simplicity, Boggi–Zalesskii gives disjointness, and Aramayona–Leininger gives finite rigidity.

Xu reconstructs the whole surface-group isomorphism. Semiauthenticity gives an algebraic automorphism of the SL_2 character variety; Marché–Simon makes it a mapping class; characteristic-cover compatibility makes the original profinite map genuine.

8.7 The closed nonfibered case

The experimental route uses the normalizer reduction of Belolipetsky–Cheetham-West and the inheritance theorem of Bridson–Reid.

Xu instead passes to a virtual crossfibered cover and proves a finite-index criterion for genuineness. His reduction therefore supports the stronger outer-automorphism theorem and works naturally for orbifold lattices.

9 A theorem-by-theorem correspondence

Problem	**Experimental draft** E	**Xu's paper** X
Regularity of the cyclic quotient	External input from Hanany, followed by reversal of the target fibration.	Theorem 10.2, proved from crossfibering, cut-and-glue, fundamental classes, and cap products.
Recognize one periodic orbit	Finite-character marked spectral inversion and exact ambient descent.	Liu trajectory alignment gives a unit power; internal regularity makes it ± 1.
Recognize arbitrary primitive elements	Exact ambient descent applies to every primitive element.	No analogous intermediate theorem; final genuineness gives the conclusion afterwards.
Obtain a simple curve	Wang–Xu after exact descent.	Crossfibered orbit lies in a facial fiber; Lemma 9.19 passes to a characteristic cover where elevations are nonseparating simple curves.
Recover the cut splitting	Cut cocycle, one-dimensional compact cohomology, fixed-chain uniqueness, affine reconstruction.	Theorem 9.2: fixed vertices, profinite PD^2, 1–acylindricity, stabilizer intersections.
Track a unit power on the edge group	Eliminated before surface reconstruction.	Retained and shown to scale the surface fundamental class by $\pm\mu$.
Control ambient mapping-torus times	Quasiconvex transporter theorem plus loxodromicity of the curve graph.	Bypassed by drilled retractions, semiauthenticity, and global realization.
Realize the surface topology	Finite rigid set in the curve complex.	Automorphism of the SL_2 character variety plus characteristic-cover descent.
Closed nonfibered case	Normalizer reduction to a fibered lattice.	Virtual crossfibering plus a finite-index genuineness criterion.
Cusped case	Corresponding long fillings, short-core recognition, drilling.	Peripheral regularity, factorial orbifold fillings, and compatibility on all characteristic quotients.
Control the specified map	Not claimed globally.	Central conclusion: every profinite isomorphism is genuine.
Outer automorphisms	No analogue.	$\mathrm{Out}(\widehat{\Gamma}) \cong \mathrm{Out}(\Gamma)$.
Representation volume	Major internal theorem.	No counterpart.

10 What a reader of Xu can learn from the experimental draft

The experimental route does not replace Xu's proof, but it offers several useful alternative viewpoints.

10.1 A conceptual model for Section 9

The cut class packages a one-edge splitting into a single cohomology class. The calculation

$$H^1(G, \mathbf{F}_2[[G/C]]) \cong \mathbf{F}_2, \qquad \mathbf{F}_2[[G/C]]^C = \mathbf{F}_2 e$$

explains conceptually why there should be only one standard cut tree with the given embedded edge subgroup, up to switching endpoints. Xu's Section 9 proves more tailored HNN information, but the cut class supplies an independent sanity check on the expected uniqueness statement and suggests extensions to separating curves and multicurves.

A referee checking Theorem 9.2 may find it useful to compare every action-theoretic step with the following cohomological expectation: once the edge subgroup is identified and the cut class is nonzero, no exotic one-edge tree should remain.

10.2 A sharper formulation of the element-marking obstruction

Xu's semiauthenticity deliberately asks for a generating semigroup, not all elements. The experimental marked-spectrum theorem shows what stronger data would remove the ambiguity for every primitive element. This clarifies exactly what is lost when one uses only rational permutation characters or cyclic subgroup data: an uncontrolled unit power in $\widehat{\mathbf{Z}}^\times$.

Even if the analytic machinery is too heavy to insert into Xu's proof, it provides a useful conceptual benchmark. Semiauthenticity is not simply a weak version of a known discrete map; it is a carefully chosen substitute for a much harder exact element-descent theorem.

10.3 The transporter theorem isolates a separate issue

After a curve is known to be discrete, there remains a different problem: the ambient conjugator can have nonintegral profinite time. The transporter theorem shows that this is a quasiconvex subgroup-separability problem, not part of spectral reconstruction. This separation is pedagogically useful when reading Xu, where the same obstruction is avoided globally through semiauthenticity and character varieties rather than solved directly.

10.4 Independent profinite invariants

The representation-volume and marked-length results in E are independent of Xu's realization theorem. A reader interested in what geometric data the finite quotients determine may learn more from these intermediate results than from the final rigidity statement alone. Xu proves that the group and map are recoverable; the experimental route attempts to explain how individual geodesics, their finite labels, and representation volumes are encoded.

11 What the experimental route can learn from Xu

The comparison also runs strongly in the other direction.

11.1 Genuineness is the natural strongest target

The experimental draft stops after recovering the manifold and conjugating the monodromy. Xu shows that the natural endpoint is stronger: identify the actual profinite isomorphism. This immediately yields the outer-automorphism theorem and better compatibility under finite extensions.

11.2 Semiauthenticity is an efficient abstraction

The experimental route attempts exact descent for every primitive element. Xu shows that one can instead isolate a generating semigroup of dynamical elements, prove elementwise discreteness only there, and let the character variety enforce global consistency. This is conceptually economical even though the geometric construction required to prove hereditary bi-semiauthenticity is long.

11.3 Character varieties solve the synchronization problem globally

Finite rigid sets synchronize finitely many curves. Xu's character-variety method synchronizes all trace functions and then all characteristic covers. It is better adapted to proving that a specified profinite map is genuine and to handling outer automorphisms.

11.4 Crossfibering internalizes regularity

The experimental proof depends on Hanany's regularity theorem and a recent normalizer reduction. Xu's crossfibered construction turns the unit-power problem into a homological calculation inside the proof. This is a substantial structural advantage and explains why Xu can treat all lattices, including orbifold lattices, in one framework.

12 Referee-oriented audit of Xu's paper

12.1 Scope of this audit

Xu's v1 is ninety-six pages and uses several deep external results. The assessment below is not a line-by-line certification. It is a structural audit of the proof, with closer attention to the new and load-bearing Sections 3, 9–14, and 16. The overall architecture is coherent, and no fatal gap was found. One definite local error and two likely typographical errors are recorded below.

Provisional verdict. Xu's main theorem appears likely to be correct. The current v1 needs a small repair in Lemma 9.19 and would benefit from several explicit clarifications. The long drilling and character-variety realization sections deserve specialist checking before the proof is treated as fully certified.

12.2 Section 3: virtual crossfibering

A referee should verify the following chain carefully.

1. The finite cover supplied by Agol/RFRS has enough first cohomology to choose the required nonzero class annihilating a rational direction in the Fried cone.
2. The passage to a cover where that class lies on the boundary of a fibered cone is exactly the form of Agol's criterion cited.
3. A rational perturbation into the interior of the fibered cone can be made while retaining a nonzero rational vector in the intersection of its kernel with the open Fried cone of the axial class.
4. Fried's periodic-orbit theorem applies to that rational homology direction and produces a periodic orbit whose class is killed by the facial fibration.
5. Vanishing of the facial class on the orbit is sufficient to put the conjugacy class into the facial fiber subgroup.

The proof as written appears coherent. Its role should not be underestimated: it is the device that places profinite trajectory information inside a surface group.

12.3 Sections 7–9: cut and glue

Theorem 9.2 is one of the principal new theorems. The following points are load-bearing.

1. The discrete surface splitting must be adequate/efficient so that its completion is the profinite fundamental group of the completed graph of groups and the expected Bass–Serre tree is obtained.
2. Profinite malnormality of the simple-curve subgroup must give the stated 1–acylindricity.
3. In Lemma 9.10, after passing to a finite cover of the complementary surface and quotienting by the closed normal subgroup generated by all boundary subgroups, the quotient must indeed be a profinite closed surface group.
4. Edge stabilizers in the quotient tree must be trivial, allowing the profinite PD^2 fixed-point theorem to be applied.
5. The symmetric argument for the inverse map must identify the full vertex stabilizers, not merely inclusions.
6. The criterion "adjacent if and only if stabilizers intersect nontrivially" depends on the exact acylindricity and edge-stabilizer statements.
7. The final HNN calculation must track the stable letter and the possible orientation reversal exactly as stated in items (1)–(3) of Theorem 9.2.

The proof is substantial and, apart from the local issues below, appears to supply these arrows.

12.4 Concrete corrections in Section 9

Lemma 9.19. Let $S'' \to S$ be the finite cover obtained after Scott's theorem and the possible additional two-fold cover. The standard characteristic subgroup at level m is the intersection of all subgroups of index at most m. To ensure that the standard characteristic cover $S^{(m)} \to S$ factors through $S'' \to S$, one must take

$$m \geq [\pi_1 S : \pi_1 S''] = \deg(S'' \to S).$$

The paper instead suggests $m = [S'' : S']$, which is only 1 or 2 in the construction and generally does not force factorization through S''. This is a definite error, but the repair is one line and does not affect the later use of the lemma.

Lemma 9.11. The statement says that a subgroup acting on T_1 fixes a vertex of $V(T_2)$. The target should be $V(T_1)$.

Remark 9.14(3). The theorem already treats a nonseparating curve, but the remark says that the proof "also works for non-separating simple closed curves." The intended word is almost certainly "separating."

12.5 Section 10: regularity

The cap-product computation is elegant. A referee should match the scalar normalizations exactly with Liu's results:

- which scalar μ acts on the trajectory or H_1 class;
- why the completed H_3 fundamental class scales by $\pm\mu^3$;
- why Theorem 9.2 and Proposition 9.15 make the facial surface class scale by $\pm\mu$;
- the precise naturality identity giving $\lambda_3 = \lambda_2\mu$;
- the deduction that a unit satisfying the resulting equation is ± 1.

The algebra in Xu's paper is consistent. The principal task is checking that the external scalars use the same normalizations and orientations.

12.6 Section 11: drilling

This is one of the longest geometric constructions. The later argument needs more than the existence of some hyperbolic orbifold: it needs exact commutative retraction diagrams and peripheral control. A referee should check:

1. the graph-of-groups descriptions of the two drilled link exteriors;
2. both applications of Theorem 9.2 and the compatibility of their edge maps;
3. the induced isomorphism on the relevant free profinite products;
4. the meridian calculation and compatibility with the retractions onto the surface groups;
5. irreducibility, boundary incompressibility, atoroidality, and anannularity of the link exterior when the two curves fill;

6. the orbifold Dehn-filling theorem in exactly the form needed to produce uniform lattices while retaining the retractions;
7. adequacy of every graph-of-groups decomposition used after completion.

No contradiction was found, but this section is a natural target for a 3–orbifold specialist.

12.7 Section 12: semiauthenticity

The abstraction is clean. The main checks are:

1. the Markov transition graph cycles generate the mapping-torus group as a group, not merely a proper subgroup;
2. Liu's trajectory correspondence plus regularity gives discrete target conjugacy classes for every element of the chosen semigroup;
3. the property survives all finite covers required by "virtually hereditarily";
4. the retraction lemmas transfer semiauthenticity through the drilled orbifold diagrams in both directions;
5. Lemma 12.16 uses the corrected form of Lemma 9.19.

Subject to those inputs, the semiauthenticity deductions are logically clear.

12.8 Section 13: character varieties

Theorem 13.5 is new and load-bearing. A referee should verify the following.

1. Every algebraic representation can be made integral at a suitable finite place and extended continuously to the profinite completion.
2. Pullback through a semiauthentic map gives a character whose values on the generating semigroup lie in the original algebraic field.
3. Trace functions attached to the generating semigroup generate the coordinate ring modulo the universal trace relations used in the paper.
4. The assignment is independent of the chosen discrete representative $\widetilde{s}$ of the profinite conjugacy class of $\Phi(s)$. An explicit useful sentence is: any two choices are profinitely conjugate, and every p-adic extension has equal trace on profinitely conjugate elements, so the resulting regular trace functions agree.
5. Equality on algebraic points is promoted to equality of regular functions in the reduced coordinate ring or affine algebraic set convention being used.
6. The construction is independent of the chosen finite place and embedding, and Proposition 13.16 really gives the functoriality required in Section 14.

The paper develops most of this carefully, especially the functoriality in Subsection 13.5. The choice-independence observation above would make the canonical nature of the map more transparent at the first point where it is defined.

12.9 Section 14: realization of the surface map

Theorem 14.1 is the final surface rigidity theorem. The critical checks are:

1. Marché–Simon's theorem applies to the precise $\mathrm{SL}_2(\mathbf{C})$ character variety and genus range used.
2. Fixing the trivial character eliminates the $H^1(S; \mathbf{Z}/2)$ central twist.
3. The genus-two kernel is genuinely avoided by passing to a proper characteristic cover of genus at least three.
4. The induced mapping classes on characteristic covers intertwine the deck actions. This relies on the functoriality of Section 13, not merely on abstract existence of character-variety automorphisms.
5. The centerless extension theorem descends the deck-equivariant mapping class from the cover to the base surface group.
6. Agreement of the induced maps on all standard characteristic quotients is sufficient to conclude that the original profinite isomorphism is genuine.

This is the least standard part of the paper. The logical chain appears complete, but it deserves independent checking by someone comfortable with both surface character varieties and profinite group extensions.

12.10 Sections 15–16: uniform and nonuniform lattices

For uniform lattices, one should check the finite-index criterion saying that genuineness on a suitable non-elementary finite-index subgroup forces genuineness on the full lattice, including orbifold torsion.

For nonuniform lattices, the factorial filling argument should be checked in the following order:

1. peripheral regularity matches the relevant cusp subgroups and slopes;
2. every fixed finite quotient eventually factors through the factorial orbifold filling;
3. sufficiently long fillings are uniform lattices to which the uniform theorem applies;
4. the resulting discrete maps can be chosen compatibly on a cofinal family of characteristic quotients;
5. the limiting criterion proves genuineness of the original cusped map.

This is stronger than the usual short-core drilling argument and appears consistent with the preceding machinery.

12.11 Issue table for a referee

Location	Issue	Status	Suggested action
Lemma 9.19	The example $m = [S'' : S']$ does not generally force the standard characteristic cover to factor through $S'' \to S$.	Definite local error	Take any $m \geq [\pi_1 S : \pi_1 S''] = \deg(S'' \to S)$.
Lemma 9.11	An action on T_1 is said to fix a vertex in $V(T_2)$.	Typo	Replace $V(T_2)$ by $V(T_1)$.
Remark 9.14(3)	Says the proof also works for nonseparating curves, already the case treated.	Probable typo	Replace by "separating," if that is the intended extension.
Theorem 13.5	Independence of the chosen discrete conjugate is more implicit than necessary.	Clarification	Add one sentence using continuous p-adic extension and trace invariance under profinite conjugacy.
Theorem 13.5	The coordinate-ring argument uses the reduced affine variety convention.	Clarification	State the reducedness and passage to functions on points where it is used.
Theorem 14.1	Functoriality on characteristic covers is load-bearing.	Specialist check	Verify every use of Proposition 13.16 in the deck-action and extension diagrams.
Section 16	One discrete isotopy class must control each fixed characteristic quotient along a suitable subsequence.	Specialist check	Check the factorial divisibility and subsequence argument together.

12.12 Overall correctness assessment

The proof architecture is well designed. The new theorems are stated precisely, and the main implications are visible rather than hidden in vague reductions. The error in Lemma 9.19 is local and repairable. The two nearby textual errors do not affect the mathematics. No fatal gap was found in this audit.

The appropriate present conclusion is therefore not "certified correct" but:

> Xu's theorem is likely correct after a minor repair. The main places requiring expert verification are the profinite PD^2 fixed-point use in Section 9, the exact retraction and

peripheral bookkeeping in Section 11, the canonicity and functoriality of the character-variety map in Section 13, and the characteristic-cover descent in Section 14.

13 Other Remarks

13.1 Length and local conceptual economy

The experimental draft is much shorter. Its cut-class proof of fixed-edge reconstruction is especially compact and conceptual. Xu's longer argument is partly explained by the stronger theorem and the need to build regularity and global map realization internally.

13.2 Analytic versus geometric risk

The most delicate part of E is analytic: based Shapiro–Artin formalism, Brauer induction, torsion normalizations, Adams–Möbius inversion, and a complex-measure uniqueness theorem.

The most delicate parts of X are geometric and algebraic: the drilled orbifold construction, hereditary semiauthenticity, the p-adic character-variety map, and finite-cover realization.

Neither proof is uniformly more elementary. They distribute the difficulty in different places.

13.3 Independent information

The experimental route says more about which geometric quantities are encoded in finite quotients: representation volume, marked complex lengths, and individual primitive conjugacy classes.

Xu says more about the full algebraic symmetry of the lattice and its profinite completion. These are complementary strengths.

14 Final synthesis

The shortest accurate comparison is the following.

Experimental route	Xu's route
Recognize holonomy using Bloch classes and volume.	Avoid holonomy recognition.
Mark each primitive geodesic using finite-group characters and torsion.	Mark a generating semigroup of periodic trajectories using crossfibering and regularity.
Descend individual primitive elements exactly.	Prove the entire profinite surface map genuine.
Reconstruct the cut tree from a cohomology class.	Reconstruct it from fixed vertices and stabilizer intersections.
Synchronize curves using a transporter theorem and finite rigid set.	Synchronize the whole map using semiauthenticity and character varieties.
Use a normalizer theorem for the general closed case.	Use virtual crossfibering and a finite-index genuineness theorem.
Recover cusps by matching fillings and drilling short cores.	Recover the specified map through factorial orbifold fillings.

The two arguments have one substantial local convergence: both must recognize the completed surface splitting obtained by cutting along a simple curve. The rest of their machinery is genuinely different. Xu's theorem is broader and, if correct, contains the headline rigidity consequence of the experimental draft. The experimental draft remains useful as an independent source of arithmetic, spectral, transporter, and cut-class ideas, and as a conceptual cross-check on Xu's Section 9.

The v1 of Xu's paper appears likely correct. A referee should repair Lemma 9.19, correct the two nearby textual errors, and devote particular attention to Sections 11, 13, and 14. It would be inaccurate either to say that the two proofs are essentially the same or to say that they have no meaningful overlap.

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