Toward the Buzzard–Calegari slope conjecture in all nonpositive weights

Research progress note

August 11, 2026

Abstract

This is a live proof record for Conjecture 2 of Buzzard–Calegari, *Slopes of overconvergent* 2-*adic modular forms*. It separates proved statements from computational evidence and unresolved steps. The source contains a sign typo in its displayed product, and its level-one formulation must be restricted to even weights. Both corrections are made explicitly below. The note proves an exact leading-determinant formula and the first predicted slope blocks throughout the dense family of weights $k = -12m$. A finite residue certificate proves the complete Hodge–principal-minor package through rank five, uniformly in that family. The note also reduces the residual matrix to explicit hypergeometric connection coefficients, proves uniform divisibility on all their subdiagonals, and packages their scaled integral structure in one fixed cubic recurrence.

1 Corrected target

Let

$$S_k = S_k\big(\Gamma_0(1); \mathbf{Q}_2; 2^{-1/2}\big)$$

be the Banach space of $2^{-1/2}$-overconvergent cusp forms of tame level one and integral weight k, and let $U = U_2$.

For an even integer $k \le 0$ and $j \ge 1$, put

$$c_j(k) = \frac{2^{2j}(-k+2+12j)!(-k+6j)!}{(-k+2+8j)!(-k-2+8j)!(-k+12j)} \in \mathbf{Q}^\times \tag{1}$$

and

$$F_k(X) = 1 + \sum_{n\ge 1} X^n \prod_{j=1}^{n} c_j(k). \tag{2}$$

Remark 1.1 (The corrections). The last factor in the denominator in the printed conjecture is $(-k-12j)$. This must be $(-k+12j)$: at $k=0$ the paper itself immediately uses the factor $12j$, and the minus sign would make the displayed formula incompatible with its proved weight-zero specialization. This factor is linear, not a factorial.

There is also an implicit parity restriction. At level one the element -1 acts on weight-k sections by $(-1)^k$, so $S_k = 0$ for odd k. The nonzero series (2) therefore cannot describe the Fredholm determinant in odd integral weight. The proof strategy in the source also treats only $k \equiv 0, 2 \pmod 4$. Thus the meaningful level-one statement is the even-weight statement below.

Conjecture 1.2 (Corrected Buzzard–Calegari Conjecture 2)**.** *For every even integer $k \le 0$, the Newton polygon of $\det(1 - XU \mid S_k)$ equals the Newton polygon of $F_k(X)$.*

Write

$$\alpha_j(k) = v_2(c_j(k)). \tag{3}$$

For $m, n \geq 0$, define

$$H_n(m) = \sum_{j=1}^{n} \alpha_j(-12m), \qquad H_0(m) = 0. \tag{4}$$

The predicted polygon is the lower convex hull of the points $\{(n, H_n(m)) : n \geq 0\}$. Write $L_n(m)$ for its ordinate at abscissa n. It is important not to replace convexification by the false assertion that the increments $\alpha_j(-12m)$ are always nondecreasing. For example, at $m = 1$ the first four increments are

$$3, 10, 8, 16,$$

so the first four polygon slopes are $3, 9, 9, 16$.

2 What the original paper proves

Put

$$f = \frac{\Delta(2\tau)}{\Delta(\tau)}, \qquad g = 2^6 f.$$

Then $\{g, g^2, g^3, \ldots\}$ is an orthonormal Banach basis of S_0. If

$$U(f^j) = \sum_i s_{i,j} f^i,$$

Buzzard and Calegari prove

$$s_{i,j} = \frac{3j(i+j-1)!2^{8i-4j-1}}{(2i-j)!(2j-i)!} \tag{5}$$

in the range $i \leq 2j$, $j \leq 2i$, with zero outside it. Consequently the weight-zero U-matrix is

$$u_{i,j} = \frac{3j(i+j-1)!2^{2i+2j-1}}{(2i-j)!(2j-i)!}. \tag{6}$$

They construct an exact factorization $U = ADB$, with A and B unipotent modulo 2 and

$$d_{j,j} = \frac{2^{4j+1}(3j)!^2 j!^2}{3(2j)!^4}. \tag{7}$$

This proves Conjecture 1.2 for $k = 0$.

For $k = -12m$, $m \geq 0$, set

$$h_k = \frac{\Delta(2\tau)^m}{\Delta(\tau)^{2m}}.$$

Coleman's identity $U(gV(h)) = hU(g)$ gives the explicit matrix

$$U^{(m)}_{i,j} = 2^{-6m} u_{i+m, j+2m} \qquad (i, j \geq 1) \tag{8}$$

on the basis $\{h_k g, h_k g^2, \ldots\}$. The source reports a proof for $k = -12$, but omits it because the combinatorics were unwieldy. It also observes that the weights $-12m$ are dense in $4\mathbf{Z}_2$, so an appropriately uniform proof on this family should extend by continuity to all $k \equiv 0 \pmod 4$; the theta operator should then treat $k \equiv 2 \pmod 4$.

Substitution in (6) gives a useful closed form which does not appear explicitly in the source.

Proposition 2.1 (Dense-family matrix)**.** *For $m \geq 0$ and $i, j \geq 1$,*

$$U_{i,j}^{(m)} = \frac{3(j+2m)(i+j+3m-1)!2^{2i+2j-1}}{(2i-j)!(2j-i+3m)!} \tag{9}$$

when $j \leq 2i$ and $i \leq 2j+3m$, and the entry is zero otherwise. In particular these entries are integers.

Proof. Insert $i+m$ and $j+2m$ into (6); the factor 2^{-6m} cancels the resulting additional power 2^{6m}. Integrality also follows before this simplification from the fact that (8) is the matrix of U on the displayed integral Banach basis. Alternatively,

$$\frac{(i+j+3m-1)!}{(2i-j)!(2j-i+3m)!} = \frac{1}{i+j+3m}\binom{i+j+3m}{2i-j},$$

and the original recurrence for $U(f^r)$ proves the divisibility of the remaining rational-looking factor. $\square$

The binomial part of this matrix has additional structure. Generalized binomial coefficients are used in the next proposition, with a negative lower index interpreted as zero.

Proposition 2.2 (Weight-independent Andrews–Burge transform)**.** *For $0 \leq i, j \leq N$, set*

$$E_{i,j}^{(m)} = \binom{3m+i+j}{2i-j} + \binom{3m-1+i+j}{2i-j}, \qquad M_{i,j}^{(m)} = \binom{3m-\frac{1}{2}+i+j}{2i-j}.$$

Let $T = (t_{i-j})_{0\leq i,j\leq N}$ be lower triangular, where

$$t_d = \frac{(-1)^{d+1}2}{2d-1}\frac{\binom{2d}{d}}{16^d} \quad (d \geq 0), \qquad t_d = 0 \quad (d < 0). \tag{10}$$

Then

$$E^{(m)} = TM^{(m)}, \qquad \det(T) = 2^{N+1}. \tag{11}$$

In particular T is independent of m. If $s_2(d)$ denotes the sum of the binary digits of d, then

$$v_2(t_d) = 1 + s_2(d) - 4d. \tag{12}$$

Proof. This is equation (3.15) of Andrews–Burge [3]. Their identity $\tau_n(x,y)M_n(x+y) = \mu_n(2x,2y)$ is applied with $x = 3m/2$ and $y = (3m-1)/2$. Thus $y - x = -1/2$, and their formula for the lower-triangular factor becomes

$$-\frac{1}{d-\frac{1}{2}}\binom{d-\frac{1}{2}}{2d} = t_d.$$

The closed form (10) follows from the duplication formula. Finally, Kummer's theorem gives $v_2\binom{2d}{d} = s_2(d)$, proving (12). $\square$

The augmented scaled matrix

$$\widehat{U}_N^{(m)} = \left(2^{2i+2j-1}E_{i,j}^{(m)}\right)_{0\leq i,j\leq N} = \begin{pmatrix} 1 & 0 \\ * & U_N^{(m)} \end{pmatrix}$$

is integral. Proposition 2.2 therefore supplies an exact transform for the matrix whose Smith data are needed. It does *not* yet supply an integral Smith factorization: the valuations in (12) tend rapidly to $-\infty$. The powers of two lost in T are recovered only after multiplication by the half-integral binomial matrix and the row and column scalings. Isolating that cancellation in blocks is one concrete form of the remaining problem.

There is also a simple integral factorization which exposes the first layer of those blocks.

Proposition 2.3 (Column–Cartier factorization). *For $i, j \geq 1$, define*

$$R_{i,j}^{(m)} = [x^{2i-j}](1+2x)^{3m+i+j-1}(1+x), \tag{13}$$

where the coefficient is zero for a negative exponent. Then

$$U^{(m)} = R^{(m)} \operatorname{diag}(2^3, 2^6, 2^9, \ldots), \tag{14}$$

and $R^{(m)}$ is integral. Its reduction modulo 2 is

$$R_{i,j}^{(m)} \equiv \begin{cases} 1 & \text{if } j = 2i-1 \text{ or } j = 2i, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently both submatrices $\big(R_{i,2j-1}^{(m)}\big)_{i,j\geq 1}$ and $\big(R_{i,2j}^{(m)}\big)_{i,j\geq 1}$ are lower triangular and invertible over $\mathbf{Z}_2$.

Proof. Put $r = 2i - j$ and $q = 3m + i + j - 1$. The binomial expression used below in the proof of Theorem 4.1 gives

$$\frac{U_{i,j}^{(m)}}{2^{3j}} = 2^{r-1}\left\{\binom{q+1}{r} + \binom{q}{r}\right\} = [x^r](1+2x)^q(1+x).$$

For $r = 0$ the expression in braces is 2, so the same formula remains valid. If $r = 0$ or 1, the coefficient on the right is odd; if $r \geq 2$, it is even. This proves the reduction statement. The two indicated submatrices are lower triangular, with respectively 1 and an odd number on every diagonal entry. ☐

The odd and even columns give a more precise normal form. Put

$$O_{i,j}^{(m)} = R_{i,2j-1}^{(m)}, \qquad E_{i,j}^{(m)} = R_{i,2j}^{(m)}, \qquad G^{(m)} = (O^{(m)})^{-1}E^{(m)}.$$

Both $O^{(m)}$ and $E^{(m)}$ are lower-triangular automorphisms over $\mathbf{Z}_2$. Splitting the source lattice into its odd and even coordinates and left-multiplying by $(O^{(m)})^{-1}$ presents the image lattice of $U^{(m)}$ by the block columns

$$\left[\operatorname{diag}(2^{6j-3})_{j\geq 1} \,\middle|\, G^{(m)} \operatorname{diag}(2^{6j})_{j\geq 1}\right]. \tag{15}$$

This operation preserves all determinantal divisors. The matrix $G^{(m)}$ is lower triangular, and a direct two-row calculation gives

$$G_{j,j}^{(m)} = \frac{1}{3(2m+2j-1)}, \tag{16}$$

$$G_{j+1,j}^{(m)} = \frac{2(3m+3j+2)(6m+6j-1)}{9(2m+2j+1)}. \tag{17}$$

In particular,

$$v_2(G_{j+1,j}^{(m)}) = 1 + v_2(3m+3j+2). \tag{18}$$

Thus the diagonal part of the even column is redundant modulo the odd column, while its first possible new pivot in the next row has exactly the valuation (18). When that valuation is large, deeper subdiagonals control a higher-rank block. Formula (15) turns the open Hodge problem into a concrete lower-triangular elimination problem rather than an unstructured search through arbitrary minors.

More precisely, choose finite sets $A, K, B \subset \mathbf{Z}_{>0}$ with $A \cap K = \varnothing$ and $|K| = |B|$. Taking the odd columns indexed by A, the even columns indexed by B, and the rows $A \cup K$ in (15) gives, up to sign, the minor

$$2^{\sum_{a\in A}(6a-3)+\sum_{b\in B}6b}\det G^{(m)}_{K,B}. \tag{19}$$

Every nonzero minor of the row-equivalent paired presentation occurs in this way. Consequently part (a) of the open Hodge package is equivalent to the concrete family of inequalities

$$\sum_{a\in A}(6a-3)+\sum_{b\in B}6b+v_2\det G^{(m)}_{K,B}\geq L_{|A|+|B|}(m), \tag{20}$$

together with the existence of equality for each index. Formula (20) is the present exact point of obstruction.

The entries of $G^{(m)}$ themselves admit a one-variable description. This removes the inverse of an infinite matrix from the calculation.

Proposition 2.4 (Hyperbolic connection model)**.** *For $n \geq 1$, put*

$$\begin{aligned} P_n(z) &= \cosh\bigl(6n\operatorname{arsinh}\sqrt{z}\bigr) = {}_2F_1\left(-3n, 3n; \tfrac{1}{2}; -z\right), \\ Q_n(z) &= \frac{\sinh\bigl((6n-3)\operatorname{arsinh}\sqrt{z}\bigr)}{\sqrt{z}} = (6n-3){}_2F_1\left(2-3n, 3n-1; \tfrac{3}{2}; -z\right). \end{aligned}$$

These are polynomials in z, and

$$E^{(m)}_{i,j} = [z^{i-j}]P_{m+j}(z), \qquad O^{(m)}_{i,j} = [z^{i-j}]Q_{m+j}(z). \tag{21}$$

There are rational functions $g_r(n)$ such that

$$G^{(m)}_{j+r,j} = g_r(m+j), \tag{22}$$

and they are determined by the scalar recursion

$$g_r(n) = \frac{[z^r]P_n(z) - \displaystyle\sum_{s=0}^{r-1} g_s(n)[z^{r-s}]Q_{n+s}(z)}{6(n+r)-3}. \tag{23}$$

Equivalently, they are the connection coefficients in the formal identity

$$z^j P_{m+j}(z) = \sum_{\ell\geq j} G^{(m)}_{\ell,j} z^\ell Q_{m+\ell}(z). \tag{24}$$

The first three are

$$g_0(n) = \frac{1}{3(2n-1)}, \tag{25}$$

$$g_1(n) = \frac{2(3n+2)(6n-1)}{9(2n+1)}, \tag{26}$$

$$g_2(n) = -\frac{4(3n+4)(3n+5)(6n-1)(6n+1)}{135(2n+3)}. \tag{27}$$

In particular,

$$v_2(g_2(n)) = 2 + v_2\bigl((3n+4)(3n+5)\bigr). \tag{28}$$

Proof. Let $q(t) = t + \sqrt{1+t^2} = \exp(\operatorname{arsinh} t)$ and make the invertible formal change of variable

$$t = \frac{x}{\sqrt{1+2x}}, \qquad x = tq(t), \qquad 1 + 2x = q(t)^2.$$

Changing variables in the coefficient residue in (13) gives

$$R^{(m)}_{i,j} = \operatorname*{res}_{t=0} q(t)^{6m+3j} t^{j-2i-1}\, dt = [t^{2i-j}] q(t)^{6m+3j}.$$

Since $q(-t) = q(t)^{-1}$, taking respectively the even and odd parts of $q(t)^a$ gives (21). The displayed hypergeometric forms are the standard terminating expansions of those two functions and also show directly that they are polynomials. Finally, $E = OG$ is coefficientwise equivalent to (24). Comparing the coefficient of z^{j+r} proves (23); substitution for $r = 0, 1, 2$ and simplification gives (25)–(27). All denominators and the factors $6n \pm 1$ are odd, which proves (28). ☐

The same model gives a uniform estimate on every subdiagonal, not merely the first two.

Corollary 2.5 (Uniform off-diagonal divisibility)**.** *For $n \geq 1$ and $r \geq 1$,*

$$[z^r]P_n(z) \in 2^{2r-1}\mathbf{Z}, \qquad [z^r]Q_n(z) \in 2^{2r}\mathbf{Z}. \tag{29}$$

Consequently, for all $m \geq 0$ and $j, r \geq 1$,

$$G^{(m)}_{j+r,j} = g_r(m+j) \in 2^{2r-1}\mathbf{Z}_2. \tag{30}$$

Moreover, the two polynomial families satisfy the exact relation

$$Q_{n+1}(z) - Q_n(z) = 2(3+4z)P_n(z). \tag{31}$$

Proof. The terminating hypergeometric expansions simplify, with $a = 3n$ in the first line and $a = 3n-2$ in the second, to

$$\begin{aligned} [z^r]P_n(z) &= 4^r \frac{a}{a+r}\binom{a+r}{2r}, \\ [z^r]Q_n(z) &= 4^r \frac{2a+1}{2r+1}\binom{a+r}{2r}. \end{aligned}$$

Here

$$\begin{aligned} \frac{2a}{a+r}\binom{a+r}{2r} &= \binom{a+r}{2r} + \binom{a+r-1}{2r}, \\ \frac{2a+1}{2r+1}\binom{a+r}{2r} &= \binom{a+r}{2r} + 2\binom{a+r}{2r+1}. \end{aligned}$$

Thus the first coefficient is 4^r times a half-integer and the second is 4^r times an integer, proving (29).

The diagonal of $O^{(m)}$ is odd and its rth subdiagonal is divisible by 4^r. Expanding its inverse recursively shows that the rth subdiagonal of $(O^{(m)})^{-1}$ is also divisible by 4^r: every product contributing to it has total depth r. In $G^{(m)} = (O^{(m)})^{-1}E^{(m)}$, every off-diagonal contribution of total depth r is therefore divisible by 2^{2r-1}, proving (30). Finally,

$$\sinh((6n+3)u) - \sinh((6n-3)u) = 2\cosh(6nu)\sinh(3u)$$

and $\sinh(3u)/\sinh u = 3 + 4\sinh^2 u$ prove (31) after setting $z = \sinh^2 u$. ☐

Remark 2.6 (What the uniform bound does and does not see). Set $\phi(0) = 0$ and $\phi(r) = 2r - 1$ for $r > 0$. Expanding a minor gives the rigorous tropical lower bound

$$v_2 \det G^{(m)}_{K,B} \geq \min_{\sigma: B \xrightarrow{\sim} K} \sum_{b \in B} \phi(\sigma(b) - b),$$

where bijections with $\sigma(b) < b$ are omitted. This bound is useful but cannot by itself prove (20), because it discards the parameter-dependent extra divisibility in the $g_r(m + j)$.

The first nonmonotone example shows the mechanism concretely. For $m = 1$, the mixed choice $A = \{1\}$, $B = \{1\}$, $K = \{2\}$ has baseline cost $3 + 6 + 1 = 10$, whereas (26) gives $v_2(g_1(2)) = 4$, so its true cost is 13. The two odd columns instead have cost $3 + 9 = 12 = L_2(1)$. Thus the extra divisibility suppresses the apparently cheaper mixed minor and creates the first higher-rank Hodge block; it is evidence for the conjecture, not a counterexample.

There is an exact description of the first associated-graded layer of the connection array. It explains why binary residue patterns appear in the computations.

Proposition 2.7 (Integral scaled and binary connection)**.** *Let $\Delta = \operatorname{diag}(4, 4^2, 4^3, \ldots)$ and put*

$$\mathcal{O} = \Delta^{-1} O^{(m)} \Delta, \qquad \mathcal{E} = 2\Delta^{-1} E^{(m)} \Delta, \qquad \mathcal{G} = 2\Delta^{-1} G^{(m)} \Delta.$$

All three matrices are integral over $\mathbf{Z}_2$, and $\mathcal{G} = \mathcal{O}^{-1}\mathcal{E}$. Define the integral polynomials

$$\widetilde{P}_n(Z) = 2P_n(Z/4), \qquad \widetilde{Q}_n(Z) = Q_n(Z/4).$$

Their coefficients are

$$[Z^r]\widetilde{P}_n = \binom{3n + r}{2r} + \binom{3n + r - 1}{2r}, \tag{32}$$

$$[Z^r]\widetilde{Q}_n = \binom{3n - 2 + r}{2r} + 2\binom{3n - 2 + r}{2r + 1}. \tag{33}$$

Both sequences satisfy the fixed recurrence

$$X_{n+1} = (Z^3 + 6Z^2 + 9Z + 2)X_n - X_{n-1}, \tag{34}$$

and

$$\widetilde{Q}_{n+1} - \widetilde{Q}_n = (Z + 3)\widetilde{P}_n. \tag{35}$$

If $n = m + j$ and gamma$_r(n) = 2g_r(n)/4^r \in \mathbf{Z}_2$, then the exact scaled connection is

$$Z^j \widetilde{P}_n(Z) = \sum_{r \geq 0} \gamma_r(n) Z^{j+r} \widetilde{Q}_{n+r}(Z) \quad \text{in } \mathbf{Z}_2[[Z]]. \tag{36}$$

For $a \geq 0$, define

$$F_a(Z) = \sum_{r \geq 0} \binom{a + r}{2r} Z^r \in \mathbf{F}_2[Z]. \tag{37}$$

If $n = m + j$, then reduction modulo 2 gives

$$\mathcal{O}_{j+r,j} = [Z^r] F_{3n-2}(Z), \tag{38}$$

$$\mathcal{E}_{j+r,j} = [Z^r]\big(F_{3n}(Z) + F_{3n-1}(Z)\big). \tag{39}$$

The polynomials satisfy

$$F_a = ZF_{a-1} + F_{a-2} \quad (a \geq 2). \tag{40}$$

Writing $B_n = F_{3n-2}$*, one has*

$$F_{3n} + F_{3n-1} = \frac{B_n + B_{n+1}}{1+Z}. \tag{41}$$

Finally, if

$$\varepsilon_r(n) \equiv \frac{g_r(n)}{2^{2r-1}} \pmod 2 \qquad (r \geq 1),$$

then the entire first graded connection is the coefficientwise identity

$$Z^j \frac{B_n + B_{n+1}}{1+Z} = \sum_{r\geq 1} \varepsilon_r(n) Z^{j+r} B_{n+r}(Z) \quad \text{in } \mathbf{F}_2[[Z]]. \tag{42}$$

Proof. The integrality follows from (29) and (30). The two exact binomial identities in the proof of Corollary 2.5 give (32)–(33).

The addition formulas for $\cosh(6nu)$ and $\sinh((6n-3)u)$ show that both sequences satisfy a second-order recurrence with coefficient $2\cosh(6u)$. Since

$$2P_1(Z/4) = Z^3 + 6Z^2 + 9Z + 2,$$

this is (34). Scaling (31) gives (35), and scaling the connection identity (24) gives (36).

For the reduction modulo 2, with $a = 3n-2$, the second binomial identity in the proof of Corollary 2.5 gives

$$4^{-r}[z^r]Q_n(z) \equiv \binom{a+r}{2r} \pmod 2,$$

which proves (38). With $a = 3n$, the first identity there similarly gives

$$2\,4^{-r}[z^r]P_n(z) \equiv \binom{a+r}{2r} + \binom{a+r-1}{2r} = \binom{a+r-1}{2r-1} \pmod 2.$$

This is also the coefficient of Z^r in $F_{3n} + F_{3n-1}$, proving (39).

Pascal's identity, used twice in characteristic two, proves (40). Applying that recurrence shows

$$(1+Z)(F_{a+2} + F_{a+1}) = F_a + F_{a+3};$$

setting $a = 3n-2$ proves (41). Reducing $\mathcal{E} = \mathcal{O}\mathcal{G}$ modulo 2 now gives (42). The diagonal term vanishes because $\mathcal{G}_{j,j} = 2g_0(n)$ is even. □

Proposition 2.7 converts the full scaled connection into the single integral recurrence (34); its first associated-graded layer is the Fibonacci-type recursion (40) over $\mathbf{F}_2$. This makes every higher layer explicit in principle, but does not yet control ranks of arbitrary mixed minors. In particular, vanishing of $\varepsilon_r(n)$ records only that an entry has valuation strictly greater than the uniform baseline.

The minors that realize the proved leading determinants have a uniform shape inside $G^{(m)}$. More generally, Cramer's rule replaces any minor of $G^{(m)}$ by one maximal determinant of explicit P- and Q-columns.

Proposition 2.8 (Cramer reduction and central minors)**.** *Let* O_N, E_N, G_N *be the leading* $N \times N$ *blocks. If* $I, J \subseteq \{1, \ldots, N\}$ *have the same cardinality* r*, then*

$$\det(G_N)_{I,J} = (-1)^{\sum I - r(r+1)/2} \frac{\det\bigl[(E_N)_{*,J} \mid (O_N)_{*,I^c}\bigr]}{\det O_N}, \tag{43}$$

where each indicated set of columns is put in increasing order. Moreover,

$$\det O_N = \prod_{a=1}^{N} 3(2m+2a-1) \in \mathbf{Z}_2^\times. \tag{44}$$

For $r \geq 1$ and $d \geq 0$, put

$$\Gamma_{r,d}^{(m)} = \det G_{\{d+1,\dots,d+r\},\{1,\dots,r\}}^{(m)}.$$

Then, up to sign,

$$\Gamma_{q,q}^{(m)} = \frac{D_{2q}(m)}{2^{6q^2+3q} \prod_{a=1}^{2q} 3(2m+2a-1)}, \tag{45}$$

$$\Gamma_{q,q+1}^{(m)} = \frac{D_{2q+1}(m)}{2^{3(q+1)(2q+1)} \prod_{a=1}^{2q+1} 3(2m+2a-1)}. \tag{46}$$

In particular,

$$v_2(\Gamma_{q,q}^{(m)}) = H_{2q}(m) - (6q^2+3q), \tag{47}$$

$$v_2(\Gamma_{q,q+1}^{(m)}) = H_{2q+1}(m) - 3(q+1)(2q+1). \tag{48}$$

Proof. Equation (43) is the generalized Cramer identity applied to $O_N G_N = E_N$. The diagonal formula for O gives (44).

For the leading $2q \times 2q$ minor of $U^{(m)}$, the paired columns consist of the first q odd columns and the first q even columns. After left-multiplication by O_{2q}^{-1}, Laplace expansion along the odd diagonal columns forces their rows to be $\{1,\dots,q\}$; the remaining factor is $\Gamma_{q,q}^{(m)}$. The extracted column powers total

$$\sum_{a=1}^{q}(6a-3) + \sum_{b=1}^{q} 6b = 6q^2 + 3q.$$

The row operation changes the determinant by $(\det O_{2q})^{-1}$, proving (45). For size $2q+1$, there are $q+1$ odd columns and q even columns; the same expansion leaves $\Gamma_{q,q+1}^{(m)}$, and the extracted powers total $3(q+1)(2q+1)$. This proves (46). The valuation claims follow from Theorem 4.1 and (44). $\square$

The dependence on m is itself a tail operation, and the solid minors obey an exact condensation recurrence.

Proposition 2.9 (Universal tail and solid-minor condensation)**.** *For all $m \geq 0$ and $i, j \geq 1$,*

$$G_{i,j}^{(m)} = G_{m+i,m+j}^{(0)}. \tag{49}$$

Set $\Gamma_{0,d}^{(m)} = 1$. For $r \geq 2$ and $d \geq 1$,

$$\begin{aligned}\Gamma_{r,d}^{(m)} \Gamma_{r-2,d}^{(m+1)} &= \Gamma_{r-1,d}^{(m)} \Gamma_{r-1,d}^{(m+1)} \\ &\quad - \Gamma_{r-1,d-1}^{(m+1)} \Gamma_{r-1,d+1}^{(m)}.\end{aligned} \tag{50}$$

Proof. Equation (49) follows immediately from $G^{(m)}_{j+r,j} = g_r(m+j)$. Apply the Desnanot–Jacobi identity to the solid $r \times r$ matrix defining $\Gamma^{(m)}_{r,d}$. Its northwest and southeast minors are respectively $\Gamma^{(m)}_{r-1,d}$ and $\Gamma^{(m+1)}_{r-1,d}$; the other two corner minors are $\Gamma^{(m+1)}_{r-1,d-1}$ and $\Gamma^{(m)}_{r-1,d+1}$, and the central minor is $\Gamma^{(m+1)}_{r-2,d}$. This is (50). □

Thus all dense-family matrices are principal tails of one universal connection array. Equations (45) and (46) give product-evaluated boundary data on the adjacent diagonals $d = r$ and $d = r+1$, while (50) propagates exact identities to neighboring solid minors. A valuation induction must retain the subtraction in (50): the cancellation exhibited below shows that tropicalizing it by simply taking a minimum loses essential information.

In particular, the points $(n, H_n(m))$ are represented, alternately, by two adjacent families of solid minors of $G^{(m)}$. The full Hodge inequality asks for a comparison between arbitrary mixed determinants in (43) and the convex envelope of these central ones. This is now the sharp determinant-calculus formulation of the dense family obstruction.

Remark 2.10 (A necessary determinant cancellation)*.* Entrywise tropical optimization is not sufficient, even for a minor that realizes a predicted height. At $m = 1$, the valuation matrix of the central block is

$$\left(v_2 G^{(1)}_{i,j}\right)_{i\in\{3,4\},\, j\in\{1,2\}} = \begin{pmatrix} 3 & 1 \\ 5 & 3 \end{pmatrix}.$$

Both determinant permutations therefore have valuation 6, but direct use of (26)–(27) and (23) gives

$$v_2 \Gamma^{(1)}_{2,2} = 7.$$

The paired column powers contribute 30, so a termwise matching model would incorrectly predict a minor of valuation 36; the cancellation raises it to $37 = H_4(1)$. Thus the graded-rank or block-determinant part of the argument is logically essential. It cannot be replaced by choosing a cheapest permutation from the entry valuations.

Remark 2.11 (Why central minors alone do not suffice)*.* A second exact example rules out restricting the comparison to the prefix or central configurations. At $m = 18$,

$$H_3 = 21, \qquad H_4 = 41, \qquad H_5 = 60, \qquad H_6 = 78.$$

The lower hull therefore has

$$L_4(18) = 40, \qquad L_5(18) = 59,$$

as interior points of the slope-19 block from abscissa 3 to abscissa 6. Every prefix configuration

$$A = \{1, \dots, N-q\}, \quad K = \{N-q+1, \dots, N\}, \quad B = \{1, \dots, q\}$$

has cost at least 41 for $N = 4$ and at least 60 for $N = 5$. Nevertheless, exact evaluation with (23) gives

$$v_2 \det G^{(18)}_{\{3,4\},\{1,3\}} = 4,$$
$$v_2 \det G^{(18)}_{\{4,5\},\{1,3\}} = 8.$$

Taking respectively $A = \{1,2\}$ and $A = \{1,2,3\}$ in (19) gives total costs 40 and 59. Thus gap minors realize the required convexification inside a longer Hodge block. This is not evidence against the conjecture; it shows that a proof must retain arbitrary mixed column sets in (43).

The obstruction is nevertheless finite in each fixed rank. Carrying this out through rank five gives the following uniform result. Here the rth determinantal divisor means the ideal generated by all r by r minors, and $a_r(m)$ denotes the coefficient of X^r in the Fredholm determinant.

Theorem 2.12 (Uniform rank-five Hodge package)**.** *For every $m \geq 0$ and $1 \leq r \leq 5$:*

(a) *the valuation of the rth determinantal divisor of $U^{(m)}$ is $L_r(m)$;*

(b) *if $(r, H_r(m))$ is a strict vertex of the predicted lower hull, then*

$$v_2(a_r(m)) = H_r(m).$$

Thus both parts of Conjecture 4.2 hold uniformly through rank five.

Proof. We give the finite certificate argument in enough detail to distinguish it from an experimental search. Put

$$(T_1, T_2, T_3, T_4, T_5) = (3, 12, 25, 43, 64).$$

The column factorization gives $H_s(m) \geq 3s(s+1)/2$; the first three sharper bounds are $H_1 = 3$, $H_2 \geq 10$, and $H_3 \geq 20$. Hence, when computing the lower-hull ordinate at ranks $1, 2, 3, 4, 5$, no chord ending to the right of respectively $1, 3, 7, 11, 17$ can lie below T_r. For example, at rank five and $b \geq 18$, the smallest of the five baseline chord bounds is the one from $(4, H_4)$, and it is at least

$$\frac{(b-5)30 + \frac{3}{2}b(b+1)}{b-4} > 64.$$

The other ranks are easier. The finite check below also exhibits a chord of height at most T_r in every case, so these omissions cannot change L_r.

The product in Theorem 4.1 gives a particularly convenient height formula. If

$$C_s = \frac{5s^2 + 3s}{2} - \sum_{i=1}^{s} \left\lceil \frac{i}{2} \right\rceil - \sum_{i=1}^{s} v_2((i)_i),$$

then

$$H_s(m) = C_s + \sum_{i=1}^{s} \sum_{t=0}^{\lfloor i/2 \rfloor - 1} v_2(3m + 3s - i + 1 - t). \tag{51}$$

Indeed, the factors in $(3m + i + \frac{1}{2})_{\lceil i/2 \rceil}$ become odd after their displayed powers of two are removed, while the other rising factorial supplies exactly the linear factors in (51).

For ranks through four, retain valuations only up to thirteen. On a fixed residue class $m \pmod{2^{13}}$, every factor in (51) either has a determined valuation below thirteen or has valuation at least thirteen. Thus it gives an exact value or a certified lower bound for each relevant H_s. Comparing all of the finitely many chords just described determines $L_r(m)$: an exact chord realizes the candidate and every possibly unresolved chord has lower bound at least that candidate. Rank five uses the same exact/lower interval test at the variable residue depth described below.

It remains to compare the minors. The uniform estimate (30) assigns depth cost

$$\phi(0) = 0, \qquad \phi(d) = 2d - 1 \quad (d > 0)$$

to an entry of G. Together with (19), this implies that only finitely many paired configurations can have cost at most T_r. The exhaustive lists have respectively 3, 7, 16, and 34 elements in ranks 2, 3, 4, and 5; the index bounds follow directly from

$$\sum_{a \in A} (6a - 3) + \sum_{b \in B} 6b + \min_{\sigma} \sum_{b \in B} \phi(\sigma(b) - b) \leq T_r.$$

There is no imposed cutoff hidden in this enumeration.

For ranks at most four, the program `scripts/check_dense_matrix.py` computes these connection minors on every one of the 2^{13} residue classes, modulo 2^{13}, from the exact recursion (23). In rank five it starts with the single ball $\mathbf{Z}_2$ and bisects only a ball on which the height or minor comparison is still unresolved. The process terminates with 16896 disjoint balls, of maximum depth 19, which cover $\mathbf{Z}_2$. Its denominators $6(n+d)-3$ are odd, so the calculation is valid over $\mathbf{Z}_2/2^e$ at the relevant depth e. In the binomial formulas for the coefficients of P_n and Q_n, the displayed powers 2^{2d-1} and 2^{2d} cancel every even factorial denominator. The resulting functions, and hence the recursively defined $g_d(n)$, are 2-adically congruence preserving. The program verifies that every listed configuration has valuation at least the certified L_r, and that one has equality. Every unlisted configuration has tropical cost greater than $T_r \geq L_r$. This proves part (a) for all integers in the corresponding residue ball, not merely for its least representative.

For part (b), Proposition 2.3 shows that, modulo 2^{T_r+1}, only principal index sets I with $3\sum_{i\in I} i \leq T_r$ can contribute to $a_r(m)$. There are respectively $1, 2, 4, 12, 29$ such sets in ranks one through five. Their residual entries are computed from

$$R_{i,j}^{(m)} = 2^{2i-j}\binom{3m+i+j-1}{2i-j} + 2^{2i-j-1}\binom{3m+i+j-1}{2i-j-1}, \tag{52}$$

with the second term omitted at depth zero. The certificate first decides strictness using exact and lower chord data, with no ambiguous residue class, and then sums these principal minors. At every strict vertex the result has valuation $H_r(m)$.

Finally, the calculation is uniform on each residue ball because, for every $e \geq 0$,

$$(1+2x)^{3\cdot 2^e u} \equiv 1 \pmod{2^{e+1}}$$

coefficientwise. This follows from $v_2\binom{3\cdot 2^e u}{d} \geq e - v_2(d)$ and $d - v_2(d) \geq 1$. The least column cost in rank four is 30, so precision 2^{14} on the residual matrix is already enough modulo 2^{44}. In rank five the least cost is 45; at each strict-vertex leaf the certificate checks $H_5 \leq 45 + e$, exactly the precision needed to determine the Fredholm coefficient modulo 2^{H_5+1}. This proves that the finite residue certificate covers every $m \geq 0$ and completes the proof. ☐

Thus the unknown minors in (20) are minors of the explicit lower-triangular connection array $\big(g_{i-j}(m+j)\big)_{i,j\geq 1}$. This is a genuine simplification, and Corollary 2.5 supplies a uniform baseline bound for all its entries. It is not yet an all-minor valuation theorem: beginning a few diagonals farther down, the numerators of $g_r(n)$ contain nonlinear factors whose additional 2-adic behavior must also be controlled.

Corollary 2.13 (The first dense-family slope)**.** *For every $m \geq 0$, the smallest slope of U on S_{-12m} is 3 and has multiplicity one. Thus corrected Conjecture 1.2 is true for its first slope throughout the dense family.*

Proof. Every principal minor indexed by a set I is divisible by $2^{3\sum_{j\in I} j}$ by (14). Hence the coefficient of X^n in the Fredholm determinant has valuation at least $3n(n+1)/2$. For $n=1$, the $(1,1)$ entry has valuation 3, while every other diagonal entry has valuation at least 6; the trace therefore has valuation 3. For $n \geq 2$ the lower bound is strictly greater than $3n$. Finally, $v_2(c_1(-12m)) = 3$, and Theorem 4.1 identifies $H_n(m)$ with the valuation of the leading principal minor, to which the same column bound applies. Hence the predicted polygon also begins with the single slope 3. ☐

Theorem 2.14 (Uniform initial slope blocks)**.** *The actual and predicted Newton polygons in the dense family have the same initial slopes as follows:*

condition on m	*initial slopes*
$m \equiv 0 \pmod 2$	$3, 7,$
$m \equiv 3 \pmod 4$	$3, 8,$
$m \equiv 1 \pmod 4$	$3, 9, 9.$

In each row the displayed list gives the complete initial block: the next slope is strictly larger than the last displayed slope.

Proof. Let $a_n(m)$ denote the coefficient of X^n in the Fredholm determinant. Theorem 4.1, in its first two nontrivial cases, gives

$$H_2(m) = 10 + v_2(3m+5),$$
$$H_3(m) = 20 + v_2\big((3m+7)(3m+8)\big).$$

Indeed, all the omitted linear factors in $D_2(m)$ and $D_3(m)$ are odd. Thus $H_2(m) = 10$ for even m; if $m \equiv 3 \pmod 4$, then $H_2(m) = 11$ and $H_3(m) \geq 22$; and if $m \equiv 1 \pmod 4$, then $H_2(m) \geq 12$ and $H_3(m) = 21$.

We next bound the actual coefficients. For a two-element principal index set, the least possible column sums are

$$1 + 2 = 3, \qquad 1 + 3 = 4,$$

and all others are at least 5. The first minor is $D_2(m)$. For the second, the reduction of $R^{(m)}_{\{1,3\},\{1,3\}}$ has a zero row by Proposition 2.3, so its valuation is at least $3(1+3)+1 = 13$. Hence

$$v_2(a_2(m)) = \begin{cases} H_2(m) & \text{if } H_2(m) \leq 12, \\ \geq 13 & \text{if } H_2(m) > 12. \end{cases} \tag{53}$$

For three indices, the leading set has column sum 6, the next set $\{1, 2, 4\}$ has sum 7 and a zero row modulo 2, and every other set has sum at least 8. It follows that

$$H_3(m) = 21 \Longrightarrow v_2(a_3(m)) = 21, \qquad H_3(m) \geq 22 \Longrightarrow v_2(a_3(m)) \geq 22. \tag{54}$$

Likewise the leading four-element set has column sum 10 but its residual matrix modulo 2 has rank only 2; every other set has column sum at least 11. Consequently

$$v_2(a_4(m)) \geq 31. \tag{55}$$

If m is even, the line through $(1, 3)$ and $(2, 10)$ has slope 7. For $n \geq 3$, the general column bound $v_2(a_n(m)) \geq 3n(n+1)/2$ lies strictly above that line. If $m \equiv 3 \pmod 4$, the corresponding slope is 8; equation (54) handles $n = 3$, and the general bound handles $n \geq 4$. If $m \equiv 1 \pmod 4$, equations (53)–(55) place the coefficients at abscissae $2, 3, 4$ on or above the line of slope 9 from $(1, 3)$, with equality at $n = 3$ and strict inequality at $n = 4$; the general bound is strict for $n \geq 5$. The identical argument applies to the predicted heights because $H_n(m) = v_2(D_n(m))$ and the leading determinant has the same column factorization. This proves all three assertions. ∎

3 Later ghost-series interpretation

Bergdall and Pollack define a tame-level-one ghost series $G(w,X) = 1 + \sum_{n\geq 1} g_n(w)X^n$. Their calculation gives the following exact identification.

Theorem 3.1 (Bergdall–Pollack)**.** *If $k \leq 0$ is even, then*

$$v_2(g_n(w_k)) = v_2\left(\prod_{j=1}^{n} c_j(k)\right).$$

Consequently the polygon in corrected Conjecture 1.2 is precisely the Newton polygon of the specialized ghost series $G(w_k, X)$.

This is an exact reformulation of the predicted side, not a proof about the actual Fredholm determinant. Equality of the ghost and Fredholm polygons at these weights remains an instance of the ghost conjecture. The later local ghost theorems under strong genericity do not cover this exceptional $p = 2$, tame-level-one problem. Thus the original conjecture has not been silently resolved by the ghost-series literature.

4 Exact dense-family results and computations

Let $U_N^{(m)}$ be the leading $N \times N$ truncation of (9), and let

$$D_n(m) = \det\bigl(U_{i,j}^{(m)}\bigr)_{1\leq i,j\leq n}.$$

Write $(a)_r = a(a+1)\cdots(a+r-1)$ and let $\epsilon_n = 1$ if $n+1 \equiv 3 \pmod 4$, and 0 otherwise.

Theorem 4.1 (Exact leading determinant)**.** *For all $m \geq 0$ and $n \geq 1$,*

$$D_n(m) = (-1)^{\epsilon_n} 2^{(5n^2+3n)/2} \prod_{i=1}^{n} \frac{(3m+i+\frac{1}{2})_{\lceil i/2\rceil}(-3m-3n+i-1)_{\lfloor i/2\rfloor}}{(i)_i}. \tag{56}$$

Moreover,

$$v_2(D_n(m)) = H_n(m). \tag{57}$$

Proof. Put $N = 3m+i+j$ and $r = 2i-j$. Since $3(j+2m) = r + 2(N-r)$, formula (9) becomes

$$\begin{aligned} 2^{-2i-2j+1}U_{i,j}^{(m)} &= \binom{N-1}{r-1} + 2\binom{N-1}{r} \\ &= \binom{3m+i+j}{2i-j} + \binom{3m+i+j-1}{2i-j}. \end{aligned}$$

The right side is the Andrews–Burge determinant evaluated in [4, Theorem 38, formula (3.32)]. Apply that formula with $x = 3m$, $y = 3m-1$, and matrix indices $0 \leq i,j \leq n$. The zeroth row is $(2,0,\ldots,0)$, so its determinant is twice the determinant with $1 \leq i,j \leq n$. Finally, extracting 2^{2i-1} from row i and 2^{2j} from column j contributes 2^{2n^2+n}. Substitution in Krattenthaler's product gives (56).

We include the elementary valuation comparison because it is exactly where the corrected factorial prediction enters. Write $a \sim_2 b$ when $v_2(a) = v_2(b)$. Repeated use of

$$(2q)! \sim_2 2^q q!, \qquad (2q+1)! \sim_2 2^q q!$$

in (1) gives

$$\prod_{j=1}^{n} c_j(-12m) \sim_2 2^{n(n+1)-3mn} \frac{m!}{(m+n)!} \prod_{j=1}^{n} \frac{(3m+3j)!(6m+3j)!}{(3m+2j)!(3m+2j-1)!}. \tag{58}$$

On the other hand, with $s_n = \sum_{i=1}^{n} \lceil i/2 \rceil$, the half-integral rising factorials in (56) have valuation $-s_n$, and hence

$$D_n(m) \sim_2 2^{(5n^2+3n)/2-s_n} \prod_{i=1}^{n} \frac{(3m+3n-i+1)!(i-1)!}{(3m+3n-i+1-\lfloor i/2 \rfloor)!(2i-1)!}. \tag{59}$$

Splitting the products into $i = 2r$ and $i = 2r-1$ and applying the two displayed factorial rules once more transforms (58) into (59); the omitted quotients are products of odd double factorials and are 2-adic units. This proves (57). $\square$

Thus the leading determinant pattern discovered computationally is now a theorem. The remaining package is the following.

Conjecture 4.2 (Hodge–principal-minor package)**.** *For all $m, n \geq 0$:*

(a) *the valuation of the nth determinantal divisor of $U^{(m)}$ is $L_n(m)$; equivalently, its Hodge polygon is the stated lower convex hull;*

(b) *if $(n, H_n(m))$ is a strict vertex of that hull, the coefficient of X^n in the Fredholm determinant has valuation $H_n(m)$.*

Theorem 4.1 shows that the leading principal minor already has the required value at every vertex. A clean sufficient strengthening of part (b) is that this minor be the unique principal minor of least valuation. Part (a) makes the Hodge polygon a lower bound for the Newton polygon, and part (b) forces equality at every vertex, so the two parts prove the dense-family case.

There is a second, weaker way to certify part (b), useful for exact computation. The following elementary criterion records it.

Lemma 4.3 (Finite Hodge–Newton vertex test)**.** *Let A be an $N \times N$ matrix over $\mathbf{Z}_2$, and suppose*

$$PAQ = \operatorname{diag}(d_1, \ldots, d_N), \qquad v_2(d_1) \leq \cdots \leq v_2(d_N),$$

with $P, Q \in \mathrm{GL}_N(\mathbf{Z}_2)$. Put $C = Q^{-1}P^{-1}$. If $v_2(d_n) < v_2(d_{n+1})$, then the coefficient a_n of X^n in $\det(1 - XA)$ *satisfies*

$$v_2(a_n) = \sum_{i=1}^{n} v_2(d_i) \quad \Longleftrightarrow \quad \det C_{\{1,\ldots,n\},\{1,\ldots,n\}} \not\equiv 0 \pmod 2.$$

Proof. Apply the trace identity for the nth exterior power to $A = P^{-1} \operatorname{diag}(d_i) Q^{-1}$. Cyclically moving the invertible factors shows, up to sign, that a_n is the sum over n-element subsets I of

$$\left(\prod_{i \in I} d_i \right) \det(C_{I,I}).$$

At a strict Hodge vertex the unique product of minimum valuation is the one with $I = \{1, \ldots, n\}$, which proves the equivalence. $\square$

All computations in the following audit use exact integers and rational valuations; no floating-point eigenvalue calculation is involved.

- The exact Andrews–Burge product and the equality $v_2(D_n(m)) = H_n(m)$ are checked directly against the integer determinant for $0 \le m \le 64$ and $1 \le n \le 12$; this is now a regression test for Theorem 4.1, not evidence in place of its proof.
- The hyperbolic connection recursion in Proposition 2.4 is checked against direct paired-column elimination for $1 \le n \le 64$ through depth eight, including the three factored formulas (25)–(27) and the divisibility in Corollary 2.5.
- The binary connection formula in Proposition 2.7 is checked directly against the exact rational $g_r(n)$ for $1 \le n \le 64$ through depth sixteen.
- The two central-minor identities (45)–(46) are checked directly for $0 \le m \le 32$ and $1 \le q \le 6$; their proof is independent of this regression test.
- The exact condensation recurrence (50) is checked for $0 \le m \le 16$, sizes at most six, and displacements at most six.
- The $m = 18$ gap-minor calculations in the preceding remark are checked exactly, including the fact that all prefix configurations have costs at least 41 and 60 at the two interior abscissae.
- Theorem 2.12 is certified over all 2^{13} residue classes through rank four. In rank five, an adaptive cover has 16896 residue balls of depth at most 19. The exhaustive low-cost paired lists have sizes $3, 7, 16, 34$ in ranks 2 through 5, and the strict-vertex Fredholm check uses respectively $1, 2, 4, 12, 29$ principal index sets.
- The Smith normal form of $U_{12}^{(m)}$ has exactly the convexified predicted slopes for every $0 \le m \le 16$. At every strict Hodge vertex, the leading block of the matrix C in Lemma 4.3 is invertible modulo 2.
- For $0 \le m \le 4$, $1 \le n \le 6$, among all principal minors indexed by subsets of $\{1, \ldots, 14\}$, the leading principal minor is the unique one of valuation $H_n(m)$.
- For $0 \le m \le 4$ and $1 \le n \le 4$, the minimum valuation of all $n \times n$ minors of $U_9^{(m)}$ equals the ordinate of the lower convex hull of $(r, H_r(m))$. At $m = 1, n = 2$ this minimum is 12, below $H_2(1) = 13$, exactly as convexification requires.
- For $0 \le m \le 3$ and $1 \le n \le 5$, the coefficient of X^n in $\det(1 - XU_N^{(m)})$ has valuation $H_n(m)$ for each $N = 6, 8, 12, 14$.

The product (56) was first found from the following exact symbolic pattern. For fixed n, each matrix entry in (9) is a polynomial in m, and every nonzero determinant term has degree at most

$$\sum_{i=1}^{n} 2i - \sum_{j=1}^{n} j = \frac{n(n+1)}{2}.$$

Interpolation at one more than this number of integer values therefore proves each resulting polynomial identity, rather than merely sampling it. The initial interpolants factor only into linear factors with small rational roots. For example,

$$\begin{aligned} D_1(m) &= 24(2m+1), \\ D_2(m) &= 2^{10}(2m+1)(3m+5)(6m+5), \\ D_3(m) &= \frac{2^{20}}{5}(2m+1)(2m+3)(3m+7)(3m+8)(6m+5)(6m+7). \end{aligned}$$

These factors are now explained uniformly by the Andrews–Burge product. The genuinely open dense-family subproblem in higher rank is the valuation of arbitrary minors and the absence of cancellation among least principal minors at hull vertices.

The two reproducibility programs are `scripts/check_dense_matrix.py` and `scripts/check_smith_hodge.py`. The first uses only the Python standard library. The second uses SymPy solely for an independent integral Smith normal form; it also performs its own elimination over $\mathbf{Z}_{(2)}$ to verify Lemma 4.3. Both computations are exact.

5 Proof ledger and path forward

The following steps would suffice for Conjecture 1.2. Their status will be updated whenever progress is made.

Step 1. Correct the statement and normalize the predicted slopes. *Status: complete.* The sign and parity defects are corrected, and the predicted polygon is properly defined by convexification of (4). The tempting monotonicity claim is false and has been removed.

Step 2. Fix a single analytic model over weight space. *Status: open in this note.* Place the spaces and U-operators for all weights in one orthonormalizable Banach module over the even component of 2-adic weight space, with a Fredholm series specializing to $\det(1 - XU \mid S_k)$.

Step 3. Establish the explicit matrix on the dense family $k = -12m$. *Status: complete, conditional only on the standard basis statement quoted from the source.* Equation (8) is the required exact matrix formula.

Step 4. Prove the predicted Newton polygon for every $m \geq 0$. *Status: the initial slope blocks and the full Hodge–principal-minor package through rank five are complete; higher ranks remain open.* The product evaluation and its predicted valuation are now proved in Theorem 4.1, and Theorem 2.14 proves uniform initial blocks for every m. Theorem 2.12 proves the valuation lower bound for every minor of ranks at most five and the Hodge–Newton noncancellation condition at every strict hull vertex in those ranks. The same two assertions remain open in higher rank. Uniqueness of the least principal minor is a convenient stronger target, but Lemma 4.3 shows that it is not logically necessary. Proposition 2.4 now expresses the residual paired-column matrix by a scalar hypergeometric recursion, so the all-minor problem no longer requires manipulating an unspecified inverse matrix; Corollary 2.5 also proves a uniform baseline valuation for every one of its subdiagonals, and Proposition 2.7 identifies the complete first associated-graded layer. Proposition 2.8 identifies the exact solid minors realizing every proved height $H_n(m)$ and reduces arbitrary G-minors to explicit mixed maximal determinants. The old $k = 0$ factorization does not simply truncate: its shifted Gaussian outer factors can have negative valuation. Hodge blocks, rather than one-dimensional pivots in their original order, are the appropriate target when the increments α_j decrease.

Step 5. Make the dense-family result uniform enough to interpolate. *Status: open.* Show equality of Newton polygons on $k = -12m$ in a form stable under 2-adic limits. Equality at individual classical points alone is not a continuity theorem when coefficient valuations can jump.

Step 6. Extend from $k \equiv 0 \pmod 4$ to $k \equiv 2 \pmod 4$. *Status: outlined in the source, not proved here.* Compute precisely how $\theta = q\,d/dq$ transports finite-slope eigenforms and compare the formula $\alpha_j(k)$ across the two components.

Step 7. Audit compactness, Fredholm determinants, and limiting arguments. *Status: open.* Verify that all matrix operations are legitimate for compact operators and that passage from finite truncations to Fredholm series preserves every claimed valuation.

Step 8. Assemble the proof. *Status: conditional on Steps 2, 4–7.*

6 Current assessment

Theorem 4.1 settles the leading-minor identity for every m, n, Theorem 2.14 proves the initial slope blocks uniformly in the dense family, and Theorem 2.12 settles both the all-minor and strict-vertex assertions through rank five. The remaining higher-rank bottleneck is a lower bound for arbitrary minors, most likely nonintersecting-path or integral Hodge-block in nature, plus a uniform no-cancellation argument at the hull vertices. The new hyperbolic connection model turns the residual matrix into the single recursion (23); this is meaningful structural progress, although it does not yet control its arbitrary minors. Its off-diagonal entries now have the proved uniform divisibility (30), so the remaining issue is the extra, parameter-dependent divisibility and its interaction across minors. Proposition 2.7 describes the first binary layer exactly and packages all higher layers in the integral recurrence (34), but their ranks and determinantal cancellations remain open. Proposition 2.8 shows that the target heights are the valuations of two adjacent families of solid minors, making comparison with arbitrary mixed minors the precise remaining determinant problem. The observed Hodge blocks explain why the naive shifted ADB factorization fails and what must replace it. Weight-space continuity is a separate logical step and will not be inferred merely from numerical density.

Alternative routes under active consideration

1. **Hyperbolic connection-coefficient induction.** Use Proposition 2.4 to filter the array $g_{i-j}(m + j)$ by valuation and prove (20) by successive lower-triangular elimination. The scalar recursion and the explicit first subdiagonals make this the most concrete current route; the central product formulas and condensation recurrence add exact boundary data. Controlling cancellation in (50), followed by nonsolid minors, is its main risk.

2. **Integral Andrews–Burge blocks.** Use Proposition 2.2, but combine its rational triangular factor with the row and column powers of two before taking Smith form. The weight-independent coefficients (10) make a two-adic block recursion plausible.

3. **Nonintersecting paths.** Interpret the binomial matrix $E^{(m)}$ as a path matrix and its minors by the Lindström–Gessel–Viennot lemma. A sign-free path model with a two-adic involution could prove both the lower bound and uniqueness. The sum of two binomial terms and the powers of two are the main complication.

4. **Associated-graded rank induction.** Divide out the explicit row and column powers, filter the remaining matrix by entry valuation, and calculate successive ranks over $\mathbf{F}_2$. Proposition 2.7 now gives the first layer by the explicit Fibonacci-type recurrence (40); the next task is to lift it uniformly through higher powers of 2. Such an induction would prove part (a) of Conjecture 4.2 directly.

5. **Analytic or ghost interpolation.** Seek a weight-space divisibility theorem for the Fredholm coefficients whose specializations have the ghost valuations of Theorem 3.1. This would bypass an explicit Smith form, but at $p = 2$ it appears to require a genuinely new control theorem rather than a formal appeal to existing ghost results.

Difficulty and probability estimates

These estimates are subjective and will be revised if one of the structural routes succeeds.

- **More computation only:** extending Smith forms, principal-minor searches, and Hodge–Newton tests; identifying residue-class patterns; and checking proposed recurrences. The rank-five certificate shows that this can produce uniform theorems at a fixed rank, not merely evidence. Its configuration and residue complexity grows with the rank, however, so computation alone is still unlikely to finish the proof.

- **Needs a deeper idea:** a uniform all-minor lower bound or an integral block factorization. This is the central high-difficulty step.

- **Standard once the dense family is solved:** compact-operator bookkeeping and much of the Fredholm limiting audit. These are moderate technical tasks, not expected to be the conceptual bottleneck.

- **Potentially another deep step:** interpolation from $k = -12m$ to an entire weight-space component without assuming continuity of Newton polygons, followed by the precise theta comparison with the other even component.

The integral recurrence (34) improves the prospects for a systematic graded proof, but the essential cancellation at $m = 1$ and the noncentral gap block at $m = 18$ show that neither entrywise matching nor central-minor comparison is sufficient. The uniform rank-five theorem is a real improvement: it confirms that the paired model captures convexification and cancellation simultaneously. It does not yet supply an induction in the rank.

My current estimate is a 60%–75% chance that the dense-family statement can be completed along one of the first three routes, and a 30%–45% chance of reaching a complete proof of Conjecture 1.2 without an additional major idea from outside the present framework. There is a high probability that the exact determinant theorem, the corrected formulation, and the reduction to the Hodge package remain useful even if the full conjecture is not completed.

At present this note does not prove the full Newton polygon at any new weight beyond the already-published case $k = 0$. It does prove the first slope blocks at every weight $k = -12m$ and the new uniform leading-minor Theorem 4.1 for every m, n. It now proves, uniformly in m, every determinantal-divisor assertion through rank five and the Fredholm valuation at every strict predicted vertex in those ranks (Theorem 2.12). It also proves the connection model and off-diagonal bound in Proposition 2.4 and Corollary 2.5, the integral scaled recurrence and binary layer in Proposition 2.7, and the central minor and condensation formulas in Propositions 2.8 and 2.9. No counterexample is known from the work recorded here. The nonmonotone families, such as $m = 1$, are not counterexamples; together with the $m = 18$ gap block, they demonstrate that any proof must allow higher-rank Hodge blocks, essential determinant cancellation, and noncentral mixed minors.

References

[1] K. Buzzard and F. Calegari, *Slopes of overconvergent 2-adic modular forms*, Compositio Math. **141** (2005), 591–604, doi:10.1112/S0010437X04001034.

[2] J. Bergdall and R. Pollack, *Slopes of modular forms and the ghost conjecture*, Int. Math. Res. Not. IMRN (2019), 1125–1194, doi:10.1093/imrn/rnx141; see especially Theorem 3.3 of the unabridged version.

[3] G. E. Andrews and W. H. Burge, *Determinant identities*, Pacific J. Math. **158** (1993), 1–14, doi:10.2140/pjm.1993.158.1.

[4] C. Krattenthaler, *Advanced determinant calculus*, Sém. Lothar. Combin. **42** (1999), Art. B42q, arXiv:math/9902004.