Central extensions of loop groups and loop algebras

§1 G: compact connected Lie group.
If G commutative, then $G \cong \mathbb{C}_1^r$ ($\mathbb{C}_1 = \{z \in \mathbb{C} : |z|=1\}$) a torus
So in general $Z(G)^\circ$ (id. comp. of center) a torus
$G' := (G,G)$ connected compact with finite center. Then $G = Z(G)^\circ . G'$, $Z(G')$ finite.
<u>Assume from now on $Z(G)$ finite ($\Leftrightarrow G = G'$).</u>

<u>Facts</u>: Then $\pi_1 G$ finite, $\pi_2 G = \{1\}$, $\pi_3 G$ free abelian, of rank r, say.
So G has a finite universal cover $\tilde{G} \to G$ which is 2-connected and $\pi_3 \tilde{G} \cong \mathbb{Z}^r$.

A connected Lie group is called <u>simple</u> if every proper normal subgroup is finite.
<u>Facts</u> 1) $\underline{G} = G_1 \cdot \ldots \cdot G_r$ with each G_i simple and $G_i \cap G_j$ ($i \neq j$) contained in $Z(G)$ hence finite.
2) $\tilde{G} = \tilde{G}_1 \times \ldots \times \tilde{G}_r$ each $\tilde{G}_i$ 2-connected and $\pi_3 \tilde{G}_i \cong \mathbb{Z}$.
So in a sense the simply connected, simple groups are the building blocks. Therefore:
<u>We now assume G simply connected and simple</u>. So G 2-connected and $\pi_3 G \cong \mathbb{Z}$.

Examples:
1) SU_n ($n \geq 2$) esp. $SU_2 \cong$ unit quaternions $\cong S^3$.
2) SO_n is simple for $n=3$, $n \geq 5$, but not simply connected: its π_1 is of order 2; its universal cover is a spin group (e.g., the univ. cover of $SO(3)$ is $\cong SU_2$)
3) quaternion analogue of SU_n.
[isom. classes of s.c. simple compact Lie groups $\Leftrightarrow$ root systems with connected Dynkin diagram]

The De Rham complex of G contains the left-invariant differential forms as a subcomplex with the same cohomology. This is also true if we consider diff-forms that are both left and right invariant, but then d becomes zero (it is like the space of harmonic forms. This can be expressed in terms of the Lie algebra $\mathfrak{g}$:

$$H^\bullet(G,\mathbb{R}) = (\Lambda^\bullet \mathfrak{g}^*)^G$$ ← G acts on $\mathfrak{g}$, hence on $\Lambda^\bullet\mathfrak{g}^*$ via adjoint action

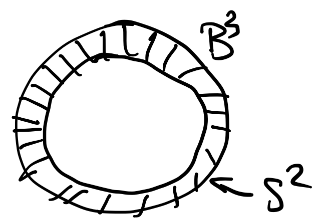
Example: Let $\mathfrak{g} \times \mathfrak{g} \xrightarrow{\langle , \rangle} \mathbb{R}$ be an inner product inv under the adjoint action (exists always: G is compact and we can average over it). The infinitesimal version is invariance under ad: $\langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle = 0$, so that

$$\omega : (X,Y,Z) \in \mathfrak{g}^3 \mapsto \langle [X,Y], Z \rangle$$

is alternating and invariant under the adjoint action of G. So this gives an element of $(\Lambda^3 \mathfrak{g}^*)^G \cong H^3(G,\mathbb{R})$. Normalize it so that we get a generator of $H^3(G;\mathbb{Z})$.

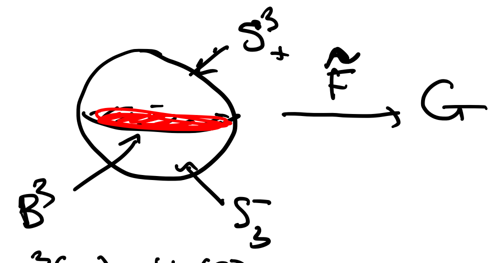
§2 M manifold $C^\infty(M,G)$ a group for pointwise multiplication.
We are going to define a remarkable group homomorphism

$$\Theta : C^\infty(S^2,G) \longrightarrow \mathbb{C}_1$$



Let $f : S^2 \xrightarrow{C^\infty} G$. Since G is 2-connected, f extends to $F : B^3 \to G$; make F constant along radii near S^2 and put

$$\Theta(F) := \exp\left(2\pi\sqrt{-1}\int_{B^3} F^*\omega\right) \in \mathbb{C}_1$$



Other choice F' for F combines with F to define (F on S^3_+; F' on S^3_-)

$H^3(G)$ $H_3(G)$

Then $$\frac{\Theta(F)}{\Theta(F')} = \exp\left(2\pi\sqrt{-1}\int_{S^3} \tilde{F}^*\omega\right) = \exp\left(2\pi\sqrt{-1}\underbrace{\langle [\omega] \mid \tilde{F}_*[S^3]\rangle}_{\in\,\mathbb{Z}}\right) = 1.$$

hence $\Theta(F)$ only depends on f.

This Θ extends immediately to $C^{[\infty]}(S^2,G) := \{$continuous piecewise diff maps $S^2 \to G\}$
Every element of $C^{[\infty]}(B^2,G)$ whose restriction to $\partial B^2 = S^1$ is constant 1 defines an elt. of $C^{[\infty]}_*(S^2,G)$ (an elt which takes base point of S^2 to $1 \in G$)
So we have an exact sequence of groups

$$1 \longrightarrow C^{[\infty]}_*(S^2,G) \longrightarrow C^{[\infty]}(B^2,G) \longrightarrow C^{[\infty]}(S^1,G) \longrightarrow 1$$

You may check that $\Theta | C^{[\infty]}_*(S^2,G)$ is invariant under conjugation with elements of $C^{[\infty]}(B^2,G)$. In particular, its kernel is normal in $C^{[\infty]}(B^2,G)$. If we divide out by this kernel (a "pushout") and write $\mathcal{L}G$ for $C^{[\infty]}(S^1,G)$ (refered to as the <u>loop group of G</u>) we get the central extension

$$1 \longrightarrow \mathbb{C}_1 \longrightarrow \widehat{\mathcal{L}G} \longrightarrow \mathcal{L}G \longrightarrow 1$$

This central extension is important because $\widehat{\mathcal{L}G}$ has a better representation theory than $\mathcal{L}G$: we have a class of irreducible reps on pre-Hilbert spaces (so-called highest weight rep's) which are in general nontrivial on $\mathbb{C}_1$, the ones which are trivial on $\mathbb{C}_1$ factoring over G (given by evaluation at $1 \in \mathbb{C}_1$). So this gives only projective representations $\mathcal{L}G$. This is not uncommon in quantum theory, where a "state" is a point of a projectivized Hilbert space.

Its counterpart for the Lie algebra ($L\mathfrak{g} := C^{[\infty]}(S^1, \mathfrak{g})$) is a so-called affine Kac-Moody algebra:

$$0 \longrightarrow \mathbb{R} \longrightarrow \widehat{L\mathfrak{g}} \longrightarrow L\mathfrak{g} \longrightarrow 0,$$
$$\widehat{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{R} \text{ as } \mathbb{R}\text{-v.sp.}$$

where the Lie bracket is given as follows: if $\alpha, \beta : S^1 \to \mathfrak{g}$, then

$$[\alpha + \mathbb{R}, \beta + \mathbb{R}] = \left([\alpha\beta], \frac{1}{2\pi}\int_{S^1} \langle \alpha(\theta), \dot{\beta}(\theta)\rangle d\theta\right)$$

This can also be done algebraically: if k is a field of char zero and K a local which contains k and has k as residue field (e.g. $K = k((t))$). We assume $\mathfrak{g}$ defined over k. We then have a central extension of $\mathfrak{g}(K) := K \otimes_k \mathfrak{g}$ by k:

$$0 \to k \to \widehat{\mathfrak{g}(K)} \to \mathfrak{g}(K) \to 0$$

$$[\alpha + k, \beta + k] = ([\alpha, \beta], \operatorname{Res}\langle \alpha, d\beta\rangle)$$

There is also an algebraic counterpart for the construction of $\widehat{LG}$: here G is a simple simply connected alg group $/k$ (i.e. $G' = G$ and G has no nontrivial covers) and one can construct an extension of $G(K)$ by k by means of Steinberg symbols

§3 Let Σ now be an oriented connected closed surface, and $P \overset{\pi}{\downarrow} \Sigma$ a G principal bundle (so G acts freely on the right of P, its orbits are the fibers of π and π is loc. trivial)

Since G is 1-connected, such a bundle admits a section s (this trivializes the bundle). If s' is another section, then since G is 2-connected, s and s' are homotopic: if s' is given by $s'(x) = s(x) g(x)$ with $g : \Sigma \to G$, then g is homotopically trivial. If $\tilde{g} : [0,1] \to G$ is a C^∞-homotopy (with $\tilde{g}(x,1) = g(x)$, $\tilde{g}(x,0) = 1 \in G$), then we put

$$\Theta(S) := \exp\left(2\pi\sqrt{-1}\int_{[0,1]\times G} \tilde{g}^* \omega\right) \in \mathbb{C}_1$$

and conclude as before that $\Theta(S)$ only depends on the pair (s, s'); let us therefore denote it by $\Theta^s_{s'}$. So if we divide out by the relation $\sigma \sim \sigma' \Leftrightarrow \Theta^s_{s'} = 1$, then the quotient is a $\mathbb{C}_1$-torsor. This also makes sense if $\partial\Sigma \neq \emptyset$ provided we have already fixed a section (= trivialization) over $\partial\Sigma$. This is even of interest when $(\Sigma, \partial\Sigma) = (B^2, S^1)$.

This has been used in Jones-Witten theory and (related to this) conformal field theory. One then considers a compact Riemann surface C and a simple simply conn. alg grp $\mathcal{G}$ over $\mathbb{C}$. The principal $\mathcal{G}$-bundles over C have a moduli stack $\mathcal{M}(C, \mathcal{G})$, and the alg counterpart constructs an ample line bundle $\mathcal{L} \downarrow \mathcal{M}(C, \mathcal{G})$ which generates $\operatorname{Pic}(\mathcal{M}(C, \mathcal{G}))$. The space of sections $\mathcal{L}^{\otimes \ell}$ is finite-dimensional - the conformal blocks of level ℓ.