M compact complex manifold of dim n.

holom line bundle $\mathcal{L}$ determines a sheaf: sheaf of holom. local sections $\mathcal{O}_M(\mathcal{L})$

This is a sheaf of $\mathcal{O}_M$-modules, locally free of rank 1 (Conversely such a sheaf defines a hol. line bundle up to isom.) If σ is a merom section of $\mathcal{L} \neq 0$ (need not exist!)

then this gives a divisor $\mathrm{div}(\sigma)$: a formal $\mathbb{Z}$-linear combination of hypersurfaces.

- σ is holom $\Leftrightarrow$ $\mathrm{div}(\sigma) \geq 0$ (all coeff ≥ 0)
- if σ, σ' are nonzero merom. sections, then $\sigma' = f\sigma$ for some merom. function f on M and then $\mathrm{div}(\sigma') = \mathrm{div}(\sigma) + \underbrace{\mathrm{div}(f)}_{\text{called a principal divisor}}$

Conversely, a divisor D defines a loc-free $\mathcal{O}_M$-module $\mathcal{O}_M(D)$ (and hence a holom. line bundle) characterized by

$$\Gamma(U, \mathcal{O}_M(D)) = \{\text{merom functions } \varphi \text{ on } U \text{ with } \mathrm{div}(\varphi) + D|U \geq 0\}$$

So if $D \geq 0$, then the function constant 1 on U defines a global section of $\mathcal{O}_M(D)$ and you may check that its divisor is precisely D. If $D' = D + \mathrm{div}(f)$ with f meromorphic, then

$$\Gamma(U, \mathcal{O}_M(D')) \xrightarrow{f\cdot} \Gamma(U, \mathcal{O}_M(D))$$ is an isom. of line bundles for

$\varphi \in \Gamma(U, \mathcal{O}_M(D')) \Leftrightarrow \mathrm{div}(\varphi) + D' \geq 0 \Leftrightarrow \mathrm{div}(f\varphi) + \underbrace{\mathrm{div}(D') - \mathrm{div}(f)}_{\mathrm{div}(D)} \geq 0$. Whence a group-hom.

$$\underset{\text{divisor class group}}{Cl(M)} := \underset{(\text{sum})}{\frac{\text{divisors on } M}{\text{principal divisors}}} \longrightarrow \underset{(\otimes)}{\left\{\begin{matrix}\text{isom cl. of holom}\\ \text{line bundles on } M\end{matrix}\right\}} =: \underset{\text{Picard group}}{Pic(M)}$$

This map is injective and has image the line bundles admitting a nonzero merom. section (these are all hol. line bundles if M is algebraic).

The first Chern class of a line bundle $\mathcal{L}$, $c_1(\mathcal{L})$, only depends on its isomorphism type so that we a map $c_1: Pic(M) \to \mathbb{Z}$. This is a group-hom. It's precomposite with $Cl(M) \to Pic(M)$ assigns to an irreducible hypersurface D in M, the Poincaré dual of its fundamental class, $PD[D]$.

<u>Remark</u>: for every $n \geq 2$ there exists a complex torus $M = \mathbb{C}^n / (\mathbb{Z}\text{-span of an } \mathbb{R}\text{-basis of } \mathbb{C}^n)$ which has no hypersurfaces at all and so $Cl(M)$ is trivial in that case. Yet $Pic(M)$ can be shown to be nonzero (it contains a copy of the dual of M)

We call $\int_M c_1(\mathcal{L})^n$ the <u>degree of $\mathcal{L}$</u>, denoted $\deg(\mathcal{L})$.

Basic fact: $\Gamma(M,\mathcal{L})$ (= space of hol. of sections of $\mathcal{L}$) is finite dimensional

Assume M connected and $\dim_{\mathbb{C}}\Gamma(M,\mathcal{L})\geq 1$, Let $(\sigma_0,\dots,\sigma_d)$ be a basis of $\Gamma(M,\mathcal{L})$. Put

$$\mathrm{Fix}(\mathcal{L}) := \{x\in M:\ \sigma(x)=0\ \ \forall \sigma\in\Gamma(M,\mathcal{L})\}$$
$$= \{x\in M:\ \sigma_i(x)=0\ \ i=0,\dots,d\}$$

This is a closed analytic subset of M, $\mathrm{Fix}(\mathcal{L})\neq M$. Collect all the irreducible components of $\mathrm{Fix}(\mathcal{L})$ of codim 1 and find a divisor $D_0\geq 0$ such that $\mathrm{div}(\sigma)\geq D_0$ for all $\sigma\in\Gamma(M,\mathcal{L})$ and which is maximal for this property. The obvious inclusion of $\mathcal{L}(-D)=\mathcal{L}\otimes\mathcal{O}_M(-D_0)$ in $\mathcal{L}$ then induces an isom. on Γ, but

$\Sigma(\mathcal{L}) := \mathrm{Fix}(\mathcal{L}(-D_0))$ is of codim ≥ 2 everywhere.

We can write $\sigma_i = h_i\sigma_0$ with h_i meromorphic, and this gives a well-defined map

$$\Phi_{\mathcal{L}} = |\mathcal{L}| :\ M\setminus \mathrm{Fix}(\mathcal{L}) \ni x \longmapsto [\sigma_0(x):\dots:\sigma_d(x)] = [1:h_1:\dots:h_d]\in\mathbb{P}^d$$

By viewing the σ_i's as sections of $\mathcal{L}(-D_0)$, we see that this map extends over $M\setminus\Sigma(\mathcal{L})$, that is, in codim ≥ 2.

Note that if $D_0=0$ ("$\mathcal{L}$ has no fixed divisors"), then $\Phi_{\mathcal{L}}$ recovers all the positive divisors of $\mathcal{L}$: each of these is the pull-back under $\Phi_{\mathcal{L}}$ of a hyperplane in $\mathbb{P}^d$. So we always have a bijection

$$\{\text{divisors}\geq 0 \text{ for } \mathcal{L}\} \longleftrightarrow \check{\mathbb{P}}^d \quad (\text{more intrinsically } \mathbb{P}H^0(M,\mathcal{L}))$$
$$D \longleftrightarrow \text{hyperplane defining } D-D_0$$

The left hand side has therefore the structure of a projective space; it is called the complete linear system defined by $\mathcal{L}$.

Of special interest:

$\mathcal{L}$ = top exterior power of the holomorphic cotangent bundle, also denote Ω^n_M

This is called the canonical bundle of M. Its complete linear system is called the canonical system of M.

The dual of the canonical bundle (= top ext. power of hol. tangent bundle) is called the anticanonical bundle of M; it gives us the anticanonical system

Def. $\mathcal{L}$ is very ample if $\Phi_{\mathcal{L}}$ is an embedding

$\mathcal{L}$ is ample if $\Phi_{\mathcal{L}^{\otimes r}}$ is an embedding for some $r\geq 1$

NB. $\mathcal{L}$ ample implies that M can be realized as a closed analytic subset of some projective space. By a theorem of Chow, M is then automatically Zariski closed. This makes M a complex projective manifold. Chow's theorem also implies that if two complex proj manifolds are biholomorphic, then they are isomorphic as algebraic varieties. So M is then a proj manifold in a unique way.

We say that M is a Fano manifold if its anticanonical bundle is ample.

Basic example: $\mathbb{P}^n$ is a Fano variety. Its degree is $(n+1)^n$.
In fact its anticanonical system consists of the hypersurfaces of degree $n+1$.

For $n=2$ this is the lin. system of cubic curves

$$\underset{\text{linear system of cubic curves}}{\mathbb{P}^2 \hookrightarrow \mathbb{P}^9} \qquad [z_0 : z_1 : z_2] \mapsto [\text{mon. of degree } 3] \qquad \text{has degree } 9.$$

In dimension 1, $\mathbb{P}^1$ is in fact the only example: if C is a compact conn. Riemann surface of genus g, then the canonical bundle of C, Ω_C is ample for $g \geq 2$ and its dual, the sheaf of holom. vector fields has no sections $\neq 0$. For $g=1$ both bundles are trivial (hence not ample).

A Fano manifold of complex dim 2 is also called a Del Pezzo surface. Most of these are obtained from blowing up points of a $\mathbb{P}^2$.

Let us see what the effect of blowing up a point is on the anticanonical system.

S a complex surface, $p \in S$ $\quad \hat{S} := \underset{\underset{E}{\cup}}{Bl_p(S)} \xrightarrow{\pi} \underset{\underset{p}{\Psi}}{S}$. $\quad E$ exc. curve of 1st kind $E \cong \mathbb{P}^1$, $E \cdot E = -1$.

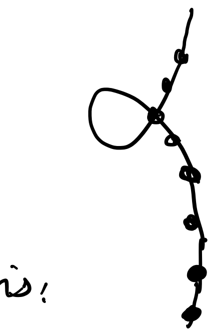
If ω is a 2-form on S with $\omega(p) \neq 0$, then $\pi^*\omega$ is a 2-form on $\hat{S}$ which vanishes with order 1 along E. So $\Omega_{\hat{S}} = \pi^*\Omega_S(E)$. Hence if we dualize $\Omega^\vee_{\hat{S}} = \pi^*\Omega^\vee_S(-E)$, and we find that

$$\Gamma(\hat{S}, \Omega^\vee_S) \cong \{ s \in \Gamma(S, \Omega_S) : s(p) = 0 \}.$$

So if $\Omega^\vee_S$ is very ample, then the antican. system of $\hat{S}$ is a hyperplane in the antican system of S.

The classification of Del Pezzo surfaces is then given by the foll. thm.

Thm. A Del Pezzo surface S is either $\cong \mathbb{P}^1\times\mathbb{P}^1$ (has degree 8) or is obtained from $\mathbb{P}^2$ by blowing up r points $p_1,\dots,p_r\in\mathbb{P}^2$ in gen. position ($0\leq r\leq 8$). When $r=3,4,\dots,8$, $\Omega_S^\vee$ is very ample and embeds S in a $\mathbb{P}^{9-r}$ as a surface of degree $9-r$. In general pos. means here



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- no 8 on a singular cubic with one of the points singular, like this:

The anticanonical proj. realizations can be described concretely as follows

$\mathbb{P}^1\times\mathbb{P}^1 \hookrightarrow \mathbb{P}^8$ embedded by polynomials in Z_0,Z_1,W_0,W_1 of degree 2 in both Z_0,Z_1 and W_0,W_1.

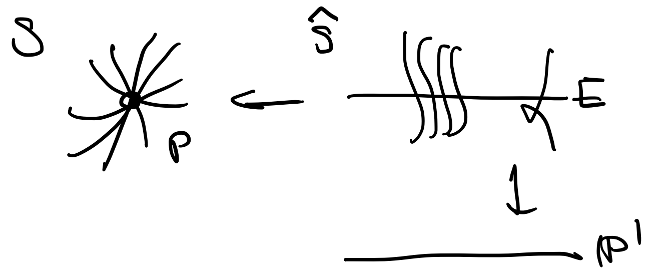
$r=6$ (deg 3): $S\hookrightarrow\mathbb{P}^3$ smooth cubic surface, each line l of S is an exceptional curve of the 1st kind: $l\cdot l=-1$ and $l\cdot c_1(\Omega_S)=-1$. (The 6 exc. curves of a blowup realization $S\to\mathbb{P}^2$ will be among them)

$r=5$ (deg 4) $S\hookrightarrow\mathbb{P}^4$ intersection of two quadrics (more intrinsically: the quadrics in $\mathbb{P}^4$ containing S form a pencil and their common intersection is S)

$r=4$ (deg 5) S obtained from blowing up the points of a coordinate system of $\mathbb{P}^2$: $[1:0:0],[0:1:0],[0:0:1],[1:1:1]$. It has however 10 exc. curves of the first kind. The group $\mathrm{Aut}(S)$ is isomorphic to the alternating group $\mathfrak{A}_5$. This group acts transitively on the coll. of exc. curves.

$r=7$ (deg 2) Then $\Omega_S^\vee$ is not ample, but the antican. system defines a morphism $S\longrightarrow\mathbb{P}^2$ which is finite of degree 2 and whose ramification locus is a smooth quartic curve. Conversely every smooth quartic curve in $\mathbb{P}^2$ determines a Del Pezzo surface of degree 2.

$r=8$ Then $\mathrm{Fix}(\Omega_S^\vee)$ is a single point $\{p\}$ and we get a morphism $S\smallsetminus\{p\}\to\mathbb{P}^1$!



If we blow up this point we get an elliptic fibration which has the exceptional divisor E as a section

Cohomological aspects. Let $p_1, \dots, p_r \in \mathbb{P}^2$ distinct and let $S \xrightarrow{\pi} \mathbb{P}^2$ be obtained from blowing up these points. We denote the corresp. exc. curves $E_1, \dots, E_r$.

They define classes $e_1, \dots, e_r \in H^2(S)$. If $l := \pi^*(\text{line in } \mathbb{P}^2) \in H^2(S)$, then $(l; e_1, \dots, e_r)$ is a basis of $H^2(S)$ whose intersection matrix is $\begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & \ddots \; -1 \end{pmatrix}$

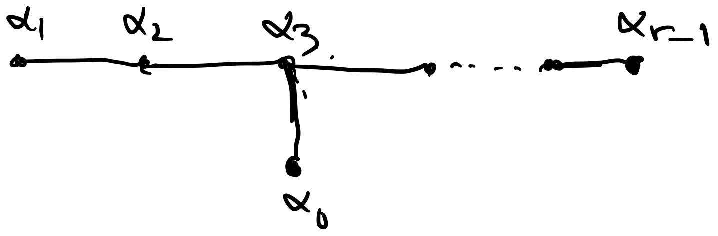
Then $\kappa := 3l - e_1 - \dots - e_r \in H^2(S)$ is the first class of the anticanonical bundle.

Note that $\kappa \cdot \kappa = 9 - r$. So $r \leq 8 \iff \kappa \cdot \kappa > 0 \iff \kappa^\perp$ pos. definite.

If we assume $r \geq 3$, then $\kappa^\perp$ has basis

$$(\underset{\alpha_0}{\underset{\parallel}{l - e_1 - e_2 - e_3}}, \underset{\alpha_1}{\underset{\parallel}{e_1 - e_2}}, \underset{\alpha_2}{\underset{\parallel}{e_2 - e_3}}, \dots, \underset{\alpha_r}{\underset{\parallel}{e_{r-1} - e_r}})$$

These elements have the remarkable property that $\alpha_i \cdot \alpha_i = -2$ and for $i \neq j$, $\alpha_i \cdot \alpha_j \in \{0, 1\}$. We can make an intersection graph with vertices labeled by $\alpha_0, \dots, \alpha_{r-1}$ and vertex α_i connected with vertex α_j $(i \neq j)$ if $\alpha_i \cdot \alpha_j = 1$:



For $r = 3, 4, 5, 6, 7, 8$ this is the Dynkin diagram of $A_1 \times A_2, A_4, D_5, E_6, E_7, E_8$. To be precise, every $\alpha \in H^2(S)$ with $\alpha \cdot \kappa = 0$, $\alpha \cdot \alpha = -2$ determines an orthogonal reflection $s_\alpha : x \in H^2(S) \mapsto x + (\alpha \cdot x)\alpha$ which fixes κ. We may regard the set $R_S \subset H^2(S)$ of such α as a root system (in some generalized sense); for $r \leq 8$ this set is finite.

For $3 \leq r \leq 8$ of the type listed, with $(\alpha_0, \dots, \alpha_{r-1})$ defining a system of simple roots and its Weyl group $W(R_S)$ (= group generated by the s_α's) is the $O(H^2(S))$-stabilizer of κ. Moreover, $W(R_S)$ acts transitively on

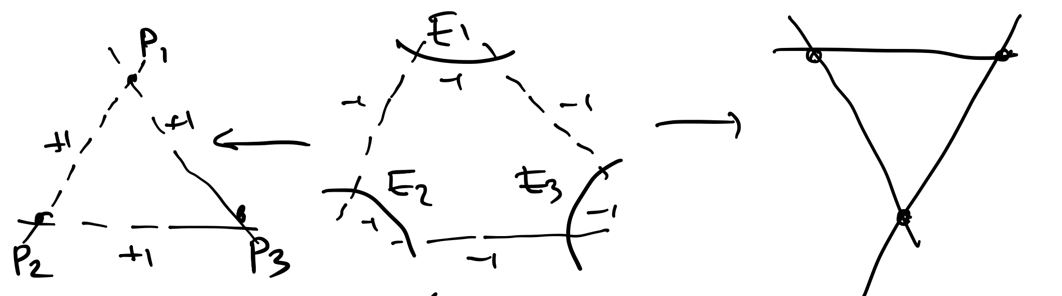
$$\mathcal{E}_S := \{\varepsilon \in H^2(S) : \varepsilon \cdot \varepsilon = -1, \varepsilon \cdot \kappa\} \quad (\text{"potential exceptional classes"})$$

and even simply transitively on the collection of sequences $(\varepsilon_1, \dots, \varepsilon_r) \in \mathcal{E}^r$ that are pairwise perpendicular: $\varepsilon_i \cdot \varepsilon_j = 0$ for all i, j.

Thm For $3 \leq r \leq 9$, S is a del Pezzo surface $\iff$ each $\varepsilon \in \mathcal{E}_S$ is the class of an exceptional curve. If S is such a surface, then each exceptional sequence $(\varepsilon_1, \dots, \varepsilon_r)$ defines a blow-down $S \longrightarrow \bar{S}(\varepsilon_1, \dots, \varepsilon_r) \cong \mathbb{P}^2$ (and so $W(R_S)$ permutes these blow downs simply transitively). The reflection in α_i $(i = 1, \dots, r-1)$ has merely the effect of renumbering (e_i and e_{i+1} are exchanged), but α_0 can be realized as a <u>Cremona transformation</u>

$$\begin{tikzcd} & S \arrow[dl, "\pi"'] \arrow[dr, "s_{\alpha_0}(\pi)"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\varphi"'] & & \mathbb{P}^2 \end{tikzcd}$$

(birat. map)



(6 ex. curves

$$[z_0 : z_1 : z_2] \overset{\varphi}{\dashrightarrow} \left[\frac{1}{z_0} : \frac{1}{z_1} : \frac{1}{z_2}\right]$$

$\#$

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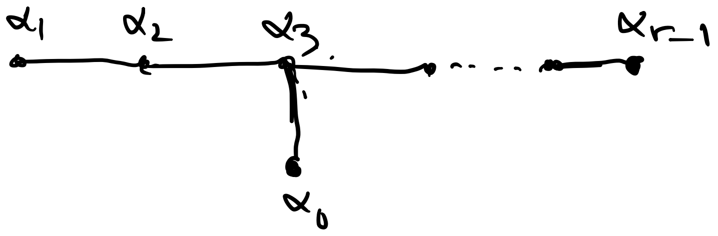

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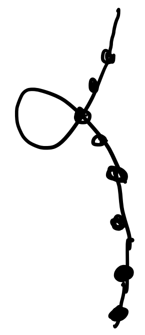

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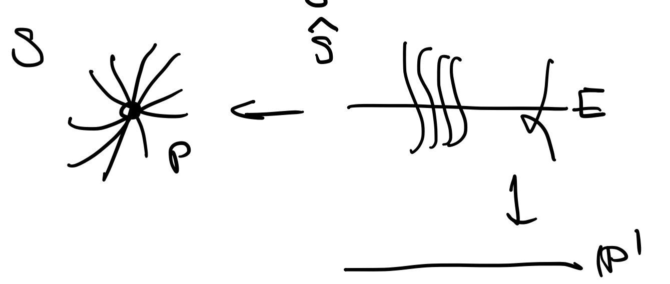

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