# Seven Trees in One

**Trevor Hyde**

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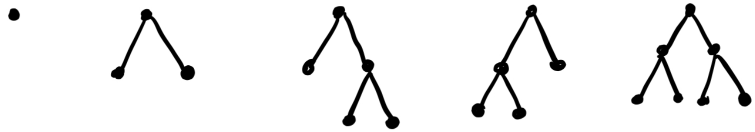
# Binary Trees

A **binary tree** is a rooted, planar tree such that every node has either 0 or 2 branches.



Let $B$ denote the set of all binary trees.

# Binary Trees

Every binary tree is either a single node or equivalent to an ordered pair of binary trees.


$$B \xrightarrow{\sim} \{*\} \sqcup B^2 \qquad \Longrightarrow \qquad B \cong 1 + B^2$$

# Binary trees as a sixth root of unity?

Recall that the sixth cyclotomic polynomial is $\Phi_6(x) = x^2 - x + 1$. The complex roots of $\Phi_6(x)$ are the primitive sixth roots of unity $\zeta_6$ and $\zeta_6^{-1}$.

Hence $B \cong 1 + B^2$ "implies"

$$\Phi_6(B) \approx 0.$$

**Q:** In what sense does $B$ behave like a sixth root of unity?

Obviously $B^6 \not\cong 1$, but...

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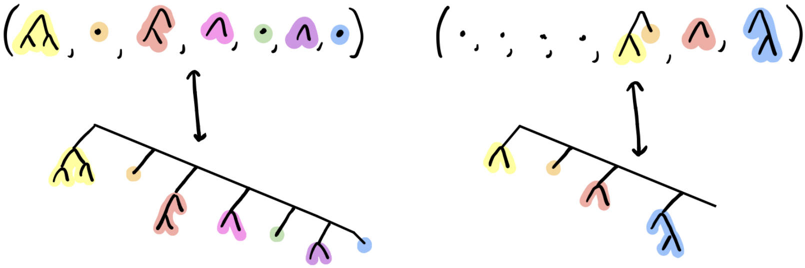
## Theorem (Blass, Lawvere)

*There exists a "very explicit bijection"* $B \xrightarrow{\sim} B^7$.



# Lifting Identities to Isomorphisms

Clearly not all identities satisfied by $\zeta_6$ lift to isomorphisms satisfied by $B$ (e.g. $B^6 \ncong 1$.)

**Q:** Which identities lift to isomorphisms?

**A:** *Semiring consequences.* The consequences of $B = 1 + B^2$ that can be deduced without subtraction or division!

In other words, if $f(x), g(x) \in \mathbb{N}[x]$ are such that $f(x) = g(x)$ in the semiring $\mathbb{N}[x]/(x = 1 + x^2)$, then there is a "very explicit bijection" $f(B) \cong g(B)$.

# Characterizing Lifts

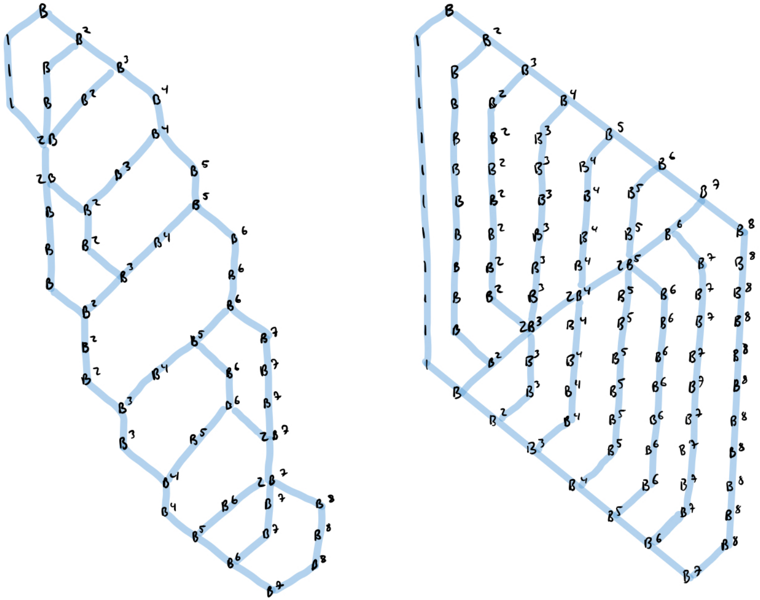
## Theorem (Fiore-Leinster)

*Suppose that $f, g_1, g_2 \in \mathbb{N}[x]$ are polynomials such that $f(x)$ has a nonzero constant term and degree at least 2. If $f(x) - x$ divides $g_1(x) - g_2(x)$ in $\mathbb{Z}[x]$ and $g_1, g_2$ are **both non-constant**, then $g_1(x) = g_2(x)$ in $\mathbb{N}[x]/(x = f(x))$.*

In particular, if $f(x) = 1 + x^2$, then we can deduce $x = x^7$ in $\mathbb{N}[x]/(x = 1 + x^2)$ but not $1 = x^6$!

$\Phi_6(x)$ divides $x^4 + x^2 + 1$, hence

$$0 \ncong B^4 + B^2 + 1$$
$$\color{red}{B} \cong B^4 + B^2 + \color{red}{B} + 1.$$

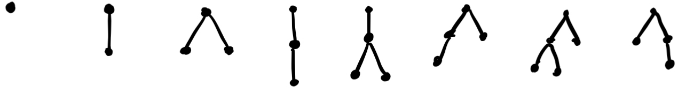


(Taken from : John Baez’s blog)

# Other Trees, Other Numbers

Using the same ideas you can construct tree types that "categorify" other complex numbers.

For example, the set $T$ of rooted, planar trees where every node has 0, 1, or 2 children satisfies $T \cong 1 + T + T^2$, hence $T \approx i$.



There is a very explicit bijection $T \cong T^5$ (Fiore-Leinster.)

# References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