Isom($\tilde{M}$) for M an aspherical manifold

papers of Farb, Weinberger, Avramidi

Ex: ① T^n $\qquad \mathrm{Isom}(\widetilde{T^n}) = \mathrm{Isom}(\mathbb{R}^n)$

$= SO(n) \ltimes \mathbb{R}^n$

Lie group

(d_{hyp}, S_g) $g \geq 2$ $\qquad \mathrm{Isom}(\tilde{S}_g) = PSL_2\mathbb{R}$

Lie group

quotient of cocompact lattice

② Generically, $\mathrm{Isom}(\tilde{M}, \tilde{d}) = \pi_1(M)$

Farb–Weinberger: What condition

$\mathrm{Isom}(\tilde{M})$ is nondiscrete?

Thm 1 (FW) M is closed (aspherical), either $\mathrm{Isom}(\tilde{M}, \tilde{d})$ is discrete, or

$$F \longrightarrow M \longmapsto B$$

$\mathrm{Isom}(\tilde{B})$ is discrete

$F \downarrow \Gamma \backslash G / K$

Γ: cocompact lattice; G: Lie group; K: maximal compact

$\mathrm{Isom}(G/K) \overset{fi}{\geq} G$

Thm 2 (F-W, Avramidi)

$M = M_g \quad \tilde{M} = \mathrm{Teich}_g \quad \pi_1(\tilde{M}) = \mathrm{Mod}(S_g)$

d Finsler metric such that $\mathrm{Vol}(\tilde{M}, \tilde{d}) < \infty$ on M_g (complete metric)

$$\mathrm{Isom}(\mathrm{Teich}(S_g), \tilde{d}) = \mathrm{Mod}^{\pm}(S_g)$$

Royden's Thm: $\mathrm{Isom}(\mathrm{Teich}_g, d_{\mathrm{Teich}}) = \mathrm{Mod}^{\pm}(S_g)$

Pf of Thm 1

$I = \mathrm{Isom}(\tilde{M})$ Lie group (possibly with infinite component)

Myer-Steenrod: I acts properly on $\tilde{M}$.

$I \supseteq \Gamma := \pi_1(M)$

$I_0 \subseteq I$ id component

$I_0 \cap \Gamma =: \Gamma_0$

Pf outline:

Step 1: $\Gamma_0 \subseteq I_0$ is a cocompact lattice Γ

Step 2: I_0 has no compact factor

Step 3: $\tilde{M} / I_0$ is contractible

Step 4: K $I_x^0 = \{g \in I_0 \mid gx = x\} \subseteq I_0$ G

is maximal compact subgroup.

Step 1, 2, 3, 4 $\Longrightarrow$ Thm.

$$\begin{array}{ccccc} I_0/I_x & \longrightarrow & \tilde{M} & \longrightarrow & \tilde{M}/I_0 \\ & & & & \text{by 3, contractible} \\ \circlearrowright & & \circlearrowright & & \circlearrowright \\ P_0 & & P & & P/P_0 \end{array}$$

$$P_0 \backslash I_0/I_x \longrightarrow \tilde{M}/P \longrightarrow \tilde{M}/I_0 \big/ (P/P_0)$$

Step 1: I_0 / P_0 is compact

$$I \curvearrowright \overbrace{\mathrm{Fr}(\tilde{M})}^{SO(n)\text{-bundle}} \quad \text{free}$$

$$I/P \longrightarrow \underbrace{\mathrm{Fr}(\tilde{M})/P}_{\substack{\text{compact} \\ \mathrm{Fr}(M)}} \longrightarrow \underbrace{\mathrm{Fr}(\tilde{M})/I}_{\text{compact manifold.}}$$

$\Rightarrow$ I/P is compact.

$$1 \to \underbrace{I_0/P_0}_{\text{compact.}} \to I/P \to \underbrace{\pi_0(I)/_{P/P_0}}_{\substack{\text{compact \& discrete} \\ \Rightarrow \text{finite}}} \to 1$$

Step 2: I_0 has no compact factor.

q.i invariant $[M] \neq 0 \in H_n^{lf}(\tilde{M}; \mathbb{Z})$

$\Rightarrow$ q.i can see the dim of aspherical manifolds

if I_0 has compact factor K.

$$\tilde{M}/K \overset{q.i}{\equiv} \tilde{M}$$

$\downarrow$

$\dim < \dim(\tilde{M})$

Step 3: $\tilde{M}/I_0$ is contractible

Oliver's Thm: G compact $\curvearrowright X$

X is contractible $\Longrightarrow$ X/G is contractible

$I_0 \curvearrowright \tilde{M}$ properly ✓. $\square$.

Step 4: $I_x^\circ = \mathrm{Stab}(x) \subseteq I_0$ maximal.
compact

pf: I_x° is compact $\subseteq SO(n)$

$I_x^\circ \subseteq K_0 \subseteq I_0$

$$\underline{\dim(\tilde{M})} = cd_{\mathbb{Q}}(P) \leq cd_{\mathbb{Q}}(P_0) + cd_{\mathbb{Q}}(P/P_0)$$

$$\leq \dim(I_0/K_0) + \dim(\tilde{M}/I_0)$$

$$\dim(\tilde{M}) \geq \dim(\tilde{M}/I_x) = \dim(\tilde{M}/I_0) + \dim(I_0/I_x)$$

$$\Rightarrow \quad \dim(K_0) = \dim(I_x^\circ) \qquad \square$$

$\underline{I_0 \text{ semisimple.}} \Rightarrow P = P_0 \times P/P_0$

$F \to M \to B \rightsquigarrow M = B \times F$

If I_0 is ~~ss~~ finite center $\Rightarrow P = P_0 \times P/P_0$

Pf: $\mathrm{Out}(I_0) = \text{finite}$

$$\begin{tikzcd}
1 \arrow[r] & P_0 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, hook] & \underline{P/P_0} \arrow[r] \arrow[d, hook, "\text{finite index}"] & 1 \\
\boxed{1 \arrow[r] & I_0 \arrow[r] & I \arrow[r] & \underline{I/I_0} \arrow[r] & 1}
\end{tikzcd}$$

as a product!

$\mathrm{Out}(I_0)$ trivial

① $\rho: P/P_0 \to \mathrm{Out}(P_0)$ trivial

② $H^2(P/P_0; Z(P_0)) = 0$

$Z(P_0) \to 0$

If P has no solvable normal subgp $\Rightarrow I_0$ is semi simple finite center.