Knots Which Behave Like the Prime Numbers

Based on the paper ([1]) by Curtis T. McMullen

Seraphina Lee

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Chebotarev, for prime numbers

A version of Chebotarev's density law says the following:

Theorem

Let $K/\mathbf{Q}$ be a finite, Galois extension, and let C be a conjugacy class in $\mathrm{Gal}(K/\mathbf{Q})$. *Then*

$$\lim_{N\to\infty} \frac{\#\{p \in \mathbf{N} : p \textit{ prime}, \, p \leq N, \mathrm{Frob}_p = C\}}{\#\{p \leq N\}} = \frac{\#C}{\#\,\mathrm{Gal}(K/\mathbf{Q})}.$$

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Remark

Frob_p only defined if p is unramified in K.

Chebotarev, for knots

Let

- M be a closed, connected 3-manifold,
- $K_1, K_2, \ldots$ be a sequence of disjoint, smooth, oriented knots in M with $L_n := \bigcup_{i=1}^{n} K_i$,
- G finite group,
- $\rho : \pi_1(M - L_n) \to G$ a surjective homomorphism.

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Remark

- $\ker \rho \leq \pi_1(M - L_n)$ determines a branched cover $\tilde{M}$ of M, possibly ramified only over L_n.
- $\tilde{M}$ has deck group G over M.

Chebotarev, for knots

Definition

We say that (K_i) obeys the Chebotarev law if for any ρ as above and any conjugacy class $C \subseteq G$,

$$\lim_{N \to \infty} \frac{\#\{n < i \leq N : [K_i] = C\}}{N} = \frac{\#C}{\#G}.$$

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From now on, let X be a closed hyperbolic surface of genus $g \geq 2$, and $T_1(X)$ its unit tangent bundle.

Theorem (McMullen)

Let $K_1, K_2, \cdots \subseteq T_1(X)$ *be the closed orbits of the geodesic flow ordered by length. Then* (K_i) *obeys the Chebotarev law.*

Remarks, about the main theorem

But not all classes of $\pi_1(T_1(X))$ come from geodesic loops!

Example

The fibers of the map $T_1(X) \to X$ are nontrivial loops in $\pi_1(T_1(X))$ but are not freely homotopic to geodesics.

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Proof technique

(1) Reduce to a problem about finite, directed graphs.

(2) Apply a Chebotarev theorem for closed orbits of flows on such graphs.

Rectangles

Definition

A rectangle R is a simply-connected subset of $T_1(X)$, which looks like:

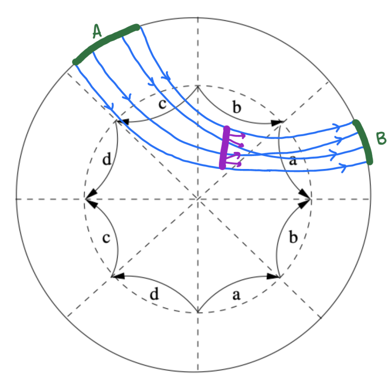

Figure: *underlying picture is Figure 1 in [2].*

Geodesics to finite graphs

- If R_i, R_j are rectangles,

$$R_{ij} = \{v \in \mathrm{int}(R_i) : \text{first return to } \bigcup R_k \text{ lands in } R_j\}.$$

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- Let (R_i) be a finite set of rectangles in $T_1(X)$, and let Γ be a directed graph with

$$\begin{aligned} V(\Gamma) &= \{R_i\}, \\ E(\Gamma) &= \{(R_i, R_j) : R_{ij} \neq \emptyset\}, \\ \Sigma(\Gamma) &= \{\text{bi-infinite paths in } \Gamma\} \\ &= \{v : \mathbf{Z} \to V(\Gamma) : (v(i), v(i+1)) \in E(\Gamma)\}, \\ \sigma(v)(i) &= v(i+1). \end{aligned}$$

Paths in Γ to geodesics

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- $p(\ldots, v_{-1}, v_0, v_1, \ldots) \in v_0$,

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Definition

The finite set of rectangles (R_i) is a Markov section if

- *for all $v \in T_1(X)$, there is some time $t > 0$ such that $t \cdot v \in \bigcup R_i$, and*
- $R_{ij} := \{v \in \text{int}(R_i)$: *first return of v lands in* $\text{int}(R_j)\}$ *looks like:*

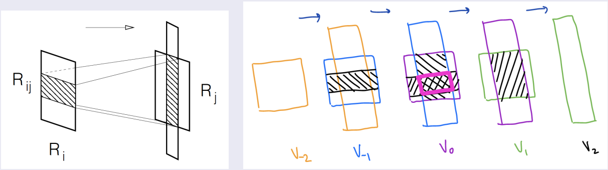

Figure: *Left: Figure 1 in [1]*

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Now we have:

- Continuous map $p : \Sigma(\Gamma) \to \bigcup_i R_i$ with $(\ldots, p(\sigma^{-1}(\gamma)), p(\gamma), p(\sigma(\gamma)), \ldots)$ lying on a geodesic.

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- Continuous map $p : \Sigma(\Gamma) \to \bigcup_i R_i$ with $(\ldots, p(\sigma^{-1}(\gamma)), p(\gamma), p(\sigma(\gamma)), \ldots)$ lying on a geodesic.
- Continuous height function $h : \Sigma(\Gamma) \to (0, \infty)$,

$$h(\gamma) := \text{time until first return of } p(\gamma) \text{ to } \bigcup R_i.$$

Flows

Definition

Given a nice function $h : \Sigma(\Gamma) \to (0, \infty)$*, the corresponding suspended subshift is defined as*

$$\Sigma(\Gamma, h) = \frac{\Sigma(\Gamma) \times \mathbf{R}}{(\gamma, t + h(\gamma)) \sim (\sigma(\gamma), t)}$$

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Can think of $\Sigma(\Gamma, h) =$ "space of (flow, time) on Γ with speed given by h." In particular,

$$\{\text{closed orbits of } \Sigma(\Gamma, h)\} \xrightarrow{c} \{\text{closed orbits of } \Gamma \text{ with speed } h\}$$

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Definition

$\Sigma(\Gamma, h)$ *is equipped with a flow,*

$$s \cdot [\gamma, t] := [\gamma, s + t] \qquad \forall s \in \mathbf{R}.$$

Chebotarev theorem from symbolic dynamics

Theorem (Parry–Pollicott, McMullen)

Let Γ_0 be a finite directed graph, and let h_0 be a "nice" function on $\Sigma(\Gamma)$. The Chebotarev law holds for (τ_i), closed orbits of $\Sigma(\Gamma_0, h_0)$ ordered by length, if the flow on $\Sigma(\Gamma_0, h_0)$ is topologically mixing.

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A flow is topologically mixing if for all open sets U, V, we have $U \cap tV \neq \emptyset$ for all $t \gg 0$.

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Corollary

The flow on $\Sigma(\Gamma, h)$ is topologically mixing, where Γ, h are the graph and function constructed in the previous slides.

Symbolic flow to geodesic flow

- Let

$$U := \left(\bigcup \mathrm{int}(R_i) \right) \cup \{\text{geodesics running from } R_i \text{ to } R_j,\ \forall i, j\}$$
$$\subseteq T_1(X).$$

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$$\pi : \Sigma(\Gamma, h) \to U$$
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- After finitely many closed orbits have been excluded, π gives a bijection between closed orbits of $\Sigma(\Gamma, h)$ and $T_1(X)$, with $L(\tau) = L(\pi(\tau))$.

Symbolic flow to geodesic flow

- Continuous map $c : \Sigma(\Gamma, h) \to \Gamma$,

$$[(\dots, v_{-1}, v_0, v_1, \dots), t] \mapsto \text{linearly onto edge } (v_0, v_1) \in \Gamma.$$

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- π and $\iota \circ c$ are homotopic to each other.

A lemma

Lemma

Let $L \subseteq \partial U$ be the union of finitely many closed geodesics. Then the map $\iota : \Gamma \to U \subseteq T_1(X) - L$ induces a surjective map

$$\iota_* : \pi_1(\Gamma) \to \pi_1(T_1(X) - L).$$

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- Let $\alpha : \mathbf{S}^1 \to T_1(X)$ based at x.
- Can assume that α in general position with $T_1(X) - U$, with α crossing ∂U transversely at finitely many points $p = \alpha(s)$.

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Proof idea, continued.

- After perturbing α, (WLOG) positive geodesic ray through $p = \alpha(s)$ is dense in $T_1(X)$.

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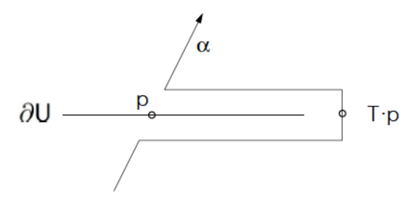

Figure: Figure 2 in [1]

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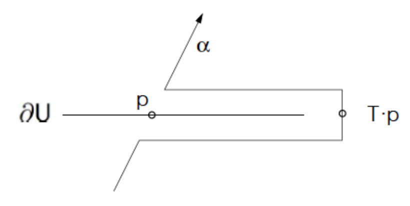

Figure: Figure 2 in [1]

- When $L \neq \emptyset$, all of this can be done away from L.

Proof

- Take $\rho : \pi_1(T_1(X) - L_n) \twoheadrightarrow G$ with $L_n = K_1 \cup \cdots \cup K_n$.

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- The composition

$$\pi_1(\Sigma(\Gamma, h)) \xrightarrow{c_*} \pi_1(\Gamma) \xrightarrow{\iota_*} \pi_1(T_1(X) - L_n) \xrightarrow{\rho} G$$

is surjective.

Proof

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- (τ_i) satisfies the Chebotarev law under $\rho \circ \iota_* \circ c_*$, so (K_i) satisfies the Chebotarev law under ρ.

References

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Javier Aramayona. *Hyperbolic Structures on Surfaces*. 2011.