Recall: V real v.sp. of dim ℓ, $\mathcal{H}$ finite set of linear hyperplanes s.t every conn. component of $U := V \setminus \bigcup_{H \in \mathcal{H}} H$ (chamber) has ℓ walls.

So if $S_V \subset V$ is the unit sphere rel some inner product (more intrinsically the space of rays in V), then $\mathcal{H}|S$ defines a triangulation Σ of S_V.

In case $\mathcal{H}$ is the set of reflection hyperplanes of a Coxeter group W this is called the Coxeter complex of W. Write B_V for the unit ball (so $\partial B_V = S_V$); more intrinsically $\mathrm{Cone}(S_V)$.

Intermezzo: To every s.s. alg group G / alg. closed field there is associated its Tits complex, a simpl. cx. on which G acts (simplicially). It is of dim $\ell = \mathrm{rk}(G) - 1$. It is a union of subcomplexes isomorphic to the Coxeter complex of the Weyl group of G; these are called apartments. Quillen proved that $\mathcal{T}(G)$ has the homotopy type of a bouquet of $(\ell-1)$-spheres; the apartments provide generators of $\pi_{\ell-1}(\mathcal{T}(G))$.

Example: Let N be a complex vector space of dim $\ell+1$. The Tits complex $\mathcal{T}(SL(N)) =: \mathcal{T}(N)$ has a vertex $v(F)$ for every proper subspace $0 \neq F \subsetneq L$. A "flag" of proper subspaces

$$0 \neq F_0 \subsetneq \cdots \subsetneq F_r \subsetneq L$$

defines an r-simplex and each r-simplex of $\mathcal{T}(N)$ is of this form. An apartment is given by a decomposition $N = L_0 \oplus \cdots \oplus L_\ell$ into 1-dim subspaces. For every $\emptyset \neq I \subsetneq \{0, .., \ell\}$, $L_I := \bigoplus_{i \in I} L_i$ defines a vertex $v(L_I)$. The full subcomplex $\Sigma(L_0, .., L_\ell) \subset \mathcal{T}(N)$ on this vertex set is an apartment. To see it is isomorphic to the Coxeter complex of $\mathfrak{S}_{\ell+1}$: $(\ell-1)$-simplex of the Coxeter complex $\longleftrightarrow$ chamber of the $\mathfrak{S}_{\ell+1}$-arrangement. Such a chamber is given by some $\sigma \in \mathfrak{S}_{\ell+1}$: $x_{\sigma(0)} < x_{\sigma(1)} < \cdots < x_{\sigma(\ell)}$). This defines the $(\ell-1)$-simplex $L_{\sigma(0)} < L_{\sigma(0)} + L_{\sigma(1)} < \cdots < L_{\sigma(0)} + \cdots + L_{\sigma(\ell-1)}$ of $\Sigma(L_0, .., L_{\ell-1})$.

Deligne's proof of his theorem is inspired (but not based) on this construction. We give the main idea. It uses induction on $\ell = \dim V$. (For $\ell = 1$ the theorem is clear.)

A simplex σ of Σ corresponds to a face $F_\sigma \subset V$ (here a face always open in its linear span so that the faces decompose V). Note: $\mathrm{Star}(\sigma)$ defines an open convex nbhd $\mathrm{Star}(F_\sigma)$ of F_σ in V.

Let $U_{\sigma,\mathbb{C}} := (\mathrm{Star}(F_\sigma) + \sqrt{-1}V) \cap U_{\mathbb{C}} = (\mathrm{Star}(F_\sigma)+\sqrt{-1}V) \setminus \bigcup_{\substack{H\in\mathcal{H} \\ H\supset F_\sigma}} H_{\mathbb{C}}$

Consider

$$U_{\sigma,\mathbb{C}} \subset U_{\mathbb{C}} \subset V_{\mathbb{C}} \setminus \bigcup\{H_{\mathbb{C}} : H\in\mathcal{H},\ H\supset F\}$$

The composite inclusion is a htp equivalence. So $U_{\sigma,\mathbb{C}}\subset U_{\mathbb{C}}$ is injective on π_1. 1

If $\langle F_\sigma\rangle$ denote the real span of F_σ, then

$$V_{\mathbb{C}}\setminus\bigcup\{H_{\mathbb{C}}: H\in\mathcal{H},\ H\supset F\} \longrightarrow V_{\mathbb{C}}/\langle F\rangle_{\mathbb{C}} \setminus \bigcup\{H_{\mathbb{C}}/\langle F\rangle_{\mathbb{C}} : H\in\mathcal{H},\ H\supset F\}$$

is also a htp equiv. The induction hypothesis therefore implies that $U_{\sigma,\mathbb{C}}$ is aspherical 2

Now let $\tilde{U}_{\mathbb{C}} \xrightarrow{\pi} U_{\mathbb{C}}$ be a universal cover. Must show that $\tilde{U}_{\mathbb{C}}$ is contractible. Will do this by finding a Leray covering $\hat{\mathcal{U}}$ of $\tilde{U}_{\mathbb{C}}$ whose nerve is contractible.

[If $\mathcal{U}$ is an open covering of a paracompact space X, then a partition of 1 subordinate to $\mathcal{U}$ defines a map $X\to \mathrm{Nerve}(\mathcal{U})$ whose homotopy type is indep of the partition of 1. If $\mathcal{U}$ is a Leray covering (meaning: every nonempty intersection is contractible), then according to A. Weil this is htp equivalence.]

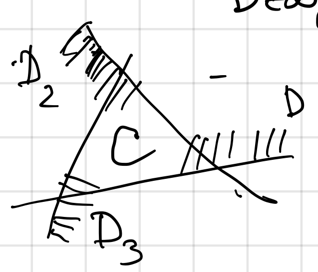
Note that $\bigcup_\sigma F_\sigma = V\setminus\{0\}$. We therefore first focus on $U_{\mathbb{C}}\setminus\sqrt{-1}U$.

1 and 2 imply that each connected component of $\pi^{-1}U_{\sigma,\mathbb{C}}$ is contractible.
Let $\mathcal{U}$ be the collection of such connected components with σ running over all simplices of Σ.
Since $\bigcup_\sigma F_\sigma = V\setminus\{0\}$, the union of these open subsets is $\pi^{-1}(U_{\mathbb{C}}\setminus\sqrt{-1}U)$. In fact $\mathcal{U}$ is a Leray covering of $\pi^{-1}(U_{\mathbb{C}}\setminus\sqrt{-1}U)$ and the construction shows that we have a projection $\mathrm{Nerve}(\mathcal{U}) \longrightarrow \Sigma$ (preimage of ~~a~~ k-simplex is a union of k-simplices).

Deligne proves that $\mathrm{Nerve}(\mathcal{U})$ has the homotopy type of a bouquet of $(\ell-1)$-spheres with each sphere mapping isomorphically onto S_V

To cover $\pi^{-1}(\sqrt{-1}U)$ also, we define for every chamber C of $\mathcal{H}$ a contractible nbhd $U_{C,\mathbb{C}}$ of $\sqrt{-1}C$ in $U_{\mathbb{C}}$: let $\mathcal{D}_C$ be the set of ℓ open half spaces in V s.t. $C=\bigcap_{D\in\mathcal{D}_C} D$. (so each ∂D defines a wall of C). We let

$$U_{C,\mathbb{C}} = \{x+\sqrt{-1}y : \text{ if } x\notin D, \text{ then } y\in D \text{ for all } D\in\mathcal{D}_C\}$$



Note $\sqrt{-1}C \subset U_{C,\mathbb{C}} \subset U_{\mathbb{C}}$. Every fiber of $(x+\sqrt{-1}y) \in U_{C,\mathbb{C}} \mapsto x \in V$ is a nonempty intersection of members of $\mathcal{D}_C \Rightarrow U_{C,\mathbb{C}}$ is contractible. Hence each connected component of $\pi^{-1}(U\ \ )$ maps iso onto $U_{C,\mathbb{C}}$. We add these to $\mathcal{U}$: this gives the Leray covering $\hat{\mathcal{U}}$. The inclusion $\mathrm{Nerve}(\mathcal{U}) \subset \mathrm{Nerve}(\hat{\mathcal{U}})$ has the effect of "coning off" the $(\ell-1)$-spheres in $\mathrm{Nerve}(\hat{\mathcal{U}})$ that prevent it from being contractible.

Question (A'Campo) Does there exist on $U_{\mathbb{C}}$ a complete metric of nonneg. curvature?

Corresponding question for $\tilde{\Sigma}$ (Charney): Is $\tilde{\Sigma}$ a CAT(1) space?

A yes answer would lead to interesting generalization:

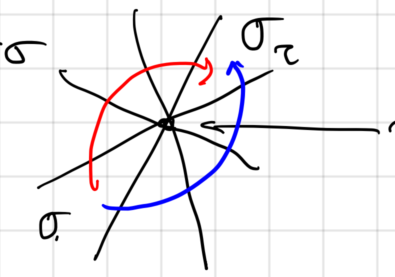
Let $\Omega \subset V$ be open convex, $\mathcal{H}$ a collection of linear hyperplanes in V, loc. finite on Ω and such that (*) is satisfied for every conn. component of $U := \Omega \setminus \bigcup_{H\in\mathcal{H}} H$. Then $U_{\mathbb{C}} := (\Omega + \sqrt{-1}V) \setminus \bigcup_{H\in\mathcal{H}} H_{\mathbb{C}}$ is aspherical.

(This would give corresponding results for arbitrary Coxeter groups and their Artin groups)

<u>Combinatorial construction of Nerve($\mathcal{U}$)</u>

A positive gallery G of length d in Σ is a sequence of max simplices $\sigma_0 \sigma_1 \dots \sigma_d$ such that σ_{i-1} and σ_i have a common $(\ell-2)$-simplex.

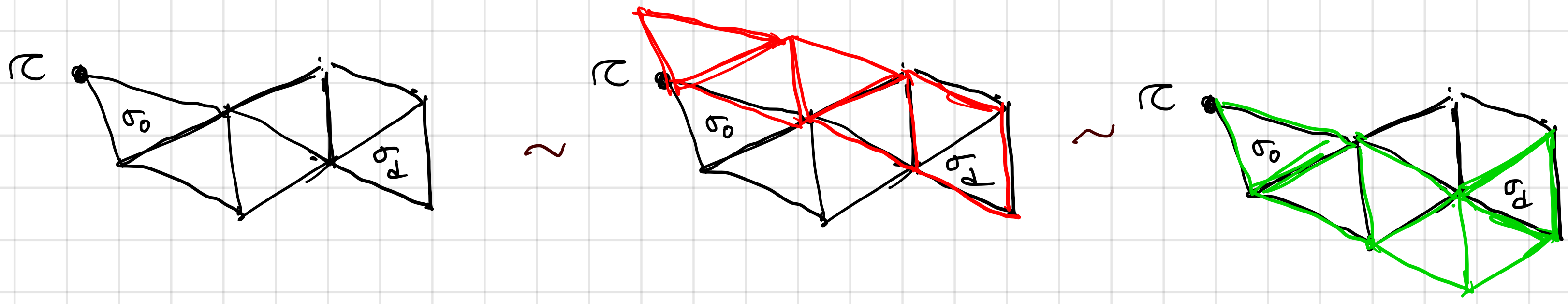
If σ is a (max) $(\ell-1)$-simplex and $\tau \subset \sigma$ is a $(\ell-3)$-simplex, σ_τ the $(\ell-1)$-simplex opposite σ w.r.t. τ.



picture on S_V

Then have two galleries connecting σ with σ_τ. We declare these to be equivalent. This generates an equivalence relation among the positive galleries (comb. version of the braid relations!) Note that this equivalence relation does not change σ_0 and σ_d.

Fix a simplex τ of Σ. We define a simplicial complex $\Sigma(\tau)$ over Σ by glueing the pos. galleries starting at τ: an $(\ell-1)$-simplex of $\Sigma(\tau)$ over an $(\ell-1)$-simplex σ of Σ is given by a positive gallery $\sigma_0 \sigma_1 \dots \sigma_d$ with τ a face of σ_0 and $\sigma_d = \sigma$.



We regard $\sigma_0' \dots \sigma_{d'}'$ as defining the same simplex over σ if they become equivalent after precomposing pos. galleries in $\mathrm{Star}(\tau)$.

Deligne proves

① $\Sigma(\tau)$ has the homotopy type of a bouquet of $(\ell-1)$-spheres

② Let $v \in \Sigma$ be a vertex. Choose a vertex $\tilde{v}$ of $\mathrm{Nerve}(\mathcal{U})$ over v.

Then we have a natural embedding $\Sigma(v) \hookrightarrow \mathrm{Nerve}(\mathcal{U})$ $\quad v \mapsto \tilde{v}$

denote image $\Sigma(\tilde{v})$

③ we can find $\tilde{v}_1, \tilde{v}_2, \dots$

such that $\Sigma(\tilde{v}_1) \subset \Sigma(\tilde{v}_2) \subset \dots \quad \bigcup_i \Sigma(\tilde{v}_i) = \Sigma_{\mathrm{Nerve}(\mathcal{U})}$.

Hence $\mathrm{Nerve}(\mathcal{U})$ has also the htp type of a bouquet $(\ell-1)$-spheres.