①

V real vector space of dim ℓ; $\mathcal{H}$ a finite set of linear hyperplanes in V

$U := V \setminus \bigcup_{H\in\mathcal{H}} H$. Call every conn. comp a chamber.

(*) Assume: every chamber C has precisely ℓ walls

Equivalently: $\exists$ linear isom $V \cong \mathbb{R}^\ell$ that takes C onto $\mathbb{R}^\ell_{>0}$.

Deligne (1972) Then $U_{\mathbb{C}} := V_{\mathbb{C}} \setminus \bigcup_{H\in\mathcal{H}} H_{\mathbb{C}}$ is aspherical ($\Leftrightarrow$ its univ. cover is contractible)

Application/motivation: Suppose $W \subset GL(V)$ finite subgroup generated by reflections (i.e., a Coxeter group) with $V^W = \{0\}$. Then (*) fulfilled and W acts freely on $U_{\mathbb{C}}$

So the universal cover of $U_{\mathbb{C}}$ is also a universal cover of $W\backslash U_{\mathbb{C}}$: $W\backslash U_{\mathbb{C}}$ aspherical.

The fund group of $W\backslash U_{\mathbb{C}}$ is called the Artin group of W, denoted A_W (also called generalized braid group, reason explained below) and $H^\bullet(A_W) = H^\bullet(W\backslash U_{\mathbb{C}})$

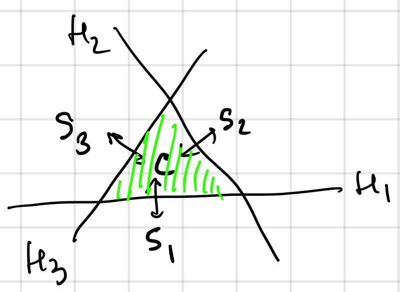
Example: $W = \mathfrak{S}_{\ell+1} \curvearrowright \mathbb{R}^{\ell+1}$. Leaves $V := \{x_0 + \dots + x_\ell = 0\}$ invariant.

The reflections are the transpositions and so the hyperplanes are $x_i = x_j$ $(i<j)$

A chamber is $x_0 < x_1 < \dots < x_\ell$

$U_{\mathbb{C}} = \{ (z_0,\dots,z_\ell) \in \mathbb{C}^{\ell+1} : z_0,\dots,z_\ell \text{ distinct}; \sum z_i = 0 \}$

$\mathfrak{S}_{\ell+1}\backslash U_{\mathbb{C}} = \{ z^{\ell+1} + a_2 z^{\ell-1} + \dots + a_{\ell+1} \in \mathbb{C}[z] \text{ separable} \}$ fundamental group is the braid group on $\ell+1$ strands.



"top view"

Return to W as above. W permutes the chambers simply transitively.

Fix a chamber C. Let X be the set of walls of C.

So for every $i \in X$ we have a supporting hyperplane H_i and corresp. reflections $s_i \in W$. Then $\{s_i\}_{i\in X}$ generate W. Clearly:

(i) $s_i^2 = 0$ $(i \in X)$

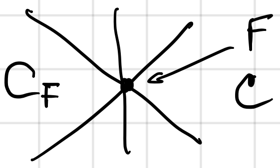
If $i \neq j$ and $s_i s_j$ has order m_{ij}, then $(s_i s_j)^{m_{ij}} = 1$ is in view of (i) equiv. to

(ii) $\underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}$ (braid relations)

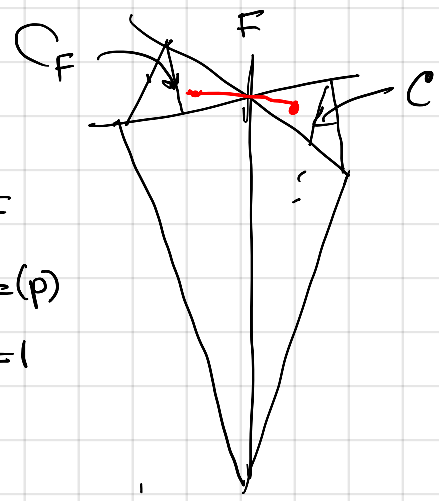
Coxeter ('34) This is a presentation for W.

Brieskorn ('71) A_W has a presentation with generators $\{t_i\}_{i\in X}$ subject to the braid relations $(\overbrace{t_it_j\cdots}^{m_{ij}} = \overbrace{t_jt_i\cdots}^{m_{ij}})$ so that $A_W \to W$, $t_i \mapsto s_i$ is the nat. hom.

Geometric realization of t_i: Let F be a face of C. Then we have a chamber C_F opposite F:

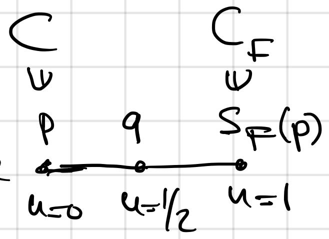


and there is a unique



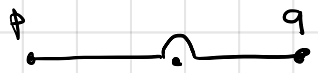
$w_F \in W$ that takes C to C_F.

Choose $p \in C$, $q \in F$ and let $\delta_F : [0,1] \to V$ be



Choose lift $\tilde{\delta}_F : [0,1] \to \mathcal{U}_{\mathbb{C}}$

$$\tilde{\delta}_F(u) = \delta_F(u) + \varphi(u)\sqrt{-1}\,p$$



Then $\tilde{\delta}_F$ closes up in $W \backslash \mathcal{U}_{\mathbb{C}}$ and defines $t_F \in \pi_1(W\backslash \mathcal{U}_{\mathbb{C}}, \bar{p}) =: A_W$.

If we let F run over the walls of C we get our generators $\{t_i\}_{i \in X}$

Rem 1. If F is the codim 2 face defined by H_i and H_j $(i \neq j)$, then $t_F = \underbrace{t_it_j\cdots}_{m_{ij}} = \underbrace{t_jt_i\cdots}_{m_{ij}}$

2. Take $F = \{0\}$ and put $\Delta := t_{\{0\}}$. Then we have an involution $i \in S \mapsto i' \in S$ such that $\Delta t_i = t_{i'}\Delta$. In particular Δ is central.

3: Property (*) is inherited if we choose some $H \in \mathcal{H}$ and consider the restriction of $\mathcal{H} \setminus \{H\}$ to H. (But if we do this for a Coxeter arrangement, then this restriction need not be a Coxeter arrangement!)

Idea of Deligne's proof. (Uses induction on ℓ.)

Let $S = S(V)$ be the sphere of rays in V and $B = B(V)$ the cone over it (can take unit sphere resp. unit ball wrt some inner product)

The collection $\mathcal{H}$ defines a triangulation Σ of S (chambers define $(\ell-1)$-simplices)

Every simplex σ of Σ defines a "face" $F_\sigma \subset V$ (open in its linear span)

Star(σ) defines an open nbhd N_σ of F_σ in V

So $\mathcal{U}_\sigma := N_\sigma \cap \mathcal{U}$ is the union of chambers having F_σ as a face