Model problem

M - closed hyperbolic n-manifold.

$N_m$ - totally geodesic, codim-1 hypersurface

- How do $N_m$ distribute inside $M$?
- What is the limit of

$$\frac{[N_m]}{\text{vol}(N_m)} \in N_{n-1}(M, M).$$

We want to construct a limit object.

$N_m \sim (n-1)$-dim. hyperbolic submanifolds

if $$M = \Gamma \backslash \frac{SO(n,1)}{SO(n)}$$

then $N_m$ correspond to closed orbits of $SO(n,1)$ on $\Gamma \backslash \frac{SO(n,1)}{}$.

$$\Gamma \circ (gSO(n,1)g^{-1})$$
Let $\mu_m$ be the normalized volume on $N_m$, lifted to $\Gamma \backslash SO(n,1)$, so it is a probability measure invariant under $SO(n-1,1)$.

Let $\nu$ be a weak-$*$ limit of $\mu_m$ as $m \to \infty$.

**Ratner classification**

- $\nu$ is a convex combination of measures supported on closed orbits of $SO(n,1,1)$ and the Haar measure on $\Gamma \backslash SO(n,1)$.

- $\nu$ doesn't give mass to closed orbits, so $\nu = \text{Haar on } \Gamma \backslash SO(n,1)$. 
1) the sequence \( N_m \) equidistributes

\[
\lim_{m \to \infty} \frac{[N_m]}{\text{vol}(N_m)} = 0 \quad \text{in} \quad H_{n-1}(M, \mathbb{R}).
\]

What if our sequence of objects does not lie in any obvious ambient space?

1. \( Y_0 \) - finite CW-complex, aspherical

\( Y_m \) - sequence of finite covers of \( Y_0 \).

2. \( X \) - symmetric space,

\[ X = G/K \]

\( G \) - semisimple Lie group

\( K \) - maximal compact subgroup

( e.g. \( H^n = \text{SO}(n,1)/\text{SO}(n) \). )
\[ (\Gamma_m \backslash X) \text{ - finite volume quotients of } X, \]

\[ (\text{locally symmetric spaces}) \]

\[ \lim_{m \to \infty} \frac{\delta_k (\Gamma_m \backslash X, \mathbb{R})}{\text{vol} (\Gamma_m \backslash X)} = ? \]

\[ \lim_{m \to \infty} \frac{\log \left| \text{Hom} (\Gamma_m \backslash X, \mathbb{Z})_{\text{tor}} \right|}{\text{vol} (\Gamma_m \backslash X)} = ? \]

3. \( G_n \) - a sequence of bounded degree finite graphs.

In (1), the finite covers \( Y_0 \) are in 1-to-1 correspondence with finite index subgroups of \( \Gamma = \Pi_{\gamma} (Y_0, g_0) \).

\[ Y_m \mapsto \Pi_{\gamma} (Y_m, \tilde{g}_0) \]
Consider

$$\text{Sub}_0(\Gamma) = \{ \Lambda \subset \Gamma \mid \text{closed subgroup} \}$$

with topology from $\{0,1,\hat{\Gamma}\}$.

$\Gamma \cap \text{Sub}_0(\Gamma)$ by conjugation.

for $a \Lambda \subset \Gamma$, $|\Gamma : \Lambda| < +\infty$

we define

$$\mu^\Lambda := \sum_{[\Gamma : \Lambda]} S g \Lambda^{-1}.$$

in this way we a well defined map

$$Y_m \to \mu_{\text{H}_2}(Y_m) =: \mu_m.$$

All those measures are invariant
under $\text{Sub}(\Gamma)$.

The limit objects will be the weak-* limits of $\mu_n$.

**Def:** An invariant random subgroup of $\Gamma$ is a probability measure $\nu$ on $\text{Sub}(\Gamma)$ invariant under conjugation.

- If $N \vartriangleleft \Gamma$ then
  
  $\delta_N$ is on $\text{Irr}$.

- If $\Lambda \lhd \Gamma$ is finite index then $\mu_\Lambda$ is on $\text{Irr}$.
G - a semisimple Lie group
( l.c.r.c group).

\[ \text{Sub}(G) = \{ N \subseteq G / N \text{ a closed subgroup} \} \]

with the topology of Hausdorff convergence on compact sets.

**Def:** IRSses are defined in the same way as before.

- trivial \( S_{\xi g} \), \( S_{\xi 1(G)} \), \( S_{\xi G'} \).
- if \( N \subseteq G \) is finite covolume

then

\[ \mu N = \frac{1}{\nu d(N G)} \int_{N \chi G} \int_{g' \chi G} dg' \]
is an IRS.

In general much more.

Some applications

Thm (Lück, Farb–Eilen)

If $Y$ is aspherical finite CW complex, $Y_n$ sequence of finite covers so that $\mu_n = \mu_{\pi_1(Y_n)}$

converges to $\delta_{\mathbb{Z}}$ then

$$\lim_{n \to \infty} \frac{\beta^k(Y_n, M)}{[\pi_1(Y) : \pi_1(Y_n)]} = \beta^k(Y)$$

where $L^2$-Betti number of $Y$. 
Thus (Abert - Bergeron - Gelander),

\[ X \text{ - a symmetric space, simple (not a product).} \]

\[ \neq H/3. \]

\( (\Gamma_m \backslash X) \) - a sequence s.t.

\[ \mu_{\Gamma_m} \to \nu \text{ then} \]

\[ \lim_{n \to \infty} \frac{C^n(\Gamma_m \backslash X)}{vd(\Gamma_m \backslash X)} \text{ exists.} \]

if \( \omega = \delta_{11} \) then the limit is \( \beta^n(X) \).

In particular if \( \omega = \frac{\text{clim}X}{2} \)

then the limit is \( 0 \).
Structure of IRS' s on

Thm (Stuck, Zimmer)

If $G$ is a higher rank simple Lie group then the ergodic IRS' s are:

1. $S_{\Gamma}$, $S_{\Gamma_{1}}$ and $\Lambda \subseteq Z(G)$, $S_{\Gamma}$.

2. $\mu_{\Gamma}$, $\Gamma$ is a lattice in $G$.

Thm (Abert, Bergeron, ..., 2015)

If $\Gamma_{n}$ is a sequence of pairwise different lattices in $G$, a higher rank simple. Then

$$\lim_{n \to \infty} \mu_{\Gamma_{n}} = S_{\Gamma_{1}}.$$
Thm: (Abert - Bergeron -- 2015)

If $G$ is a simple Lie group, $\mu$ is an ergodic IRS, non-trivial, then $\mu$-almost every subgroup is Zariski dense in $G$. 

**Tuñez (F. - Hurtado - Raini Kant)**

if $G$ - semisimple, adjoint

$\Gamma_n$ - a sequence of congruence arithmetic lattices, then

$$\mu_{\Gamma_n} = \bigcup_{\{1\}}$$

if $\mu_{\Gamma_n} = \bigcup_{\{1\}}$
More exotic examples

\( \text{Fin}(\mathbb{N}) \) - group of finitely supported permutations on \( \mathbb{N} \).

Verchik classified all ergodic IRS in \( \text{Fin}(\mathbb{N}) \).

- choose \( \alpha_0, \alpha_1, \alpha_2, \ldots \geq 0 \)
  \[ s.t. \quad \sum_{i=0}^{\infty} \alpha_i = 1. \]

- on \( \mathbb{N} \) choose i.i.d. random label from \([0,1]\) (uniform)

\[ l(i) \] - is the label of \( i \in \mathbb{N} \)

then the color of \( i \)
is $= m$

we produce a coloring of $N$ by 0, 1, 2,

\[ c_1, \ldots, c_4 \]

define $S \leq \text{Fin}(\mathbb{N})$ as the subgroup that fixes all the numbers colored by 0, and preserves all other colors.