Combing Lattices in Semisimple Lie Groups

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Abstract

Let Γ be an irreducible lattice in a semisimple linear Lie group $G \neq SO(n,1)$. Then Γ is bicombable if and only if Γ is uniform [1]. If $rank(G) = 1$ then Γ is automatic if and only if Γ is uniform.

1 Introduction

The connection between the curvature of a manifold and combinatorial properties of its fundamental group goes back to the work of Dehn at the start of this century ([De]). Today, notions of curvature in groups are central to the theory of infinite discrete groups. Many manifestations of negative curvature are captured by Gromov's theory of hyperbolic groups ([Gr, GdlH]). The main candidate for the property of nonpositive curvature in groups has been Thurston's (bi)combable, or "semi-hyperbolic" groups; these groups have been studied extensively by Alonso-Bridson and others ([AB]).

It has been an open problem to determine precisely which lattices in Lie groups are (bi)combable or (bi)automatic (for precise definitions, see below). Thurston-Epstein have shown ([E et al.]) that $SL_n(\boldsymbol{Z})$ is not combable for $n > 2$. Using completely different techniques, Gersten-Short have shown ([GS, Sh]) that $SL_2(\mathcal{O})$ is not bicombable, where $\mathcal{O} = \mathcal{O}(\sqrt{d})$ is the ring of algebraic integers in the real quadratic number field $\boldsymbol{Q}(\sqrt{d})$. The groups $SL_2(\mathcal{O})$ and $SL_n(\boldsymbol{Z})$ are nonuniform, irreducible lattices in semisimple linear

[1]Recall that a lattice in a Lie group is *uniform* if it is cocompact.

Lie groups of (real) rank 2 and $n-1$, respectively. In this paper we give a complete solution to the above problem for irreducible lattices in semisimple linear Lie groups. The main result is the following :

Theorem 1.1 *Let Γ be an irreducible lattice in a semisimple linear Lie group $G \neq SO(n,1)$. Then Γ is bicombable if and only if Γ is uniform.*

Remark: Our results easily extend to irreducible lattices in (not necessarily linear) semisimple Lie groups with finitely many connected components and finite center, as the kernel of the adjoint representation of G is the center of G (also see Lemma 2.1 below). Further, by results in [Ra], one can easily deal with reducible lattices in these semisimple Lie groups.

The proof of Theorem 1.1 falls into two cases, depending on rank:

Theorem 1.2 *Let Γ be a lattice in a rank one Lie group $G \neq SO(n,1)$. Then the following are equivalent :*

1. *Γ is word-hyperbolic.*
2. *Γ is (bi)combable.*
3. *Γ is (bi)automatic.*
4. *Γ is uniform.*

It should be noted that every lattice in $SO(n,1)$ is biautomatic. This follows from Epstein's result that geometrically finite hyperbolic groups are biautomatic ([E et al.]), and from the fact that lattices in $SO(n,1)$ are geometrically finite ([Bo]).

The fact that (3) and (4) are equivalent in Theorem 1.2 can be used to give an algorithm to determine when a lattice in a rank 1 Lie group is uniform.

Theorem 1.3 *Let Γ be an irreducible lattice in a semisimple linear Lie group G with $rank(G) \geq 2$. Then Γ is bicombable if and only if Γ is uniform.*

Note that the lattices $SL_2(\mathcal{O})$ and $SL_n(\boldsymbol{Z})$ are covered by Theorem 1.3. The essential ingredient in the proof of Theorem 1.3 is the recent proof by Lubotzky-Mozes-Raghunathan ([LMR]) of the Kazhdan Conjecture, which provides so-called "U-elements" for nonuniform, irreducible lattices in semisimple linear groups of rank ≥ 2. By definition, U-elements generate exponentially distorted (in the lattice) cyclic subgroups. On the other hand, by showing that every element of infinite order in a bicombable group has positive translation number, one sees that cyclic subgroups of bicombable groups embed quasi-isometrically, hence have linear distortion.

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2 Combings and Bicombings

In this section we review the definition of (bi)combings and (bi)automatic structures, and recall a few basic facts which we shall need in the proof of the main theorem.

Let Γ be a finitely generated group with generating set X. Let $\mathcal{A} = X \cup X^{-1}$, and let $\mathcal{A}^*$ denote the free monoid on $\mathcal{A}$. For a word $w \in \mathcal{A}^*$, we denote by $\overline{w}$ the image of w under the natural map $\mathcal{A}^* \longrightarrow \Gamma$. For $g \in \Gamma$, we let $||g||_\Gamma = d_\Gamma(1, g)$, where d_Γ denotes the word metric in Γ (with respect to X). We identify Γ with its Cayley graph (with respect to X), and think of each word $w \in \mathcal{A}^*$ as a path $w : [0, \infty) \longrightarrow \Gamma$ which is eventually constant.

The group Γ is *word-hyperbolic* if there exists $\delta > 0$ so that geodesic triangles in Γ are uniformly δ-thin; that is, the δ-neighborhood of any two sides of a geodesic triangle contains the third side. For an introduction to word-hyperbolic groups, see [GdlH].

A *combing* is a section $\sigma : \Gamma \longrightarrow \mathcal{A}^*$ of the natural map $\mathcal{A}^* \longrightarrow \Gamma$ so that

1. $\sigma(g)$ is a quasi-geodesic in Γ for all $g \in \Gamma$; and
2. There is a constant $K > 0$ so that $d_\Gamma(\sigma(g)(t), \sigma(g \cdot x)(t)) \leq K$ for all $g \in \Gamma, x \in X, t > 0$.

In other words, a combing on Γ is a choice of paths from $1 \in \Gamma$ to each element of Γ, so that paths which end close stay uniformly close together. A *bicombing* is a combing $\sigma : \Gamma \longrightarrow \mathcal{A}^*$ which is "almost-equivariant" in the following sense : there exists $K' > 0$ so that

$$d_\Gamma(x \cdot \sigma(g)(t), \sigma(x \cdot g)(t)) \leq K'$$

for all $g \in \Gamma, x \in X, t > 0$. Note that the $x \cdot \sigma(g)$ refers to the action (on the left) of Γ on the set of paths in its Cayley graph. The group Γ is said to be *(bi)combable* if there exists a (bi)combing on Γ. The group Γ is *(bi)automatic* if there is a (bi)combing $\sigma : \Gamma \longrightarrow \mathcal{A}^*$ such that $\sigma(\Gamma)$ is a regular language.

(Bi)combable and (bi)automatic groups are among the central examples studied in geometric group theory. We refer the reader to [E et al.],[Fa1],[Sh] for background and elementary properties of these groups.

Warning : Different authors mean different things by "(bi)combing"; in particular what we call "combing" certain others call "bounded width combing by quasi-geodesics". What we call a bicombable group, Alonso-Bridson call a "semi-hyperbolic group" ([AB]).

Note the following list of implications :

$$\begin{array}{ccc} \text{biautomatic} & \Longrightarrow & \text{automatic} \\ \Downarrow & & \Downarrow \\ \text{bicombable} & \Longrightarrow & \text{combable} \end{array}$$

It is not known whether or not any of the arrows is reversible, although it is widely believed that there are combable groups which are not automatic. Word-hyperbolic groups are biautomatic ([E et al.]). Groups which act cocompactly and properly by isometries on a $CAT(0)$ space (e.g., a complete, simply connected manifold of nonpositive curvature) are bicombable ([AB]); indeed such groups were the motivation behind the definition of bicombable.

We will need the following property of bicombable groups, which is proved in [AB] :

Lemma 2.1 *Let $1 \longrightarrow A \longrightarrow G \longrightarrow C \longrightarrow 1$ be a short exact sequence of finitely generated groups. If A is finite and C is bicombable then G is bicombable.*

Definition: Let (X, d) and (X', d') be metric spaces with metrics d, d'. A K-quasi-isometry is a (not necessarily continuous) map $f : X \longrightarrow X'$ such that every point of X' lies in a K-neighborhood of the image of f, and so that

$$\frac{1}{K}d(x,y) - K \leq d'(f(x), f(y)) \leq Kd(x,y) + K$$

for all $x, y \in X$.

It is a fundamental observation (made by Svarc and Milnor ([Mi])) that if M is a closed Riemannian manifold and $m \in M$, then the map $\gamma \mapsto \gamma \cdot m$ is a quasi-isometry from $\pi_1(M)$ to the universal cover $\widetilde{M}$. Milnor's observation holds more generally: let X be a proper geodesic metric space and let Γ be a discrete group of isometries acting properly discontinuously on X. If $\Gamma\backslash X$ is compact, then Γ is finitely generated and is quasi-isometric to X.

A path $\alpha : [0,p] \longrightarrow X$ is a *K-quasi-geodesic* if it is a K-quasi-isometric embedding of $[0,p]$ into X.

3 Quasiconvexity and Translation Numbers

All of the results in this section are due to Gersten and Short, and can be found in [GS] and [Sh]. We include them here for completeness, as they will be used in the proofs of Theorems 1.2 and 1.3.

Let $\sigma : \Gamma \longrightarrow \mathcal{A}^*$ be a combing, and let $L = \sigma(\Gamma)$. A subgroup N of Γ is *L-quasiconvex* if there exists a $C > 0$ so that for all $g \in N$, $\sigma(g)$ lies in a C-neighborhood of N in Γ. We will need the following:

Theorem 3.1 *If $\sigma : \Gamma \longrightarrow L$ is a (bi)combing, and if N is an L-quasiconvex subgroup of Γ, then N is (bi)combable. Furthermore, if this (bi)combing is the (bi)combing of a (bi)automatic structure, then N is (bi)automatic.*

The *translation number* of an element $g \in \Gamma$ is defined to be

$$\tau(g) = \liminf_{n\to\infty} \frac{||g^n||_\Gamma}{n}.$$

Although τ depends on the choice of generators for Γ, the vanishing of $\tau(g)$ for a given element $g \in \Gamma$ does not depend on this choice, since changing generators gives a Lipschitz equivalent metric on Γ. We will need the following two facts :

Proposition 3.2 *If Γ is bicombable and $g \in \Gamma$ has infinite order then $\tau(g) > 0$.*

Proposition 3.2 is proved by showing that the center of the centralizer of g in Γ is quasiconvex in Γ; note that this subgroup is virtually free abelian. It is this first fact that uses the "almost-equivariance" of the bicombing. It is then shown that for any quasiconvex subgroup N of Γ, an element $g \in N$ has positive translation number in N if and only if g has positive translation number in Γ. The argument concludes by noting that every nontrivial element of a free abelian group has nonzero translation number. For details, see [Sh].

Proposition 3.3 *Let Γ be a finitely generated group, and let $g \in \Gamma$ have positive translation number. Then $||g^n||_\Gamma = O(n)$.*

Proof: Clearly $||g^n||_\Gamma \leq ||g||_\Gamma \cdot n$. For the reverse inequality, note that since $\tau(g)$ is a lim inf, there exists $N > 0$ so that $||g^n||_\Gamma / n \geq \tau(g)/2$ for all $n \geq N$; that is, $||g^n||_\Gamma \geq (\tau(g)/2) \cdot n$ for $n \geq N$, and we are done since $\tau(g) > 0$. $\square$

4 Proofs

In this section we present proofs of Theorem 1.2 and Theorem 1.3, from which Theorem 1.1 immediately follows.

Our proof of Theorem 1.2 requires the following proposition concerning the geometry of what has come to be known as "the neutered space".

Proposition 4.1 *Let H be a complete, simply connected Riemannian manifold with sectional curvatures satisfying $-\infty < -\delta^2 \leq K(H) \leq -1$, where $\delta \geq 1$. Let Γ be a discrete group of isometries of H, and let X be the result of removing the interiors of a Γ-invariant set of closed disjoint horoballs from H. Give X the path metric d_X. Then d_X is at most exponentially distorted with respect to d_H; namely,*

$$d_H(x, y) \leq d_X(x, y) \leq \sinh(\delta \cdot d_H(x, y))$$

for all $x, y \in X$.

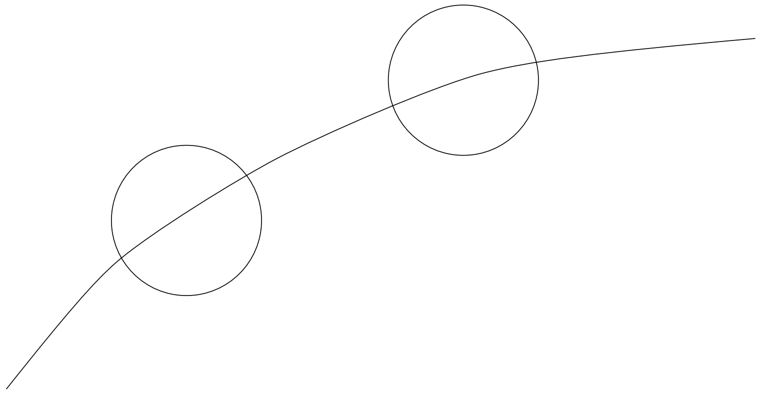

Figure 1: A path α' in X of length $\sinh(\delta \cdot d_H(x,y))$ between $x, y \in X$.

We will only apply Proposition 4.1 to globally symmetric spaces H, so that the metric is known explicitly and the proof is a direct computation. We provide a proof in the more general case here since this proposition is basic to the study of fundamental groups of finite volume, (not necessarily symmetric) negatively curved manifolds.

Proof: The first inequality is clear since any path in X is a path in H. To prove the second inequality, let $x, y \in X$ be given, and let α be the H-geodesic from x to y. To bound the X-distance from x to y, do the following replacement: if α penetrates a horoball S_i at $\alpha(t_i)$ and exits at $\alpha(s_i)$, replace the subsegment $\alpha([t_i, s_i])$ by a geodesic α'_i on the horosphere from $\alpha(t_i)$ to $\alpha(s_i)$. Do this for each horoball that α penetrates, and call the resulting path α' (see figure 1).

Let β be the length of that part of α (and of α') which lies outside every horoball.

Now

$$\begin{aligned} \sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \\ &\geq \sinh(x) + \sinh(y) \end{aligned}$$

as $\cosh(x) \geq 1$ for all $x \geq 0$. Furthermore, if $h(p,q)$ denotes the distance between two points p, q on a horosphere, then

$$h(p,q) \le \frac{2}{\delta}\sinh(\frac{\delta}{2}\cdot d_H(p,q))$$

for any such p, q ([HI], Theorem 4.6). From these two facts it follows that

$$\begin{aligned}
d_X(x,y) &\le \ell(\alpha') \\
&= \beta + \sum \ell(\alpha_i') \\
&\le \beta + \sum \tfrac{2}{\delta}\sinh(\tfrac{\delta}{2}\cdot \ell(\alpha_i)) \\
&\le \sinh(\beta) + \tfrac{2}{\delta}\sinh(\tfrac{\delta}{2}\cdot \sum \ell(\alpha_i)) \\
&\le 2\sinh(\beta + \tfrac{\delta}{2}\cdot \sum \ell(\alpha_i)) \\
&\le 2\sinh(\tfrac{\delta}{2}\cdot d_H(x,y)) \\
&\le \sinh(\delta \cdot d_H(x,y))
\end{aligned}$$

since $\delta \ge 1$, completing the proof. $\square$

Proof of Theorem 1.2: Let Γ be a lattice in a rank 1 Lie group $G \neq SO(n,1)$; in other words $G = SU(n,1), SP(n,1)$, or F_4^{-20}. We will show that a uniform lattice in G is word-hyperbolic and that a nonuniform lattice in G is not combable. By the discussion in Section 2, this suffices to prove the theorem.

By Selberg's Lemma, Γ contains a torsion-free subgroup Γ' of finite index. Since Γ is word-hyperbolic (resp. combable) iff Γ' is word-hyperbolic (resp.combable)([GdlH, E et al.]), we assume without loss of generality that Γ is torsion free. Then Γ is the fundamental group of a finite volume Riemannian manifold M whose sectional curvatures are pinched between -1 and -4; M is the quotient of an exotic hyperbolic space (the symmetric space associated to G) by Γ acting freely as a discrete group of isometries. If Γ is uniform then M is a compact, negatively curved manifold. It follows that Γ is word-hyperbolic ([GdlH]).

Now suppose that Γ is a nonuniform lattice in G, so that M is a finite volume, (pinched) negatively curved, locally symmetric manifold with cusps; the universal cover $\widetilde{M}$ is either complex or quaternionic hyperbolic space or the hyperbolic Cayley plane, according to whether $G = SU(n,1), SP(n,1)$, or F_4^{-20}. Now Γ acts freely by isometries on $\widetilde{M}$ with finitely many orbits of parabolic fixed points. Choose a Γ-invariant set of disjoint horoballs centered on the parabolic fixed points; these horoballs can be thought of as lifts of the cusps of M. By shrinking these horoballs sufficiently (i.e., looking at lifts of the cusps further and further out), we may assume that any two translates of any of the horoballs are a distance of at least 1 from each other. Let X be the space formed by deleting the interiors of the Γ-orbits of these horoballs, and give X the path metric. This makes each component of the boundary of X a totally geodesic horosphere. Choose some component S of the boundary of X.

Now Γ acts freely and cocompactly by isometries on X, there is a maximal parabolic subgroup N of Γ which preserves S, and N acts on S cocompactly. Picking a basepoint $s \in S$, we see that the map $f : \Gamma \longrightarrow X$ defined by $f(\gamma) = \gamma \cdot s$ is a K-quasi-isometry for some $K > 0$, and that the restriction of f to N is a quasi-isometry of N and S. Let $\sigma : \Gamma \longrightarrow \mathcal{A}^*$ be a combing of Γ, and let $L = \sigma(\Gamma)$. Now there exists $C > 0$ so that for all $g \in N$, $\sigma(g)$ is a C-quasi-geodesic in Γ which begins and ends in N. Hence $f \circ \sigma(g)$ is a KC-quasigeodesic in X which begins and ends in S.

We now show that there exists a $K' > 0$ so that $f \circ \sigma(g)$ lies in a K'-neighborhood of S; this argument is similar to the familiar proof that quasi-geodesics track geodesics in a simply connected manifold of (pinched) negative curvature. It will then follow that $\sigma(g)$ lies in a KK'-neighborhood of N in Γ, so that N is L-quasiconvex in Γ.

So choose a number P so big that $\cosh(2p) > KC$. Suppose that $\alpha : [0,1] \longrightarrow X$ is a segment of $f \circ \sigma(g)$ which lies completely outside a P-neighborhood of S, and let ℓ be the length of α. Let x be the point at which α exits the P-neighborhood of S, and let y be the point at which α returns to the P-neighborhood of S. We bound ℓ universally.

Let x' (resp. y') be the point in S which is the image of x (resp. y) under orthogonal projection onto S (see figure 2).

Since the universal cover $H = \widetilde{M}$ is a simply connected Riemannian manifold with sectional curvatures pinched between -1 and -4, Proposition 4.1 gives

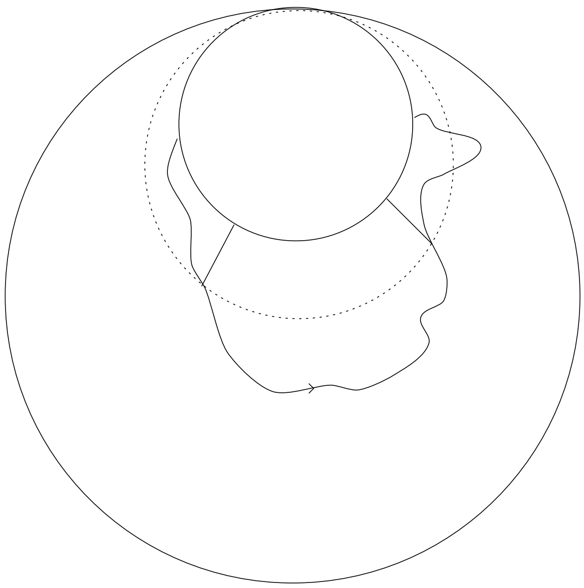

Figure 2: Why quasi-geodesics beginning and ending on a horosphere must stay uniformly close to the horosphere.

$$d_H(x,y) \leq d_X(x,y) \leq \sinh(2 \cdot d_H(x,y))$$

for all $x, y \in X$. So we have

$$d_H(x,x') = d_H(x,S) \leq d_X(x,S) = P$$

so that $d_X(x,x') \leq \sinh(2P)$; similarly $d_X(y,y') \leq \sinh(2P)$.

Now ignore all of the deleted horospheres. Orthogonal projection onto S shows that $d_X(x',y') \leq \ell/\cosh(2P)$. Hence there is a path in X from x to y of length $2\sinh(2P) + \ell/\cosh(2P)$. Since $f \circ \sigma(g)$, hence α, is a KC-quasi-geodesic, this gives :

$$\ell \leq KC(2\sinh(2P) + \ell/\cosh(2P))$$

which gives

$$\ell \leq \frac{2KC\sinh(2P)}{(1 - KC/\cosh(2P))}$$

which gives a (universal) bound on ℓ. Note that $1 - KC/\cosh(2P) > 0$ by our choice of P. Now let

$$K' = 2\sinh(2P) + \frac{KC\sinh(2P)}{(1 - KC/\cosh(2P))}.$$

Then $f \circ \sigma(g)$ lies in a K'-neighborhood of S, as required.

Since N is L-quasiconvex, it follows from Theorem 3.1 that the combing $\sigma : \Gamma \longrightarrow L$ descends to a combing on N. But $G = SU(n,1), SP(n,1)$, or F_4^{-20}, so that N is not virtually abelian. This gives a contradiction, for Thurston has shown that these groups are not combable ([Th]). $\square$

Remark: There are two alternatives to using Thurston's unpublished result in the proof of Theorem 1.2; each of these gives a slightly weaker conclusion. By our same argument, any automatic structure on Γ descends to an automatic structure on N; but it is a result of Epstein-Holt that nilpotent groups are not automatic ([E et al.]). Hence Γ is not automatic. It is a result of Gersten-Short ([GS, Sh]) that any nilpotent subgroup of a bicombable group is virtually abelian; hence Γ is not bicombable. It is this latter fact that we really use in the proof of Theorem 1.1.

Remark: Although many of the nonuniform lattices in rank 1 Lie groups are not hyperbolic, automatic, or bicombable, much can still be said. Every such lattice is hyperbolic (and automatic) relative to its cusp subgroups; this gives precise combinatorial information about these lattices ([Fa2]).

The $rank(G) \geq 2$ case rests wholly on the following result of Lubotzky-Mozes-Raghunathan :

Theorem 4.2 ([LMR]) *Let Γ be an irreducible lattice in a semisimple linear Lie group G, and let $\gamma \in \Gamma$ be an element of infinite order. Then $||\gamma^n||_\Gamma = O(log(n))$ if and only if γ is virtually unipotent and $rank(G) \geq 2$.*

The proof of Theorem 4.2 makes an essential use of the Margulis arithmeticity theorem, which states that irreducible lattices are arithmetic for $rank(G) \geq 2$. Lubotzky-Mozes-Raghunathan apply Theorem 4.2 to prove a conjecture of Kazhdan: for semisimple linear Lie groups G with $rank(G) \geq 2$, the word metric on an irreducible lattice Γ in G is Lipschitz equivalent to the restriction to Γ of the left invariant Riemannian metric on G.

Proof of Theorem 1.3: Let G be a semisimple linear Lie group with $rank(G) \geq 2$, and let Γ be an irreducible lattice in G. Suppose that Γ is uniform. Let U be a maximal connected, normal, compact subgroup of G. Then the short exact sequence

$$1 \longrightarrow U \longrightarrow G \longrightarrow G/U \longrightarrow 1$$

gives a short exact sequence

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow \Gamma' \longrightarrow 1$$

where A is finite and Γ' is a uniform lattice in G/U. By Lemma 2.1, to show that Γ is bicombable, it suffices to show that Γ' is bicombable. Hence without loss of generality, we assume that G is a semisimple linear Lie group whose symmetric space is a product of irreducible symmetric spaces of noncompact type.

Let K be a maximal compact subgroup of G. Since the symmetric space G/K associated to G is a product of irreducible symmetric spaces of noncompact type, we may apply the classical result of Cartan (see [He], Theorem 3.1) which states that G/K has nonpositive curvature. Since Γ acts properly and cocompactly by isometries on G/K , it follows that Γ' is bicombable.

Now suppose that Γ is nonuniform and bicombable. Let $\gamma \in \Gamma$ be a unipotent element of Γ of infinite order. The existence of such an element for irreducible lattices in linear semisimple Lie groups G with $rank(G) \geq 2$ follows from the Compactness Criterion for arithmetic lattices ([Ra], Theorem 10.19); for arbitrary irreducible lattices, this follows from the Margulis Arithmeticity Theorem ([Ma]). By Theorem 4.2, γ satisfies $||\gamma^n||_\Gamma = O(log(n))$. Since Γ is bicombable and $\gamma \in \Gamma$ has infinite order, it follows from Proposition 3.2 that γ has nonzero translation number. Hence $||\gamma^n||_\Gamma = O(n)$ by Proposition 3.3, which gives a contradiction. $\Box$

Note: The arguments which show that an element of infinite order in a bicombable group generates a quasi-isometrically embedded cyclic subgroup depend heavily on the "almost-equivariance" of the bicombing; hence these arguments do not extend to combable groups. It is not known whether infinite cyclic subgroups of combable groups must embed quasi-isometrically.

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