Groups of real-analytic diffeomorphisms of the circle

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1 Introduction

Let $\mathrm{Homeo}_+(S^1)$ (resp. $\mathrm{Diff}^\infty_+(S^1)$) denote the group of orientation-preserving homeomorphisms (resp. C^∞ diffeomorphisms) of the circle. While these groups have been thought of as the simplest examples of "infinite-dimensional Lie groups", examples show (see §6) that the standard algebraic properties of linear groups generally do not hold for subgroups of $\mathrm{Homeo}_+(S^1)$ or even $\mathrm{Diff}^\infty_+(S^1)$.

The theme of this paper is that, in contrast, arbitrary groups $G < \mathrm{Diff}^\omega_+(S^1)$ of orientation-preserving, real-analytic diffeomorphisms actually have some algebraic properties in common with linear groups. In fact such groups G behave in some ways like subgroups of $\mathrm{PSL}(2, \mathbf{R})$, which acts projectively on S^1. Hence the increase in smoothness from C^∞ to C^ω has strong implications for the algebraic structure of groups of diffeomorphisms.

Theorem 1.1 (Residual Tits alternative) *Let G be any subgroup of $\mathrm{Diff}^\omega_+(S^1)$. Then either*

1. *G contains a nonabelian free group, or*
2. *G is residually solvable. In fact G admits a homomorphism ϕ onto a finite cyclic group so that the commutator subgroup of ker(ϕ) is residually torsion-free nilpotent.*

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Remarks:

1. It is an old result of Magnus [Ma] that free groups are residually torsion-free nilpotent. Hence the "alternatives" in Theorem 1.1 are not mutually exclusive.

2. A group G satisfying the second alternative is residually finite.

Recall that a group G is *perfect* if the commutator subgroup $[G, G]$ is all of G. An immediate corollary of Theorem 1.1 is the following.

Corollary 1.2 (Perfect groups) *Any nontrivial perfect subgroup of* $\mathrm{Diff}_+^\omega(S^1)$ *contains a nonabelian free group.*

Remark. Theorem 1.1 and Corollary 1.2 become false if $\mathrm{Diff}_+^\omega(S^1)$ is replaced by $\mathrm{Diff}_+^\infty(S^1)$. A counterexample is Thompson's group. See §6.

Let G be any subgroup of $\mathrm{PSL}(2, \mathbf{R})$. It is a basic fact that the centralizer $C_G(g)$ of any nontrivial element $g \in G$ is virtually abelian. The following theorem generalizes this property to arbitrary groups of real-analytic diffeomorphisms of S^1.

Theorem 1.3 (Centralizers are virtually abelian) *Let G be any subgroup of* $\mathrm{Diff}_+^\omega(S^1)$. *Then the centralizer $C_G(g)$ of any infinite-order element $g \in G$ is virtually abelian. In fact $C_G(g)$ admits a homomorphism onto a finite cyclic group with abelian kernel.*

Remark. Theorem 1.3 becomes false if $\mathrm{Diff}_+^\omega(S^1)$ is replaced by $\mathrm{Diff}_+^\infty(S^1)$. Again the counterexample is Thompson's group. See §6.

Our proof of Theorem 1.3 extends an argument given by Ghys in [Gh]. Ghys proves that every solvable subgroup of $\mathrm{Diff}_+^\omega(S^1)$ is metabelian, and that every nilpotent subgroup of $\mathrm{Diff}_+^\omega(S^1)$ is virtually abelian. The torsion-free case of the latter result is included in Theorem 1.3.

In §5.2, as another application of Theorem 1.3, we give an extension of one of the results of [FS], which asserts that certain higher-rank lattices do not admit infinite, real-analytic actions on S^1.

Let $\mathrm{Mod}(g, r)$ denote the mapping class group of the oriented surface $\Sigma_{g,r}$ of genus g with r punctures. The group $\mathrm{Mod}(g, 1)$ is a subgroup of $\mathrm{Homeo}_+(S^1)$ for all $g \geq 0$ (see §6). The following result implies, however, that $\mathrm{Mod}(g, 1)$ is not isomorphic to any subgroup of $\mathrm{Diff}_+^\omega(S^1)$ if $g \geq 4$.

Theorem 1.4 (Mapping class groups) *For any $g \geq 4, r \geq 0$, any homomorphism $\psi : \mathrm{Mod}(g,r) \to \mathrm{Diff}^{\omega}_{+}(S^1)$ is trivial.*

The case $r = 0$ is also of interest since it is an open question whether or not $\mathrm{Mod}(g,0)$ is a subgroup of $\mathrm{Homeo}_{+}(S^1)$ for $g \geq 2$.

Note that $\mathrm{Mod}(g,r)$ has many infinite quotient groups with infinite kernel. Examples include the quotients by the Torelli group $T_1(g,r)$ and the higher Torelli groups $T_n(g,r)$. By Theorem 1.4, none of the quotients $\mathrm{Mod}(g,r)/T_n(g,r)$ admits a nontrivial, real-analytic actions on S^1 for $g \geq 4, r \geq 0$. This can be viewed as a nonlinear generalization of the fact (which is a special case of a theorem of Witte [Wi]) that the linear symplectic groups $\mathrm{Sp}(2g,\mathbf{Z}) = \mathrm{Mod}(g,0)/T_1(g,0)$ do not have an infinite, real-analytic action on S^1 for $g \geq 2$.

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2 Some basic facts

2.1 Groups fixing a point

Proposition 2.1 (Analytic groups fixing a point) *Let G be a group of real-analytic diffeomorphisms of S^1 fixing a point $x \in S^1$. Then the commutator subgroup of G is residually a torsion-free nilpotent group. Further, any finitely-generated subgroup of G is residually finite.*

Proof. This is standard. The action of G on the tangent space T_xS^1 gives a representation $G \to \mathrm{GL}(1,\mathbf{R}) = \mathbf{R}$. The commutator subgroup $[G,G]$ has trivial image under this representation. Hence for every $i > 0$ the action of $[G,G]$ on the space of i-jets at x factors through the action of a torsion-free nilpotent group. Because the action of G is real-analytic, the only element of G that acts trivially on the i-jets for every i is the identity. $\diamond$

The following lemma gives a property of groups of orientation-preserving homeomorphisms which is quite special to the circle.

Lemma 2.2 *Let G be a group of orientation-preserving homeomorphisms of S^1. Suppose F is a finite subset of S^1 which is invariant under the action* *of G. Then the restricted action of G on F factors through an action of the cyclic group of order $|F|$.*

Proof. The action of G on F preserves the natural cyclic ordering on F inherited from the orientation on S^1. The result follows easily. $\diamond$

2.2 Groups acting freely on $\mathbf{R}$

The following theorem is due to Sacksteder. However, as we have been unable to find a proof in the published literature, we include one here for completeness.

Theorem 2.3 (Sacksteder Theorem) *Any group G acting freely on $\mathbf{R}$ is abelian.*

Proof. If G has a free action on $\mathbf{R}$ then define an order on G by declaring $g < h$ if $gx < hx$ for some (hence for every) $x \in \mathbf{R}$. It is easy to check that $<$ gives a total order on G which is invariant under both left and right multiplication. Since the action is free, it is also easy to check that $(G, <)$ is an *archimedean group*: for every $g, h \in G$ there exists $n \in \mathbf{Z}$ with $g^n > h$. Theorem 2.3 now follows immediately from the following algebraic theorem, which apparently goes back to Hölder. $\diamond$

Theorem 2.4 (Hölder Theorem) *Let G be an archimedean totally ordered group. Then G is abelian.*

Proof. Let $a, b \in G$ be arbitrary. We want to show that a and b commute. Without loss of generality take $a > 0$ and $b > 0$. Let $x > 0$ be arbitrary. By the archimedean property (and considering "smallest powers" satisfying it) there exist $m, n > 0$ so that

$$x^m < a < x^{m+1}$$

$$x^n < b < x^{n+1}$$

Hence

$$x^{m+n} < ab < x^{m+n+2}$$

$$x^{m+n} < ba < x^{m+n+2}$$

Note that the above claims use the 2-sided order on G. From these inequalities it is easy to see that

$$x^{-2} < [a,b] < x^2$$

where $[a,b] = aba^{-1}b^{-1}$. Now we want to show $[a,b] = 0$. Suppose not. Then we can assume $[a,b] > 0$ (otherwise apply the following argument to the inverse). Note that x above was arbitrary.

There are 2 cases:

Case (i): G has a least positive element z. In this case we claim that G is the cyclic group generated by z. To prove this, note that for any $a > 0 \in G$ we have $z^n < a < z^{n+1}$ for some n by the archimedean property. Hence $0 < az^{-n} < z$, which means $a = z^n$ or $a = z^{n+1}$.

Case (ii): G has no least positive element. Let $c = [a,b]$. Then there exists $d < c$. By the above $c < d^2$. Hence $1 < cd^{-1} < d$ and so $d^{-1}cd^{-1} < 1$, from which it follows that

$$(cd^{-1})^2 = c(d^{-1}cd^{-1}) < c(1) = c$$

This gives a positive element whose square is less than c. This contradicts the first part of the argument above, since x was arbitrary.

Hence either G is cyclic or the assumption $[a,b] \neq 0$ leads to a contradiction. In either case $[a,b] = 0$. $\diamond$

In fact it is possible to show that G is order isomorphic to a subgroup of $\mathbf{R}$.

3 Proof of the Residual Tits alternative

We begin with an elementary but useful lemma.

Lemma 3.1 (Coset lemma) *Suppose a group G can be written as a finite union of cosets $g_1A_1 \cup \cdots \cup g_rA_r$ of subgroups A_i. Then there exists j with $1 \leq j \leq r$ such that A_j has finite index in G.*

Proof. We proceed by induction on the number $k \leq r$ of distinct subgroups A_i. Write the coset decomposition as

$$G = S_1A_1 \cup \cdots \cup S_kA_k$$

where each S_i is a finite set, the A_i's are distinct, and S_iA_i denotes the product $\{sa : s \in S_i, a \in A_i\}$.

If $k = 1$ then we are clearly done. So suppose the theorem is true for all $k \leq n$, and that we are given a coset decomposition $G = S_1A_1 \cup \cdots \cup S_{n+1}A_{n+1}$ as above.

If $S_1A_1 = G$ then A_1 has finite index in G and we are done. If this is not the case then there exists $x \in G$ with $xA_1 \cap S_1A_1 = \emptyset$. Then

$$xA_1 \subset S_2A_2 \cup \cdots \cup S_{n+1}A_{n+1}$$

so that

$$S_1A_1 \subset T_2A_2 \cup \cdots \cup T_{n+1}A_{n+1}$$

where T_i denotes the finite set $S_1x^{-1}S_i$. Hence

$$G = T_2A_2 \cup \cdots \cup T_{n+1}A_{n+1} \cup S_2A_2 \cup \cdots S_{n+1}A_{n+1}$$

and so

$$G = (T_2 \cup S_2)A_2 \cup \cdots \cup (T_{n+1} \cup S_{n+1})A_{n+1}$$

where each $T_i \cup S_i$ is a finite set and the A_i's are distinct. Hence by the inductive hypothesis there exists j with $2 \leq j \leq n+1$ such that A_j has finite index in G. $\diamond$

Lemma 3.1 could have been used in Tits's proof of his alternative for linear groups. Tits avoided this by reducing the proof to the case where the given group is connected in the Zariski topology; this ingenious device is less elementary than the use of Lemma 3.1, and is not available in the proof of Theorem 1.1, as there is no analogue of the Zariski topology for subgroups of $\mathrm{Diff}^{\omega}_{+}(S^1)$.

3.1 Proof of Theorem 1.1

If G acts freely on S^1, then the group Λ of all lifts of all elements of G to $\mathbf{R}$ acts freely on $\mathbf{R}$. There is an obvious exact sequence

$$1 \to \mathbf{Z} \to \Lambda \to G \to 1$$

By Theorem 2.3 the group Λ is abelian, so G is abelian and we are clearly done.

If the action of G on S^1 is not free, then there is a nontrivial element $g \in G$ with $\mathrm{Fix}(g) \neq \emptyset$. For any such g, since g is real-analytic, $\mathrm{Fix}(g)$ is a finite set of points. Let $A(g)$ be a neighborhood of $\mathrm{Fix}(g)$ in S^1 which is chosen so that each component of $A(g)$ is a closed arc containing a unique point of $\mathrm{Fix}(g)$.

If J is any component of $S^1 - \mathrm{Fix}(g)$ then $J \cap A(g)$ consists of two half-open arcs U_J^+ and U_J^-. Since g has no fixed point in J, we may take these arcs to be labeled in such a way that for each $x \in J$ we have $g^n(x) \in U_J^+$ and $g^{-n}(x) \in U_J^-$ for all sufficiently large positive n. It follows that for any large enough $n > 0$ we have $g^n(J - U_J^-) \subset U_J^+$.

Set

$$A^+(g) = \mathrm{Fix}(g) \cup \bigcup_J U_J^+$$

and

$$A^-(g) = \bigcup_J U_J^-,$$

where J ranges over all components of $S^1 - \mathrm{Fix}(g)$. Then for all sufficiently large $n > 0$ we have

$$g^n(S^1 - A^-(g)) \subseteq A^+(g).$$

Note also that $A(g)$ is the disjoint union of the sets $A^+(g)$ and $A^-(g)$.

We claim that if g and h are two elements of G such that $\mathrm{Fix}(h)$ and $\mathrm{Fix}(g)$ are non-empty and disjoint, then for some $n > 0$, the group generated by g^n and h^n is a nonabelian free group. To see this, note that we may choose the neighborhoods $A(g)$ and $A(h)$ of $\mathrm{Fix}(g)$ and $\mathrm{Fix}(h)$ to be disjoint from each other. Hence the four sets $A^+(g)$, $A^-(g)$, $A^+(h)$ and $A^-(h)$ are pairwise disjoint. If we choose n large enough so that

$$g^n(S^1 - A^-(g)) \subseteq A^+(g)$$

and

$$h^n(S^1 - A^-(h)) \subseteq A^+(h),$$

it follows from the Ping-Pong Lemma (see, e.g., [H]) that the group generated by g^n and h^n is a free group on 2 generators.

Suppose that we have chosen an element $g \in G$ with $\mathrm{Fix}(g)$ nonempty. If there exists $b \in G$ with $b(\mathrm{Fix}(g)) \cap \mathrm{Fix}(g) = \emptyset$, then we can set $h = bgb^{-1}$ and apply the argument just given to obtain a nonabelian free subgroup of G. Now suppose that no such b exists. Then for every $c \in G$ there exist $x, y \in \mathrm{Fix}(g)$ with $cx = y$. Thus we may write

$$G = \bigcup_{x,y \in \mathrm{Fix}(g)} T_{x,y}$$

where $T_{x,y}$ denotes the set of all $c \in G$ such that $cx = y$. Note that $T_{x,y}$ is either empty or is a left coset of the stabilizer of x in G. We have therefore exihibited G as a finite union of cosets of stabilizers of points of $\mathrm{Fix}(g)$, so by Lemma 3.1 one of these stabilizers H of some $z \in \mathrm{Fix}(g)$ must be of finite index in G. Since the orbit Gz is finite, Lemma 2.2 implies that G/H is a finite cyclic group. By Proposition 2.1 the commutator subgroup of H is residually torsion-free nilpotent. $\diamond$

4 Centralizers

Proof of Theorem 1.3. Let $f : \mathbf{R} \to \mathbf{R}$ denote any lift of g. Recall ([CFS], Theorem 1, p.74) that the limit

$$\lim_{n\to\infty} f^n(x)/n = \alpha$$

exists and does not depend on the choice of the lift f or the point $x \in \mathbf{R}$. Furthermore, this *rotation number* α of g is rational if and only if g^r has a fixed point for some $r > 0$.

Suppose first that α is irrational. Then since g is a C^∞ diffeomorphism, the Denjoy Theorem (see, e.g,. [CFS], Theorem 1, p.84) tells us that there exists $\phi \in \mathrm{Homeo}_+(S^1)$ such that $\phi^{-1} \circ g \circ \phi = T_\alpha$, where T_α is the rotation of the circle by angle α.

The cyclic group $\langle T_\alpha \rangle$ generated by T_α is dense in the group R of rotations of S^1 since α is irrational. Since $\phi^{-1}C_G(g)\phi$ commutes with $\phi^{-1}g\phi = T_\alpha$, hence with $\langle T_\alpha \rangle$, and since $\langle T_\alpha \rangle$ is dense in R, by continuity we have that $\phi^{-1}C_G(g)\phi$ commutes with R. But any homeomorphism of S^1 which commutes with every rotation must be a rotation, so $\phi^{-1}C_G(g)\phi$ lies in R, hence is abelian. Hence $C_G(g)$ is abelian.

Now suppose that α is rational. Then g^r has a fixed point for some $r > 0$. Since g is real-analytic, the fixed-point set $F = \mathrm{Fix}(g^r) \neq \emptyset$ is a finite set. Clearly $C_G(g) \subset C_G(g^r)$ leaves F invariant. Hence by Lemma 2.2 the action of $C_G(g)$ on F factors through a homomorphism

$$\phi : C_G(g) \to \mathbf{Z}/n\mathbf{Z}$$

where $n = |F|$. Let N denote the kernel of this homomorphism, so that $N \cdot x = x$ for all $x \in F$. Hence N leaves each of the (finitely many) components of $S^1 - F$ invariant. Let Y be any one of these components.

We claim that the action of N on Y is free. Since Y is homeomorphic to $\mathbf{R}$, it then follows from Theorem 2.3 that N is abelian, and we are done.

To prove the claim, consider any $h \in N$. Let $\mathrm{Fix}_Y(h) = \mathrm{Fix}(h) \cap Y$. Since h commutes with g^r and $g^r \cdot Y = Y$, we have $g^r \cdot \mathrm{Fix}_Y(h) = \mathrm{Fix}_Y(h)$. But since g^r is orientation-preserving on Y, Y is homeomorphic to $\mathbf{R}$, and $\mathrm{Fix}_Y(h)$ is a finite set, it must be that g^r fixes every point of $\mathrm{Fix}_Y(h)$. Thus $\mathrm{Fix}_Y(h) \subseteq \mathrm{Fix}(g^r) = F$. Since $Y \subseteq S^1 - F$, it follows that $\mathrm{Fix}_Y(h) = \emptyset$. $\diamond$

5 Two applications of Theorem 1.3

In this section we apply Theorem 1.3 to two different problems.

5.1 Actions of mapping class groups

Proof of Theorem 1.4. We will need to consider mapping class groups of surfaces-with-boundary. Let $\Sigma_{g,r,n}$ denote the oriented surface of genus g with r distinguished points and n boundary components. The mapping class group $\Gamma_{g,r,n} = \mathrm{Mod}(g,r,n)$ is $\pi_0(\mathrm{Homeo}_+(\Sigma_{g,r,n}))$ where $\mathrm{Homeo}_+(\Sigma_{g,r,n})$ is the topological group of orientation-preserving homeomorphisms of $\Sigma_{g,r,n}$ which fix the r distinguished points and restrict to the identity on $\partial\Sigma_{g,r,n}$. We will prove Theorem 1.4 for $\Sigma_{g,r,n}$ with $g \geq 4, r, n \geq 0$.

Let $\psi : \Gamma_{g,r,n} \to \mathrm{Diff}^\omega_+(S^1)$ be given, where $g \geq 4, r, n \geq 0$. Let β be any nonseparating, simple closed curve in $\Sigma_{g,r,n}$, and let $T_\beta \in \Gamma_{g,r,n}$ denote the Dehn twist about β.

If $\psi(T_\beta)$ has infinite order then Theorem 1.3 implies that $C_{\psi(\Gamma_{g,r,n})}(\psi(T_\beta))$ has a homomorphism onto a finite cyclic group with abelian kernel. Let N be the subgroup of $\Gamma_{g,r,n}$ of those mapping classes having representatives which restrict to the identity on some tubular neighborhood of β .

It is clear that N is a subgroup of $C_{\Gamma_{g,r,n}}(T_\beta)$. It follows that $\psi(N)$ has a homomorphism onto a finite cyclic group with abelian kernel. On the other hand there is an obvious surjective homomorphism $\Gamma_{g-1,r,n+2} \to N$. Powell and Harer (see [Ha]) have shown that $H_1(\Gamma_{g,r,n}, \mathbf{Z}) = 0$ for $g \geq 3$ and any $r, n \geq 0$. Hence our assumption that $g \geq 4$ implies that $H_1(N, \mathbf{Z}) = 0$. It now follows that $\psi(N) = 1$.

Note that $T_\beta \in N$ since β is isotopic to a simple closed curve lying in the complement of its tubular neighborhood. Hence $\psi(T_\beta) = 1$, contradicting that $\psi(T_\beta)$ has infinite order.

Hence for every nonseparating, simple closed curve $\beta \in \Sigma_{g,r,n}$, the element $\psi(T_\beta)$ has some finite order $d \geq 1$. Now for any nonseparating, simple closed curve α in $\Sigma_{g,r,n}$, the element T_α is conjugate to T_β in $\Gamma_{g,r,n}$, so that $\psi(T_\alpha)$ has order d.

It is an old result of Dehn and Lickorish (see, e.g. [Bi], Theorem 4.6)that there is a finite collection $\mathcal{C}$ of nonseparating, simple closed curves in $\Sigma_{g,r,n}$ with the following properties:

1. The set $\{T_\alpha : \alpha \in \mathcal{C}\}$ generates $\Gamma_{g,r,n}$,
2. Given any $\alpha, \alpha' \in \mathcal{C}$, there exists $\alpha'' \in \mathcal{C}$ with $\alpha \cap \alpha'' = \alpha' \cap \alpha'' = \emptyset$. In particular $T_{\alpha''}$ commutes with both T_α and $T_{\alpha'}$.

For each $\alpha \in \mathcal{C}$ let Z_α denote the cyclic group of order d generated by $\psi(T_\alpha)$. We claim that if α and α' are disjoint curves in $\mathcal{C}$, then $Z_\alpha = Z_{\alpha'}$. To see this note that $\psi(T_\alpha)$ and $\psi(T_{\alpha'})$ generate a finite abelian group of homeomorphisms of S^1. But any such group is cyclic. Since a cyclic group contains at most one subgroup of order d the claim is proved.

In view of property (2) above, it follows from this claim that $Z_\alpha = Z_{\alpha'}$ for any $\alpha, \alpha' \in \mathcal{C}$. By property (1) it then follows that $\psi(\Gamma_{g,r,n})$ is cyclic. Since $H_1(\Gamma_{g,r,n}, \mathbf{Z}) = 0$ it must be that $\psi(\Gamma_{g,r,n}) = 1$. $\diamond$

5.2 Lattice actions

Let Γ be a *higher-rank lattice*, that is, an irreducible lattice in a semisimple Lie group of $\mathbf{R}$-rank at least two. It is conjectured that any smooth action of a higher-rank lattice on S^1 must factor through a finite group.

The conjecture was proved by Witte [Wi] for lattices of $\mathbf{Q}$-rank at least two. In [FS] the conjecture was proved for real-analytic actions of higher-rank lattices Γ which were called *1-big*. For a uniform lattice Γ, this means that there exists an infinite-order element $\gamma \in \Gamma$ such that $C_\Gamma(\gamma)$ contains a higher-rank lattice. We can apply Theorem 1.3 to extend this last result.

Theorem 5.1 (Lattice actions) *Let Γ be a higher-rank lattice which contains an infinite-order element γ such that $C_\Gamma(\gamma)$ is not virtually abelian. Then any action $\psi : \Gamma \to \mathrm{Diff}^\omega_+(S^1)$ factors through a finite group.*

Proof. By the Margulis Finiteness Theorem (see, e.g. Theorem 8.1 of [Zi]) any normal subgroup of Γ is finite or has finite index, so we may assume

$\ker(\psi)$ is finite. Then $\psi(\gamma)$ is infinite, so Theorem 1.3 gives that $C_{\psi(\Gamma)}(\psi(\gamma))$ is virtually abelian. There is an exact sequence

$$1 \to N \to C_\Gamma(\gamma) \to C_{\psi(\Gamma)}(\psi(\gamma)) \to 1$$

where $N \leq \ker(\psi)$ is finite. Since $C_\Gamma(\gamma)$ is linear hence residually finite, it follows that $C_\Gamma(\gamma)$ has a finite-index subgroup which is virtually abelian. Thus $C_\Gamma(\gamma)$ is virtually abelian, contradicting the hypothesis. $\diamond$

Examples covered by Theorem 5.1 which are not covered by [FS] include many lattices in low-rank semisimple groups. As a concrete example, let $K = \mathbf{Q}(\sqrt{2})$, and for positive integers p, q let $\Gamma_{p,q}$ be the group of K-integral unimodular matrices preserving the form

$$\Phi = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} \sqrt{2} x_i^2$$

Then for $p + q > 4$, the group $\Gamma_{p,q}$ is an irreducible, uniform lattice in a semisimple Lie group containing $\mathrm{O}(p,q)$ as a factor. Furthermore, $\Gamma_{p,q}$ has an infinite-order element whose centralizer contains $\Gamma_{p-1,q-1}$ (see §2.2 of [FS]), so that $\Gamma_{p,q}$ is 1-big if $(p,q) \geq (3,4)$. Although $\Gamma_{2,3}$ is not 1-big, it has an infinite-order element whose centralizer contains $\Gamma_{1,2}$, which is easily seen to contain a free group. Thus Theorem 5.1 applies in this case, showing that any real-analytic action of $\Gamma_{2,3}$ on S^1 is finite.

6 Counterexamples and examples

In this section we give a more detailed description of the examples mentioned in the introduction.

6.1 Thompson's group

We fix an identification of S^1 with $\mathbf{R}/\mathbf{Z}$. Let T denote the group of all piecewise-linear homeomorphisms of S^1 with finitely many break points, each occuring at a dyadic rational, and with slopes powers of 2. Let F denote the subgroup of T fixing $0 \in S^1$. We can (and will) also think of F as a group of piecewise-linear homeomorphisms of the interval $[0,1]$.

The group F has the following remarkable properties (see [CFP] for a survey):

- F is finitely presented and torsion-free.
- Every proper quotient of F is abelian.
- $[F, F]$ is an infinite simple group.
- F has infinite virtual cohomological dimension [BG].
- F does not contain a free group of rank 2.
- T (hence F) is conjugate in $\mathrm{Homeo}_+(S^1)$ to a subgroup of $\mathrm{Diff}_+^\infty(S^1)$ [GS].

Proposition 6.1 (F contains big centralizers) *There exists an infinite-order element $g \in F$ such that the centralizer $C_F(g)$ has a subgroup isomorphic to F.*

Proof. For $i = 1, 2$ let $[a_i, b_i]$ be disjoint, closed subintervals of $[0, 1]$ so that a_i, b_i are dyadic rationals with $0 < a_i < b_i < 1$, and $b_i - a_i$ is a power of 2. Let F_i be the subgroup of F consisting of those elements whose support lies in $[a_i, b_i]$. Clearly F_1 and F_2 commute. It is also easy to check (see [CFP]) that F_i is isomorphic to F via conjugation by the linear map taking $[a_i, b_i]$ to $[0, 1]$. The proposition follows. $\diamond$

From the above properties it is easy to see that Theorem 1.1, Corollary 1.2, and Theorem 1.3 all become false if $\mathrm{Diff}_+^\omega(S^1)$ is replaced by $\mathrm{Diff}_+^\infty(S^1)$:

Proposition 6.2 (Counterexamples) *The group $[F, F]$ is a perfect subgroup of $\mathrm{Diff}_+^\infty(S^1)$ but doesn't contain a nonabelian free group. The group F is a subgroup of $\mathrm{Diff}_+^\infty(S^1)$ which has infinite-order elements whose centralizers are not virtually abelian.*

6.2 Once-punctured mapping class groups

In this section we show that Theorem 1.4 becomes false if $\mathrm{Diff}_+^\omega(S^1)$ is replaced by $\mathrm{Homeo}_+(S^1)$.

For any $g \geq 2$, let $\Sigma_{g,1}$ denote the oriented surface of genus g with a distinguished point $x \in \Sigma_{g,1}$. Let $\mathrm{Mod}(g, 1)$ be $\pi_0(\mathrm{Homeo}_+(\Sigma_{g,1}))$ where $\mathrm{Homeo}_+(\Sigma_{g,1})$ is the topological group of orientation-preserving homeomorphisms of $\Sigma_{g,1}$ which fix x. Pick a single lift y of x in the universal cover $\mathbf{H}^2$ of $\Sigma_{g,1}$. Each $f \in \mathrm{Homeo}_+(\Sigma_{g,1})$ has a unique lift to a homeomorphism

$\tilde{f}$ with $\tilde{f}(y) = y$. The homeomorphism $\tilde{f}$ extends to a homeomorphism $\hat{f} \in \mathrm{Homeo}_+(S^1)$.

For any $h \in \mathrm{Homeo}_+(\Sigma_{g,1})$ with $[f] = [h] \in \mathrm{Mod}(g,1)$, the fact that f and h are isotopic via an isotopy which fixes x shows that $\hat{f} = \hat{g}$. Hence $\hat{f}$ only depends on $[f] \in \mathrm{Mod}(g,1)$. Thus we can define a map

$$\psi : \mathrm{Mod}(g,1) \to \mathrm{Homeo}_+(S^1)$$

via $\psi([f]) = \hat{f}$. It is straightforward to check that ψ is a homomorphism. Note that choosing a different lift y' of x results in a homomorphism ψ' which is conjugate to ψ via an element of $\pi_1(\Sigma_{g,1}) \leq \mathrm{PSL}(2,\mathbf{R})$.

We claim that ψ is injective. To see this, let γ be any primitive element of $\pi_1(\Sigma_{g,1}, x)$. Then $\hat{f}$ takes the attracting (resp. repelling) fixed point of the deck tranformation corresponding to γ to the attracting (resp. repelling) fixed point of the deck transformation corresponding to $f_*(\gamma) \in \pi_1(\Sigma_{g,1}, x)$. If $\psi([f]) = \hat{f} = 1$ then γ and $f_*(\gamma)$ are primitive and have the same attracting and repelling fixed points, hence coincide. This shows that f_* is the identity. It follows easily (see, e.g. [Ep]) that $[f] = 1$.

Thus we have shown that $\mathrm{Mod}(g,1)$ is a subgroup of $\mathrm{Homeo}_+(S^1)$ for every $g \geq 2$. In contrast, Theorem 1.4 implies that $\mathrm{Mod}(g,1)$ is not isomorphic to any subgroup of $\mathrm{Diff}_+^{\omega}(S^1)$.

Question: Is $\mathrm{Mod}(g,1), g \geq 2$ isomorphic to a subgroup of $\mathrm{Diff}_+^{\infty}(S^1)$?

While this may at first seem unlikely, the example of Thompson's group mentioned above gives one pause.

6.3 Hyperbolic 3-manifold groups

There is a well-known exact sequence (see, e.g. [Bi], §4.1)

$$1 \to \pi_1(\Sigma_{g,0}, x) \to \mathrm{Mod}(g,1) \xrightarrow{\pi} \mathrm{Mod}(g,0) \to 1$$

Let $[\phi] \in \mathrm{Mod}(g,0)$ denote the class of a pseudo-Anosov homeomorphism. Then the inverse image under π of the cyclic group generated by $[\phi]$ gives a group $\Gamma < \mathrm{Homeo}_+(S^1)$ for which

$$1 \to \pi_1(\Sigma_{g,0}, x) \to \Gamma \to \mathbf{Z} \to 1$$

is exact. It is easy to check that the action of $\mathbf{Z}$ on $\pi_1(\Sigma_{g,0}, x)$ is given by $[\phi]$. By Thurston's Hyperbolization Theorem (see, e.g. [Ot]), the group $\Gamma <$

$\mathrm{Homeo}_+(S^1)$ is isomorphic to the fundamental group of a closed hyperbolic 3-manifold.

Question: Can Γ be conjugated into $\mathrm{Diff}^\infty_+(S^1)$? into $\mathrm{Diff}^\omega_+(S^1)$?

Note also that $\mathrm{Mod}(0,4)$ admits an action on the space of projective measured laminations on $\Sigma_{0,4}$, which is a circle. This gives an action of $\mathrm{Mod}(0,4)$ on S^1 by piecewise integral projective transformations. We do not know whether $\mathrm{Mod}(0,4)$ admits an effective real-analytic action on S^1.

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