Some open problems related to mapping class groups of del Pezzo manfiolds

Seraphina Lee

August 7, 2022

The following problems are open, to the best of my knowledge. For any 4-manifold M, let $\mathrm{Mod}(M)$ denote the topological mapping class group $\pi_0(\mathrm{Homeo}^+(M))$. Throughout, "isotopic" means "topologically isotopic" unless specified otherwise.

Problem 1. What is the relationship between $\mathbb{H}^n$ and the manifold M_n? What is a space of structures of M_n on which $\mathrm{Mod}(M_n)$ acts, like Teichmüller space for $\mathrm{Mod}(\Sigma_g)$ for 2-dimensional surfaces?

Problem 2. Dehn twists are not isotopic to any finite order diffeomorphism in K3 manifolds (Farb–Looijenga [FL21]) or more generally, in spin 4-manifolds with non-zero signature (Konno [Kon22]). What about Dehn twists in non-spin manifolds? What about other order-2 elements in non-spin manifolds?

Problem 3. Are Dehn twists in $\mathrm{Mod}(M_n)$ *smoothly* isotopic to any order-2 diffeomorphism in M_n? To any finite-order diffeomorphism in M_n?

Problem 4. Let $n \leq 8$. All order-2 elements of $\mathrm{Mod}(M_n)$ are isotopic to an order-2 diffeomorphism of M_n by [Lee22]. Which (if any) order-2 elements of $\mathrm{Mod}(M_n)$ are isotopic to a unique order-2 diffeomorphism, up to conjugacy in $\mathrm{Diff}^+(M_n)$?

Problem 5. Let $n \leq 8$ and $m \geq 3$. Are order-m elements of $\mathrm{Mod}(M_n)$ realizable by diffeomorphisms of order m? By diffeomorphisms of finite order?

Problem 6. The finite subgroups of $\mathrm{Mod}(M_2)$ that have lifts to $\mathrm{Diff}^+(M_2)$ are classified by [Lee21]. What about for $\mathrm{Mod}(M_n)$ for any $n \geq 3$?

Problem 7. Do any finite subgroups of $\mathrm{Mod}(M_2)$ that do not have lifts to $\mathrm{Diff}^+(M_2)$ have lifts to $\mathrm{Homeo}^+(M_2)$?

References

[FL21] Benson Farb and Eduard Looijenga. The Nielsen realization problem for K3 surfaces. *Jour. Diff. Geom.*, 2021. to appear.

[Kon22] Hokuto Konno. Dehn twists and the Nielsen realization problem for spin 4-manifolds. 2022. arXiv:2203.11631.

[Lee21] Seraphina Eun Bi Lee. The Nielsen realization problem for high degree del Pezzo surfaces. 2021. arXiv:2112.13500.

[Lee22] Seraphina Eun Bi Lee. Isotopy classes of involutions of del pezzo surfaces. 2022. arXiv:2202.10616.