Mod(M^4) note

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Abstract

(Personal) Log of records for 'Mod(M^4) minicourse' by Prof. Benson Farb.

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1.1 The Projective Space

Recall the *projective space*, the space of lines through origin in $\mathbb{C}^{n+1}$.

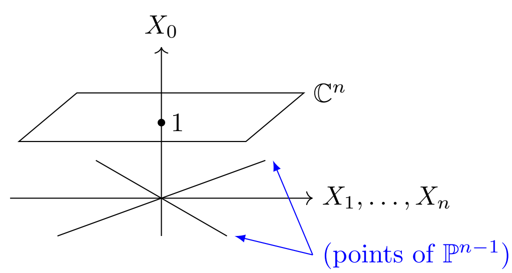
$$\mathbb{P}^n = \mathbb{CP}^n = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\times.$$

This is a complex n-dimensional manifold, which is also a real $2n$-dimensional oriented closed manifold. Few first remarks:

- When $n = 1$, then $\mathbb{P}^1 = \mathbb{S}^2$ is a sphere.
- For $V \subset \mathbb{C}^{n+1}$ a linear hyperplane (i.e., dimension n linear subspace), then we also have a subset $H = \mathbb{P}(V) \subset \mathbb{P}^n$, by the image of the natural quotienting $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. It is also conventionally called a *hyperplane* in $\mathbb{P}^n$.
- The space $\mathbb{P}^n$ has the natural *homogeneous coordinate* $[X_0 : X_1 : \ldots : X_n]$, which decomposes $\mathbb{P}^n$ as

  $$\begin{aligned} \mathbb{P}^n &= \{[X_0 : X_1 : \ldots : X_n] : X_0 \neq 0\} \cup \{[0 : X_1 : \ldots : X_n]\} \\ &= \{[1 : x_1 : \ldots : x_n] : x_i \in \mathbb{C}\} \cup \mathbb{P}^{n-1} = \mathbb{C}^n \cup \mathbb{P}^{n-1}. \end{aligned}$$

  Here, this $\mathbb{P}^{n-1}$ is called the *hyperplane at infinity*, and its complement $\mathbb{C}^n$ is called the *affine plane*. Sketch of their configurations:



Remark. A differential form on $\mathbb{P}^n$ is thus written by either

(i) $\mathbb{C}^\times$-invariant form on $\mathbb{C}^{n+1} \setminus \{0\}$, or

(ii) written on each affine plane $\mathbb{C}^n$, but compatible with transitions.

An example of the first kind is $\dfrac{dX_0}{X_0}$. An example of the second kind is the Fubini–Study metric ω_{FS}: on an affine plane $\{X_i \neq 0\}$ with coordinates $(x_0, \ldots, \widehat{x}_i, \ldots, x_n)$, we define

$$\omega_{FS} = \frac{1}{2} dd^c \log \left(1 + \sum_{j \neq i} |x_i|^2\right).$$

- For the homologies of $\mathbb{P}^n$, we have (recall that $\mathbb{P}^n$ is a $2n$-manifold)

$$H_i(\mathbb{P}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & (i = 0, 2, \ldots, 2n), \\ 0 & (\text{else}). \end{cases}$$

  Here, each $H_{2k}(\mathbb{P}^n;\mathbb{Z})$ is generated by $[\mathbb{P}^k]$, where $\mathbb{P}^1 \subset \mathbb{P}^2 \subset \ldots \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$ are standard copies of $\mathbb{P}^k$'s.

  Remark. The notation $[\mathbb{P}^k]$ is for the image of *any* $(k+1)$-dimensional linear subspace of $\mathbb{C}^{n+1}$. This is well-defined, since any two such images are homologous to one another.

  We specifically name the class $[\mathbb{P}^{n-1}]$ as the *hyperplane class* and also may denote $[H]$ for it. A *line* in $\mathbb{P}^n$ refers to a linear copy of $\mathbb{P}^1 \subset \mathbb{P}^n$.

- (Dual view) The *Fubini–Study metric* ω_{FS} is Poincaré dual of the hyperplane class $[H]$. This is one of 'the' Kähler class of $\mathbb{P}^n$.

  Remark. To elaborate on the Poincaré duality, we think of the dual class $\mathrm{PD}([H]) \in H^2(\mathbb{P}^n;\mathbb{Z}) = \mathrm{Hom}(H_2(\mathbb{P}^n;\mathbb{Z}),\mathbb{Z})$ by

$$\begin{aligned} H_2(\mathbb{P}^n;\mathbb{Z}) &\to \mathbb{Z} \\ [X] &\mapsto [X].[H], \end{aligned}$$

  the *algebraic intersection number* between the curve X and the hyperplane H. That $[\omega_{FS}] = \mathrm{PD}([H])$ means we have the identity

$$[X].[H] = \int_X \omega_{FS},$$

  the integration of a 2-form on a (closed) 2-manifold X. In fact, we have $\mathrm{PD}([H_1 \cap \ldots \cap H_r]) = [\omega_{FS}]^r$. We thus observe that, via Poincaré duality, we have algebraic intersection $\bullet$ corresponding to the wedging $\wedge$.

The above observations conclude the cohomology ring isomorphism

$$H^\bullet(\mathbb{P}^N;\mathbb{Z}) = \mathbb{Z}[\omega_{FS}]/(\omega_{FS}^{n+1} = 0).$$

1.2 Projective Varieties

Let $\mathbb{C}[X_0, \ldots, X_n]_{(d)}$ be the linear space of homogeneous polynomials of degree d. That is, the set of polynomials $F \in \mathbb{C}[X_0, \ldots, X_n]$ such that

$$F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d \cdot F(x_0, \ldots, x_n).$$

For $F \in \mathbb{C}[X_0, \ldots, X_n]_{(d)}$, the set

$$Z(F) := \{[X_0 : \ldots : X_n] \in \mathbb{P}^n : F(X_0, \ldots, X_n) = 0\}$$

is well-defined; this is a feature which is only admitted for homogeneous polynomials.

Definition 1.1. A *projective variety* is a set of the form $Z(F_1, \ldots, F_m) = \bigcap_{i=1}^m Z(F_i)$ for some homogeneous polynomials $F_1, \ldots, F_m$ (possibly with various degrees).

Any projective variety, as a closed subset of $\mathbb{P}^n$, is compact. Some adjectives for projective varieties:

Definition 1.2. A projective variety X is

- a *hypersurface* of degree d, if $X = Z(F)$ for some F of homogeneous of degree d.
- *smooth* if X is also a manifold (i.e., a submanifold of $\mathbb{P}^n$).

By Implicit Function Theorem, the projective variety $X = Z(F_1, \ldots, F_m)$ is smooth of codimension m iff for each $a \in X$, the $m \times (n+1)$ matrix

$$\left(\frac{\partial F_i}{\partial X_j}(a)\right)_{i,j}$$

has rank m. This smoothness criterion is generic, in the space of all degree d hypersurfaces.

Example 1.3. Let $F(X_0, \ldots, X_n) = X_0^d + \cdots + X_n^d$, with $d \geq 2$. The variety $Z(F)$, called (degree d) *Fermat hypersuface*, is smooth. Indeed, the matrix

$$\left(\frac{\partial F}{\partial X_j}\right) = \begin{bmatrix} dX_0^{d-1} & dX_1^{d-1} & \cdots & dX_n^{d-1} \end{bmatrix} = 0$$

iff $X_0 = \ldots = X_n = 0$ (which does not define any point in the projective space).

There are two themes of the study of projective varieties.

Theme 1 The main property that distinguishes the topology of a smooth projective r-dimensional variety X among all smooth $2r$-dimensional closed oriented manifold is that

- X comes with an inclusion map $i \colon X \hookrightarrow \mathbb{P}^n$, and
- the classes $i^*\omega_{FS}^k \in H^{2k}(X; \mathbb{Z})$ are found. Equivalently, we find the classes

$$[X \cap H] \in H_{2(\dim X - 1)}(X; \mathbb{Z}),$$
$$[X \cap H \cap H'] \in H_{2(\dim X - 2)}(X; \mathbb{Z}),$$
$$\cdots$$

  with $H, H', \ldots$ a generic choice of hyperplanes.

Theme 2 The nature of solutions of polynomial equations are governed by the topology of the zero sets.

Example 1.4. Recall the Fundamental Theorem of Algebra: any nonconstant $f \in \mathbb{C}[z]$ has $(\deg f)$ zeroes, counted with multiplicities. (So $f(z) = z^3(z-1)^2$ has 5 zeroes $0, 0, 0, 1, 1$.)

So for instance, $x^2 = a$ has 2 solutions, unless $a = 0$ where we have a double solution.

Such an observation may extend to the curves in $\mathbb{P}^2$, as discussed below.

1.3 Complex Curves in the projective plane

Question 1.5. How many points $(x, y) \in \mathbb{C}^2$ are common solutions to

$$7x^3y - (4\pi + \mathrm{i})xy + 2y^5 = 0, \quad x^7 + y^3 - xy^2 = 0?$$

To answer this, we follow the steps below.

1. *Homogenize* the polynomials. For the question above, define

$$f(X, Y, Z) = 7X^3YZ - (4\pi + \mathrm{i})XYZ^3 + 2Y^5,$$
$$g(X, Y, Z) = X^7 + Y^3Z^4 - XY^2Z^4.$$

2. Observe that putting $Z = 1$, we get back to the original locus. All new intersection points are coming from $\{Z = 0\} = \mathbb{P}^1 \subset \mathbb{P}^2$.

   So we have lifted the intersection problem of complex curves $Z(f)$ and $Z(g)$ within $\mathbb{P}^2$.

3. We ask,

   (a) What is $|Z(f) \cap Z(g)| =$? Is the intersection transverse?

   (b) For which genus g we can say $Z(f) \cong \Sigma_g$? (Will show that $g = \frac{1}{2}(d_f - 1)(d_f - 2)$ in the next lecture, where $d_f = \deg f$.)

For the first question, we split into some cases. When f, g have a (nonconstant) common factor, i.e., $f = \alpha\beta$ and $g = \alpha\gamma$, then as $Z(f) \cap Z(g) \supset Z(\alpha)$, we have an infinite intersection. But otherwise, there is a theorem that *algebraically* counts the intersection:

Theorem 1.6 (Bezout)**.** *Suppose $f, g \in \mathbb{C}[X, Y, Z]$ are homogeneous of degrees d_1, d_2 respectively. Assume f and g have no common factor. Then $Z(f) \cap Z(g)$ is finite, and has d_1d_2 elements counted with multiplicities. For generic f, g, we furthermore have $|Z(f) \cap Z(g)| = d_1d_2$.*

Proof. First, we show the 'linear Bezout theorem.' As $Z(f) \subset \mathbb{P}^2$ is a closed manifold, we have its homology class, $[Z(f)] \in H_2(\mathbb{P}^2; \mathbb{Z})$, homologous to $r[H]$ for some $r \in \mathbb{Z}$.

For a generic line $L \subset \mathbb{P}^2$, $|Z(f) \cap L| = d_1$; if counted with multiplicities, this intersection equals d_1. For example, if we set $L = \mathbb{P}^1 = \{X = 0\}$ the line at infinity, then the intersection counts the zeroes $Z(f(0, Y, Z)) \subset \mathbb{P}^1_{YZ}$, which reduces to the question that how many solutions do we have for $f(0, 1, z) = 0$ (for generic f).

On the other hand, one has the algebraic intersection computed as (with L generic)

$$[L].[Z(f)] = |L \cap Z(f)|,$$

since for *complex* submanifolds $A \pitchfork B$ intersecting transversally, the sign of each intersection locus is 1. As discussed earlier, this number equals to

$$= \deg f.$$

But we also know that $[Z(f)] = r[L]$. Thus $[L].[Z(f)] = r([L].[L]) = r$. (Note that any (nonequal) two lines in $\mathbb{P}^2$ meets at a unique point!) Therefore $[Z(f)] = (\deg f)[L]$ follows.

For the general case, we first find the algebraic multiplicity

$$[Z(f)].[Z(g)] = d_1[L].d_2[L] = d_1 d_2.$$

If f, g were generic, then $Z(f) \pitchfork Z(g)$, so $|Z(f) \cap Z(g)| = [Z(f)].[Z(g)] = d_1 d_2$ also follows. $\square$

Example 1.7. For our example case,

$$\begin{aligned} f(X,Y,Z) &= 7X^3YZ - (4\pi + \mathrm{i})XYZ^3 + 2Y^5, \\ g(X,Y,Z) &= X^7 + Y^3Z^4 - XY^2Z^4, \end{aligned}$$

Bezout's theorem thus gives $[Z(f)].[Z(g)] = 5 \cdot 7 = 35$. Furthermore, $Z(f) \pitchfork Z(g)$ in this case, because the common solutions to $f = 0$, $g = 0$, and "$\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix}$ has rank 1" does not have any nonzero solution. (Programs like `Macaulay2` helps us to verify this fact.)

Finally, we think of the points $Z(f) \cap Z(g)$ within $\mathbb{P}^1 = \{Z = 0\}$. It turns out that $f(X,Y,0) = 2Y^5 = 0$ and $g(X,Y,0) = X^7 = 0$ has $(X,Y) = (0,0)$ as the only common zero, so we do not need to take care of it.

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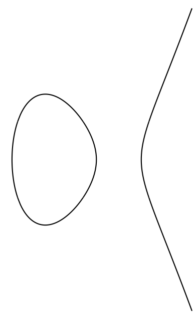
We continue about the following

Question 2.1. Let $Z(F) \subset \mathbb{P}^2$ be a degree d smooth projective plane curve, which is a closed oriented 2-manifold. What is its genus?

A somewhat related question is this. Think of the plane curve $y^2 = x^3 - 3x + 1$, an elliptic curve, which is known to be isomorphic to a torus Σ_1. But in its real picture, we only see some circles (including the point $[0 : 1 : 0]$ at infinity). What happened?



Theorem 2.2 (Degree-Genus formula)**.** *Let $C = Z(F)$ be a smooth degree $d \geq 1$ curve. Then $Z(F) \cong \Sigma_g$, where $g = \frac{1}{2}(d-1)(d-2)$.*

The values of g for first d's goes as follows: $g\colon 0, 0, 1, 3, 6, 10, 15, 21, 28, 36, \ldots$. So,

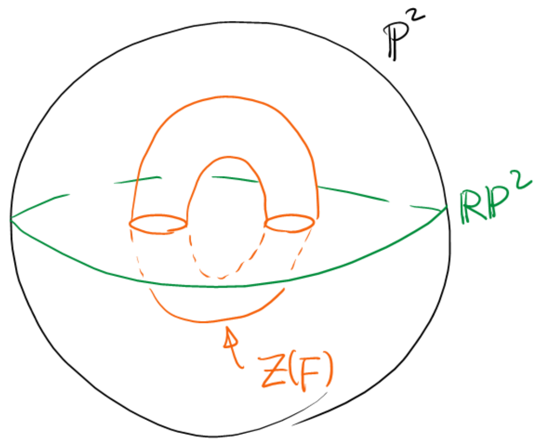
- for $d = 1$, when C is a *line* $\mathbb{P}^1$, we have $C \cong \mathbb{S}^2$.
- for $d = 2$, when C is a *conic*, this is also $\cong \mathbb{S}^2$.
- for $d = 3$, when C is a *cubic curve*, this has the shape of a torus.
- for $d = 4$, when C is a *quartic curve*, this appears as a genus 3 surface.

Recall that the Fermat curves of various degrees are generic examples of smooth planar curves.

Remark. Most algebraic curves (projective Riemann surfaces) C are *not* planar, i.e., does not admit an embedding $C \hookrightarrow \mathbb{P}^2$. The degree-genus formula provides one of the obstructions; so say for genus 5 curves, as there is no d such that $\frac{1}{2}(d-1)(d-2) = 5$, we fail to have a planar embedding. (Of course there may be bigger projective spaces that can contain such one.)

Even if the degree-genus issue is cleared (which admits us to have a topological embedding $C \hookrightarrow \mathbb{P}^2$ with a complex submanifold image), whether we can embed into $\mathbb{P}^2$ really depends on the complex structure that C carries. That is, planar curves is a distinguished subset of the moduli $\mathcal{M}_g$ of closed genus g surfaces.

Real curves Suppose $F \in \mathbb{R}[X, Y, Z]_{(d)}$, i.e., F has real coefficients. Think of the complex conjugation $\sigma \curvearrowright \mathbb{P}^2$, whose fixed points are $\mathsf{Fix}(\sigma) = \mathbb{RP}^2$. Obviously we have $Z(F)$ invariant under σ, with its fixed locus $\mathsf{Fix}(\sigma|Z(F)) = Z(F) \cap \mathbb{RP}^2$. We can view this as an order 2 diffeomorphism on $Z(F) = \Sigma_g$ as well.



For instance, if $F = Y^2Z - X^3 + 3XZ^2 - Z^3$ (so that $F(x, y, 1) = 0$ gives the elliptic curve mentioned above), then $\mathsf{Fix}(\sigma|Z(F))$ is a union of two loops. That is what we see in the "Weierstrass picture" of an elliptic curve defined over $\mathbb{R}$.

Note that the intersection $Z(F) \cap \mathbb{RP}^2$ morally has to be points, while in reality it is not. This is because for generic $p \in Z(F) \cap \mathbb{RP}^2$, $T_pZ(F)$ admits an action by σ whose fixed locus is a real 1-dimensional subspace $\subset T_p\mathbb{RP}^2$. So transversality often fails in this intersection.

Genus is not the end Because we classify closed surfaces diffeomorphically according to their genus, we conclude that the diffeomorphic type of $Z(F)$ is solely determined by the degree. However, the detailed polynomial still has something to do; depending on the polynomial F, we have different *complex structures* of Σ_g realized as $Z(F)$.

2.1 Adjunction Formula

Theorem 2.3 (Adjunction Formula)**.** *Let $L = \mathbb{P}^1$ be any line in $\mathbb{P}^2$, say $L = Z(X) = \{[0 : Y : Z]\}$. For any smooth curve $C = Z(F) \subset \mathbb{P}^2$, we have its topological Euler characteristic*

$$\chi(C) = -[C].[C] + 3[L].[C].$$

Note that the formula computes an intrinsic invariant $\chi(C)$ by the extrinsic information $[C].[C]$, determined by the embedding $C \subset \mathbb{P}^2$, plus a correction term $3[L].[C]$, determined by the 'canonical bundle' $K_{\mathbb{P}^2} = -3[L]$.

Corollary 2.4. *The degree-genus formula.*

Proof. Set $C = Z(F)$. By Bezout theorem, we have $[L].[C] = d(= \deg F)$, and $[C].[C] = d^2$ (if we have trouble applying the theorem, simply replace $[C]$ to $d[L]$ to do the computation). Now by adjunction formula,

$$\chi(C) = -[C].[C] + 3[L].[C]$$

$$\Rightarrow 2(1-g) = -d^2 + 3d.$$
$$\therefore g = \frac{1}{2}(d^2 - 3d + 2) = \frac{(d-1)(d-2)}{2}.$$

The formula is proven. □

There was a plenty of efforts to generalize the adjunction formula above, for C not holomorphic but smooth. (If we see the topological proof below, this may sound tempting.) This was studied by Kronheimer, Mrowka, etc. and one of the results are:

Theorem 2.5 (Thom's conjuecture; Kronheimer–Mrowka theorem)**.** *The minimal genus* smooth *surface* $\subset \mathbb{P}^2$ *that represents the homology class* $d[L] \in H_2(\mathbb{P}^2; \mathbb{Z})$*,* $d \geq 1$*, is* $\frac{1}{2}(d-1)(d-2)$*. That is, algebraic curves attain the minimal genus.*

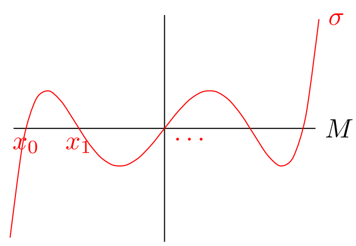
2.2 Proof of the Adjunction Formula

In most versions of the proof, the key observation is that

$$T\mathbb{P}^2|_C = TC \oplus \mathcal{N}_{\mathbb{P}^2}(C),$$

as vector bundles over C. (Notations to be clarified later.) To emphasize the topological proof, we present some theory of characteristic classes. (cf. Milnor–Stasheff, *Characteristic classes.*) Let M^n be a closed oriented (real) n-manifold throughout the section.

1. Consider any rank n vector bundle $E \to M$ (model eg: $E = TM$). Pick a section $\sigma\colon M \to E$ (model eg: vector field). Let the section be generic enough, so that points $x_i \in M$ with $\sigma(x_i) = 0$ are finite and transverse.



If $E = TM$, then Poincaré–Hopf index theorem will assert that

$$\chi(M) = \sum_{x_i \in \sigma^{-1}(0)} \pm 1,$$

where ± 1 refers to the index of the zero x_i of the vector field σ. This 'number' version may be improved as follows.

Consider 0-cycles $x = \sum_{x_i \in \sigma^{-1}(0)} \pm [x_i] \in H_0(M^n; \mathbb{Z}) = \mathbb{Z}$, where $\pm$ is determined according to the index of vanishing σ at x_i. Then the Poincaré dual of x is a cohomology class $e(E) \in H^n(M; \mathbb{Z})$ called the *Euler class* of E.

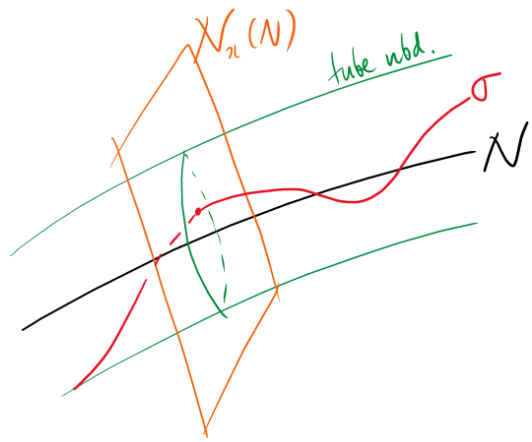
Exercise 2.6. Recover Poincaré–Hopf index theorem, esp. using $\chi(M) = \langle e(TM), [M] \rangle$.

Normal bundle Another interesting example, other than the tangent bundle, is the normal bundle. Consider closed oriented (real) manifolds $N^n \subset M^{2n}$. By the tubular neighborhood of N, we define a *normal bundle* $\mathcal{N}_M(N) \to N$ of N in M; it is a vector bundle consisting of fibers

$$\mathcal{N}_x(N) = T_xM/T_xN.$$

Proposition 2.7. *We have* $e(\mathcal{N}_M(N)) = [N].[N]$.

Proof. By the Tubular Neighborhood Theorem, a generic section of the normal bundle gives a section σ which is transverse to the 0-section, N.



Hence the graph of σ gives a perturbation of N which is transverse to N. Now compare the definition of $e(\mathcal{N}_M(N))$ and the self-intersection $[N].[N] = [N].[\mathsf{Graph}(\sigma)]$. □

2. Let $\dim_{\mathbb{R}} M = 2n$, and let $E \to M$ be a *complex* vector bundle of rank n. As a real vector bundle of rank $2n$, this has its Euler class, $e(E) \in H^{2n}(M, \mathbb{Z})$, which is same as its n-th *Chern class* $c_n(E)$.

Now suppose $\dim_{\mathbb{R}} M = 4$, i.e., $n = 2$. From the complex vector bundle E, one has its *first Chern class* $c_1(E) \in H^2(M, \mathbb{Z})$, defined with two generic sections $\sigma_1, \sigma_2 \colon M \to E$ as follows.

$$\begin{aligned} c_1(E) &:= \mathrm{PD}(\{m \in M : \sigma_1(m), \sigma_2(m) \text{ are proportional}\}) \\ &:= \mathrm{PD}(\{m \in M : \sigma_1(m) \wedge \sigma_2(m) = 0\}). \end{aligned}$$

(For general n, we generalize this with n sections, and apply $c_1(\bigwedge^n E) = c_1(E)$. See [MSE1][MSE2].)

As noted in Milnor–Stasheff, we have the following.

Proposition 2.8. *We have* $c_1(T\mathbb{P}^2) = -c_1(T^*\mathbb{P}^2) = 3\mathrm{PD}([L]) \in H^2(\mathbb{P}^2; \mathbb{Z})$.

Proof. See Exercise below. □

3. Let ξ_1, ξ_2 be $\mathbb{C}^2$-bundles over M^4. Then we have $c_1(\xi_1 \oplus \xi_2) = c_1(\xi_1) + c_1(\xi_2)$. (By the Whitney product formula; this is not specific for 4D manifolds.)

If we apply this to $C = Z(F) \subset \mathbb{P}^2$, together with the decomposition

$$T\mathbb{P}^2|_C = TC \oplus \mathcal{N}_{\mathbb{P}^2}(C),$$

we have

$$\begin{aligned} c_1(T\mathbb{P}^2|_C) &= c_1(TC) + c_1(\mathcal{N}_{\mathbb{P}^2}(C)) \\ &= \chi(C) + [C].[C]. \end{aligned}$$

On the other hand, if we write $i\colon C \hookrightarrow \mathbb{P}^2$ for the inclusion, we can compute $c_1(T\mathbb{P}^2|_C)$ as

$$\begin{aligned} c_1(T\mathbb{P}^2|_C) &= c_1(i^*(T\mathbb{P}^2)) = i^* c_1(T\mathbb{P}^2) \\ &= i^*(3\mathrm{PD}([L])) = 3[L].[C]. \end{aligned}$$

Combining the two, we get the adjunction formula.

2.3 Next

Next, we will see some interesting visual aspects of projective varieties. For instance, a classical fact that cubic surfaces exhibit 27 lines, which admits a visualization as in this [AMS blog], may be discussed.

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3.1 Space of hypersurfaces

Recall that a degree d hypersurface $X \subset \mathbb{P}^n$ is given by $X = Z(F)$, where $F \in \mathbb{C}[X_0, X_1, \ldots, X_n]_{(d)}$. Name the space

$$V_{d,n} = \mathbb{C}[x_0, X_1, \ldots, X_n]_{(d)}$$

of homogeneous degree d polynomials in $(n+1)$ variables. Then we have $\dim V_{d,n} = \binom{d+n}{d}$, and the space of degree d hypersurfaces $Z(F) \subset \mathbb{P}^n$ may be written as $\mathbb{P}(V_{d,n})$ (as $Z(F) = Z(\lambda F)$ for $\lambda \in \mathbb{C}^\times$). This projective space is called the *parameter space of degree d hypersurfaces $\subset \mathbb{P}^n$*.

Note that the holomorphic automorphism group $\mathsf{Aut}_{\mathcal{O}}(\mathbb{P}^n) = \mathrm{PGL}(n+1, \mathbb{C})$ acts on $\mathbb{P}(V_{d,n})$ by coordinate transforms.

Remark. Let $F, G \in V_{d,n}$. It is in general not true that $Z(F) = Z(G)$ implies $F = \lambda G$ for some $\lambda \in \mathbb{C}^\times$; cf. $F = X_0 X_1^2$, $G = X_0^2 X_1$. The claim is true, however, when F, G are both square-free. [MSE]

Proof. If F, G were square-free, then they are determined (up to a constant) by their prime factors. By Nullstellensatz, $Z(F) = Z(G)$ entails $\sqrt{(F)} = \sqrt{(G)}$. So we have polynomials P, Q and integers $r, s > 0$ with $FP = G^r$ and $GQ = F^s$. If $rs = 1$ then $r = s = 1$ and P, Q must be constant.

If $rs > 1$, we have $PQ^r = F^{rs-1}$ and $P^sQ = G^{rs-1}$. For any prime factor π of F, if we valuate each side of $PQ^r = F^{rs-1}$ by π, we have $\nu_\pi(P) + r\nu_\pi(Q) = (rs-1)\nu_\pi(F) > 0$. So either P or Q has π. By $P^sQ = G^{rs-1}$, this entails $\nu_\pi(G) > 0$. Arguing symmetrically, we have $\nu_\pi(F) > 0$ iff $\nu_\pi(G) > 0$. Thus prime factors of F and G are the same. $\square$

Question 3.1. Think of two random smooth degree d hypersurfaces, say $Z(\sum_{i=0}^n X_i^d)$ and $Z(X_0X_1X_2^{d-2} + (\pi + \mathrm{i})X_1^{d-5}X_2^5 - 17X_n^d)$. What are the common properties we expect from these?

One conclusion is that they are *diffeomorphic*, which will be proven during this lecture.

Call $\sigma_{d,n}$ for the space of singular degree d hypersurfaces $\subset \mathbb{P}^n$. That is,

$$\Sigma_{d,n} = \{[F] \in \mathbb{P}(V_{d,n}) : \exists a \underline{\in Z(F)} \forall i \ \frac{\partial F}{\partial X_i}(a) = 0\}.$$

Remark. By Euler's identity, $(\deg F) \cdot F = \sum_{i=0}^n x_i \frac{\partial}{\partial x_i} F$, (we obtain this by differentiating $F(\lambda x) = \lambda^{\deg F} F(x)$ at $\lambda = 1$) that $\frac{\partial}{\partial x_i} F(a) = 0$ for all i implies $a \in Z(F)$. Thus the underlined '$\in Z(F)$' may be replaced to '$\in \mathbb{P}^n$'.

Although we have defined $\Sigma_{d,n}$ by a statement $Z(\frac{\partial}{\partial X_0}F, \ldots, \frac{\partial}{\partial X_n}F) \neq \varnothing$, this is in fact an algebraic condition by the coefficients of F, thanks to an amazing classical result below.

Theorem 3.2 (Resultant)**.** *Given N and $d_1, \ldots, d_r \geq 1$, there exists a polynomial* Res *in $\mathbb{Z}$-coefficient (called* resultant*) and variables a_α^i ($i \in \{1, \ldots, r\}$ and $\alpha \in \mathbb{Z}_{\geq 0}^N$ with $|\alpha| \leq d_i$) with the following property.*

For any polynomials $F_1, \ldots, F_r$ in $\mathbb{C}[x_1, \ldots, x_n]$, with $d_j = \deg F_j$, write $F_i = \sum_{|\alpha| \leq d_i} a_\alpha^i x^\alpha$. Then we have

$$Z(F_1, \ldots, F_r) \neq \varnothing \iff \mathrm{Res}(a_\alpha^1, \ldots, a_\alpha^r) = 0.$$

Proof. (cf. [Wiki]) For $r = 2$, denote P_k for the space of degree $\leq k$ polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. Define a map

$$\begin{aligned} \varphi \colon P_{d_2} \times P_{d_1} &\to P_{d_1+d_2} \\ (A, B) &\mapsto F_1A + F_2B. \end{aligned}$$

Then $Z(F_1, F_2) = \varnothing$ iff φ is a linear isomorphism. Thus writing the matrix of φ and taking its determinant, we get a $\mathbb{Z}$-coefficient matrix of a_α^1's ($|\alpha| \leq d_1$) and a_β^2's ($|\beta| \leq d_2$).

The matter is tricky for $r > 2$; see [Wiki]. $\square$

Example 3.3. For $F(x) = ax^2 + bx + c$, we have $\mathrm{Res}(F, F') = -a(b^2 - 4ac)$. Sylvester's theorem gives that the following determinant is used to find this:

$$\begin{vmatrix} a & 2a & 0 \\ b & b & 2a \\ c & 0 & b \end{vmatrix} = -a(b^2 - 4ac).$$

Corollary 3.4. *The set $\Sigma_{d,n}$ described above is a subvariety of $\mathbb{P}(V_{d,n})$.*

Proof. Simply think of $\mathrm{Res}(\partial_0 F, \dots, \partial_n F) = 0$. □

Definition 3.5. We define $X_{d,n} := \mathbb{P}(V_{d,n}) \setminus \Sigma_{d,n}$. This is a Zariski dense open subset of $\mathbb{P}(V_{d,n})$; hence a smooth manifold.

Remark. If $[F] \in X_{d,n}$, then F must be square-free. (If $P^2 \mid F$, P nonconstant, then putting $Q = F/P^2$, we have $\nabla F = 2PQ\nabla P + P^2\nabla Q$. Thus any $p \in Z(P)$ has $p \in Z(F)$ as well as $\nabla F(p) = 0$. Thus $[F] \notin X_{d,n}$.) Therefore by a remark above, if $[F], [G] \in X_{d,n}$, we have $Z(F) = Z(G)$ iff $[F] = [G]$. Hence we identify each element of $X_{d,n}$ to its zero set.

Question 3.6 (Open, except for smaller d, n's)**.** What can be said about the topology of $\Sigma_{d,n}$ and $X_{d,n}$? Say, what are the cohomology rings $H^\bullet(\Sigma_{d,n}; \mathbb{Q})$ or $H^\bullet(X_{d,n}; \mathbb{Q})$?

3.2 Application

Theorem 3.7. *Let $n, d \geq 1$. Any two smooth degree d hypersurface in $\mathbb{P}^n$ are diffeomorphic.*

Note that, for $n = 1$, this is more or less just the Fundamental Theorem of Algebra. For $n = 2$, we have seen degree-genus formula and classification of closed surfaces gives the result.

Proof. Declare the set

$$U_{d,n} := \{(M, p) : M \in X_{d,n},\ p \in M\}.$$

This is called the 'universal smooth degree d hypersurface in $\mathbb{P}^n$,' although technically $U_{d,n} \subset X_{d,n} \times \mathbb{P}^n$ (and obtain topology from it as well). As no $M \in X_{d,n}$ is empty, we have a surjective smooth map $\pi\colon U_{d,n} \to X_{d,n}$ by the projection:

$$\begin{tikzcd} M \arrow[r] & U_{d,n} \arrow[d, "\pi(M,p):=M"] \\ & X_{d,n} \end{tikzcd}$$

1. The map π is a proper map.

   Because $U_{d,n}$ is within $X_{d,n} \times$ (compact), so any sequence $(M_i, p_i)_{i \geq 0}$ in $U_{d,n}$ exiting to infinity cannot have p_i's to go to infinity. So we must have $(M_i)_{i \geq 0}$ in $X_{d,n}$ exiting to infinity.

2. The map π is a submersion.

   To see this, think of a C^1 family $(Z(F_t))_{|t|<\epsilon} \in X_{d,n}$. Then we claim that for any $p_0 \in Z(F_0)$, we can extend this to a path $(p_t)_{|t|<\epsilon'}$ in $\mathbb{P}^n$, such that $p_t \in Z(F_t)$.

   Choose an affine chart U containing p_0, say $U = \{X_0 \neq 0\}$. Denote $f_t(x_1, \dots, x_n) = F_t(1, x_1, \dots, x_n)$. By smoothness, we may assume that $\partial_{x_1} f_0(p_0) \neq 0$. Define $G(t,x) := f_t(x, p_{0,2}, \dots, p_{0,n})$. Because $\partial_x G(0, p_{0,1}) = \partial_{x_1} f_0(p_0) \neq 0$, we can solve $G(t,x) = 0$ to have $x = x(t)$, with $x(0) = p_{0,1}$, for $|t| < \epsilon'$. Declare $p_t = (x(t), p_{0,2}, \dots, p_{0,n})$ on the affine chart, so that $f_t(p_t) = G(t, x(t)) = 0$.

3. (Ehresmann Theorem) tells that any smooth proper submersion is a fiber bundle. So $\pi \colon U_{d,n} \to X_{d,n}$ is a fiber bundle.

4. We have $X_{d,n}$ path connected.

   This is essentially because its complement, $\Sigma_{d,n}$, has real codimension 2 (cf. complex codimension 1). Although $\Sigma_{d,n}$ itself is not a manifold, as singularities of $\Sigma_{d,n}$ has further real codimension ≥ 2, this does not matter while deforming a path to a 'safe position.'

Combining the Ehresmann Theorem, together with path-connectedness, we see that every fiber of $U_{d,n} \to X_{d,n}$ is diffeomorphic. □

3.3 27 Lines on a Smooth Cubic Surface

Let $S = Z(F) \subset \mathbb{P}^3$, with $\deg F = 3$, be a smooth cubic surface. So S is a closed complex 2-manifold, which is also a smooth projective variety with $\dim_{\mathbb{R}} S = 4$. Their moduli space, $X_{3,3}$, has dimension $\binom{3+3}{3} - 1 = 19$.

A *line* L in $\mathbb{P}^3$ is a linear copy of $\mathbb{P}^1$. One writes $L = Z(F_1, F_2)$ for $F_1, F_2 \in \mathbb{C}[X_0, \dots, X_3]_{(1)}$, and F_1, F_2 linearly independent.

The following is quoted as a 'start of the modern algebraic geometry.'

Theorem 3.8 (Cayley–Salman)**.** *Any smooth cubic surface S has exactly 27 distinct lines.*

Cayley showed that there is a fixed number k such that all smooth cubic surfaces contain k lines; it is a work of Salman who computed $k = 27$, especially in comparison with the Fermat cubic.

Remark. Finding 27 lines *algorithmically* is still open.

Proof. **Step 1** We first check the theorem for the Fermat cubic. (Exhibiting 27 lines is not hard. Showing that the list is exhaustive is precisely the question of finding $v, w \in S$, $v = [V_0 : V_1 : V_2 : V_3]$, etc. such that $\sum_{i=0}^3 V_i^2 W_i = \sum_{i=0}^3 V_i W_i^2 = 0$, which can be done via standard computational algebraic geometry means.)

Step 2 (The Parameter Space) Set $X_{3,3} = \mathbb{P}^{19} \setminus \Sigma_{3,3}$. We have seen that this is path connected.

Step 3 (The Incidence Variety) Recall that the moduli of lines in space, $\{\mathbb{P}^1 \hookrightarrow \mathbb{P}^3\} = \mathrm{Gr}_{\mathbb{C}}(2,4)$, forms a smooth projective variety (in $\mathbb{P}^5$, by the Plücker embedding).

The *incidence variety* of lines on a smooth cubic surface is

$$M = \{(S, L) : S \in X_{3,3},\ L \subset S \text{ a line}\}.$$

This has natural projection $\pi\colon M \to X_{3,3}$, and $\pi^{-1}(S)$ is precisely the set of all lines in S.

A key fact is that, one can compute $D\pi$ in coordinates, which turns out to be a linear isomorphism (on each point). So π is a local diffeomorphism. Because π is proper, π is a covering map.

Step 4 (Endgame—"Method of Continuity") By Step 3 and Step 2, we conclude that the size of the fiber $|\pi^{-1}(S)|$ is constant, in $S \in X_{3,3}$. That $|\pi^{-1}(S)| = 27$ is from Step 1.

Step 5 (Last hole) What is proven up to here is that π is a cover on $\pi(M) \subset X_{3,3}$, of degree 27. The image, the set of cubic surfaces $S \in X_{3,3}$ that has a line in it, forms a nonempty clopen subset of $X_{3,3}$, as follows.

- The set $\pi(M) \subset X_{3,3}$ is closed.

  For a cubic homogeneous polynomial $F \in \mathbb{C}[X_0, \ldots, X_3]_{(3)}$, one can expand

  $$F(V + tW) = F(V) + tF^{(1)}(V, W) + t^2 F^{(1)}(V, W) + t^3 F(W),$$

  for some homogeneous polynomial $F^{(1)} \in \mathbb{C}[V_i, W_j]$ in bidegree $(2, 1)$; in fact, $F^{(1)}(V, W) = D_V F(W)$.

  A line $L = \mathbb{P}(\mathbb{C}.\{v, w\})$ is in S if and only if $v, w \in S = Z(F)$ and $F^{(1)}(v, w) = F^{(1)}(w, v) = 0$. Denoting $F = \sum_{|\alpha|=3} c_\alpha X^\alpha$, the resultant $R(c) = \mathrm{Res}(F(V), F^{(1)}(V, W), F^{(1)}(W, V), F(W))$ is a polynomial in $(c_\alpha)_{|\alpha|=3}$'s. Therefore the zero set (evidently closed) $Z(R) \cap X_{3,3} \subset X_{3,3}$ precisely collects the cubic surfaces with a line in it, i.e., equals to $\pi(M)$.

- The set $\pi(M) \subset X_{3,3}$ is open. This is general property of a submersion: Lemma 3.9.

- The set $\pi(M)$ is nonempty thanks to the Fermat cubic.

Hence $\pi(M) = X_{3,3}$ follows. More algebro-geometric proof of this surjectivity may be found in [Reid, UAG, Prop 7.2]. □

Lemma 3.9. *Let $f\colon N \to M$ be a C^1 submersion, i.e., each $p \in N$ has $D_p f\colon T_p N \to T_{f(p)} M$ a surjection. Then any continuous map $\gamma\colon (-\epsilon, \epsilon)^k \to M$ with $\gamma(0) = f(p)$ has a local lift $\Gamma\colon (-\epsilon', \epsilon')^k \to N$ with $\Gamma(0) = p$. That is, $f(\Gamma(t)) = \gamma(t)$ whenever $\|t\| < \epsilon'$.*

Proof. Let $\dim M = m$ and $\dim N = m+\ell$. Because f is a submersion, there is a coordinate (x^m, y^ℓ) near p such that $\partial_x f(p)$ is invertible. Define a function $G\colon U \subset \mathbb{R}^m \to M$ by

$$G(x^m) = f(x^m, y^\ell(p)).$$

Then $D_{x^m(p)}G$ is an invertible map, hence has a local inverse $G^{-1}\colon V \subset M \to U$ such that $G^{-1}(\gamma(0)) = G^{-1}(f(p)) = x^m(p)$. Thus shrinking the domain of γ (to $\gamma^{-1}(V)$), we have a lift $\Gamma(t) = (G^{-1}(\gamma(t)), y^\ell(p))$. □

Corollary 3.10 (Not so trivial). *For $S \in X_{3,3}$, this is diffeomorphically $\mathbb{P}^2$ blown up at 6 points. That is, $S \cong \mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$.*

To see this, play with a birational map $S \dashrightarrow L_1 \times L_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and some blow-ups on that base.

Corollary 3.11. *For $S \in X_{3,3}$, we have $H_2(S;\mathbb{Z}) = \mathbb{Z}^7$ (by Mayer–Vietoris), with the intersection form $[1] \oplus [-1]^{\oplus 6}$.*

Therefore $\mathrm{Mod}(S)$ has a representation

$$\begin{aligned} \rho\colon \mathrm{Mod}(S) &\to \mathrm{Isometry}(H_2(S;\mathbb{Z}), \cdot) = \mathrm{O}(1,6)(\mathbb{Z}), \\ [f] &\mapsto f_* \circlearrowright H_2. \end{aligned}$$

By Freedman (by his Fields-medal theorem), this is surjective; by Quinn, this is injective. So this leads us to think of the orbifold $\mathbb{H}^6/\mathrm{O}(1,6)(\mathbb{Z})$ versus the moduli space $X_{3,3}$ of S.

4 220804-1: 2-manifolds

As the first day of the minicourse, we announce our goal as follows: recover 'Zariski-style algebraic geometry,' focused on the study of algebraic varieties to answer fundamental questions like "write down homology representatives of a K3 surface."

4.1 Overview

Let M^n be a closed oriented manifold. We define its *mapping class group*

$$\begin{aligned} \mathrm{Mod}(M) &= \pi_0(\mathsf{Homeo}^+(M)) \\ &= \mathsf{Homeo}^+(M)/\mathsf{Homeo}^0(M), \end{aligned}$$

where $\mathsf{Homeo}^+(M)$ equips the compact-open topology. There the identity component carries the homeomorphisms f isotopic to the identity, i.e., those homeomorphisms that admits a continuous map $F\colon M \times [0,1] \to M$ such that

- each $F(-,t)$, $t \in [0,1]$, is a homeomorphism,

- $F(-,0) = \mathrm{Id}_M$, and
- $F(-,1) = f$.

Remark. For surfaces, it is well-known that two homeomorphisms are isotopic iff they are homotopic. Because surfaces of finite type are $K(\pi,1)$-spaces, this is equivalent of inducing the same homomorphisms on the fundamental groups.

The study of $\mathrm{Mod}(M)$ is risen from the question, "what are the self-homeomorphisms of M?" And for the state of art today, *our understanding of* $\mathrm{Mod}(M^4)$ *today is in no different status than that of* $\mathrm{Mod}(\Sigma_g)$ *in 1974,* when it waited the contributions of William Thurston and many.

Outline The outline of the whole course will be as follows.

1. Recap of 2-manifold stories.
2. About 4-manifolds.
3. Mapping class groups of 4-manifolds.
4. Case studies.

4.2 The story of 2-manifolds

The primary source of the facts stated below are from [FM12] B. Farb, D. Margalit. *A Primer on Mapping Class Groups*. Princeton University Press. 2012.

Topological or Smooth classification We all know that all closed oriented surfaces are homeomorphic / diffeomorphic to Σ_g, for some $g \geq 0$. We further have the following zoo of surface models:

Topological	Metric	$\widetilde{\Sigma}_g$	Holomorphic 1-form	Set of complex structures
$\Sigma_0 = \mathbb{S}^2$	$K \equiv 1$	$\widehat{\mathbb{C}}(= \mathbb{S}^2)$	$\omega \equiv 0$	Unique
$\Sigma_1 = \mathbb{T}^2$	$K \equiv 0$	$\mathbb{C}$	dz (vanishes nowhere)	∞ many
Σ_g, $g \geq 2$	$K \equiv -1$	$\triangle$	any ω has $2g-2$ zeroes	∞ many

Some quick facts about the surface mapping class groups:

- We have $\mathrm{Mod}(\mathbb{S}^2) = (1)$, $\mathrm{Mod}(\mathbb{T}^2) = \mathrm{SL}_2(\mathbb{Z})$.
- A theorem of Dehn:

Theorem 4.1 (Dehn, 1922)**.** *Let* $g \geq 0$*. Then* $\mathrm{Mod}(\Sigma_g)$ *is generated by a finitely many Dehn twists.*

By a *Dehn twist* we refer to the map (here, $I = [0,1]$ and $S^1 = \{z \in \mathbb{C} : |z| = 1\}$)

$$\begin{aligned} S^1 \times I &\to S^1 \times I \\ (z,t) &\mapsto (z \cdot e^{2\pi \mathrm{i} t}, t). \end{aligned}$$

On surfaces this is determined by the simple closed curve (SCC) α, so that we can define this map on a tubular neighborhood of α. We denote T_α for the Dehn twist constructed in this way.

Tales after 1974 Now we trace a panorama of results about mapping class groups after 1974, the year when Thurston developed the theory of measured foliations on surfaces and used this to prove Nielson–Thurston trichotomy stated below.

4.2.1 Normal Forms

Nielson–Thurston trichotomy For simplicity, let $g \geq 3$.

Theorem 4.2 (Nielson–Thurston)**.** *For each mapping class $\varphi \in \mathrm{Mod}(\Sigma_g)$ there exists $F \in \mathsf{Homeo}^+(\Sigma_g)$ in the class φ, denoted $F \in \varphi$, such that one of the following holds.*

1. *Map F has finite order: $F^d = \mathrm{Id}$.*[i]
2. *Map F is reducible: there exists a finite set $\{\alpha_1, \ldots, \alpha_n\}$ of SCCs such that F shuffles α_i's.*[ii]
3. *Map F is pseudo-Anosov: there exists pairs $(\mathcal{F}_u, \mu_u)$ and $(\mathcal{F}_s, \mu_s)$ of F-invariant transverse measured foliations, on which F acts by multiplications by λ (for $\mathcal{F}_u$) and λ^{-1} (for $\mathcal{F}_s$) for some $\lambda > 1$.*

Thurston Normal Form (See Ivanov's book [Iva02], and also [BLM83].)

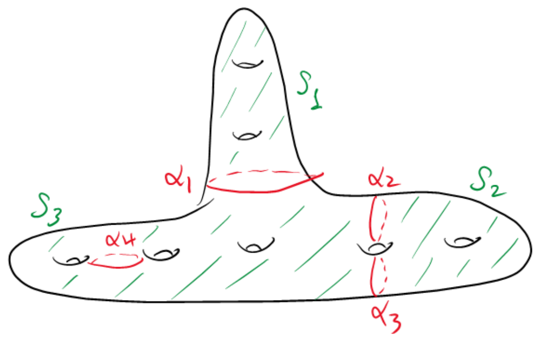
Theorem 4.3. *For all $\varphi \in \mathrm{Mod}(\Sigma_g)$, $\varphi \neq 1$, there exists $N \in \mathbb{Z}_{>0}$*[iii]*, $F \in \varphi^N$, and a finite set of SCCs $\{\alpha_j\}$ (possibly empty) that has the following represention of F.*

---

[i] Technically speaking, that $\varphi^d = 1$ is different from having an actual homeomorphism $F \in \varphi$ of order d. But if we know the isometric action of φ on the Teichmüller space of Σ_g, it is easy to guess there *does* exists an order d representative of φ.

[ii] Typical example of this is the Dehn twist.

[iii] To remove any combinatorial concerns that comes from e.g. rotations



Denote $S_1, \ldots, S_k$ for the component surfaces of $\Sigma_g \setminus \bigcup \alpha_i$. Then we can write[iv]

$$F = \prod T_{\alpha_j}^{n_j} \circ F|_{S_1} \circ F|_{S_2} \circ \ldots \circ F|_{S_k},$$

where each $F|_{S_i}$ is either the identity or a pseudo-Anosov map composed several time.

This result is analogous to the Jordan normal form in the linear algebra. Furthermore, we have the set $\{\alpha_j\}$—called the *canonical reduction system* (CRS)—being canonical, in the sense that conjugations $\mathrm{CRS}(fgf^{-1}) = f.\mathrm{CRS}(g)$ give rise to the natural group action.

To extend the theory of mapping class groups to 4D, we ask:

Question 4.4. Is there any 4-manifold analogues to the trichotomy or the normal form?

4.2.2 Preserving Structures

Theorem 4.5. *Let $g \geq 2$. For each $\varphi \in \mathrm{Mod}(\Sigma_g)$, we have $F \in \varphi$ preserving...*

1. *a hyperbolic metric on Σ_g iff φ has a finite order,*
2. *a complex structure on Σ_g iff φ has a finite order,*
3. *a proper compact submanifold $N \subset \Sigma_g$*[v] *iff φ is reducible.*

4.2.3 Realization Problems

Think of the natural projection $\pi \colon \mathrm{Diff}^+(\Sigma_g) \to \mathrm{Mod}(\Sigma_g)$. A natural question rises:

Question 4.6. Is there a section of π? That is, is there a copy $\sigma \colon \mathrm{Mod}(\Sigma_g) \to \mathrm{Diff}^+(\Sigma_g)$ of $\mathrm{Mod}(\Sigma_g)$ that inverts π in the sense that $\pi\sigma = \mathrm{Id}_{\mathrm{Mod}}$?

The answer is found by Morita [Mor87] as *no*.

Question 4.7. What about a subgroup $G < \mathrm{Mod}(\Sigma_g)$?

[iv] Note that T_{α_j}'s and $F|_{S_i}$'s are commutating, so the order does not matter much here.

[v] This can have a boundary, and not necessarily connected. Think of the annulus preserved under the Dehn twist.

The Kerkhoff–Nielson realization states that this problem is affirmative when $|G| < \infty$. We may further extend the question for $G < \mathrm{Mod}(\Sigma_g)$ with finite index, but this is already implied by Morita [Mor87] to be false.

4.2.4 Best Representation

A pseudo-Anosov map F has *minimal* topological entropy [KH95, §3.1.b] in its homotopy class.

4.2.5 Relations with bundles

Consider a surface bundle

$$\begin{tikzcd} \Sigma_g \arrow[r] & E \arrow[d] \\ & B. \end{tikzcd}$$

This induces the *monodromy representation* $\rho\colon \pi_1(B) \to \mathrm{Mod}(\Sigma_g)$, by stacking infinitesimal deformations of Σ_g going along a loop in B.

Theorem 4.8 (Earle–Eells 1969)**.** *There exists a bijection between the set of surface bundles over B, modulo bundle isomorphism, and the set $\mathrm{Hom}(\pi_1(B), \mathrm{Mod}(\Sigma_g))$ modulo conjugate homomorphisms.*

4.2.6 Relations with Moduli spaces

There are many ways to view the *moduli space* $\mathfrak{M}_g$ ($g \geq 2$) of the surface Σ_g: $\mathfrak{M}_g$ is

- the set of hyperbolic metrics on Σ_g modulo isometries,
- the set of Riemannian metrics on Σ_g modulo conformal diffeomorphisms,
- the set of complex structures on Σ_g modulo biholomorphisms,
- the set of smooth genus g complex curves modulo algebraic isomorphisms,
- the set of singular flat structures of genus g[vi],

etc. Now all these equivalent definitions gives rise to a complex orbifold, whose orbifold fundamental group is $\mathrm{Mod}(\Sigma_g) = \pi_1^{\mathrm{orb}}(\mathfrak{M}_g)$.

This comes from the isometric action $\mathrm{Mod}(\Sigma_g) \to \mathsf{Isom}(\mathrm{Teich}(\Sigma_g))$ on the Teichmüller space of Σ_g. As $\mathrm{Teich}(\Sigma_g) \subset \mathbb{C}^{3g-3}$ is a contractible open subset and $\mathfrak{M}_g = \mathrm{Teich}(\Sigma_g)/\mathrm{Mod}(\Sigma_g)$, the claim follows.

[vi]Keyword: translation surfaces. But this apparently carries more structure, so seems to be more than what $\mathfrak{M}_g$ is.

5 220804-2: 4-manifolds

5.1 A quick overview

Theorem 5.1 (Markov, 1956)**.** *No algorithm exists whose input is a pair of triangulations of a closed 4-manifolds M_1, M_2, and the output is whether M_1 and M_2 are homeomorphic or not.*

Proof. (idea) Every finitely presented group $\Gamma = \langle \alpha_1, \ldots, \alpha_m \mid R_1, \ldots, R_n \rangle$ is $\Gamma = \pi_1(M)$ for some closed orientable 4-manifold M.

Proof. To see this, think of the π_1 of the connected sum, $\pi_1(\#_{j=1}^m (S^1 \times S^3))$. This is the free group of m letters, by van Kampen thoerem.

Pick embedded loops $\alpha_1, \ldots, \alpha_m$ that generates the free group; set $\alpha_i \cap \alpha_j = \{p\}$ (basepoint) whenever $i \neq j$. Denote $N = \#_{j=1}^m (S^1 \times S^3) \setminus \bigcup N(\alpha_i)$, where $N(\alpha_i)$ is a tubular neighborhood of α_i. Then N is a compact manifold whose boundary ∂N is $\bigcup_{i=1}^m \alpha_i \times S^2 (= \bigsqcup_{i=1}^m S^1 \times S^2)$.

For each relation R_i, we glue $S^2 \times D^2$, with boundary $S^2 \times S^1$, via the word given as R_i. By van Kampen we have the π_1 demanded. $\square$

By Markov, there is no algorithm that determines isomorphicity of two finitely presented groups. This finishes. $\square$

Hence we see that there is no convenient way to classify 4-manifolds, unlike what we had seen for surfaces. However, there are two big classes that interests us.

1. Simply connected 4-manifolds: $\pi_1(M) = 0$.
2. "Algebraic surfaces," i.e., smooth complex projective varieties of $\dim_{\mathbb{C}} = 2$.

Furthermore, there are two main topogical invariants for 4-manifolds M:

- the fundamental group $\pi_1(M)$, and
- the intersection lattice $(H_2(M, \mathbb{Z}), Q_M)$.

5.2 The Intersection Form

Recall that, for a closed oriented $2n$-manifold M, we have a bilinear form, called the *intersection form*,

$$\begin{aligned} Q_M \colon H^n(M; \mathbb{Z})_{\text{free}} \times H^n(M; \mathbb{Z})_{\text{free}} &\to \mathbb{Z} \\ (\alpha, \beta) &\mapsto \langle \alpha \smile \beta, [M] \rangle \\ &= \int_M \alpha \wedge \beta. \end{aligned}$$

Here, $H^n(M;\mathbb{Z})_{\text{free}}$ is $H^n(M;\mathbb{Z})$ quotiented out the torsion subgroup; this naturally embeds into the de Rham cohomology $H^n_{\text{dR}}(M)$, thus the notation $\int_M \alpha \wedge \beta$.

The Poincaré dual of this form is described as

$$\begin{aligned} H_n(M;\mathbb{Z})_{\text{free}} \times H_n(M;\mathbb{Z})_{\text{free}} &\to \mathbb{Z} \\ (a,b) &\mapsto \langle \mathrm{PD}(a) \smile \mathrm{PD}(b), [M] \rangle, \end{aligned}$$

which will be also called the *intersection form* on M.

Remark.

1. The form Q_M is a nondegenerate. That is, for each $\alpha \neq 0$ we have β such that $Q_M(\alpha,\beta) \neq 0$.
2. Note that $a \smile b = (-1)^n b \smile a$. So if n is odd, then Q_M is skew-symmetric (i.e., symplectic), but if n is even, Q_M is symmetric.

We will conventionally denote $v^2 = Q_M(v,v)$ and $u \cdot v = Q_M(u,v)$. Furthermore, if n is even, one can represent

$$Q_M(u,v) = uAv^T,$$

where A is a symmetric $n \times n$ $\mathbb{Z}$-valued matrix. If we fix a basis $v_1, \ldots, v_b$ of $H^n(M;\mathbb{Z})_{\text{free}}$, then we write $A = (a_{ij})$ where $a_{ij} = Q_M(v_i, v_j)$.

Q_M as an intersection number One can add a geometric taste to the intersection form Q_M. The following is a geometric preliminary to it.

Proposition 5.2. *Let M be any (oriented) 4-manifold, and $\xi \in H_2(M;\mathbb{Z})$. Then for some $g \geq 0$ we have an embedding $i \colon \Sigma_g \hookrightarrow M$ such that $i_*[\Sigma_g] = \xi$.*

Proof 1. We have an immersion $i \colon \Sigma_g \looparrowright M$ (see [Hatcher]) such that $i_*[\Sigma_g] = \xi$. This has at most finitely many self-intersections, and one can perturb i so that each self-intersections are locally of the form $\{z_1 z_2 = 0\} \subset \mathbb{C}^2$. This can be smoothed to $\{z_1 z_2 = \epsilon\}$; see [Scorpan, Fig 3.2 in p.113].

Each smoothing costs of (1) increasing the genus of the model surface, or (2) admitting a possibility to have a disconnected surface. Thus we have an *embedded*, possibly disconnected surface representation of ξ.

To remedy the disconnectedness, we make surface connected sums through some tubes connecting the components. $\square$

Proof 2. There is a bijection between $H^2(M;\mathbb{Z})$ with $[M, K(\mathbb{Z},2)]$, where $[M, K(\mathbb{Z},2)]$ is the set of homotopy classes of the maps $M \to K(\mathbb{Z},2) = \mathbb{CP}^\infty$. Note that $H^\bullet(\mathbb{CP}^\infty;\mathbb{Z}) = \mathbb{Z}[[\mathbb{P}^1]](\cong \mathbb{Z}[x])$. Now given $\xi \in H_2(M;\mathbb{Z})$, construct $\mathrm{PD}(\xi) \in H^2(M;\mathbb{Z})$, realize this as $F_\xi \colon M^4 \to \mathbb{CP}^3$, reduced to $M^4 \to \mathbb{CP}^2$, and we can perturb F_ξ so that this $\pitchfork \mathbb{P}^1$. Thus $F_\xi^{-1}(\mathbb{P}^1)$ is the demanded embedded 2-manifold representing ξ. $\square$

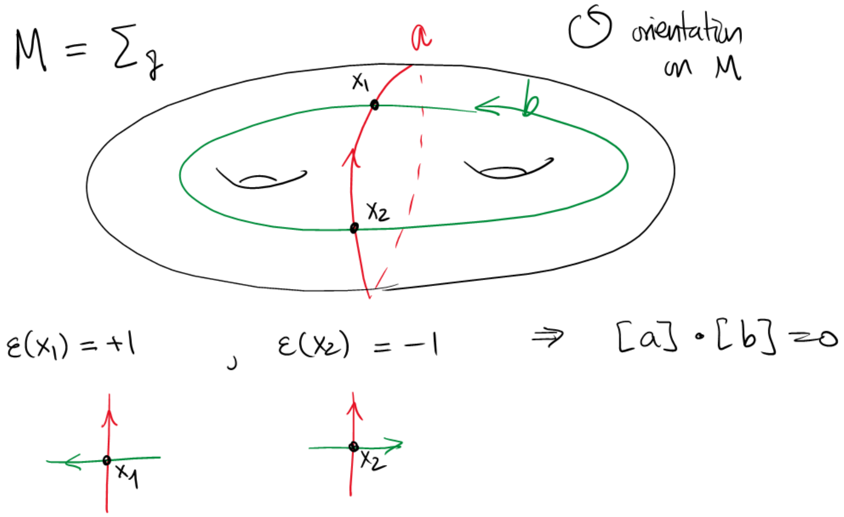
Now we are ready to define the *algebraic intersection numbers*.

Definition 5.3. Let $A^n, B^n \subset M^{2n}$ be closed embedded submanifolds, $A \pitchfork B$. The *algebraic intersection number* between A and B is defined as

$$[A] \cdot [B] := \sum_{x \in A \cap B} \epsilon(x),$$

where $\epsilon(x) = \pm 1$ according to the orientation of $T_x A \oplus T_x B$ within $T_x M$.

Example 5.4 ($n = 1$)**.**



Note that $[a] \in H_1(M; \mathbb{Z})$ is zero. So $[a] \cdot [b]$ in the intersection form is zero *a priori*.

Theorem 5.5 (Fundamental Theorem of the Intersection Theory)**.** *The intersection form measures the algebraic intersection number. That is,* $Q_M([A], [B]) = [A] \cdot [B]$.

More on 4D intersection forms Let $\dim M = 4$, and M be closed and oriented. Denote $d = b_2(M)$ be the 2nd Betti number, i.e.,

$$H_2(M; \mathbb{Z})_{\text{free}} = H^2(M; \mathbb{Z})_{\text{free}} = \mathbb{Z}^d.$$

As discussed earlier, we have Q_M a symmetric form.

Definition 5.6. A *lattice* Λ is a free abelian group $\mathbb{Z}^d$, $d \geq 0$, equipped with a nondegenerate symmetric bilinear form Q_Λ.

For two lattices Λ, Γ to be *isometric*, this means we have an isomorphism $f \colon \Lambda \xrightarrow{\sim} \Gamma$ of free abelian groups such that $f^* Q_\Gamma = Q_\Lambda$, i.e., $Q_\Gamma(f(u), f(v)) = Q_\Lambda(u, v)$ holds for all $u, v \in \Lambda$.

We say the *rank* of a lattice Λ as its Betti number, d. We say a lattice is *unimodular* if one has a basis $v_1, \ldots, v_d$ of Λ such that the matrix $A = (Q_\Lambda(v_i, v_j))$ has determinant 1.

Proposition 5.7. *For closed and oriented M^4, let $H_M = (H_2(M;\mathbb{Z})_{\text{free}}, Q_M)$ be the intersection lattice. This is a unimodular lattice.*

Proof. The curried map $\tilde{Q}_M : H_2(M;\mathbb{Z})_{\text{free}} \to \text{Hom}(H_2(M;\mathbb{Z})_{\text{free}}, \mathbb{Z})$ is same as the Poincaré dual map, thus an isomorphism. $\square$

Example 5.8. First examples of intersection lattices:

1. Let $M = \mathbb{S}^4$. Then $H_M \equiv 0$.

2. Let $M = \mathbb{CP}^2$. Then $H_M \equiv (\mathbb{Z}, (1))$, i.e., we have $Q_M(1,1) = 1$.

   Proof. We know that $H_2(M;\mathbb{Z})$ is generated by a line $[L]$. By Bezout's theorem, we have $[L] \cdot [L] = 1$. $\square$

3. Let $M = S^2 \times S^2$. Then $H_M = (\mathbb{Z}^2, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right])$.

   Proof. We know that $H_2(M;\mathbb{Z})$ is generated by $\xi_1 = [S^2 \times \{*\}]$ and $\xi_2 = [\{*\} \times S^2]$. Now $\xi_1 \cdot \xi_1 = 0$ because one can set the point $*$ in different places and make the representations disjoint; likewise, $\xi_2 \cdot \xi_2 = 0$. We have $\xi_1 \cdot \xi_2 = 1$ by looking at the only obvious intersection point. $\square$

4. Let $M = M_1 \# M_2$ (connected sum). Then $Q_M = Q_{M_1} \oplus Q_{M_2}$; that is, the intersection lattices H_{M_1} and H_{M_2} orthogonally join.

5.3 Integral Quadratic Forms

We study the isomorphism invariants of lattices (Λ, Q_Λ) here.

- The rank $\text{rank}(\Lambda)$.
- The signature $\sigma(Q_\Lambda)$.

  Given a matrix representation A of Q_Λ,[vii] one can diagonalize A over the reals and list $\lambda_1 \geq \ldots \geq \lambda_d$. If p is the number of positive eigenvalues, then $\sigma(Q_\Lambda) := p - (n-p) = 2p - n$.
- Definiteness of Q_Λ.

  That is, asking whether Q_Λ is a positive definite, negative definite, or indefinite. This is equivalent of asking whether $\sigma(Q_\Lambda) = n$, $= -n$, or $\neq \pm n$ respectively.

[vii] That is, the matrix $A = (Q_\Lambda(v_i, v_j))$ for a basis $v_1, \ldots, v_d$ of Λ

- Parity of Λ.

  We say Λ is *even* if $Q_\Lambda(v,v) \in \mathbb{Z}$ is even for all $v \in \Lambda$; *odd* otherwise. (**Exercise**: it suffices to check this whether $Q_\Lambda(v_i, v_i)$ is even for each basis vector v_i.)[viii]

Such isomorphism invariants, a number-theoretic invariant, restrict what kind of lattices can be possible. This gives a beautiful restriction of topological invariants, the intersection lattice, by the number theory.

Example 5.9.
1. Let $\Lambda = \mathbb{Z}^d$ and Q_Λ is given by the matrix I_d, the $d \times d$ identity matrix. This gives a lattice (Λ, Q_Λ), denoted $d(1)$, which has rank d, is positive definite, and is odd.

   The negative counterpart, denoted $d(-1)$, is a rank d *negative* definite and odd lattice.

2. For $Q = p(1) \oplus q(-1)$, $p, q \geq 1$, this is a rank $p+q$, signature $p-q$, and odd indefinite lattice.

3. (The hyperbolic lattice) Denote $U = (\mathbb{Z}^2, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right])$. That is, $\Lambda_U = \mathbb{Z}.\{e_1, e_2\}$ has $e_1^2 = e_2^2 = 0$ and $e_1 \cdot e_2 = e_2 \cdot e_1 = 1$. This is a rank 2, signature 0 (thus indefinite), even lattice.

   (This U is diagonalizable over $\mathbb{R}$. Think of the index 4 sublattice $\Lambda' = \mathbb{Z}.\{e_1 - e_2, e_1 + e_2\}$. Then by manual computation, Q_U restricted to Λ' has the matrix $\mathsf{diag}(-2, 2)$. This indirectly gives unimodularity, as $[\Lambda_U : \Lambda'] = \pm \det(Q_U|\Lambda')$.)

4. (The E_8 lattice) Let $\{e_1, \ldots, e_8\}$ be the standard basis of $\mathbb{R}^8$, equipped with the standard inner product. Set

   $$\Lambda_{E_8} := \mathbb{Z}.\{e_2 - e_3, e_3 - e_4, \ldots, e_7 - e_8, e_7 + e_8, \frac{1}{2}(e_1 + e_8 - e_2 - \cdots - e_7)\}.$$

   Set $Q = Q_{E_8}$ for the standard inner product restricted to Λ_{E_8}. Check: (i) Q is positive definite, (ii) Q is integral (i.e., $Q_{E_8}(\Lambda_{E_8}, \Lambda_{E_8}) \subset \mathbb{Z}$), and (iii) Q is even, and (iv) Q is unimodular.

   If explicitly written as a matrix, Q is represented by

   $$A_Q = \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{bmatrix},$$

   which is associated to the E_8 dynkin diagram with vertex weights 2.

   The lattice (Λ_{E_8}, Q_{E_8}) is named E_8, and its negation is denoted $E_8(-1)$.

[viii] Hint: $(\sum v_i)^2 \equiv \sum v_i^2 \pmod 2$.

Example 5.10 (Fermat quartic, or K3 surface)**.** Let $S = Z(F) \subset \mathbb{P}^3$ be given by a smooth quartic surface, say $S = Z(X_0^4 + X_1^4 + X_2^4 + X_3^4)$.

A non-obvious use of the Lefschetz hyperplane theorem gives $\pi_1(S) = 0$, and some insane characteristic class computations give that $H_S := (H_2(S;\mathbb{Z}), Q_S)$ is isometric to $E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ (rank 22, even, signature $-16 = 3 - 19$).

Now here is a big theorem: classification of indefinite lattices.

Theorem 5.11 (Hasse–Minkowski)**.** *Let (Λ, Q) be a unimodular lattice (i.e., Q is an integral unimodular quadratic form). Suppose Q is* indefinite*. Then (Λ, Q) is isometric to a lattice, desribed below.*

- *If (Λ, Q) is odd, then it is isometric to $a(1) \oplus b(-1)$, $a, b \geq 1$.*
- *If (Λ, Q) is even, then it is isometric to $a\,E_8(\pm 1) \oplus b\,U$, $a \geq 0$ and $b \geq 1$.*

Proof. See Serre. *A Course in Arithmetic*, or Husemoller–Milnor. □

Remark.

1. For positive or negative definite cases, there are only finitely many classes for each fixed rank. But the number of isometric classes grows infinite, as the rank goes infinity. For ranks 8, 16, 24, 40, then the corresponding numbers of isometric classes are 1, 2, 24, $\sim 10^{51}$.
2. One can have $(1) \oplus 9(-1) = E_8(-1) \oplus (1) \oplus (-1)$. This, thus, in a sense breaks the uniqueness. Another example of this sort is $E_8 \oplus E_8(-1) = 8U$.

5.4 Simply connected 4-manifolds

Let M^4 be a closed simply connected manifold. Then by universal coefficient theorem and the Poincaré duality, we have

$$H_i(M;\mathbb{Z}) = \begin{cases} \mathbb{Z} & (i = 0, 4) \\ 0 & (i = 1, 3) \\ \mathbb{Z}^d & (i = 2). \end{cases}$$

(By the universal coefficient theorem, we have $\dfrac{H^2(M;\mathbb{Z})}{\mathrm{Ext}^1(H_1(M;\mathbb{Z}),\mathbb{Z})} = \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}) = H^2(M;\mathbb{Z})_{\text{free}}$. This equals to $H^2(M;\mathbb{Z})$ when $\pi_1 M = 1$. So H^2, and H_2 too by Poincaré duality, is torsion-free.) So we are safe to discuss $H_M = (H_2(M;\mathbb{Z}), Q_M)$ without taking further quotients.

Freedman's Theorem Now here is a cornerstone theorem, that gave the Fields medal to Freedman.

Theorem 5.12 (Freedman 1982)**.** *1. (Existence) For every symmetric unimodular integral bilinear form B there exists a closed (topological) 4-manifold M which is simply connected and $Q_M \cong B$.*

2. (Uniqueness) If B is an even unimodular form, then M is unique. If B is odd, then there are two homeomorphism classes of 4-manifolds, where at most one is smoothable.

Corollary 5.13 (4D Poincaré conjecture)**.** *For a closed orientable 4-manifold M^4, M is homotopy equivalent to S^4 iff M is homeomorphic to S^4.*

The question that whether we can have a diffeomorphism with S^4 is still open.

Proof. The intersection form $Q_{S^4} = 0$. Thus $Q_M \cong Q_{S^4}$ is an even lattice, thus belongs in the unique homeomorphism class. □

Example 5.14. 1. Recall that $Q_{S^2 \times S^2} = U$. This is even (and indefinite), thus has the unique homeomorphism class. (That is, if $\pi_1 M = 1$ and $Q_M = U$, then $M \cong S^2 \times S^2$ by a homoeomorphism.)

2. Let $M = a\mathbb{CP}^2 \# b\overline{\mathbb{CP}^2}$, so that $Q_M = a(1) \oplus b(-1)$. Thus there is a 'non-smooth M.' (If $M = \mathbb{P}^2$, this means we have a 'fake $\mathbb{P}^2$' which is never smooth!)

Corollary 5.15 (Exotic 4-manifold)**.** *There exists a symmetric unimodular integral bilinear form B such that* any *closed simply connected 4-manifold M with $Q_M \cong B$ cannot be a smooth manifold.*

Proof. By (Rochlin 1952), if $\pi_1 M^4 = 1$ and Q_M is even, and if M is *smooth*, we have $16 \mid \sigma(Q_M)$. (Recall that $\sigma(Q_M)$ is the signature.)(If Q_M is even, then we always have $8 \mid \sigma(Q_M)$.)

By (Freedman), there exists a closed, simply connected, topological 4-manifold M^4 such that $Q_M = E_8$, called "E_8 manifold." This has $\sigma(Q_{E_8}) = 8$. *So any 4-manifold homeomorphic to M cannot be smoothed.* Note that our example rises with an *even* lattice B. □

Of course the above construct is not the only one; there are many more examples of this property.

Theorem 5.16 (Donaldson 1983)**.** *Let M^4 be a closed simply connected 4-manifold. Suppose M is smooth.*

1. *If Q_M is definite, then Q_M is either $n(1)$ and $n(-1)$. In particular, a $E_8(\pm 1)$ never appears as a direct summand of Q_M.*

2. *If $Q_M = a\, E_8(\pm 1) \oplus b\, U$, with $a > 0$, we have $b \geq 3$.*

6 220805-1: Example: K3 surfaces

Remark. Almost nothing is known for the mapping class group of K3 surfaces, at least just as what we know about the surface mapping classes. An ad: there is a project ongoing (Farb–Looijenga) for Nielson realization for K3 surfaces.

K3 surfaces does give a vast range of new researches, giving constant challenges for whoever gets interested in it.

Definition 6.1. Let M be a closed complex surface. We say M is a *K3 surface* if it satisfies the followings.

1. M is simply connected: $\pi_1(M) = 0$.
2. M admits a nowhere vanishing holomorphic 2-form ω. (That is, at each point $p \in M$ we have a holomorphic coordinate (z_1, z_2) that lets us to write $\omega = dz_1 \wedge dz_2$.)

Remark. Although (as it will turn out later) there is only one diffeomorphic model for K3 surfaces, distinguishing the possible complex structure is still meaningful in its study. Analogously, it is meaningful to study various complex 1D tori $\mathbb{C}/\mathbb{Z}.\{1, \tau\}$, with $\tau \in \{\Im\tau > 0\}$ varying, as they exhibit various geometries.

At first glance one might feel wondered why we are making such a complicated definition. An analogous candidate for the surface is $X = \mathbb{T}^2$, where X admits a nonvanishing holomorphic 1-form on it.

Theorem 6.2 (Enriques–Kodaira trichotomy)**.** *We have a classification of closed complex surfaces, as follows.*

1. *Rational / Ruled surfaces: those biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$, ... (Another example: nontrivial S^2-bundle over S^2)*[ix]
2. *Complex tori: $\mathbb{C}^2/\Lambda$; K3 surfaces!*[x]
3. *General Type. For example, $\mathbb{CH}^2/\Gamma$.*[xi]

Plus blow-ups of them; the above list is the list of (birationally) minimal surfaces.

Example 6.3. Let M_d be a smooth degree d hypersurface in $\mathbb{P}^3$. That is, $M_d = \{[X_0 : X_1 : X_2 : X_3] : F(X_0, X_1, X_2, X_3) = 0\}$, where $F \in \mathbb{C}[X_0, X_1, X_2, X_3]_{(d)}$, and $\nabla F = 0$, $F = 0$ do not have a common root.

Few facts about M_d's: (i) $M_1 = \mathbb{P}^2$, (ii) $M_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ (exercise), (iii) $M_3 = \mathrm{Bl}_{\{p_1,\ldots,p_6\}}\mathbb{P}^2$, (iv) $M_4 =$ (K3 surface), and (v) M_d, $d \geq 5$, is of general type.

[ix] This correponds to genus 0 surface.
[x] These correspond to genus 1 surface.
[xi] This correponds to genus ≥ 2 surfaces.

Remark. About M_3, we have seen that this carries 27 lines in it. We describe these 27 lines in the viewpoint of $S = \mathrm{Bl}_{\{p_1,\dots,p_6\}}\mathbb{P}^2$. For that, we further assume that $p_1, \dots, p_6$ are in general position, meaning that (a) no two indices collide $p_i \neq p_j$, (b) no three are on a line, and (c) no conic exists to contain all 6 points.

Get the first 6 lines $e_1, \dots, e_6$ by $e_i = \pi^{-1}(p_i)$, where $\pi\colon S \to \mathbb{P}^2$ is the natural blow-down map. We get 15 more lines coming from pairs $\hat{\ell}_{ij}$, the strict transform of ℓ_{ij} the line in $\mathbb{P}^2$ connecting p_i and p_j. (That is, take the Zariski closure of $\pi^{-1}(\ell_{ij} \setminus \{p_i, p_j\})$.)

The last 6 lines are obtained as follows. Denote $C_1, \dots, C_6$ for the conic that passes through $\complement\{p_1\}, \dots, \complement\{p_6\} \subset \{p_1, \dots, p_6\}$ respectively.[xii] The strict transforms of C_i's are the last lines that we have seeked for.

6.1 Topology of K3 surfaces

1. Because M is complex, M is orientable, so $H_0(M;\mathbb{Z}) = H_4(M;\mathbb{Z}) = \mathbb{Z}$. By Poincaré duality, we have Q_M (intersection lattice) unimodular thus.

Furthermore, as $\pi_1 M = 0$, we have $H_1(M) = H_3(M) = 0$.

2. By the Hodge theory, we have the followings about the intersection lattice Q_M:

a. Q_M is even,

b. Q_M is indefinite, and

c. Q_M has signature $(3, 19)$.

Proposition 6.4. *Suppose M is a K3 surface.*[xiii] *Then M has a* unique *holomorphic 2-form, up to scaling.*

Proof. Let ω_1, ω_2 be two holomorphic 2-forms, where $\omega_2 = \omega$ is the holomorphic 2-form required by the definition of K3 surfaces. Then we can locally write

$$\omega_1 = f_1(z_1, z_2)\, dz_1 \wedge dz_2,$$
$$\omega_2 = f_2(z_1, z_2)\, dz_1 \wedge dz_2,$$

where f_1, f_2 are holomorphic functions. Note that f_2 is nowhere vanishing. By this we have the function

$$\frac{\omega_1}{\omega_2}\colon M \to \mathbb{C}$$
$$(z_1, z_2) \mapsto \frac{f_1(z_1, z_2)}{f_2(z_1, z_2)},$$

a well-defined holomorphic function on all of M. Liouville's theorem applies, and we have ω_1/ω_2 constant. ☐

[xii] $\complement A = X \setminus A$ is the 'Bourbaki notation' for the set complement.
[xiii] More generally, M has a nowhere vanishing holomorphic 2-form.

If this proposition translates to a fact in the Hodge decomposition

$$H^2_{\mathrm{dR}}(M) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2},$$

what the proposition says is that $H^{2,0} = H^{0,2} = \mathbb{C}$.

3. Recall the classification of even, unimodular, indefinite quadratic forms: $Q_M = a\,E_8(\pm 1) \oplus b\,U$. As the right hand side has the signature $(b, 8a+b)$ or $(8a+b, b)$, the only way to match this with $(3, 19)$ is to have $Q_M = 2E_8(-1) \oplus 3U$.

6.2 Examples of K3 surfaces

1. Smooth quartic surfaces in $\mathbb{P}^3$. For example, the Fermat quartic $Z(\sum_0^3 X_i^4)$.
2. 2-sheeted Branched covers of $\mathbb{P}^2$, branched over a smooth sextic curve. That is, M in the diagram below is a K3 surface.

$$\begin{tikzcd} M \arrow[d, "2:1"] & \\ \mathbb{P}^2 & C \arrow[l, hook', "\supset"'] \left(\begin{array}{l}\text{branch}\\ \text{locus}\end{array}\right) \end{tikzcd}$$

3. The Kummer surface.

Let $A = \mathbb{C}^2/\Lambda$ be a complex torus, which is also an abelian variety. Think of 16 2-torsion points $A[2] = \frac{1}{2}\Lambda/\Lambda$. Let $\widehat{M} = \mathrm{Bl}_{A[2]}A$.

There is an involution $i\colon A \to A$, $(z_1, z_2) \mapsto (-z_1, -z_2)$, where $\langle i \rangle$ acts on A with $\mathsf{Fix}(i) = A[2]$. One can lift i to $\widehat{M}$ and induce an involution $i\colon \widehat{M} \to \widehat{M}$ as well.

Thus the quotient $M := \widehat{M}/\langle i \rangle = \mathrm{Kum}(A)$ is a smooth complex manifold, although i does not act freely on $\widehat{M}$. (Say, construct a coordinate near each 'dangerous' zone.) There, M turns out to be a K3 surface.

Remark. This construction is in the analogy of the Lattès example in complex dimension 1. That is, by the hyperelliptic involution on a torus $T = \mathbb{C}/\Lambda$, we obtain $\mathbb{S}^2(2,2,2,2)$, dismiss orbifold points, and induce any affine-linear map on T to a map on $\mathbb{S}^2$.

An interesting geometry feature of the Kummer surface is, once we have a point $e_i \in A[2]$, the corresponding line in M has self-intersection -2. That is definitely rare in 4-manifolds!

4. Let $M = X_2 \cap X_3$, where $X_j \subset \mathbb{P}^4$'s are smooth hypersurface of degree j. This is also a K3 surface.

Theorem 6.5 (Kodira 1964; conjectured by A. Weil?)**.** *All K3 surfaces are diffeomorphic.*

Proof is not explicit in the sense that we do not construct an explicit diffeomorphism between K3 surfaces. (Rather, the proof first (1) deforms every K3 surfaces to a quartic surface, and (2) show that all quartic surfaces are diffeomorphic from one another.)

Another way to illuminate this is to refer to Freedman's theorem. As the intersection lattice is even for every K3 surfaces, and they are all isometric from one another, the theorem only admits one homeomorphic class. (A weaker theorem proven in a complicated way!)

In fact, there are 'fake K3 surfaces' that does not admit complex structure on it, but still homeomorphic to any complex K3 surfaces. This gives a subtlety while talking about the moduli of 'homeomorphic K3 surfaces,' but practicaly, they can be ignored.

7 220805-2: Mapping Class Groups of 4-manifolds

7.1 Mapping Class Groups of 3-manifolds

This is the last dimension where $\mathrm{Mod}(M^3)$ defined in topological and differentiable category equals. Key results:

- (Johannson 1979) If M^3 is closed, irreducible, and atoroidal, then $\mathrm{Mod}(M^3)$ is finite.
- If M^3 is Seifert fibered—i.e., is a S^1-bundle over a 2D orbifold Σ—then we have a short exact sequence

  $$1 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Mod}(M) \longrightarrow \mathrm{Mod}(\Sigma) \longrightarrow 1.$$

  (Exception: $M = \mathbb{T}^3$.)
- The group $\mathrm{Mod}(M_1 \# \ldots \# M_r)$ can be interesting; writing $M = \#_{i=1}^r M_i$, this admits a natural homomorphism to $\mathrm{Out}(\pi_1(M))$. In case if each $M_i = S^2 \times S^1$, then $\pi_1(M) = F_r$. This perspective of studying $\mathrm{Out}(F_r)$ is focused by Lei Chen, Bena Tshishiku, etc.

7.2 The Kinds of things known about the Mapping Class Groups of 4-manifolds up until today

Here we set $\mathrm{Mod}(M^4) = \pi_0(\mathsf{Homeo}^+(M))$, i.e., define it in the topological category.

Example 7.1 (First examples)**.** a. If M is a complex surface, then the biholomorphic automorphism group $\mathsf{Aut}(M)$ is usually finite. (Esp. when M is a projective variety of the general type or K3.)

Mukai classified which finite groups can occur as $\mathsf{Aut}(\mathrm{K3})$. On contrary, there are infinite-order complex automorphisms on some K3 surfaces[xiv]; see McMullen, Filip, etc.

[xiv] which is necessarily not algebraic

b. Suppose $M = \mathbb{P}^2$ and $\sigma\colon \mathbb{P}^2 \to \mathbb{P}^2$ given by $[X : Y : Z] \mapsto [-X : Y : Z]$. Note that σ is isotopic to the identity. (Consider $[X : Y : Z] \mapsto [e^{\mathrm{i}t}X : Y : Z]$—by Y. Minsky. Another approach—use the connectivity of $\mathrm{PGL}_3(\mathbb{C})$.)

Let $\tau\colon \mathbb{P}^2 \to \mathbb{P}^2$ be the complex conjugation, i.e., $[X : Y : Z] \mapsto [\overline{X} : \overline{Y} : \overline{Z}]$. This cannot be isotoped to the identity, because $\tau_* = -\mathrm{Id} \circlearrowright H_2(\mathbb{P}^2; \mathbb{Z}) = \mathbb{Z}$. We furthermore note that τ, and $\tau^2 = \mathrm{Id}$, are both orientation-preserving.

Exercise 7.2. Any diffeomorphism $F \circlearrowright \mathbb{P}^2$ is orientation-preserving. (Also true for K3 surfaces!)

Proof. We have $c \in \mathbb{Z} \setminus \{0\}$ such that $F_*[\mathbb{P}^1] = c[\mathbb{P}^1]$. Thus $F_*[\mathbb{P}^2] = F_*[\mathbb{P}^1]^2 = c^2[\mathbb{P}^1]^2 = c^2[\mathbb{P}^2]$, $c^2 > 0$, should preserve the orientation class. □

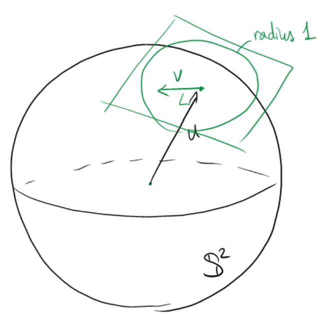
c. Dehn twists about (-2) 2-spheres. That is, by the embedded sphere $i\colon \mathbb{S}^2 \hookrightarrow M$ such that $v = i_*[\mathbb{S}^2]$ has $v^2 = -2$.

To elaborate, we recall the Dehn twists about a SCC $\gamma \subset \Sigma_g$. Find a tube (or 'collar') neighborhood N of γ, view $N = S^1 \times [0,1]$, and map

$$\begin{aligned} S^1 \times [0,1] &\to S^1 \times [0,1] \\ (z,t) &\mapsto (z \cdot e^{2\pi \mathrm{i} t}, t). \end{aligned}$$

We note that this map not only preserves the boundary and the annulus between, it also preserves the circle foliation of N.

Now let $\mathbb{S}^2 \subset M^4$ be an embedded (-2) 2-sphere. We claim that, a tubular neighborhood $N = \mathrm{Nbhd}_M(\mathbb{S}^2)$, viewed as a D^2-bundle over $\mathbb{S}^2$, is isomorphic (as a D^2-bundle) to the unit disk bundle $T\mathbb{S}^2_{\leq 1} = \{(u,v) \in \mathbb{S}^2 \times \mathbb{R}^3 : u \perp v,\ \|v\| \leq 1\}$. Name the isomorphism i.



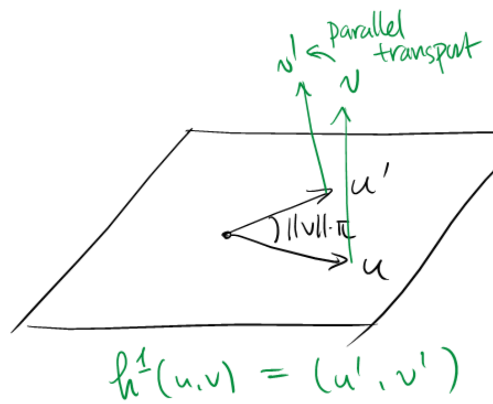
$$i\colon \quad \begin{array}{ccc} D^2 \longrightarrow & N = \mathrm{Nbhd}_M(\mathbb{S}^2) \\ & \downarrow \\ & \mathbb{S}^2 \end{array} \quad \cong \quad \begin{array}{ccc} D^2 \longrightarrow & T\mathbb{S}^2_{\leq 1} \\ & \downarrow \\ & \mathbb{S}^2 \end{array}$$

Exercise 7.3. Prove the claim.

Now construct a map $h' \in \mathrm{Diff}^+(T\mathbb{S}^2_{\leq 1})$, which is identity on the boundary. (After that, extend (by identity) to M^4, and call that the *Dehn twist along* $\mathbb{S}^2$.) The map h' is defined by the time π map of the geodesic flow on $\mathbb{S}^2$:

$$\varphi\colon T\mathbb{S}^2 \times \mathbb{R} \to T\mathbb{S}^2,$$
$$((u,v),t) \mapsto \begin{pmatrix} \text{parallel translate of } v \\ \text{along the unique geodesic} \\ \text{through } u \text{ and tangent to } v \end{pmatrix}.$$

So $h'(u,v) := \varphi((u,v),\pi) = (u',v')$ is pictorially described as follows.



Define $h(u,v) = h'(-u,-v)$. With this gadget, we can clarify what the Dehn twist $T_{\mathbb{S}^2}$ is:

$$T_{\mathbb{S}^2}(m) = \begin{cases} m & (m \notin N) \\ i \circ h \circ i^{-1} & (m \in N) \end{cases}$$

Proposition 7.4. *The Dehn twist h constructed above has the following properties.*

- *The map h is the identity at the unit (co)tangent bundle $T\mathbb{S}^2_{=1}$.*
- *The map h is the antipodal map on the zero section $T\mathbb{S}^2_{=0} = \mathbb{S}^2$.*
- *The map h commutes with the involution $(u,v) \mapsto (-u,-v)$ on $T\mathbb{S}^2_{\leq 1}$.*

Remark. Technically we demand $T\mathbb{S}^2_{\leq 1}$ to be a subbundle of the *cotangent bundle* $T^*\mathbb{S}^2$, to talk about sympletic strucures. Too late for a fix though!

We have $T^2_{\mathbb{S}^2}$ isotopic to the identity. (cf. The nontrivial element of SO(3)!) Furthermore, $T_{\mathbb{S}^2}$ induces a reflection on $H_2(M;\mathbb{Z})$. To elaborate, let $v = [\mathbb{S}^2] \in H_2(M;\mathbb{Z})$. Within $H^2(M;\mathbb{R})$,[xv] $(T_{\mathbb{S}^2})_*$ is perserving $v^\perp$ and flipping $v \mapsto -v$.

Remark. One can make the twist $T_{\mathbb{S}^2}$ to be a symplectomorphism.

More pioneering works can be found under the names C. T. C. Wall, or J. Milnor.

[xv] remove the torsion if necessary

7.2.1 Mapping Class Group, for Simply Connected cases

Theorem 7.5 (Freedman 1982; Quinn 1986)**.** *Let M^4 be a closed oriented simply connected manifold. Then the natural homomorphism*

$$\begin{aligned} \mathrm{Mod}(M) &\to \mathrm{O}(H_M) \\ [f] &\mapsto f_* \circlearrowleft H_2(M;\mathbb{Z}) \end{aligned}$$

is an isomorphism. Here, we include mapping classes that reverses the orientation.

Here, $\mathrm{O}(H_M)$ is the group of isometries $H_2(M;\mathbb{Z}) \to H_2(M;\mathbb{Z})$ that preserves the intersection form.

Recall a result of Borel: if $G = G(\mathbb{R})$ is a semisimple real Lie group, like $\mathrm{O}(p,q)(\mathbb{R})$, and $G(\mathbb{Z}) < G(\mathbb{R})$ is an 'arithmetic group,'[xvi] has a cofinite volume and is discrete.

Corollary 7.6. *The group* $\mathrm{Mod}(M)$ *is arithmetic in* $\mathrm{O}(H_M \otimes \mathbb{R})$*.*

Example 7.7.

1. Consider $Q_{\mathbb{P}^2} = (1)$. Because $\mathbb{P}^2$ is simply connected, we have $\mathrm{Mod}(\mathbb{P}^2) \cong \mathrm{Aut}(\mathbb{Z},(1)) = \mathbb{Z}/2\mathbb{Z}$. This is generated by τ the complex conjugation map.

   (But what about $\pi_0(\mathrm{Diff}^+(\mathbb{P}^2))$?—M. Klug)

2. Consider $M = \mathbb{S}^2 \times \mathbb{S}^2$ has $H_M = U$. Its isometry group is $\mathrm{O}(U) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the Klein group.

   Exercise 7.8. Find a diffeomorphism in each mapping class.

3. Let $M = a\,\mathbb{P}^2 \# b\,\overline{\mathbb{P}^2}$. Then $\mathrm{Mod}(M) = \mathrm{O}(H_M) = \mathrm{O}(a,b)(\mathbb{Z})$.

4. For a K3 surface M, $\mathrm{Mod}(M) = \mathrm{O}(2E_8(-1) \oplus 3U) \subset \mathrm{O}(3,19)(\mathbb{R})$.

   (By the way, any trace of E_8 manifolds within a K3 surface?[xvii])

Remark. It seems like no progress past the Freedman–Quinn isomorphism is made on the study of topological mapping class groups in 4-manifolds.

7.2.2 Smooth versus Topological

Now we compare the $\mathrm{Mod}(M)$ defined in differential and topological categories.

Theorem 7.9 (Ruberman, 1990s)**.** *Let $M = a\,\mathbb{P}^2 \# b\,\overline{\mathbb{P}^2}$, where $a = 2n$ and $b = 10n+1$, $n > 1$ (for instance, $4\mathbb{P}^2 \# 21\overline{\mathbb{P}^2}$ for $n = 2$). Then the kernel*

$$\ker(\pi_0(\mathrm{Diff}^+(M)) \to \pi_0(\mathsf{Homeo}^+(M)))$$

is nontrivial, and in fact infinitely generated.

This is the only example known of the property, i.e., having a 'big' kernel.

[xvi] **Theorem** (Margulis)**.** If $G(\mathbb{Z}) < G(\mathbb{R})$ is a lattice, the commensurator subgroup of $G(\mathbb{Z})$ in $G(\mathbb{R})$ is dense iff $G(\mathbb{Z}) < G(\mathbb{R})$ is arithmetic.

[xvii] ... digging out a fossil?

7.2.3 Symplectic versus Smooth

Let (M, ω) be a symplectic 4-manifold.

Theorem 7.10 (Siedel, 1997)**.** *For certain special M, and $S \subset M$ a Lagrangian (-2) 2-sphere, let T_S denote the Dehn twist along S. Then even though $T_S^2 = 1 \in \pi_0(\mathrm{Diff}^+(M))$, the map T_S^2 has infinite order in $\pi_0(\mathrm{Symp}^+(M, \omega))$.*

Remark. We do not even know what $\mathrm{Diff}_c(\mathbb{R}^4)$ (compactly supported diffeomorphism group) is, i.e., the 'local' mapping classes.

7.3 What are left to do for Mapping Class Groups of simply connected 4-manifolds

As proposed in the "Farb–Looijenga program."

Topological Problems

Problem 1 (The Realization Problem)**.** *For all matrices $A \in \mathrm{O}(H_M)$, give an explicit diffeomorphism (or homeomorphism) $f : M \to M$ such that $f_* = A$.*

Here, there is no guarantee that a smooth representative exists. So the problem essentially throws in that too. Whether we have an topological mapping class that cannot be smoothed is not (yet) transparent.

The problem is find a method (algorithm?) whose input is the action of $H_2(M; \mathbb{Z})$, and the output is an action on M.

We note that Freedman already proved surjectivity (the injectivity is attributed to Quinn), but in a very non-explicit way. His classification theorem, some abstract machinery, (h-)cobordism theory, ...

Problem 2 (Thurston-type Normal Form)**.**

The essence is to go beyond "everything is a product of Dehn twists."

Problem 3 (Nielson Realization Problem)**.** *Also to be called section problems. For which subgroups $G < \mathrm{Mod}(M)$, do we have a section $G \to \mathrm{Diff}(M)$ of $\pi : \mathrm{Diff}(M) \to \mathrm{Mod}(M)$?*

Geometric—Differential or Algebraic

Problem 4 (Preserving Structures)**.** *Characterize which $A \in \mathrm{O}(H_M)$ has a representative $f \in \mathrm{Diff}(M)$ (i.e., $f_* = A$) that preserves some structures. Examples:*

- *Some complex structures.*
- *Some special (Ricci-flat, Einstein, Kähler–Einstein, ...) metrics.*

- *Some foliations (possibly with singularities).*
- *…more to be introduced later.*

One typical way to view all this is to invoke some fixed point theorem.

Problem 5 (Best Representative)**.** *For each mapping class, what is the representative that minimizes the dynamical entropy?*

8 220805-3: Case Study: the Mapping Class Group of the Blow-ups (of a plane)

8.1 (Complex) Blowups

Consider $\mathbb{C}^2$ and $x \in \mathbb{C}^2$, $x \neq 0$. Then there exists a unique line ℓ passing the origin and x. Then we define $B = \mathrm{Bl}_0\mathbb{C}^2$ (the *blow-up of the plane at a point*) as

$$B = \{(x, \ell) \in \mathbb{C}^2 \times \mathbb{P}(\mathbb{C}^2) : x \in \ell\} \subset \mathbb{C}^2 \times \mathbb{P}^1.$$

Call the *blow-down map* $\pi\colon B \to \mathbb{C}^2$, $(x, \ell) \mapsto x$.

First to mention is that B is a 2-dimensional complex manifold. Next to mention is that we have a dictionary for the fibers: for $x \in \mathbb{C}^2$,

$$\pi^{-1}(x) = \begin{cases} x & (x \neq 0), \\ \mathbb{CP}^1 & (x = 0). \end{cases}$$

The exceptional fiber, $e = \pi^{-1}(0)$, is called the *exceptional divisor* in B. The restriction $\pi|_{B \setminus \pi^{-1}(0)}\colon B \setminus \pi^{-1}(0) \to \mathbb{C}^2 \setminus \{0\}$ is a biholomorphism.

Мы же видели, что при надлежащем выборе L в левой части получается любая точка из $\xi \times \mathbf{P}^{n-1}$.

При $n = 2$ можно наглядно представить себе отображение $\sigma\colon \Pi \to \mathbf{P}^2$ и его действие на прямые L: кривые

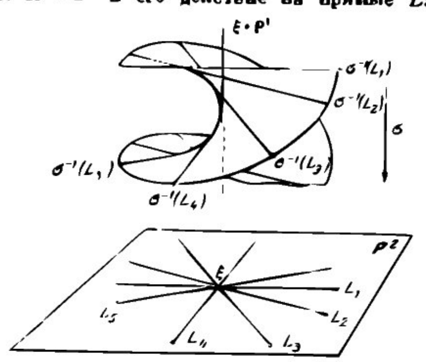
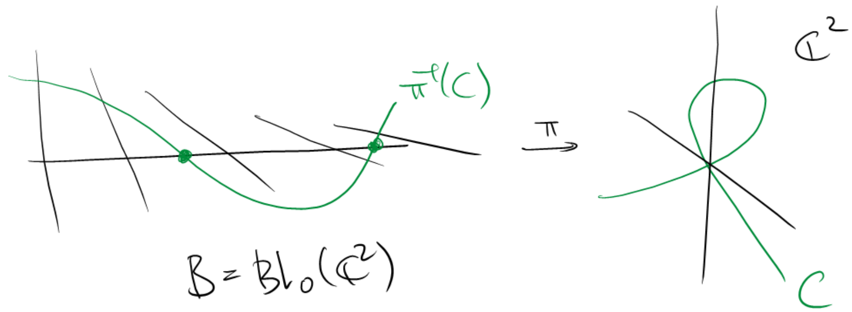


Рис. 10.

$\sigma^{-1}(L)$ пересекают прямую $\xi \times \mathbf{P}^1$ в точках, которые [illegible] по мере того, как L поворачивается в $\mathbf{P}^2$ вокруг ξ. Таким образом, Π выглядит похоже на один виток винта (рис. 10).

(A classical picture of a blow-up by I. Shafarevich)

Suppose we have a nodal curve $C \subset \mathbb{C}^2$, lifted to the blow-up $\pi^{-1}(C) \subset B$. Then (removing e and taking the closure) we get a *smooth* curve that only crosses e at two points (that corresponds to the nodal directions). So blowing up is a way to remedy singularities.



General setting Let M be a complex surface and $p \in M$ be any point. Pick a coordinate neighborhood (U, z) centered at p, so that $z(p) = 0$ and $z(U) \subset \mathbb{C}^2$. Let

$$U' = \{(x, \ell) \in U \times \mathbb{P}^1 : x \in \ell\}$$

and let $\pi\colon U' \to U$ be the first projection. Then $\pi | U' \backslash \{p\} \times \mathbb{P}^1 \to U \backslash \{p\}$ is a biholomorphism.

Replacing U to U' we have the *blow-up of M at p* by

$$\mathrm{Bl}_p(M) = (M \setminus U) \cup_{\partial U = \partial U'} U',$$

and define $\pi\colon \mathrm{Bl}_p(M) \to M$ extending $\pi\colon U' \to U$. Quick properties:

- We have $\mathrm{Bl}_p(M)$ a complex manifold.
- If M is smooth projective variety, then so is $\mathrm{Bl}_p(M)$.
- We have the *exceptional divisor* $\pi^{-1}(p) \cong \mathbb{P}^1 \subset \mathrm{Bl}_p(M)$.
- As a complex manifold, $\mathrm{Bl}_p(M)$ does not depend on the choice of neighborhood U of p. (See [Griffith–Harris].)

Proposition 8.1. *Let M be a complex surface, $p \in M$. Let $M' = \mathrm{Bl}_p(M)$. Then as a smooth manifold, we have the followings.*

1. *We have M' diffeomorphic to $M \# \overline{\mathbb{CP}^2}$.*
2. *We have $Q_{M'} = Q_M \oplus (-1)$.*

The (-1) summand is generated by the exceptional divisor. That is, if $e = \pi^{-1}(p)$ is the exeptional divisor, we have $e^2 = -1$ (cf. a note in the course webpage).[xviii]

[xviii] Essentially, we use the fact that $[e]$ (current of integration, viewed as a de Rham class) is cohomologous to $-\omega_{FS}$ (the Fubini–Study metric on $e = \mathbb{P}^1$), and evaluate $e^2 = \int_e (-\omega_{FS}) = -\mathsf{vol}(e) = -1$. Another way to view $-\omega_{FS}$ is think this as the Chern class of the tautological line bundle over $e = \mathbb{P}^1$.

Example 8.2. Let $M = \mathrm{Bl}_{\{p_1,p_2\}}(\mathbb{P}^2)$. Denote $e_i = \pi^{-1}(p_i)$, so that we have $e_1^2 = e_2^2 = -1$. Then we have $[e_1] - [e_2] \in H_2(M;\mathbb{Z})(= \mathbb{Z}^3)$. Then one evaluates

$$(e_1 - e_2) \cdot (e_1 - e_2) = e_1^2 + e_2^2 = -2,$$

and it is not very hard to see that $[e_1] - [e_2]$ is represented by a sphere $[\mathbb{S}^2]$,[xix] and this $\mathbb{S}^2$ is thus a (-2) 2-sphere.

9 220806: (Seraphina Lee) Mapping Class Groups of del Pezzo Manifolds

We will focus on the topology and mapping class groups of the del Pezzo manifolds; known, unknown, and open results.

Definition 9.1. A *del Pezzo manifold* M is either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathrm{Bl}_n\mathbb{P}^2$, where $0 \leq n \leq 8$.

Remark.

1. As projective varieties, the list is precisely those smooth projective surfaces whose anticanonical bundle is ample. (Provided that the blowing up points are in general position, if exist any.)
2. Topological behaviors change once we hit $n = 9$. (We will come back to this threshold later.)

9.1 As a Smooth Manifold

Lemma 9.2. *There is a diffeomorphism* $\mathrm{Bl}_n\mathbb{P}^2 \cong \mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$.

Proof. (sketch) We know that exceptional divisors E has $E \cdot E = -1$. Taking a normal neighborhood of each E, we have $\overline{\mathbb{CP}^2} \setminus *$ diffeomorphically.

This concludes us that, locally, blowup constructions are removing a neighborhood of a point and glue $\overline{\mathbb{CP}^2} \setminus *$ at there. □

Definition 9.3. We denote $M_n := \mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$.

9.2 Algebraic Topology on del Pezzo manifolds

First to note is that $\pi_1(M_n) = 0$. This is done by the van Kampen theorem; apply $M_n = (M_{n-1} \setminus *) \cup (\overline{\mathbb{CP}^2} \setminus *)$ inductively.

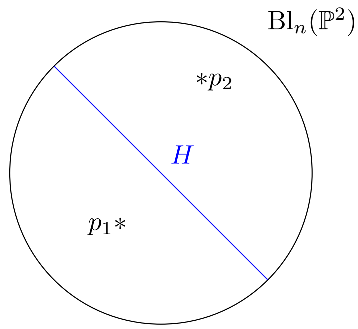
Second to note is that $H_2(M_n) \cong H_2(\mathbb{CP}^2) \oplus H_2(\overline{\mathbb{CP}^2})^{\oplus n}$, which comes from the Mayer–Vietoris sequence. We denote $H_2(\mathbb{CP}^2) = \mathbb{Z}.\{H\}$ and $H_2(\overline{\mathbb{CP}^2})^{\oplus n} = \mathbb{Z}.\{E_1, \ldots, E_n\}$ for the generators.

[xix] Make a connected sum between spheres e_1, e_2, by a tube placed along a curve connecting $p_1 \sim p_2$.

Derived from this is the description of the intersection form:

$$\begin{aligned} Q_{M_n} &= Q_{\mathbb{CP}^2} \oplus nQ_{\overline{\mathbb{CP}^2}} \\ &= (1) \oplus n(-1) = \mathsf{diag}(1, \underbrace{-1, \dots, -1}_{n}). \end{aligned}$$

A schematic picture of M_n:



There we 'by picture' know that $H \cdot E_k = 0$ for any k, and $E_j \cdot E_k = 0$ whenever $i \neq j$.

9.3 Mapping Class Group

By Freedman–Quinn isomorphism, as $\pi_1(M_n) = 0$, we know that

$$\mathrm{Mod}(M_n) \cong \mathrm{Aut}(H_2(M_n), Q_{M_n}) = \mathrm{O}(1, n)(\mathbb{Z}).$$

To elaborate, we recall that

$$\mathrm{O}(1, n)(\mathbb{Z}) = \left\{ A \in \mathrm{GL}_{n+1}(\mathbb{Z}) : A \begin{bmatrix} 1 & 0 \\ 0 & -I_n \end{bmatrix} A^T = \begin{bmatrix} 1 & 0 \\ 0 & -I_n \end{bmatrix} \right\}.$$

Example 9.4. Some elements of $\mathrm{O}(1, n)(\mathbb{Z})$ may be described as follows. Take any $v \in H_2(M_n; \mathbb{Z})$ such that $v^2 = \pm 1$ or ± 2.[xx] Define the reflection

$$\mathsf{Ref}_v(w) = w - \frac{2(w \cdot v)}{v \cdot v} v.$$

This map Ref_v is in $\mathrm{O}(1, n)(\mathbb{Z})$, and has order 2. In case if $v^2 = -2$, we can reduce to

$$\mathsf{Ref}_v(w) = w + (w \cdot v)v.$$

Example 9.5. Now think of $v = E_1 - E_2 \in H_2(M_n)$. Then we have a linear map by the reflection $\mathsf{Ref}_v \circlearrowright H_2(M_n)$. There we have an invariant space decomposition:

$$H_2(M_n) = \mathbb{Z}.\{H, E_3, \dots, E_n\} \oplus \mathbb{Z}.\{E_1, E_2\},$$

where the first summand collects elements that are $\perp v$, and thus Ref_v acts as the identity. For the second summand, we have $\mathsf{Ref}_v(E_1) = E_2$ and vice versa.

We had talked that $E_1 - E_2$ is represented by a 2-sphere $\mathbb{S}^2$ in M_n. In fact, Ref_v is the homology action of the Dehn twist of that sphere.

[xx]For higher self-intersections, we have no guarantee that Ref_v sends integral classes to integral classes.

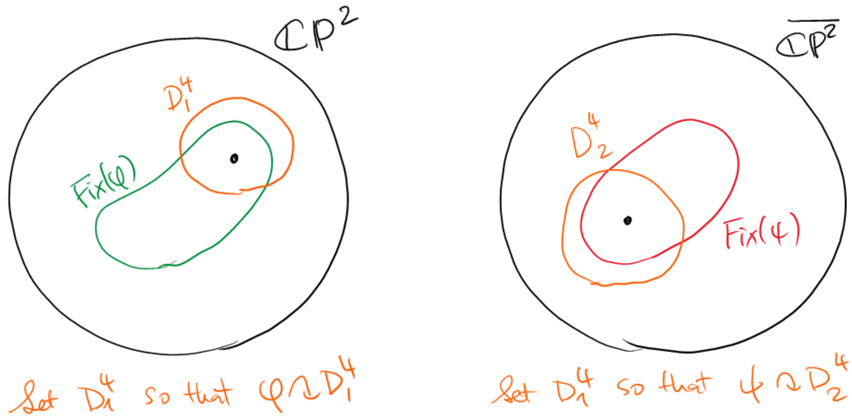
9.4 Diffeomorphisms of del Pezzos

Example 9.6 (Linear diffeomorphisms on M_0)**.** Let $g \in \mathsf{Aut}(\mathbb{CP}^2) = \mathrm{PGL}_3(\mathbb{C})$. Then $g_* = 1$ on $H_2(\mathbb{CP}^2)$. But for the complex conjugation $f \circlearrowright \mathbb{CP}^2$, we have $f_* = -1$ on $H_2(\mathbb{CP}^2)$ (as seen earlier).

Example 9.7 (Glueing for M_1)**.** Suppose we glue $\varphi \in \mathrm{Diff}^+(\mathbb{CP}^2)$ and $\psi \in \mathrm{Diff}^+(\overline{\mathbb{CP}^2})$ (to be specified later). Then we have $\Phi \in \mathrm{Diff}^+(M_1 = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2})$ constructed as follows.

Take $\varphi[X:Y:Z] = [-X:Y:Z]$, and $\psi[X:Y:Z] = [\overline{X}:\overline{Y}:\overline{Z}]$. We know how they act on $H_2(\mathbb{P}^2)$ and $H_2(\overline{\mathbb{P}^2})$ respectively, and we further know that φ, ψ have order 2.

Note also that $\mathsf{Fix}(\varphi) = \mathbb{CP}^1$ and $\mathsf{Fix}(\psi) = \mathbb{RP}^2$. They are all 2-dimensional manifolds, and we can take 4-balls D_1^4 and D_2^4 passing through these fixed loci, and each are φ- and ψ-invariant, respectively.



Then we have

$$M_1 = (\mathbb{CP}^2 \setminus D_1) \cup_{\partial D_1 \sim \partial D_2} (\overline{\mathbb{CP}^2} \setminus D_2),$$

where the gluing is done in an orientation reversing way and $\mathbb{Z}/2\mathbb{Z}$-equivariantly. Then $\Phi \in \mathrm{Diff}^+(M_1)$, with this picture, is described as

$$\Phi = \begin{cases} \varphi & (\text{on } \mathbb{CP}^2 \setminus D_1), \\ \psi & (\text{on } \overline{\mathbb{CP}^2} \setminus D_2). \end{cases}$$

Then again as a diffeomorphism, $\Phi^2 = \mathrm{Id}$, and the homology action is $\Phi_* = \varphi_* \oplus \psi_* \circlearrowright H_2(\mathbb{CP}^2) \oplus H_2(\overline{\mathbb{CP}^2})$. Because $\varphi_* = 1$ and $\psi_* = -1$ on H_2's, we conclude that $\Phi_* = \mathsf{Ref}_{E_1}$, where E_1 is the generator of $H_2(\overline{\mathbb{CP}^2})$.

9.5 A Rudiment of Hyperbolic Spaces and its Isometries

One standard model of the hyperbolic space is to use the *hyperboloid model*. This is a subset of $\mathbb{R}^{n+1}$ characterized with the symmetric bilinear form $Q_n = \mathsf{diag}(1, -I_n)$, and define

$$\mathbb{H}^n := \{p = (x, y_1, \ldots, y_n) \in \mathbb{R}^{n+1} : Q_n(p,p) = 1,\ x > 0\}.$$

On $T_p\mathbb{H}^n$, $p \in \mathbb{H}^n$ any, $Q_n|_{T_p\mathbb{H}^n}$ (resp. $-Q_n|_{T_p\mathbb{H}^n}$) defines a negative-definite (resp. positive-definite) form. By this we define a *hyperbolic metric* on $\mathbb{H}^n$, whose isometry group is clear:

$$\mathsf{Isom}^+(\mathbb{H}^n) = \mathrm{O}^+(1,n)(\mathbb{R}),$$

the set of $(n+1) \times (n+1)$ matrices which preserves Q_n and preserves the upper hyperboloid sheet.

By this we obtain

$$\mathrm{O}^+(1,n)(\mathbb{Z}) := \mathrm{O}(1,n)(\mathbb{Z}) \cap \mathrm{O}^+(1,n)(\mathbb{R}),$$

a group that naturally acts on $\mathbb{H}^n$. The group $\mathrm{O}^+(1,n)(\mathbb{Z})$ is an index 2 subgroup of $\mathrm{O}(1,n)(\mathbb{Z}) \cong \mathrm{Mod}(M_n)$.

The reflection map Ref_v acts by a *hyperbolic* reflection across $v^\perp$ on $\mathbb{H}^n \subset \mathbb{R}^{n+1}$, and virtually dominates the group $\mathrm{O}^+(1,n)(\mathbb{Z})$ for small n's:

Theorem 9.8 (Vinberg 1972)**.** *For any $n \leq 17$, the group $\mathrm{O}^+(1,n)(\mathbb{Z})$ has a finite index subgroup generated by hyperbolic reflections.*

The theorem is no longer true for bigger n's.

Classification of Isometries of $\mathbb{H}^n$ is done as follows.

1. Elliptic isometries. This refers to isometries that has a fixed point in $\mathbb{H}^n$ (e.g., finite-order isometries).

2. Hyperbolic isometries. This refers to isometries that acts by translation along an axis, thus fixing 2 points on $\partial\mathbb{H}^n$.

3. Parabolic isometries. This refers to isometries which are neither elliptic nor hyperbolic. This fixes one point on $\partial\mathbb{H}^n$.

Remark (B. Farb)**.** This trichotomy is one of the motivating remarks of the Nielson–Thurston classification, for $n = 2$. This lures us to guess similar case occurs for the del Pezzo manifolds M_n's!

Question 9.9 (open; B. Farb)**.**

- What is the relation between the hyperbolic space $\mathbb{H}^n$ and M_n, other than one designated by Freedman–Quinn?

- What is a space of structures on M_n on which $\mathrm{Mod}(M_n)$ acts, analogous to the isometric action of $\mathrm{Mod}(\Sigma_g)$ in $\mathrm{Teich}(\Sigma_g)$?

9.6 Diffeomorphisms versus Mapping Classes

Denote the natural quotient map $q_n\colon \mathsf{Homeo}^+(M_n) \to \mathrm{Mod}(M_n)$, which is surjective ('by definition'.) We further may ask, whether each mapping class can be realized by a *diffeomorphism.*

Theorem 9.10 (Wall 1964)**.** *For all $0 \leq n \leq 9$, we have $q_n|\mathrm{Diff}^+(M_n)$ surjective.*

Proof. (sketch) For n's in the range, we can generate the mapping classes by reflections:

Theorem 9.11 (Wall 1964)**.** *For $n = 2$, we have $\mathrm{O}^+(1,2)(\mathbb{Z}) = \langle \mathsf{Ref}_{H-E_1-E_2}, \mathsf{Ref}_{E_1-E_2}, \mathsf{Ref}_{E_2}\rangle$.*

For $3 \leq n \leq 9$, we have $\mathrm{O}^+(1,n)(\mathbb{Z}) = \langle \mathsf{Ref}_{H-E_1-E_2-E_3}, \mathsf{Ref}_{E_k-E_{k+1}}, \mathsf{Ref}_{E_n}\rangle$ where $k = 1, 2, \ldots, n-1$.

For $n \geq 3$, one can show that $H - E_1 - E_2 - E_3$[xxi] and $E_k - E_{k+1}$ can be represented by (-2) 2-spheres in M_n. Thus Ref_v's of this sort are given as Dehn twists. The reflection Ref_{E_n} can be realized using the gluing construct above, and can be smoothed to a diffeomorphism. Thus we have lifted all mapping class generators to diffeomorphisms! □

For large n's, the stituation is very different.

Theorem 9.12 (Friedman–Morgan, 1988)**.** [xxii] *For $n \geq 10$, the image $q_n(\mathrm{Diff}^+(M_n)) \subset \mathrm{Mod}(M_n)$ is an infinite-index subgroup.*

Remark. Compare this with the result of Vinberg for $n \leq 17$. This means there are some reflections that cannot be realized as a diffeomorphism!

9.7 Elliptic Diffeomorphisms

Now let us focus on elliptic matrices in $\mathrm{O}^+(1,n)(\mathbb{Z})$, which corresponds to finite-order mapping classes in $\mathrm{Mod}(M_n)$ for $n \leq 8$. Recall the

Problem 6 (Nielson Realization Problem)**.** *Let $G \leq \mathrm{Mod}(M_n)$ be finite. Does there exist a lift of G to $\mathrm{Diff}^+(M_n)$ under the homomorphism $q_n\colon \mathsf{Homeo}^+(M_n) \to \mathrm{Mod}(M_n)$?*

As a simplest case, consider $G = \mathbb{Z}/2\mathbb{Z} \leq \mathrm{Mod}(M_n)$. For smaller n's, it is verified to be affirmative.

Theorem 9.13 (L. 2022)**.** *For $0 \leq n \leq 8$, any mapping class $g \in \mathrm{Mod}(M_n)$ of order 2 is represented by an order 2 diffeomorphism.*

Moreover, we can find a representative of g in one of the three classical involutions (Geiser, Bertini, or de Jonquieres) seen in the birational geometry if we have the followings.

1. *The class g is in $\mathrm{O}^+(1,n)(\mathbb{Z}) \subset \mathrm{Mod}(M_n)$.*

[xxi] First merge $H - E_1 - E_2$ to a sphere, and then merge E_3.
[xxii] Beware the spelling, Friedman vs. Freedman.

2. *The class g does* not *preserve any isometric decomposition $H_2(M_n;\mathbb{Z}) = H_2(M) \oplus H_2(\#k\overline{\mathbb{CP}^2})$, where $k \geq 1$ and M_n is del Pozzo.*

Remark. The involutions mentioned preserve some complex structure on M_n. In fact,

- the Geiser involution is defined on $\mathrm{Bl}_7(\mathbb{CP}^2)$,[xxiii]
- the Bertini involution is defined on $\mathrm{Bl}_8(\mathbb{CP}^2)$, and
- the de Jonquieres involution is defined on $\mathrm{Bl}_{2n+1}(\mathbb{CP}^2)$ $(n \geq 2)$.

Corollary 9.14. *All Dehn twists on M_n with $n \leq 8$ are topologically isotopic to an order 2* diffeomorphism*.*

This result is more like an application of the Freedman–Quinn isomorphism, thus the isotopy is less explicit. The existence of smooth isotopy is not yet known.

Remark. Dehn twists are not isotopic to any diffeomorphism of finite order for the following cases.

1. K3 surfaces (Farb–Looijenga 2021).
2. Spin 4-manifolds (Konno 2022).

The landscape awaits more discovery. For instance, we do not know analogous result for order 3 classes!

9.8 Nonrealizability

We will now suggest a contrary result from above. For instance, we will show that the finite group $G = \langle \mathsf{Ref}_{H-E_1-E_2}, \mathsf{Ref}_{E_1-E_2} \rangle \subset \mathrm{Mod}(M_2)$ has no lift to $\mathrm{Diff}^+(M_2)$.

The reflection generators mentioned above are commutative as mapping classes. Thus the puncline of the nonrealizability in question is that *any lift of the reflections are not commutative.*

We introduce some tools to discuss this.

Theorem 9.15 (Edmonds 1989)**.** *Suppose $G := \mathbb{Z}/p\mathbb{Z} \curvearrowright M^4$, where M is a closed oriented simply connected 4-manifold, and p is a prime < 23. Then as a G-representation, we have*

$$H_2(M) \cong \mathbb{Z}^t \oplus \mathbb{Z}[\zeta_p]^c \oplus \mathbb{Z}[G]^r,$$

where $\mathbb{Z}^t$ is the trivial representation, $\mathbb{Z}[\zeta_p]$ is the ζ_p-representation of G, and $\mathbb{Z}[G]$ is the standard representation.

[xxiii] There are more contexts where Geiser and Bertini involutions are well-defined, perhaps with smaller blow-ups.

Furthermore, if we say $F = \mathsf{Fix}(G) \subset M$, then if $F \neq \varnothing$, we can compute mod p Betti numbers $\beta_k(F) = \dim_{\mathbb{F}_p} H_k(F; \mathbb{F}_p)$ as

$$\beta_1(F) = c, \quad \beta_0(F) + \beta_2(F) = 2 + t.$$

Remark. If a finite group G acts smoothly and orientation preserving on M^4, then $\mathrm{Fix}(G)$ will be the disjoint union of surfaces and points. (This is a general fact about smooth finite-group actions on a manifold. In this context, the punchline is that we are removing circles in the ist.)

Theorem 9.16 (Hirzebruch G-signature Theorem)**.** *Let $G = \mathbb{Z}/p\mathbb{Z} \curvearrowright M^4$ smoothly, where M^4 is a closed oriented 4-manifold. Then we have the identity*

$$p \cdot \sigma(M/G) = \sigma(M) + \sum_{\substack{C \subset \mathsf{Fix}(G) \\ \dim_{\mathbb{R}} C = 2}} \mathsf{def}_C + \sum_{\substack{\{z\} \subset \mathsf{Fix}(G) \\ \dim_{\mathbb{R}} \{z\} = 0}} \mathsf{def}_z,$$

where

- *$\sigma(M/G)$ is the signature of Q_M on $H_2(M)^G$ (the G-fixed homologies),*
- *$\sigma(M)$ is the signature of Q_M,*
- *$\mathsf{def}_C := \dfrac{p^2 - 1}{3}[C]^2$,[xxiv][xxv] and*
- *def_z is a quantity that vanishes when $p = 2$, determined by the action of $G \curvearrowright T_z M$.*

Remark. The theorem is meant to be a formula analogous to $p \cdot \chi(M/G) = \chi(M)$ (Euler characteristics), but with many correction terms.

Now we sketch the nonrealizability with the above gadgets.

Proof. (sketch) Suppose that there exists $f, g \in \mathrm{Diff}^+(M_2)$ so that $[f] = \mathsf{Ref}_{E_1 - E_2}$ and $[g] = \mathsf{Ref}_{H - E_1 - E_2}$, while $\langle f, g \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

By Edmonds's theorem, we have $\mathsf{Fix}(f)$ be a disjoint union of $\mathbb{S}^2$ and a point, or just 3 points. By the G-signature theorem, we restrict $\mathsf{Fix}(f)$ to be a union of $\mathbb{S}^2$ and a point, where $[\mathbb{S}^2]^2 = 1$.

Because $fg = gf$, we have g acting on $\mathsf{Fix}(f) = \mathbb{S}^2 \sqcup *$. By dimension concerns, we see that g leaves $\mathbb{S}^2$ invariant, and furthermore we have $g_*[\mathbb{S}^2] = \pm[\mathbb{S}^2]$.

Now note that $g_* \circlearrowright H_2(M_2)$, that has ± 1 eigenspaces

$$H_2(M_2) = \mathbb{Z}.\{H - E_1 - E_2\} \oplus \mathbb{Z}.\{H - E_1, H - E_2\}.$$

One can check that neither eigenspace contains a class with self intersection 1. But then where do we find $[\mathbb{S}^2]$ such that $[\mathbb{S}^2]^2 = 1$ and $g_*[Ss^2] = \pm[\mathbb{S}^2]$? ☐

[xxiv] def stands for 'defect.'

[xxv] We note that only orientable C appears here.

9.9 Parabolic Diffeomorphisms

Recall that a parabolic hyperbolic isometry fixes a unique point of $\partial\mathbb{H}^n$. In the view of the hyperboloid model, this unique point is represented by a line $\mathbb{R}.v$ where $v \in \mathbb{R}^{n+1}$ has $Q_n(v, v) = 0$.

If we bring this to homology actions, a parabolic mapping class fixes a line spanned by $v \in H_2(M_n; \mathbb{R})$. One can elaborate this in a context more closer to the mapping class group.

- In $\mathrm{O}^+(1, n)(\mathbb{Z})$, a parabolic class fixes some $v \in H_2(M_2; \mathbb{Z})$, $v^2 = 0$.
- In M_n, v is represented by some embedded surface $S \subset M_n$.

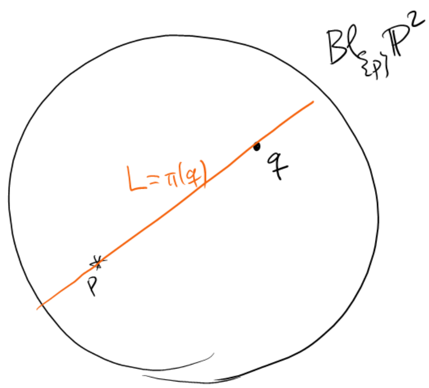
Now ideally, we want a diffeomorphism representing a parabolic mapping class that preserves the surface, not just the class v. (For that purpose, we construct a fiber bundle that will sketch the preservation of the surface better.)

One way in which this class $v \in H_2(M_n; \mathbb{Z})$ arises is the 'conic bundle.' To speak of it, we are going to construct a map (a *pencil*!) $\pi\colon M_n \to \mathbb{CP}^1$ where each fiber $\pi^{-1}(p)$ is homologous to $\mathbb{CP}^1$ (except for finitely many p). The homology class of the generic fiber $\pi^{-1}(p)$ is in fact v, the class that we are interested in.

Example 9.17 $(n = 1)$**.** We set the map

$$\pi\colon M_1 = \mathrm{Bl}_{\{p\}}(\mathbb{CP}^2) \to \mathbb{CP}^1,$$

viewing that $\mathbb{CP}^1$ as the set of lines in $\mathbb{CP}^2$ that passes through p. For $q \in M_1$, we define $\pi(q)$ to be the line passing p and q. (For q in the exceptional divisor, we just load the line from p in that direction.)



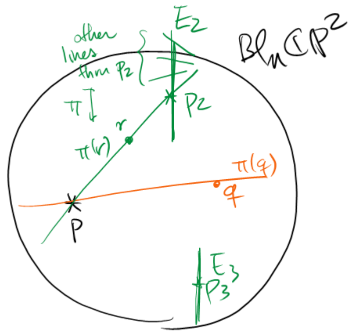
Then we have $\pi^{-1}(L) = L \subset M_1$, for L any line passing p.

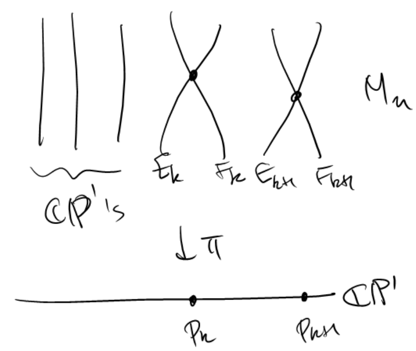
The pencil π turns M_1 into a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$. Topologically, recall the unique nontrivial $\mathbb{S}^2$-bundle over $\mathbb{S}^2$. Furthermore, the class $v = [\pi^{-1}(L)] \in H_2(M_1)$ has $v^2 = 0$.

Example 9.18 ($n \geq 2$)**.** We sketch how to generalize this construct for M_n. View $M_n = \mathrm{Bl}_{n-1}(\mathrm{Bl}_{\{p\}}(\mathbb{CP}^2))$. Then we define a map

$$\pi \colon M_n \to \mathrm{Bl}_{\{p\}}(\mathbb{CP}^2) \to \mathbb{CP}^1,$$

as follows. If $q \in M_n$ is not in one of the $(n-1)$ exceptional divisor, then we let $\pi(q)$ for the line passing p and q. If q lies on the $(n-1)$ exceptional divisors, say $q \in E_k$ (corresponding to p_k), then we simply map to the line $\overline{pp_k}$.





If we write $F_k = \overline{pp_k}$, then we see that $\pi^{-1}(\overline{pp_k}) = E_k \cup F_k (= \mathbb{CP}^1 \cup_* \mathbb{CP}^1)$. This describes all the finite exceptional points where the fiber is not $\mathbb{CP}^1$, and will be called singular fibers.

Now we can describe the parabolic diffeomorphisms. Some auxiliary constructs (applicable for odd n's):

1. There exists a complex automorphism Φ on M_n such that
   - smooth fibers $\mathbb{CP}^1$ of $\pi \colon M_n \to \mathbb{CP}^1$ are restricted to an order 2 diffeomorphism, and
   - singular fibers $\mathbb{CP}^1 \cup_* \mathbb{CP}^1$ of π are swapping the components.
2. There exists a diffeomorphism Ψ on M_n such that, on each normal neighborhood of E_k ($k \geq 2$), we act by Ref_{E_k}, and identity elsewhere.[xxvi]

Then the map $F = \Psi \circ \Phi$ (1) preserves all fibers of π outside of the normal neighborhood of E_k, and (2) has the infinite order action on $H_2(M_n)$. From (1), we see that F_* is the demanded parabolic mapping class fixing v.

9.10 What about the product of lines?

It is known that $\mathrm{Bl}_{\{*\}}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathrm{Bl}_2(\mathbb{CP}^2)$. So after the blow-ups, we see the pictures that we had already seen. (And as $Q_{\mathbb{P}^1 \times \mathbb{P}^1} = U$, the topological considerations are rather trivial before the blow-ups.)

[xxvi] A moment to recall the Dehn twist, except that the spine of the twist is a (-1) 2-sphere.

10 220807-1: Case Study—Rational Elliptic Surfaces

The content of this section is due to a joint work with Eduard Looijenga.

10.1 Setups

Let $M = \mathrm{Bl}_{\{p_1,\dots,p_9\}}(\mathbb{P}^2) \cong \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$[xxvii] be the manifold of our interest, a result of blowing up 9 points of a projective plane. Let $\pi\colon M \to \mathbb{P}^2$ be the natural blow-down map. Denote $e_i = \pi^{-1}(p_i) \cong \mathbb{CP}^1$ for the exceptional divisors.

The intersection form of M is $Q_M = (1) \oplus 9(-1)$. As $\pi_1(M) = 0$, by Freedman–Quinn isomorphism, we have an isomorphism

$$\mathrm{Mod}(M) \xrightarrow{\sim} \mathrm{Aut}(H_2(M;\mathbb{Z}), Q_M) \cong \mathrm{O}(1,9)(\mathbb{Z}).$$

A natural question, asked in Seraphina Lee's lecture yester day, is as follows.

Question 10.1. For each $A \in \mathrm{O}(1,9)(\mathbb{Z})$, do we have a *diffeomorphism* $f \in \mathrm{Diff}^+(M)$ such that $f_* = A$?

10.2 Hyperbolic isometry perpective

Because $\mathrm{O}(1,9)(\mathbb{Z}) \subset \mathrm{O}(1,9)(\mathbb{R}) = \mathsf{Isom}(\mathbb{H}^9)$, we have three cases for $A \in \mathrm{O}(1,9)(\mathbb{Z})$.

1. Matrix A has a finite order.
2. Matrix A is parabolic type (i.e., A fixes a unique point on $\partial\mathbb{H}^9$).[xxviii]
3. Matrix A is of hyperbolic type (translating along a unique geodesic).

Question 10.2 (Open)**.** Explicitly construct f in cases 1 and 3 above.

An easy note is that, if we have a finite-order diffeomorphism, this should preserve a metric. (Take the Birkhoff average.) So that will restrict the candidates for the case 1.

Today, we will focus on explicitly constructing f with $f_* = A$ in case 2.

Some Fact from Hyperbolic Geometry There exists a unique, up to a conjugation in $\mathrm{O}(1,9)(\mathbb{Z})$, maximal parabolic subgroup $\Gamma < \mathrm{O}(1,9)(\mathbb{Z})$, given as the stablizer of a zero-intersection vector $v \in (1) \oplus 9(-1)$ (i.e., $v^2 = 0$ and $\gamma.v = v$ for all $\gamma \in \Gamma$).

Proof. A general fact about hyperbolic finite-volume noncompact manifold $\mathbb{H}^n/\Lambda$ is, it contains finitely many cusps bounded by a flat orbifold $\mathbb{E}^{n-1}/\Delta$ (and a compact piece outside of the cusps).

So we study $\mathbb{H}^9/\Gamma$, and study the number of cusps; should be one. □

[xxvii] This manifold, or manifolds of this sort, is called a *rational elliptic surface.*

[xxviii] Picture to have in mind: horocycles flows, and the 'center' of the horocycles.

Becaues $\Gamma = \mathsf{Stab}_{\mathrm{O}(1,9)(\mathbb{Z})}(v)$, we have the following exact sequence.

$$1 \longrightarrow v^{\perp}/\mathbb{Z}.v \longrightarrow \mathsf{Stab}_{\mathrm{O}(1,9)(\mathbb{Z})}(v) \longrightarrow \mathrm{O}(v^{\perp}/\mathbb{Z}.v) \longrightarrow 1$$

Here, the group $v^{\perp}/\mathbb{Z}.v \cong \mathbb{Z}^8$ (in fact, $E_8(-1)$ as a lattice) represents the sheering map.

Example 10.3. Think of the action $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right] \curvearrowright \mathbb{H}^2$. At each line $y = c$, we observe that the action of this matrix is given by $x \mapsto x + 1$. If we view this in $\mathbb{H}^2/\mathrm{SL}(2,\mathbb{Z})$, then we see that this 'invariant locus' $y = c$ appears as smaller circles (of length $O(e^{-c})$).

Likewise, we consider the action $A = \left[\begin{smallmatrix}1&a+b\mathrm{i}\\0&1\end{smallmatrix}\right] \curvearrowright \mathbb{H}^3 = \{(x^0, x^1, x^2) \in \mathbb{R}^3 : x^0 > 0\}$, with $A \in \mathrm{PSL}_2\mathbb{Z}[\mathrm{i}]$. We have invariant (Euclidean!) planes $x^0 = c$ that (1) A acts as sheerings on each plane, while (2) giving *square* tori in the quotient $\mathbb{H}^3/\mathrm{PSL}_2\mathbb{Z}[\mathrm{i}]$.

The group $\mathrm{O}(v^{\perp}/\mathbb{Z}.v) = \mathrm{O}(0,8)(\mathbb{Z})$ sits in $\mathrm{O}(8)$, a compact group. As a discrete subgroup of a compact group, this is necessarily a finite group! That is same as the Weyl group $W(E_8)$ of the E_8 Dynkin diagram.

Exercise 10.4. Given $A \in \mathsf{Stab}(v)$, *construct* a diffeomorphism $f \in \mathrm{Diff}^+(M)$ with $f_* = A$.

10.3 Classical Construction

Take two (generic) smooth cubic curves $\{P = 0\}$ and $\{Q = 0\}$ in $\mathbb{CP}^2$. (That is, P, Q are cubic homogeneous polynomials here.) Consider the pencil $E_{[s:t]} = \{sP + tQ = 0\}$, where $[s : t] \in \mathbb{P}^1$.

By Bezout, we have the intersection $E_{[1:0]} \cap E_{[0:1]}$ consisting of 9 points $p_1, \dots, p_9$ (counting multiplicities).[xxix] But as $P(p_i) = Q(p_i) = 0$, we see that these points are all in every $E_{[s:t]}$.

Furthermore, $E_{[s:t]}$'s cover $\mathbb{P}^2$: for each point $p \in \mathbb{P}^2$, we have $p \in E_{[-Q(p):P(p)]}$ unless $(\exists i)(p = p_i)$. In fact this readily defines a map

$$\begin{aligned} \pi : \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1, \\ p &\mapsto [-Q(p) : P(p)], \end{aligned}$$

where $\dashrightarrow$ means we do not define π at some points or curves. In our context, this exception set is precisely $\{p_1, \dots, p_9\}$.

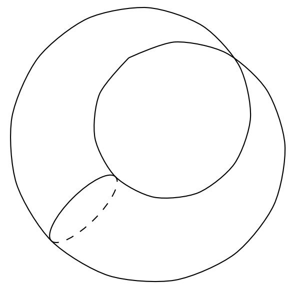
No worries, as the map π can be extended to $M = \mathrm{Bl}_{\{p_1,\dots,p_9\}}\mathbb{P}^2 \to \mathbb{P}^1$ which is globally well-defined! Properties (given that P, Q are generic):

- We have $\pi^{-1}([s : t]) = E_{[s:t]}$ except for 12 values[xxx] of $[s : t] \in \mathbb{P}^1$. (Recall that $E_{[s:t]}$ is a cubic curve and thus diffeomorphic to a torus.)

[xxix] Pick P and Q generic enough so that our list of 9 points are all distinct.

[xxx] Count the number of lines $[s : t]$ such that $E_{[s:t]}$ is a *singular* cubic. That is, solve the resultant $\mathrm{Res}(s\nabla P + t\nabla Q) = 0$; this has degree 12.

- At each 12 singular fibers, $E_{[s:t]}$ is a rational nodal curve, also called a 'fishtail fiber.' This can be pictured as a degenerate torus.



This construction gives us M to be an *elliptic fiberation.*

Remark. The space $\mathfrak{M}_1 = \mathbb{H}^2/\mathrm{PSL}_2\mathbb{Z}$ is the classifying space of the torus bundle. That is, we have a bijection

$$\left\{ \begin{tikzcd} \mathbb{T}^2 \arrow[r] & Y \arrow[d] \\ & B \end{tikzcd} \right\} \xleftrightarrow{\text{bij.}} [B, \mathfrak{M}_1].$$

For the elliptic fibration discussed above, we can find this in $[\mathbb{P}^1, \overline{\mathfrak{M}}_1]$, where $\overline{\mathfrak{M}}_1 = \mathfrak{M}_1 \cup \{\infty\}$.

The fibration $\pi\colon M \to \mathbb{P}^1$ corresponds to a degree 12 map $\mathbb{P}^1 \to \overline{\mathfrak{M}}_1$. (Simply because the preimage of the singular torus $\infty \in \overline{\mathfrak{M}}_1$ has size 12. Or could be found by comparing Euler classes?)

Remark (A secret recipe)**.** One can pullback $M \xrightarrow{\pi} \mathbb{P}^1$ by the map $sq\colon \mathbb{P}^1 \to \mathbb{P}^1$, $[1:z] \mapsto [1:z^2]$, to build a new elliptic fibration $M' \to \mathbb{P}^1$. This M' is a K3 surface.

$$\begin{tikzcd} M' \arrow[r] \arrow[d, "sq^*\pi"] & M \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "z\mapsto z^2"] & \mathbb{P}^1 \end{tikzcd}$$

10.4 Local Monodromy

Now we study the fibration $\pi\colon M \to \mathbb{P}^1$ in terms of the monodromy. To model the behavior of π near the singular fibers, we consider a fibration

$$\begin{tikzcd} Y \arrow[d, "\varpi"] \\ \triangle, \end{tikzcd}$$

where Y is a complex 2-manifold and $\triangle \subset \mathbb{C}$ is the unit disk. Suppose $\varpi^{-1}(z) = \mathbb{T}^2$ for $z \neq 0$ and $\varpi^{-1}(0)$ is a singular torus ('croissant').

Becaues $Y \setminus \varpi^{-1}(0) \xrightarrow{\varpi} \triangle^*$ is a smooth $\mathbb{T}^2$-fiber bundle, we have the natural monodromy

$$\pi_1(\triangle^*) \to \mathrm{Mod}(\mathbb{T}^2).$$

But since all the tori we see are staying near the cusp $\infty \in \overline{\mathfrak{M}}_1$, we get interested in parabolic elements in $\mathrm{SL}_2(\mathbb{Z})$ fixing $\infty \in \partial\mathbb{H}^2$, we have the following claim.

Claim. A local monodromy $\varpi \colon Y \to \triangle$ is conjugate to $\left[\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{SL}_2\mathbb{Z} = \mathrm{Mod}(\mathbb{T}^2)$, for some $N \neq 0$.

Remark (Side note)**.** Recall the family of elliptic curves

$$E_\lambda \colon y^2 = x(x-1)(x-\lambda).$$

The hyperelliptic involution $(x, y) \mapsto (x, -y)$ then folds E_λ to $\widehat{\mathbb{C}}$ 2-to-1, with singular values $0, 1, \infty, \lambda$. We can also think of the family

$$M = \mathsf{closure}_{\mathbb{P}^2 \times (\mathbb{C} \setminus \{0,1\})}(\{([x : y : 1], \lambda) : y^2 = x(x-1)(x-\lambda)\}),$$

which gives a $\mathbb{T}^2$-fiber bundle.

Exercise 10.5. Study the monodromy $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to \mathrm{Mod}(\mathbb{T}^2) = \mathrm{SL}_2\mathbb{Z}$.

(Relevant to this is the Modular lambda function $\lambda \colon \mathbb{H}^2 \to \mathbb{C} \setminus \{0, 1\}$ such that the complex torus $\mathbb{C}/\mathbb{Z}.\{1, \omega\}$ is identified with the elliptic curve $E_{\lambda(\omega)}$. The map λ is invariant under $\Gamma(2) = \{A \in \mathrm{SL}_2(\mathbb{Z}) : A \equiv I_2 \pmod 2\}$ action, and as $\mathbb{H}^2/\Gamma(2) = \mathbb{P}^1 \setminus \{0, 1, \infty\} = \mathbb{C} \setminus \{0, 1\}$ (known, from study of triangle groups), the function λ is a universal cover.)

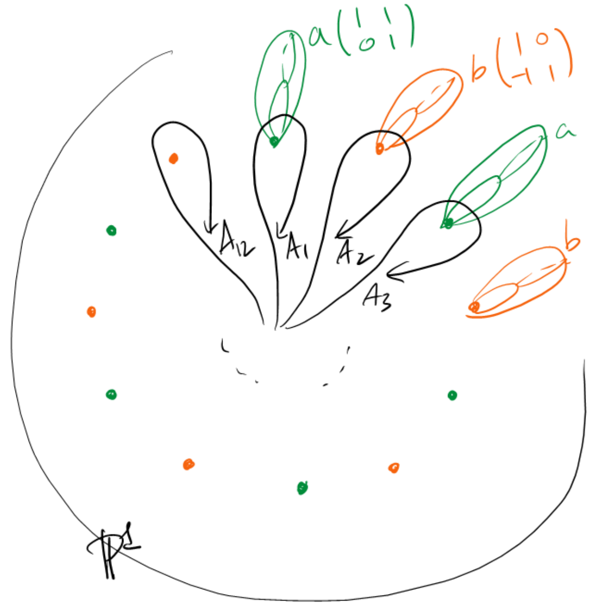
Theorem 10.6 (Moshiezon)**.** *Suppose an elliptically fibered* $\mathrm{Bl}_9(\mathbb{P}^2)$ *has the following local monodromy data.*

- *Matrices* $A_1, \ldots, A_{12} \in \mathrm{SL}_2\mathbb{Z}$*, each conjuate to* $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$*, and*
- *we have the product* $\prod_{i=1}^{12} A_i = I$*.*

Then by braid moves

$$(T_1, \ldots, T_{12}) \mapsto (T_1, \ldots, T_{i-1}, T_{i+1}, T_{i+1} T_i T_{i+1}^{-1}, T_{i+2}, \ldots, T_{12}),$$

we can set the matrices $(A_1, \ldots, A_{12}) = (\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right], \ldots)$.



Now let us get back to the elliptic fibration $\pi\colon M \to \mathbb{P}^1$, $p \mapsto E_{[-Q(p):P(p)]}$ constructed with a pencil. We claim:

Claim. Each exceptional divisor e_i gives a (holomorphic) section $\sigma([s:t]) = e_i \cap E_{[s:t]}$[xxxi] of $\pi\colon M \to \mathbb{P}^1$, which gives a 2-sphere in M.

Proof left as an exercise.

Remark. Another interesting construction comes from a line $\ell_{ij} = \overline{p_i p_j}$. Then for each $E_{[s:t]}$, we intersect ℓ_{ij} on 3 points; call $\sigma_{ij}([s:t])$ to be the third point (in M).

Similarly, we can construct a section from a conic $C_{ijklm} = \mathsf{Conic}(p_i, p_j, p_k, p_l, p_m)$. For each $E_{[s:t]}$, we intersect $C_{ijklm} \cap E_{[s:t]}$ on 6 points; call $\sigma_{ijklm}([s:t])$ to be the sixth point (in M).

10.5 Back to Parabolic realization

Recall that we had a zero intersection class v and to realize $f \in \mathrm{Diff}^+(M)$ so that $f_* \in \mathsf{Stab}(v)$.

Remark. Any section σ of $\pi\colon M \to \mathbb{P}^1$ gives rise to the group structure on each fiber, by setting $\sigma([s:t]) \in E_{[s:t]}$ as the neutral element[xxxii] of the addition. On singular fibers, except that our group structure is now $\mathbb{C}^\times$, the same story happens.

[xxxi] We know that $E_{[s:t]}$ passes p_i, in a direction v_i. The intersection $e_i \cap E_{[s:t]}$ is precisely (p_i, v_i).
[xxxii] a.k.a. additive identity

Given two sections σ_1, σ_2, we define the map $\mathrm{Trans}_{\sigma_1-\sigma_2} \in \mathrm{Diff}(M)$, defined on fibers of π as follows. On $E_{[s:t]}$, map

$$\mathrm{Trans}_{\sigma_1-\sigma_2}(z \in E_{[s:t]}) = z +_{\sigma_2[s:t]} +\sigma_1[s:t],$$

where $+_p$ means the group law on E whose neutral element is $p \in E$.[xxxiii] The map is called the *Mordell–Weil translations.*

Now we pick an isotropic (zero-intersection) vector v as $v = [F]$, where F is a fiber in an elliptic fibration. We claim that any Mordell–Weil translation is a nontrivial map(ping class) that preserves v. Indeed, for sphere classes $[\sigma_i(\mathbb{P}^1)] \in H_2(M;\mathbb{Z})$ we have

$$(\mathrm{Trans}_{\sigma_1-\sigma_0})_*[\sigma_0(\mathbb{P}^1)] = [\sigma_1(\mathbb{P}^1)].$$

We may set $[\sigma_1(\mathbb{P}^1)] \neq [\sigma_0(\mathbb{P}^1)]$, by setting σ_i's coming from the sections risen from different exceptional divisors; thus $\mathrm{Trans}_{\sigma_1-\sigma_0}$ gives a nontrivial class. Evidently $\mathrm{Trans}_{\sigma_1-\sigma_0}$ preserves fibers, so we see that v is kept (literal invariance).

Claim. Recall the sections σ_i risen from exceptional divisors e_i. We have the mapping classes $[\mathrm{Trans}_{\sigma_i-\sigma_j}] \in v^\perp/\mathbb{Z}.v$.

Exercise 10.7. Prove that, in terms of the basis $[H], [e_1], \ldots, [e_9]$ (recall: $[H]\cdot[e_i] = 0$, $[e_i]\cdot[e_j] = -\delta_{ij}$), the fiber class is found as

$$[F] = 3[H] - \sum_{i=1}^{9}[e_i].$$

Exercise 10.8 (Mordell–Weil translation)**.** Let E be a smooth cubic curve in $\mathbb{P}^2$. For $p, q \in E$, let $j(p,q)$ denote the 3rd point of the intersection $\overline{pq} \cap E$.

(1) Let $s \in E$ be a point. Define $p +_s q = j(s, j(p,q))$. Show that $(E, +_s)$ is an abelian group with the identity element s.

(2) Fix $0 \in E$. Show that $\mathrm{Trans}_{\sigma_1-\sigma_2}(p) := p +_{\sigma_2} \sigma_1 = p +_0 (\sigma_1 -_0 \sigma_2)$. [Best done by pictures!] By this the notation $\sigma_1 - \sigma_2$ should make sense in the definition of the translation map.

Summarizing the observations, we announce a

Theorem 10.9 (F.–Looijenga)**.** *Let $M = \mathrm{Bl}_9(\mathbb{P}^2)$. Let $\varphi \in \mathrm{Mod}(M)$ such that φ is of parabolic type, with the following property. We have $v \in H_M$, $v \neq 0$, $v^2 = 0$, and $\varphi(v) = v$; and $\varphi \in v^\perp/\mathbb{Z}.v$.*

Then there exists a smooth elliptic fibration (unique up to toplogical isotopy) $\pi\colon M \to \mathbb{P}^1$ and a diffeomorphism $f\colon M \to M$ such that f is in the class φ. Furthermore, f preserves each fiber of π and f acts by translation on each fiber.

[xxxiii] For $q, r \in E$, declare $j(q,r)$ for the third intersection point of the line $\overline{qr}$ and E. Then we define $q +_p r = j(p, j(q,r))$.

10.6 Anywhere further?

The number 9 is chosen since this is the smallest n^2 other than 1 and 4 (which are already discussed in Seraphina Lee's lectures). For higher n^2, then we will have fibrations by higher genus surfaces, and we can play with fiberwise maps to build parabolic mapping classes.

Not only the exceptional divisors, the conics also give rise to sections. (**Exercise.** Why? Verify.)

For hyperbolic mapping classes, the situation is a bit desperate. Say, can you construct a diffeomorphism on $\mathrm{Bl}_2(\mathbb{P}^2)$ representing a hyperbolic mapping class (in $\mathrm{O}(1,2)(\mathbb{Z})$)?

11 220807-2: Case Study—K3 surfaces

We continue the K3 version of the elliptic fibration story. Recall that the pullback of $\pi\colon \mathrm{Bl}_9(\mathbb{P}^2) \to \mathbb{P}^1$ along $z \mapsto z^2$ map, gives a K3 surface.

Another geometric construct is to start with a blow-up $\mathrm{Bl}_9(\mathbb{P}^2 \setminus (\text{disk}))$, that has an elliptic fibration. Make another copy, and glue them (along the fibers?).

Let M be a K3 surface. Fix $v \in H_M$, $v^2 = 0$. Then we can still construct an exact sequence

$$1 \longrightarrow v^\perp/\mathbb{Z}.v \longrightarrow \mathsf{Stab}_{\mathrm{Aut}(H_M)}(v) \longrightarrow \mathrm{O}(v^\perp/\mathbb{Z}.v) \longrightarrow 1$$

with $v^\perp/\mathbb{Z}.v = \mathbb{Z}^{20}$ and $\mathrm{O}(v^\perp/\mathbb{Z}.v) = \mathrm{O}(2,18)(\mathbb{Z})$ this time.

11.1 Summary of what we know

For a K3 surface M, we know that:

- $\pi_1 M = 0$, and there is nowhere vanishing holomorphic 2-form;
- examples of M include Kummer surfaces, smooth quartic surfaces in $\mathbb{P}^3$, branched cover constructions, elliptically fibered examples, ...;
- (Kodaira) there is only one diffeomorphic model of M; and
- the intersection lattice $H_M = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$.

By Freedman–Quinn, we have $\mathrm{Mod}(M) = \mathrm{Aut}(H_M) \subset \mathrm{O}^+(3,19)(\mathbb{R})$; this can be seen as an isometry group of nonpositively curved space.

Given $\varphi \in \mathrm{Mod}(M) = \mathrm{O}(H_M)$, we have

- φ has finite order (see [Farb–Looijenga] on Nielson realization for K3 surfaces),
- $\varphi(v) = v$ for some v (parabolic; similar to the rational elliptic surface case), and
- φ is semisimple and has infinite order (examples from C. McMullen).

11.2 Finite order

Question 11.1. For which finite order $\varphi \in \mathrm{Mod}(\mathrm{K3})$ is realizable by a finite group of diffeomorphisms? Biholomorphisms? Isometry of a Ricci-flat metric (done by Yau)?

Answers to this is all different in this case [Farb–Looijenga]. A caveat is that $T_{\mathbb{S}^2}$ is not realizable for any finite order diffeomorphisms.

Proof idea? Think of the Teichmüller space X of Ricci-flat metrics on a K3 surface. Equivalently, this is a moduli space of positive definite 3-planes in $H_M \otimes \mathbb{R}$.

Then we have $\mathrm{O}(3,19)(\mathbb{R})$ is the isometry group of the space X. So $\mathrm{Mod}(\mathrm{K3}) = \mathrm{O}(H_M)$ acts isometrically on X.

... No. Actually the moduli of positive definite 3-planes has many loci by fixed sets of Ref_v, $v^2 = -2$. Away from these loci we have smooth K3 surfaces, but on the loci, we observe singular K3 surfaces.

There are some ways to view this Teichmüller space. Geometrically, PDE viewpoints, dynamically, ... Further works are needed to understand this space.

11.3 A Dictionary

Table 1 works on the algebraic features of the intersection lattice H_M to a geometric feature of the manifold M. Each line is a very nontrivial fact!

H_M	$\longleftrightarrow$	M	Comment
$H_M = V_1 \oplus V_2$		$M = M_1 \# M_2$	Freedman
$\mathrm{O}(H_M)$		$\mathrm{Mod}(M)$	Freedman–Quinn
$v \in H_M$		$\Sigma \hookrightarrow M$	
Ref_v, $v^2 = -2$		$T_{\mathbb{S}^2}$	Lefschetz?, Wall
(Below, for K3 surfaces)			
$v \in H_M$, $v^2 = 0$		(a fiber in smooth elliptic fibration (∃! up to top. isotopy))	Farb–Looijenga
$U \hookrightarrow H_M$ hyperbolic lattice		(elliptic fibration with section)	Farb–Looijenga
$\varphi \in \mathsf{Stab}(v)$, $v^2 = 0$		($f \in \mathrm{Diff}^+(M)$ preserving elliptic fibers)	Farb–Looijenga
($A \in \mathrm{O}(H_M)$ A leaves invar. some 2-plane $P > 0$, not $\perp$ to any (-2)-vector)		($(\exists f)(f_* = A)$, preserves some complex structure)	Farb–Looijenga; Torelli Thm.

Table 1: Dictionary of correspondences, $\pi_1 M = 0$