# Conjugacy classes of parabolic subgroups of $\mathrm{O}^+(1,9)(\mathbb{Z})$

Seraphina Lee

August 14, 2022

Let $\mathbb{Z}^{1,9} = \mathbb{Z}\{H, e_1, \ldots, e_9\}$ with the summetric, bilinear form $Q$ of type $(1,9)$:

$$Q = \mathrm{diag}(1, -1, \ldots, -1).$$

Throughout, when we consider a $\mathbb{Z}$-submodule $W$ of $\mathbb{Z}^{10}$, we mean the pair $(W, Q|_W)$, which may or may not be a lattice.

In this note, we show that there are exactly two conjugacy classes of maximal parabolic subgroups $\Gamma = \mathrm{Stab}_{\mathrm{O}^+(1,9)(\mathbb{Z})}(u)$, where $u \in \mathbb{Z}^{1,9}$ is a primitive, isotropic, nontrivial vector corresponding to a point of $\partial\mathbb{H}^9$. It suffices to compute that there are exactly two $\mathrm{O}(1,9)(\mathbb{Z})$-orbits of primitive, isotropic, nontrivial vectors of $\mathbb{Z}^{1,9}$.

**Lemma 1.** There are exactly two $\mathrm{O}(1,9)(\mathbb{Z})$-orbits of primitive, isotropic vectors in $\mathbb{Z}^{1,9}$. They are the orbits of $v := H - e_1$ and $w := 3H - e_1 - \cdots - e_9$.

The main tool of the proof of Lemma 1 is the following theorem.

**Theorem 2** (Mordell [Mor38])**.** There are exactly two isomorphism classes of positive-definite, unimodular lattices of rank 8; they are given by $8(1)$ and $E_8$.

Equivalently, any such negative-definite lattice is isomorphic to $8(-1)$ or $-E_8$. Next, we record a computation that is used in the proof of Lemma 1.

**Lemma 3.** Let

$$L := \mathbb{Z}\{w,\, 2H - e_1 - e_2 - e_3 - e_4 - e_5\} \cong (1) \oplus (-1)$$

be a sublattice of $\mathbb{Z}^{1,9}$. Then $L^\perp \cong -E_8$.

*Proof.* Note that $L^\perp \subseteq \mathbb{Z}\{w\}^\perp$ where

$$\mathbb{Z}\{w\}^\perp = \mathbb{Z}\{H - e_1 - e_2 - e_3,\, e_1 - e_2,\, \ldots,\, e_8 - e_9\} \cong \mathbb{Z}^9.$$

The restriction of $Q$ to $\mathbb{Z}\{w\}^\perp$ is even so the restriction of $Q$ to $L^\perp$ is even. Because $L$ is odd and indefinite, $L \cong (1) \oplus (-1)$, so a signature calculation shows that $L$ is negative definite of rank 8. Theorem 2 implies that $L^\perp \cong -E_8$. □

Now we prove the main lemma.

*Proof of Lemma 1.* First, we show that $v$ and $w$ are not in the same $\mathrm{O}(1,9)(\mathbb{Z})$-orbit. Observe that the restriction of $Q$ to

$$\mathbb{Z}\{w\}^\perp = \mathbb{Z}\{H - e_1 - e_2 - e_3,\, e_1 - e_2,\, \ldots,\, e_8 - e_9\} \cong \mathbb{Z}^9$$

is even, while the restriction of $Q$ to

$$\mathbb{Z}\{v\}^\perp = \mathbb{Z}\{v, e_2, e_3, \ldots, e_9\} \cong \mathbb{Z}^9$$

is odd. For any $g \in \mathrm{O}(1,9)(\mathbb{Z})$, the restriction of $Q$ to $g(\mathbb{Z}\{w\}^\perp)$ is also even, so $g(\mathbb{Z}\{w\}^\perp) \neq \mathbb{Z}\{v\}^\perp$. Therefore, $g(w) \neq v$ for all $g \in \mathrm{O}(1,9)(\mathbb{Z})$.

Next, we show that any primitive, isotropic, nontrivial vector $u \in \mathbb{Z}^{1,9}$ lies in the $\mathrm{O}(1,9)(\mathbb{Z})$-orbit of $v$ or $w$. By unimodularity of $Q$, there exists $u^* \in \mathbb{Z}^{1,9}$ such that $Q(u,u^*) = 1$. Then the restriction of $Q$ to $\mathbb{Z}\{u,u^*\}$ is unimodular and indefinite, and so the restriction of $Q$ to $\mathbb{Z}\{u,u^*\}^\perp$ is unimodular and negative definite of rank 8. By Theorem 2, this gives two cases

$$\mathbb{Z}\{u,u^*\}^\perp \cong -E_8 \qquad \text{or} \qquad \mathbb{Z}\{u,u^*\}^\perp \cong 8(-1)$$

depending on the parity of $Q|_{\mathbb{Z}\{u,u^*\}^\perp}$.

1. If $\mathbb{Z}\{u,u^*\}^\perp \cong -E_8$ then $Q$ restricted to $\mathbb{Z}\{u,u^*\}$ must be odd since $\mathbb{Z}^{1,9}$ is odd. Therefore, $\mathbb{Z}\{u,u^*\} \cong (1) \oplus (-1)$. There exists an automorphism $A \in \mathrm{O}(1,9)(\mathbb{Z})$ such that

$$A(\mathbb{Z}\{u,u^*\}) = L \cong (1) \oplus (-1), \qquad A(\mathbb{Z}\{u,u^*\}^\perp) = L^\perp \cong -E_8.$$

   Up to automorphisms of $L$, there exists a unique primitive, isotropic, nontrivial vector in $L$, i.e. $w$. So perhaps after applying an automorphism of $L$, we can ensure that

$$A(u) = w = 3H - e_1 - \cdots - e_9 \in L.$$

2. If $\mathbb{Z}\{u,u^*\}^\perp \cong 8(-1)$ then depending on the parity of $Q|_{\mathbb{Z}\{u,u^*\}}$,

$$\mathbb{Z}\{u,u^*\} \cong (1) \oplus (-1) \qquad \text{or} \qquad \mathbb{Z}\{u,u^*\} \cong \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

   There exists an automorphism $A \in \mathrm{O}(1,9)(\mathbb{Z})$ such that

$$A(\mathbb{Z}\{u,u^*\}) = \mathbb{Z}\{H, e_1\}, \qquad A(\mathbb{Z}\{u,u^*\}^\perp) = \mathbb{Z}\{e_2, \ldots, e_9\}$$

   or

$$A(\mathbb{Z}\{u,u^*\}) = \mathbb{Z}\{H - e_1, H - e_2\}, \qquad A(\mathbb{Z}\{u,u^*\}^\perp) = \mathbb{Z}\{H - e_1 - e_2, e_3, \ldots, e_9\}$$

   respectively. In either case, up to automorphisms of $A(\mathbb{Z}\{u,u^*\})$, there's a unique primitive, isotropic, nontrivial vector in $A(\mathbb{Z}\{u,u^*\})$, i.e. $H - e_1$. So perhaps after applying an automorphism of $A(\mathbb{Z}\{u,u^*\})$, we can ensure that

$$A(u) = v = H - e_1. \qquad \square$$

# References

[Mor38] Mordell, L. J. The definite quadratic forms in eight variables with determinant unity. *J. Math. Pures Appl.*, 17:41–46, 1938.

Seraphina Eun Bi Lee
seraphinalee@uchicago.edu