Chapter 3

Getting Acquainted with Intersection Forms

We define the intersection form of a 4–manifold, which governs intersections of surfaces inside the manifold. We start by representing every homology 2–class by an embedded surface, then, in section 3.2 (page 115), we explore the properties of the intersection form. Among them is unimodularity, which is essentially equivalent to Poincaré duality. An important invariant of an intersection form is its signature, and we discuss how its vanishing is equivalent to the 4–manifold being a boundary of a 5–manifold. After listing a few simple examples of 4–manifolds and their intersection form, in section 3.3 (page 127) we present in some detail the important example of the $K3$ manifold.

Given any closed oriented 4-manifold M, its **intersection form** is the symmetric 2–form defined as follows:

$$Q_M \colon H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$Q_M(\alpha,\beta) = (\alpha \cup \beta)[M]\,.$$

This form is bilinear[1] and is represented by a matrix of determinant ± 1. While over $\mathbb{R}$ this is a recipe for boredom, since this intersection form is defined over the *integers* (and thus changes of coordinates must be made only through integer-valued matrices), our Q_M is a quite far-from-trivial object.

1. Notice that Q_M vanishes on any torsion element, and thus can be thought of as defined on the free part of $H^2(M;\mathbb{Z})$; since our manifolds are assumed simply-connected, torsion is not an issue.

For convenience, we will often denote $Q_M(\alpha,\beta)$ by $\alpha\cdot\beta$. Further, we will identify without comment a cohomology class $\alpha\in H^2(M;\mathbb{Z})$ with its Poincaré-dual homology class $\alpha\in H_2(M;\mathbb{Z})$.

For defining Q_M more geometrically,[2] we will represent classes α and β from $H_2(M;\mathbb{Z})$ by embedded surfaces S_α and S_β, and then equivalently define $Q_M(\alpha,\beta)$ as the intersection number of S_α and S_β:

$$Q_M(\alpha,\beta)=S_\alpha\cdot S_\beta\ .$$

First, though, we need to argue that any class $\alpha\in H_2(M;\mathbb{Z})$ can indeed be represented by a smoothly embedded surface S_α:

3.1. Preparation: representing homology by surfaces

It is known from general results[3] that every homology class of a 4–manifold can be represented by embedded submanifolds. Nonetheless, we present a direct argument for the case of 2–classes, owing to the useful techniques that it exhibits.

Simply-connected case. Assume first that M is simply-connected. Then by Hurewicz's theorem $\pi_2(M)\approx H_2(M;\mathbb{Z})$, and hence all homology classes of M can be represented as images of maps $f\colon S^2\to M$. The latter can always be perturbed to yield immersed spheres, whose only failures from being embedded are transverse double-points. These double-points can be eliminated at the price of increasing the genus.

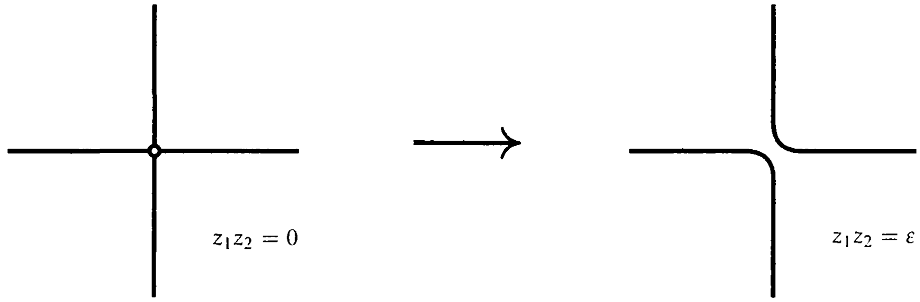
For example, by using complex coordinates, a double-point is isomorphic to the simple nodal singularity of equation $z_1z_2=0$ in $\mathbb{C}^2$: the complex planes $z_1=0$ and $z_2=0$ meeting at the origin. It can be eliminated by perturbing to $z_1z_2=\varepsilon$, as suggested in figure 3.1 on the facing page. (A simple change of coordinates transforms the situation into perturbing $w_1^2+w_2^2=0$ to $w_1^2+w_2^2=\varepsilon$.)

More geometrically, imagine two planes meeting orthogonally at the origin of $\mathbb{R}^4$. Their traces in the 3–sphere S^3 are two circles, linking once.[4] We can eliminate the singularity if we discard the portions contained in the open 4–ball bounded by S^3, and instead connect the two circles in S^3 by an annular

2. *"Think with intersections, prove with cup-products."*

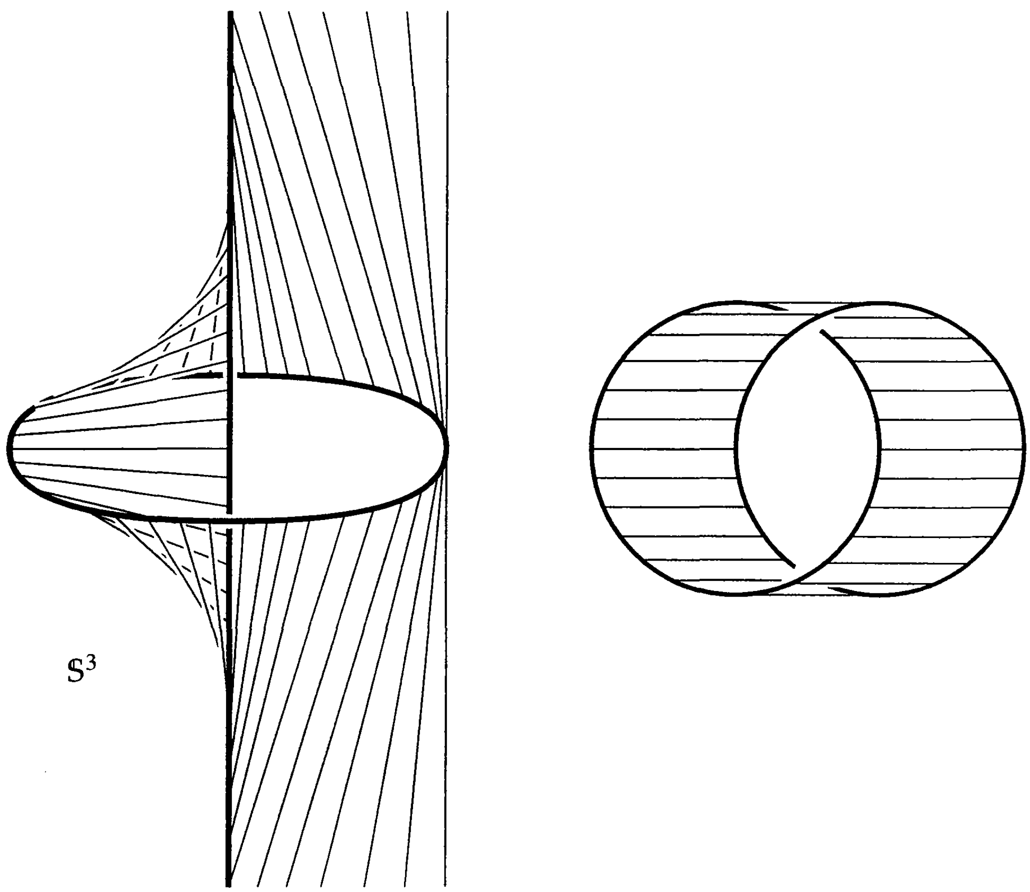
3. For example, for any smooth oriented X^m and any $\alpha\in H^*(X;\mathbb{Z})$, there is some integer k so that $k\alpha$ can be represented by an embedded submanifold; if α has dimension at most 8 or codimension at most 2, then it can be represented directly by a submanifold; if X^m is embedded in $\mathbb{R}^{m+2}$, then X is the boundary of an oriented smooth $(m+1)$–submanifold in $\mathbb{R}^{m+2}$. These results were announced in **R. Thom**'s ***Sous-variétés et classes d'homologie des variétés différentiables*** **[Tho53a]** and proved in his celebrated ***Quelques propriétés globales des variétés différentiables*** **[Tho54]**.

4. Think: fibers of the Hopf map $S^3\to\mathbb{CP}^1$; the Hopf map will be recalled in footnote 34 on page 129.



3.1. Eliminating a double-point, I: complex coordinates

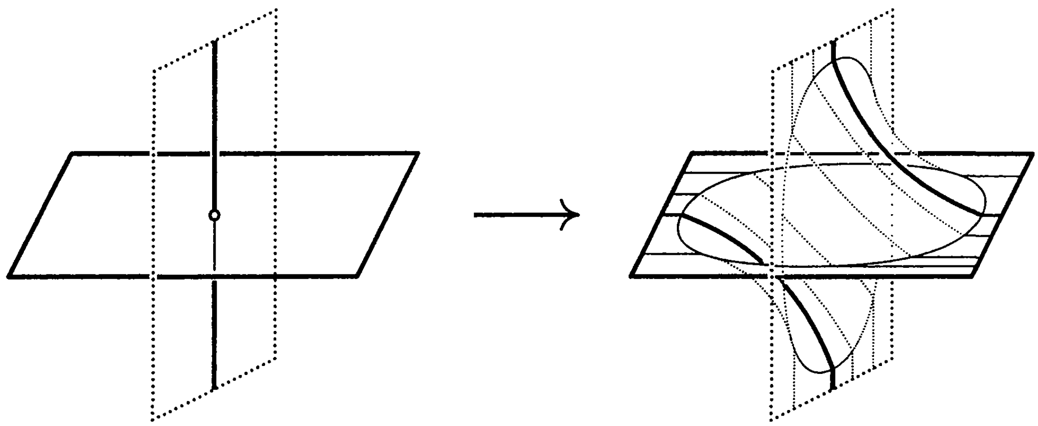
sheet, as suggested in figure[5] 3.2. Thus, we replaced two disks meeting at the double-point by an annulus. A 4–dimensional image is attempted in figure[6] 3.3 on the following page.



3.2. Eliminating a double-point, II: annulus

5. On the left of figure 3.2, one circle is drawn as a vertical line through ∞, after setting $S^3 = \mathbb{R}^3 \cup \infty$.

6. As usual, in figure 3.3, dotted lines represent creatures escaping in the fourth dimension.



3.3. Eliminating a double-point, III

Either way, we can eliminate all double-points of the immersed sphere, and the result is then an embedded surface representing that homology class. Thus, all homology classes can be represented by embedded surfaces, but rarely by spheres.

The failure to represent homology classes by smoothly embedded spheres is of course related to the failure of smoothly embedding disks. The natural question to ask is then: what is the minimum genus needed to represent a given homology class? We will come back to this question later.[7]

In general. The method above only works for simply-connected M^4's. An argument for general 4–manifolds has two equivalent versions:

(1) Since $\mathbb{CP}^\infty$ is an Eilenberg–Maclane $K(\mathbb{Z},2)$–space,[8] it follows that the elements of $H^2(M;\mathbb{Z})$ correspond to homotopy classes of maps $M \to \mathbb{CP}^\infty$. Since M is 4–dimensional, such maps can be slid off the high-dimensional cells of $\mathbb{CP}^\infty$ and thus reduced to maps $M \to \mathbb{CP}^2$. For any class $\alpha \in H^2(M;\mathbb{Z})$, pick a corresponding $f_\alpha \colon M \to \mathbb{CP}^2$ and arrange it to be differentiable and transverse to $\mathbb{CP}^1 \subset \mathbb{CP}^2$. Then $f_\alpha^{-1}[\mathbb{CP}^1]$ is a surface Poincaré-dual to α.

(2) Equivalently, since $\mathbb{CP}^\infty$ coincides with the classifying space[9] $\mathscr{B}U(1)$ of the group $U(1)$, classes in $H^2(M;\mathbb{Z})$ correspond to complex line bundles on M, with α being paired to L_α whenever $c_1(L_\alpha) = \alpha$. If we pick a

7. See ahead, chapter 11 (starting on page 481).

8. An **Eilenberg–Maclane $K(G,m)$–space** is a space whose *only* non-zero homotopy group is $\pi_m = G$; such a space is unique up to homotopy-equivalence. It can be used to represent cohomology as $H^m(X;G) = [X;\,K(G,m)]$, where $[A;B]$ denotes the set of homotopy classes of maps $A \to B$.

9. A **classifying space** $\mathscr{B}G$ for a topological group G is a space so that $[X;\mathscr{B}G]$ coincides with the set of isomorphisms classes of G–bundles over X. A bit more on classifying spaces is explained in the end-notes of the next chapter (page 204).

generic section σ of L_α, then its zero set $\sigma^{-1}[0]$ will be an embedded surface Poincaré-dual to α.

3.2. Intersection forms

Given a closed oriented 4-manifold M, we defined its **intersection form** as

$$Q_M\colon H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z} \qquad Q_M(\alpha,\beta) = S_\alpha \cdot S_\beta\,,$$

where S_α and S_β are any two surfaces representing the classes α and β.

Notice that, if M is simply-connected, then $H_2(M;\mathbb{Z})$ is a free $\mathbb{Z}$–module and there are isomorphisms $H_2(M;\mathbb{Z}) \approx \oplus m\,\mathbb{Z}$, where $m = b_2(M)$. If M is not simply-connected, then $H_2(M;\mathbb{Z})$ inherits the torsion of $H_1(M;\mathbb{Z})$, but by linearity the intersection form will always vanish on these torsion classes; thus, when studying intersection form, we can safely pretend that $H_2(M;\mathbb{Z})$ is always free.

Lemma. *The form $Q_M(\alpha,\beta) = S_\alpha \cdot S_\beta$ on $H_2(M;\mathbb{Z})$ coincides modulo Poincaré duality with the pairing $Q_M(\alpha^*,\beta^*) = (\alpha^* \cup \beta^*)[M]$ on $H^2(M;\mathbb{Z})$.*

Proof. Given any class $\alpha \in H_2(M;\mathbb{Z})$, denote by α^* its Poincaré-dual from $H^2(M;\mathbb{Z})$; we have $\alpha^* \cap [M] = \alpha$. We wish to show that the pairing

$$Q_M(\alpha^*,\beta^*) = (\alpha^* \cup \beta^*)[M]$$

on $H^2(M;\mathbb{Z})$ defines the same bilinear form as the one defined above.

We use the general formula[10] $(\alpha^* \cup \beta^*)[M] = \alpha^*\,[\beta^* \cap [M]]$, from which it follows that $Q_M(\alpha^*,\beta^*) = \alpha^*[\beta]$, or

$$Q_M(\alpha^*,\beta^*) = \alpha^*[S_\beta]\,.$$

Therefore, we need to show that

$$\alpha^*[S_\beta] = S_\alpha \cdot S_\beta\,.$$

Since Q_M vanishes on torsion classes, it is enough to check the last formula by including the free part of $H^2(M;\mathbb{Z})$ into $H^2(M;\mathbb{R})$ and by interpreting the latter as the de Rham cohomology of exterior 2–forms.

Moving into de Rham cohomology translates cup products into wedge products and cohomology/homology pairings into integrations. We have, for example,

$$Q_M(\alpha^*,\beta^*) = \int_M \alpha^* \wedge \beta^* \qquad \text{and} \qquad \alpha^*[S_\beta] = \int_{S_\beta} \alpha^*$$

for all 2–forms α^*, $\beta^* \in \Gamma\big(\Lambda^2(T_M^*)\big)$.

10. More often written in terms of the Kronecker pairing as $\langle \alpha^* \cup \beta^*,\, [M] \rangle = \langle \alpha^*,\, \beta^* \cap [M] \rangle$.

In this setting, given a surface S_α, one can find a 2–form α^* dual to S_α so that it is non-zero only close to S_α. Further, one can choose some local oriented coordinates $\{x_1, x_2, y_1, y_2\}$ so that S_α coincides locally with the plane $\{y_1 = 0;\, y_2 = 0\}$, oriented by $dx_1 \wedge dx_2$. One can then choose α^* to be locally written $\alpha^* = f(x_1, x_2)\, dy_1 \wedge dy_2$, for some suitable bump-function f on $\mathbb{R}^2$, supported only around $(0, 0)$ and with integral $\int_{\mathbb{R}^2} f = 1$.

If S_β is some surface transverse to S_α and we arrange that, around the intersection points of S_α and S_β, we have S_β described by $\{x_1 = 0;\, x_2 = 0\}$, then clearly

$$\int_{S_\beta} \alpha^* = S_\alpha \cdot S_\beta\,,$$

with each intersection point of S_α and S_β contributing ± 1 depending on whether $dy_1 \wedge dy_2$ orients S_β positively or not.[11] □

Unimodularity and dual classes

The intersection form Q_M is *$\mathbb{Z}$–bilinear* and *symmetric*. As a consequence of Poincaré duality, the form Q_M is also **unimodular**, meaning that the matrix representing Q_M is invertible over $\mathbb{Z}$. This is the same as saying that

$$\det Q_M = \pm 1\,.$$

Unimodularity is further equivalent to the property that, for every $\mathbb{Z}$–linear function $f\colon H_2(M; \mathbb{Z}) \to \mathbb{Z}$, there exists a unique $\alpha \in H_2(M; \mathbb{Z})$ so that $f(x) = \alpha \cdot x$.

Lemma. *The intersection form Q_M of a 4–manifold is unimodular.*

Proof. The intersection form is unimodular if and only if the map

$$\begin{array}{rcl} \widehat{Q}_M\colon H_2(M;\mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(H_2(M;\mathbb{Z}),\, \mathbb{Z}) \\ \alpha & \longmapsto & x \longmapsto \alpha \cdot x \end{array}$$

is an isomorphism. We will argue that this last map coincides with the Poincaré duality morphism. Indeed, Poincaré duality is the isomorphism

$$\begin{array}{rcl} H_2(M;\mathbb{Z}) & \xrightarrow[\approx]{} & H^2(M;\mathbb{Z}) \\ \alpha & \longmapsto & \alpha^*\,, \end{array}$$

with α^* characterized by $\alpha^* \cap [M] = \alpha$. Assume for simplicity that $H_2(M; \mathbb{Z})$ is free.[12] Then the universal coefficient theorem[13] shows that

11. See **R. Bott** and **L. Tu**'s ***Differential forms in algebraic topology*** **[BT82]** for more such play with exterior forms.

12. If not free, a similar argument is made on the free part $H^2(M;\mathbb{Z}) / \mathrm{Ext}(H_1(M;\mathbb{Z});\mathbb{Z})$ of $H^2(M;\mathbb{Z})$, which is all that matters since Q_M vanishes on torsion.

13. The universal coefficient theorem was recalled on page 15.

we have an isomorphism

$$\begin{array}{ccc} H^2(M;\mathbb{Z}) & \xrightarrow[\approx]{} & \mathrm{Hom}\big(H_2(M;\mathbb{Z}),\ \mathbb{Z}\big) \\ \alpha^* & \longmapsto & x \mapsto \alpha^*[x]\ . \end{array}$$

Combining Poincaré duality with the latter yields the isomorphism

$$\begin{array}{ccc} H_2(M;\mathbb{Z}) & \xrightarrow[\approx]{} & \mathrm{Hom}\big(H_2(M;\mathbb{Z}),\ \mathbb{Z}\big) \\ \alpha & \longmapsto & x \mapsto \alpha^*[x]\ . \end{array}$$

However, as argued in the preceding subsection, we have $Q_M(\alpha, x) = \alpha^*[x]$, and therefore the above isomorphism coincides with the map $\widehat{Q}_M$. That proves that the intersection form Q_M is unimodular. □

Further, the unimodularity of Q_M is equivalent to the fact that, for every basis $\{\alpha_1, \ldots, \alpha_m\}$ of $H_2(M;\mathbb{Z})$, there is a unique **dual basis** $\{\beta_1, \ldots, \beta_m\}$ of $H_2(M;\mathbb{Z})$ so that $\alpha_k \cdot \beta_k = +1$ and $\alpha_i \cdot \beta_j = 0$ if $i \neq j$.

To see this, start with the basis $\{\alpha_1, \ldots, \alpha_m\}$ in $H_2(M;\mathbb{Z})$, pick the familiar dual basis[14] *$\{\alpha_1^*, \ldots, \alpha_m^*\}$ in the dual $\mathbb{Z}$–module* $\mathrm{Hom}(H_2(M;\mathbb{Z}), \mathbb{Z})$, *then transport it back to $H_2(M;\mathbb{Z})$ by using Poincaré duality (or $\widehat{Q}_M$) and hence obtain the desired basis $\{\beta_1, \ldots, \beta_m\}$.*

In particular, for every *indivisible* class α (*i.e.*, not a multiple), there exists at least one **dual class** β such that $\alpha \cdot \beta = +1$: complete α to a basis and proceed as above. (Of course, such β's are *not* unique: once you find one, you can obtain others by adding any γ with $\alpha \cdot \gamma = 0$.)

Intersection forms and connected sums

The simplest way of combining two 4–manifolds yields the the simplest way of combining two intersection forms. First, a bit of review:

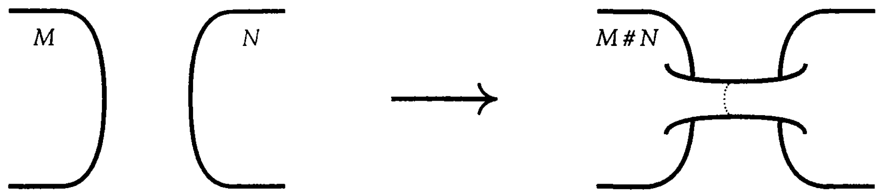
Remembering connected sums. The **connected sum** of two manifolds M and N, denoted by

$$M \# N\,,$$

is the simplest method for combining M and N into one connected manifold, by joining them with a tube as sketched in figure 3.4 on the next page. Notice that the 4–sphere is an identity element for connected sums: $M \# S^4 \cong M$.

Connected sums are described more rigorously by choosing in each of M and N a small open 4–ball and removing it to get two manifolds M° and N°, each with a 3–sphere as boundary, then identifying these 3–spheres to obtain the closed manifold $M \# N$.

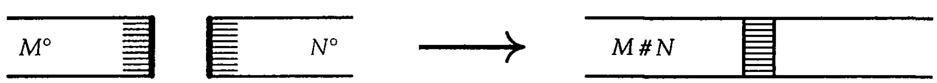
14. Recall that, given a basis $\{e_1, \ldots, e_m\}$ in a module Z, the dual basis $\{e_1^*, \ldots, e_m^*\}$ in Z^* is specified by setting $e_k^*(e_k) = 1$ and $e_i^*(e_j) = 0$ for $i \neq j$.



3.4. The connected sum of two manifolds, I

More about connected sums. *The identification of the two 3–spheres must be made through an orientation-reversing diffeomorphism* $\partial M^\circ \cong \overline{\partial N^\circ}$*, as was mentioned on page 13. Indeed, if M and N are oriented, then the new boundary 3–spheres will inherit orientations. In order that the orientations of M and N be nicely compatible with an orientation of M # N, we must identify the 3–spheres with an orientation flip.*

Furthermore, to ensure that M # N is a smooth manifold, this gluing must be done as follows: Choose open 4-balls in M and N, then remove them. Embed copies of $S^3 \times [0,1]$ *as collars to the new boundary 3–spheres. Take care to embed these collars so that, on the side of M, the sphere* $S^3 \times 1$ *be sent onto* ∂M°*, with* $S^3 \times [0,1)$ *going into the interior of* M°*. On the N side,* $S^3 \times 0$ *should be sent onto* ∂N° *and* $S^3 \times (0,1]$ *into the interior of* N°*. Now identify the two collars* $S^3 \times [0,1]$ *in the obvious manner and thus obtain M # N, as in figure 3.5. This automatically forces the boundary-spheres to be identified "inside-out", reversing orientations, and further makes it clear that M # N is smooth.*[15] *See figure 3.6 on the next page. The equivalence of this procedure with "joining by a tube" is explained in figure 3.7 on the facing page.*



3.5. Gluing by identifying collars

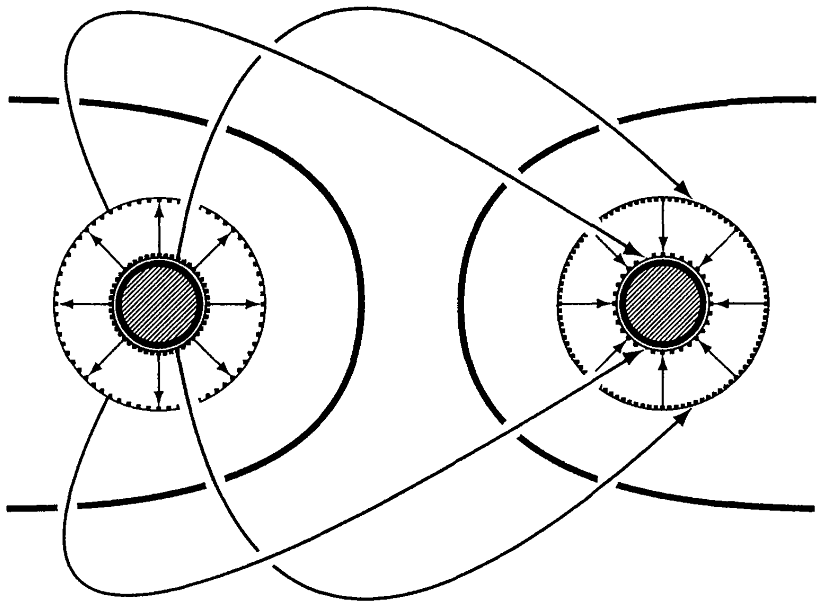
Sums and forms. This connected sum operation is nicely compatible with intersection forms:

Lemma. *If M and N have intersection forms* Q_M *and* Q_N*, then their connected sum M # N will have intersection form*

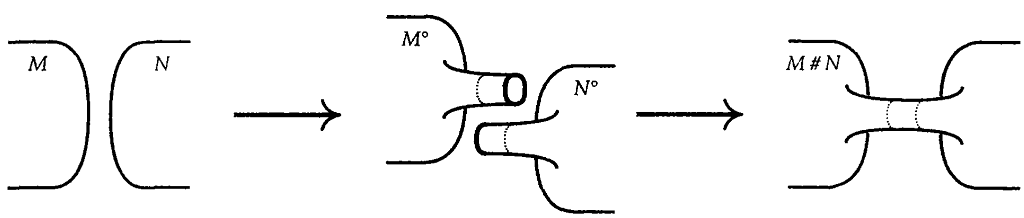
$$Q_{M\#N} = Q_M \oplus Q_N\,.$$

Proof. Since M° and N° can be viewed as M and N without a 4–handle (or a 4–cell), and since 2–homology is influenced only by 1–, 2– and 3–handles, it follows that the 2–homology of $M\#N$ will merely be the friendly gathering of the 2–homologies of M and N, intersections and all. □

15. In fact, each time you read *"A and B both have the same boundary, so we glue A and B along it"*, you should understand that the "gluing" is done *via* an orientation-reversing diffeomorphism $\partial A \cong \overline{\partial B}$, and that a collaring procedure as above is used. This was already explained on page 13. For more on the foundation of these gluings, read from **M. Hirsch's *Differential topology*** [**Hir94**, sec 8.2].



3.6. The connected sum of two manifolds, II



3.7. The connected sum of two manifolds, III

Topological heaven. For topological 4*–manifolds a converse is true:*

Theorem (M. Freedman). *If M is simply-connected and Q_M splits as a direct sum $Q_M = Q' \oplus Q''$, then there exist* topological 4*–manifolds N' and N'' with intersection forms Q' and Q'' such that $M = N' \# N''$.* □

This is a direct consequence of Freedman's classification that we will present later.[16] *Such a result certainly fails in the smooth case, and its failure spawns exotic*[17] *$\mathbb{R}^4$'s.*

Invariants of intersection forms

To start to distinguish between the various possible intersection forms, we define the following simple algebraic invariants:

16. See ahead section 5.2 (page 239). For a more refined topological sum-splitting result, we refer to **M. Freedman** and **F. Quinn**'s ***Topology of 4–manifolds*** [**FQ90**, ch 10].

17. See ahead section 5.4 (page 250).

— The **rank** of Q_M:

It is the size of Q_M's domain, defined simply as

$$\operatorname{rank} Q_M = \operatorname{rank}_{\mathbb{Z}} H^2(M;\mathbb{Z}) ,$$

or $\operatorname{rank} Q_M = \dim_{\mathbb{R}} H^2(M;\mathbb{R})$. In other words, the rank is the second Betti number $b_2(M)$ of M.

— The **signature** of Q_M:

It is obtained as follows: first diagonalize Q_M as a matrix over $\mathbb{R}$ (or $\mathbb{Q}$), separate the resulting positive and negative eigenvalues, then subtract their counts; that is

$$\operatorname{sign} Q_M = \dim H^2_+(M;\mathbb{R}) - \dim H^2_-(M;\mathbb{R}) ,$$

where $H^2_\pm$ are any maximal positive/negative-definite subspaces for Q_M. We can set partial Betti numbers $b_2^\pm = \dim H^2_\pm$, and thus we can read $\operatorname{sign} Q_M = b_2^+(M) - b_2^-(M)$.

— The **definiteness** of Q_M (*definite* or *indefinite*):

If for all non-zero classes α we always have $Q_M(\alpha,\alpha) > 0$, then Q_M is called **positive definite.**

If, on the contrary, we have $Q_M(\alpha,\alpha) < 0$ for all non-zero α's, then Q_M is called **negative definite.**

Otherwise, if for some α_+ we have $Q_M(\alpha_+, \alpha_+) > 0$ and for some α_- we have $Q_M(\alpha_-, \alpha_-) < 0$, then Q_M is called **indefinite.**

— The **parity** of Q_M (*even* or *odd*):

If, for all classes α, we have that $Q_M(\alpha,\alpha)$ is even, then Q_M is called **even.** Otherwise, it is called **odd.** Notice that it is enough to have *one* class with odd self-intersection for Q_M to be called odd.

Signatures and bounding 4-manifolds

A first remark is that signatures are *additive:* $\operatorname{sign}(Q' \oplus Q'') = \operatorname{sign} Q' + \operatorname{sign} Q''$. In particular,[18]

$$\operatorname{sign}(M \# N) = \operatorname{sign} M + \operatorname{sign} N .$$

Another remark is that changing the orientation of M will change the sign of the signature:

$$\operatorname{sign} \overline{M} = -\operatorname{sign} M ,$$

since it obviously changes the sign of its intersection form: $Q_{\overline{M}} = -Q_M$.

18. The additivity of signatures still holds for gluings $M \cup_\partial N$ more general than connected sums. This result *(Novikov additivity)* and an outline of its proof can be found in the the end-notes of the next chapter (page 224).

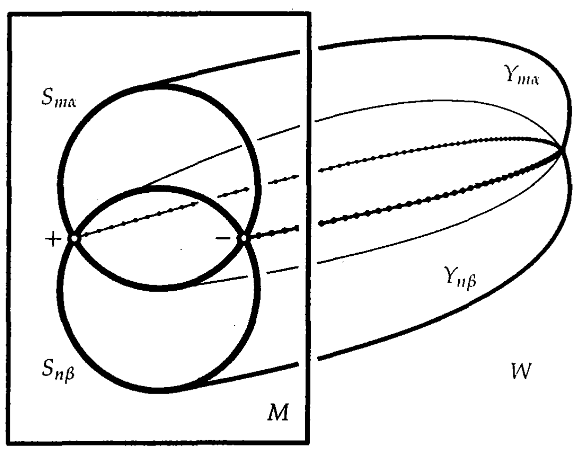
The signature vanishes for boundaries. More remarkably, the vanishing of the signature of a 4–manifold M has a direct topological interpretation:

Lemma. *If M^4 is the boundary of some oriented 5–manifold W^5, then*

$$\operatorname{sign} Q_M = 0 .$$

Proof. Since the signature appears after diagonalizing over some field, we will work here with homology with rational coefficients. Thus, denote by $\iota\colon H_2(M;\mathbb{Q}) \to H_2(W;\mathbb{Q})$ the morphism induced from the inclusion of M^4 as the boundary of W^5.

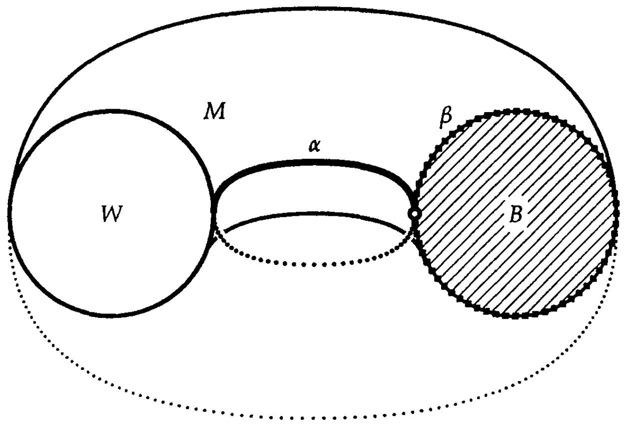
If bounding. First, we claim that if both $\alpha, \beta \in H_2(M;\mathbb{Q})$ have $\iota\alpha = 0$ and $\iota\beta = 0$ then their intersection must be $\alpha \cdot \beta = 0$. Indeed, since α and β are rational, some of their multiples $m\alpha$ and $n\beta$ will be integral. Then $m\alpha$ and $n\beta$ can be represented by two embedded surfaces $S_{m\alpha}$ and $S_{n\beta}$ in M. Since $\iota\alpha = 0$ and $\iota\beta = 0$, this implies that $S_{m\alpha}$ and $S_{n\beta}$ will bound two oriented 3–manifolds $Y_{m\alpha}$ and $Y_{n\beta}$ inside W. The intersection number $\alpha \cdot \beta$ is determined by counting the intersections of the surfaces $S_{m\alpha}$ and $S_{n\beta}$, then dividing by mn. However, the intersection of $Y^3_{m\alpha}$ and $Y^3_{n\beta}$ inside W^5 consists of arcs, which connect pairs of intersection points of $S_{m\alpha}$ and $S_{n\beta}$ with opposite signs, as pictured in figure 3.8. It follows that $S_{m\alpha} \cdot S_{n\alpha} = 0$, and therefore $\alpha \cdot \beta = 0$, as claimed.



3.8. Bounding surfaces have zero intersection

If not bounding. Second, we claim that for every $\alpha \in H_2(M;\mathbb{Q})$ with $\iota\alpha \neq 0$ there must be some $\beta \in H_2(M;\mathbb{Q})$ so that $\alpha \cdot \beta = +1$ but $\iota\beta = 0$.

To see that, we notice that, since $\iota\alpha \neq 0$ in $H_2(W;\mathbb{Q})$, there exists a 3–class $B \in H_3(W, \partial W; \mathbb{Q})$ that is *dual*[19] to our $\iota\alpha \in H_2(W;\mathbb{Q})$, *i.e.*, has $\alpha \cdot B = +1$ in W^5. Its boundary $\partial B = \beta$ is a class in $H_2(M;\mathbb{Q})$, and we have that $\alpha \cdot \beta = \iota\alpha \cdot B = +1$ and also that $\iota\beta = 0$. See figure 3.9.



3.9. A non-bounding class has a bounding dual

Unravel the form. Finally, we are ready to attack the actual intersection form of M. Any class α that bounds in W, *i.e.*, has $\iota\alpha = 0$, must have zero self-intersection $\alpha \cdot \alpha = 0$. We are thus more interested in classes α that do not bound.

Assume we choose some $\alpha \in H_2(M;\mathbb{Q})$ so that $\iota\alpha \neq 0$. Then there is some $\beta \in H_2(M;\mathbb{Q})$ so that $\alpha \cdot \beta = +1$, while $\iota\beta = 0$, and thus $\beta \cdot \beta = 0$. Therefore the part of Q_M corresponding to $\{\alpha, \beta\}$ has matrix

$$Q_{\alpha\beta} = \begin{bmatrix} * & 1 \\ 1 & 0 \end{bmatrix},$$

which has determinant -1 and diagonalizes over $\mathbb{Q}$ as $[+1] \oplus [-1]$.

Since Q_M is unimodular, this means that Q_M must actually split as a direct sum $Q_M = Q_{\alpha\beta} \oplus Q^{\perp}$ for some unimodular form $Q^{\perp}$ defined on a complement of $\mathbb{Q}\{\alpha, \beta\}$ in $H_2(M;\mathbb{Q})$. Since the signature is additive and one can see that $\operatorname{sign} Q_{\alpha\beta} = 0$, we deduce that we must have $\operatorname{sign} Q_M = \operatorname{sign} Q^{\perp}$.

We continue this procedure for $Q^{\perp}$, splitting off 2–dimensional pieces until there are no more classes α with $\iota\alpha \neq 0$ left. Then whatever is still there has to bound in W, and hence cannot contribute to the signature. Therefore $\operatorname{sign} Q_M = 0$. □

19. A reasoning analogous to the one we made earlier for Q_M applies to the intersection pairing $H_2(W;\mathbb{Z}) \times H_3(W, \partial W; \mathbb{Z}) \to \mathbb{Z}$. In particular, it is unimodular, and thus we have dual classes; since we work over $\mathbb{Q}$, the indivisibility of α is not required.

A consequence of this result is that, whenever two 4–manifolds can be linked by a cobordism, they must have the same signature. Indeed, if $\partial W = \overline{M} \cup N$, then $0 = \operatorname{sign}(\overline{M} \cup N) = -\operatorname{sign} M + \operatorname{sign} N$. That is:

Corollary. *If two manifolds are cobordant, then they have the same signature. Signature is a cobordism invariant.* □

The signature vanishes only for boundaries. A result quite more difficult to prove is the following:

Theorem (*V. Rokhlin*)**.** *If a smooth oriented 4–manifold M has*

$$\operatorname{sign} Q_M = 0\,,$$

then there is a smooth oriented 5–manifold W such that $\partial W = M$.

> ***Idea of proof.*** A classic result of Whitney assures that any manifold X^m can be immersed in $\mathbb{R}^{2m-1}$; in particular, our M^4 can be immersed in $\mathbb{R}^7$. By performing various surgery modifications, we then arrange that M be cobordant to a 4–manifold M' that embeds in $\mathbb{R}^6$. Furthermore, a result of R. Thom[20] implies that M' must bound a 5–manifold W' inside $\mathbb{R}^6$. Attaching W' to the earlier cobordism from M to M' creates the needed W^5. A few more details for such a proof will be given in an inserted note on page 167. □

Therefore, the signature of M is zero if and only if M bounds. And hence:

Corollary (*Cobordisms and signatures*)**.** *Two 4–manifolds have the same signature if and only if they are cobordant. Signature is the complete cobordism invariant.* □

A consequence is that, unlike h–cobordisms, simple cobordisms are not very interesting: *Every 4–manifold M is cobordant to a connected sum of* $\mathbb{CP}^2$*'s or of* $\overline{\mathbb{CP}}^2$*'s or to* S^4. Indeed, assume that $\operatorname{sign} M = m > 0$; then, since $\operatorname{sign} \mathbb{CP}^2 = 1$, it follows that M and $\#m\,\mathbb{CP}^2$ must be cobordant; if $m < 0$, use $\overline{\mathbb{CP}}^2$'s instead.

Simple examples of intersection forms

Since the first example of a 4–manifold that comes to mind, namely the sphere S^4, does not have any 2–homology, it has no intersection form worth mentioning. Thus, we move on:

20. The result was quoted back in footnote 3 on page 112.

The complex projective plane. The complex projective plane $\mathbb{CP}^2$ has intersection form

$$Q_{\mathbb{CP}^2} = [+1] \ .$$

Indeed, since $H_2(\mathbb{CP}^2;\mathbb{Z}) = \mathbb{Z}\{[\mathbb{CP}^1]\}$ where $[\mathbb{CP}^1]$ is the class of a projective line, and since two projective lines always meet in a point, the equality above follows.

The oppositely-oriented manifold $\overline{\mathbb{CP}}^2$ has

$$Q_{\overline{\mathbb{CP}}^2} = [-1] \ .$$

Sphere bundles. The manifold $\mathbb{S}^2 \times \mathbb{S}^2$ has intersection form

$$Q_{\mathbb{S}^2\times\mathbb{S}^2} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} .$$

We will denote this matrix by H (from "hyperbolic plane").

Reversing orientation does not exhibit a new manifold: there exist orientation-preserving diffeomorphisms $\mathbb{S}^2 \times \mathbb{S}^2 \cong \overline{\mathbb{S}^2 \times \mathbb{S}^2}$, and they correspond algebraically to isomorphisms $H \approx -H$.

The twisted product $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ is the unique nontrivial sphere-bundle[21] over $\mathbb{S}^2$. It is obtained by gluing two trivial patches (*hemisphere*) $\times\, \mathbb{S}^2$ along the equator of the base-sphere, using the identification of the $\mathbb{S}^2$–fibers that rotates them by 2π as we travel along the equator. The intersection form is

$$Q_{\mathbb{S}^2\tilde{\times}\mathbb{S}^2} = \begin{bmatrix} 1 & 1 \\ 1 & \end{bmatrix} .$$

A simple change of basis in $H_2(\mathbb{S}^2 \tilde{\times} \mathbb{S}^2; \mathbb{Z})$ exhibits the intersection form as

$$Q_{\mathbb{S}^2\tilde{\times}\mathbb{S}^2} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} = [+1] \oplus [-1] \ .$$

Even more, it is not hard to argue that in fact we have a diffeomorphism[22]

$$\mathbb{S}^2 \tilde{\times} \mathbb{S}^2 \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \ ,$$

and so we have not really encountered anything essentially new.

21. Since an $\mathbb{S}^2$-bundle over $\mathbb{S}^2 = \mathbb{D}^2{}_1 \cup \mathbb{D}^2{}_2$ is described by an equatorial gluing map $\mathbb{S}^1 \to SO(3)$, and $\pi_1 SO(3) = \mathbb{Z}_2$, it follows that there are only two topologically-distinct sphere-bundles over a sphere.

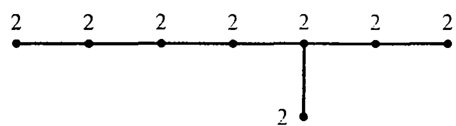
22. Quick argument: The equatorial gluing map $\mathbb{S}^1 \to SO(3)$ of $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ can be imagined as follows: as we travel along the equator of the base-sphere, it fixes the poles of the fiber-sphere and rotates the equator of the fiber-sphere by an angle increasing from 0 to 2π. Then these fiber-equators describe a circle-bundle of Euler number 1, which thus has to be the Hopf circle-bundle $\mathbb{S}^3 \to \mathbb{S}^2$. Hence the sphere-bundle is cut into two halves by a 3–sphere. Each of these halves is a disk-bundle of Euler number 1 and can therefore be identified with a neighborhood of $\mathbb{CP}^1$ inside $\mathbb{CP}^2$, but the complement of such a neighborhood is just a 4–ball. Taking care of orientations yields the splitting $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2 = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.

Connected sums. Of course, through the use of connected sums we can build a lot of boring examples, such as $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \# S^2 \times S^2$, whose intersection form is the sum $[+1] \oplus [-1] \oplus H$. (Incidentally, notice that this manifold has signature zero, and thus must be the boundary of some 5–manifold.)

The E_8–manifold. More interesting, though rather exotic, is Freedman's E_8–manifold $\mathcal{M}_{E_8} = P_{E_8} \cup_{\Sigma_P} \Delta$. This topological 4–manifold was built earlier[23] by plumbing on the E_8 diagram and capping with a fake 4–ball. Its intersection form can be read from the plumbing diagram to be

$$Q_{\mathcal{M}_{E_8}} = \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}.$$

From now on, we will denote this matrix[24] by E_8, and succinctly write $Q_{\mathcal{M}} = E_8$. The E_8–manifold does not admit any smooth structures.[25]



3.10. The E_8 diagram, yet again

An alternative algebraic description of this most important E_8–form is the following: Consider the form $Q = [-1] \oplus 8\,[+1]$, with corresponding basis $\{e_0, e_1, \ldots, e_8\}$. The vector $\kappa = 9e_0 + e_1 + \cdots + e_8$ has $\kappa \cdot \kappa = -1$; therefore its Q–orthogonal complement must be unimodular. This complement is the E_8–form. In particular, we have $E_8 \oplus [-1] \approx [-1] \oplus 8\,[+1]$.

Lemma. *The E_8–form is positive-definite, even, and of signature 8.*

Unexpectedly, proof. We will perform elementary operations on the rows and columns of the E_8–matrix. This will be fun.

23. See section 2.3 (page 86).

24. Various people have slightly different favorite choices for their E_8–matrix, for example, the negative of the above matrix. A brief discussion is contained in the end-notes of this chapter (page 137).

25. This is a consequence of Rokhlin's theorem, see section 4.4 (page 170) ahead.

First off, notice that these operations must be applied symmetrically, corresponding to changes of basis in $H_2(M;\mathbb{Z})$. That is to say, when for example we subtract $3/2$ times the first row from the third, we must afterwards also subtract $3/2$ times the first column from the third column. Indeed, since the matrix A of a bilinear form acts on $H_2 \times H_2$ by $(x,y) \mapsto x^t Ay$, any elementary change of basis $I + \lambda E_{ij}$ on H_2 will transform A into $(I + \lambda E_{ji})A(I + \lambda E_{ij})$.

Denote by $(1), (2), (3), (4), (5), (6), (7), (8)$ the eight rows/columns of the E_8–matrix, and let us start: We write down the E_8–matrix, then subtract $1/2 \times (1)$ from (2):

$$\begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} 2 & & & & & & & \\ & 3/2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}.$$

Subtract $2/3 \times (2)$ from (3), then subtract $3/4 \times (3)$ from (4):

$$\begin{bmatrix} 2 & & & & & & & \\ & 3/2 & & & & & & \\ & & 4/3 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} 2 & & & & & & & \\ & 3/2 & & & & & & \\ & & 4/3 & & & & & \\ & & & 5/4 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}.$$

Subtract $4/5 \times (4)$ from (5), then subtract $1/2 \times (8)$ from (5):

$$\begin{bmatrix} 2 & & & & & & & \\ & 3/2 & & & & & & \\ & & 4/3 & & & & & \\ & & & 5/4 & & & & \\ & & & & 6/5 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} 2 & & & & & & & \\ & 3/2 & & & & & & \\ & & 4/3 & & & & & \\ & & & 5/4 & & & & \\ & & & & 7/10 & 1 & & \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & & & & 2 \end{bmatrix}.$$

Subtract $10/7 \times (5)$ from (6), then subtract $7/4 \times (6)$ from (7):

$$\begin{bmatrix} 2 & & & & & & & \\ & 3/2 & & & & & & \\ & & 4/3 & & & & & \\ & & & 5/4 & & & & \\ & & & & 7/10 & & & \\ & & & & & 4/7 & 1 & \\ & & & & & 1 & 2 & \\ & & & & & & & 2 \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} 2 & & & & & & & \\ & 3/2 & & & & & & \\ & & 4/3 & & & & & \\ & & & 5/4 & & & & \\ & & & & 7/10 & & & \\ & & & & & 4/7 & & \\ & & & & & & 1/4 & \\ & & & & & & & 2 \end{bmatrix}.$$

We have diagonalized E_8, and its signature is 8. It is positive-definite. Its determinant is $\det E_8 = 2 \cdot 3/2 \cdot 4/3 \cdot 5/4 \cdot 7/10 \cdot 4/7 \cdot 1/4 \cdot 2 = 1$ and hence E_8 is unimodular, as claimed. □

A few more examples. *(1) The intersection form of* $\mathcal{M}_{E_8} \# \overline{\mathcal{M}}_{E_8}$ *is* $E_8 \oplus -E_8$. *Algebraically, we have* $E_8 \oplus -E_8 \approx \oplus 8\, H$ *through a suitable change of basis. As it turns out, this corresponds to an actual homeomorphism*[26]

$$\mathcal{M}_{E_8} \# \overline{\mathcal{M}}_{E_8} \simeq \#8\, \mathbb{S}^2 \times \mathbb{S}^2 \,.$$

Hence the smooth manifold $\#8\, \mathbb{S}^2 \times \mathbb{S}^2$ *can be cut into two non-smoothable topological 4–manifolds, along a topologically-embedded 3–sphere.*

(2) The intersection form of $\mathcal{M}_{E_8} \# \overline{\mathbb{CP}}^2$ *is* $[-1] \oplus 8\,[+1]$, *same as the intersection form of* $\overline{\mathbb{CP}}^2 \# 8\, \mathbb{CP}^2$. *The two 4–manifolds, though, are* not *homeomorphic, and the manifold* $\mathcal{M}_{E_8} \# \overline{\mathbb{CP}}^2$ *does not admit any smooth structures.*[27]

(3) The manifold $\mathcal{M}_{E_8} \# \mathcal{M}_{E_8}$, *with intersection form* $E_8 \oplus E_8$, *is not smooth.*[28] *Neither is* $\mathcal{M}_{E_8} \# \mathcal{M}_{E_8} \# \mathbb{S}^2 \times \mathbb{S}^2$, *nor is* $\mathcal{M}_{E_8} \# \mathcal{M}_{E_8} \# 2\, \mathbb{S}^2 \times \mathbb{S}^2$. *However, suddenly* $\mathcal{M}_{E_8} \# \mathcal{M}_{E_8} \# 3\, \mathbb{S}^2 \times \mathbb{S}^2$ does *admit smooth structures, and in what follows we will display such a smooth structure:*

3.3. Essential example: the $K3$ surface

A less exotic example (than the E_8–manifold) of a 4–manifold whose intersection form contains E_8's is the remarkable $K3$ complex surface that we build next:

The Kummer construction

Take the 4-torus

$$\mathbb{T}^4 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$$

and think of each $\mathbb{S}^1$–factor as the unit-circle inside $\mathbb{C}$. Consider the map

$$\sigma \colon \mathbb{T}^4 \to \mathbb{T}^4 \qquad \sigma(z_1, z_2, z_3, z_4) = (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)$$

given by complex-conjugation in each circle-factor, as in figure 3.11 on the next page. The involution σ has exactly $16 = 2^4$ fixed points, and thus the quotient

$$\mathbb{T}^4 / \sigma$$

will have sixteen singular points where it will fail to be a manifold. Small neighborhoods of these singular points are cones[29] on $\mathbb{RP}^3$.

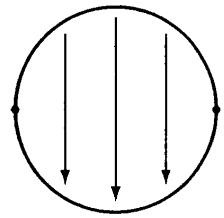
We wish to surger away these singular points of $\mathbb{T}^4/\sigma$ in order to obtain an actual 4–manifold. For that, we consider the *complex cotangent* bundle $T^*_{\mathbb{S}^2}$

26. This homeomorphism follows from Freedman's classification, see section 5.2 (page 239). A direct argument can also be made, starting with the observation that $\mathcal{M}_{E_8} \# \overline{\mathcal{M}}_{E_8}$ is the boundary of $(\mathcal{M}_{E_8} \setminus ball) \times [0,1]$.

27. This follows, again, from Freedman's classification.

28. This is a consequence of Donaldson's theorem, section 5.3 (page 243).

29. Remember that the **cone** C_A of a space A is simply the result of taking $A \times [0,1]$ and collapsing $A \times 1$ to a single point (the "vertex").



3.11. Conjugation, acting on $\mathbb{S}^1$

of the 2–sphere. It is the 2–plane bundle over $\mathbb{S}^2$ with Euler number -2 (it has *opposite orientation*[30] to the tangent bundle $T_{\mathbb{S}^2}$, whose Euler number is $+2$). Its unit-disk subbundle $\mathbb{D}T^*_{\mathbb{S}^2}$ is a 4–manifold bounded by $\mathbb{RP}^3$.

Since a neighborhood of a singular point in $\mathbb{T}^4/\sigma$ has the same boundary as $\mathbb{D}T^*_{\mathbb{S}^2}$, we can cut the former out of $\mathbb{T}^4/\sigma$ and replace it by a copy of $\mathbb{D}T^*_{\mathbb{S}^2}$. The result of this maneuver is essentially to remove the singular point and replace it with a sphere of self-intersection -2 (the zero-section of $\mathbb{D}T^*_{\mathbb{S}^2}$). We do this for all sixteen singular points.

Such a desingularization of $\mathbb{T}^4/\sigma$ yields a *simply-connected* smooth 4–manifold. This manifold admits a complex structure (thus it is a complex surface) and is called the ***K*3 surface.** The name comes from Kummer–Kähler–Kodaira.[31] The construction above is due to Kummer, which is why this manifold used to be known merely as the *Kummer surface*.

Homology. The $K3$ surface has homology $H_2(K3;\mathbb{Z}) = \oplus\, 22\mathbb{Z}$ (superficially, from 6 tori surviving from $\mathbb{T}^4$, plus the 16 desingularizing spheres). Its intersection form is

$$Q_{K3} = -\begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix} \oplus -\begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}$$
$$\oplus \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \oplus \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \oplus \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$$

and clearly it is better kept abbreviated as

$$Q_{K3} = \oplus\, 2(-E_8) \oplus 3H\ .$$

30. For a discussion of orientations for complex-duals, see the end-notes of this chapter (page 134).

31. A. Weil wrote that, besides honoring Kummer, Kodaira and Kähler, the name "*K*3" was also chosen in relation to the famous K2 peak in the Himalayas: *"[Surfaces] ainsi nommées en l'honneur de Kummer, Kähler, Kodaira, et de la belle montagne K2 au Cachemire."*

Even if this manifold does not seem simple at all, it is in many ways as simple as it gets. We will see that $K3$ is indeed the *simplest*[32] simply-connected *smooth* 4–manifold that is not $\mathbb{S}^4$ nor a boring sum of $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$'s.

The desingularization, revisited. Let us take a closer look at the desingularization of $\mathbb{T}^4/\sigma$ that created $K3$ and try to better visualize it.

Consider first a neighborhood inside $\mathbb{T}^4$ of a fixed point x_0 of σ. It is merely a 4–ball, which can be viewed as a cone over its boundary 3–sphere $\mathbb{S}^3$, with vertex at x_0. The action of σ on this cone can itself be viewed as being the cone[33] of the antipodal map $\mathbb{S}^3 \to \mathbb{S}^3$ (which sends w to $-w$). Therefore, the quotient of this neighborhood of x_0 by σ must be a cone on the quotient of $\mathbb{S}^3$ by the antipodal map, in other words, a cone on $\mathbb{RP}^3$.

Furthermore, $\mathbb{S}^3$ is fibrated by the Hopf map,[34] which makes it into a bundle with fiber $\mathbb{S}^1$ and base $\mathbb{S}^2$. Then its quotient $\mathbb{RP}^3$ inherits a structure of $\mathbb{RP}^1$–bundle over $\mathbb{S}^2$:

$$\begin{array}{ccccc} \mathbb{S}^1 & \subset & \mathbb{S}^3 & \longrightarrow & \mathbb{S}^2 \\ {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow & & \| \\ \mathbb{RP}^1 & \subset & \mathbb{RP}^3 & \longrightarrow & \mathbb{S}^2. \end{array}$$

However, $\mathbb{RP}^1$ is simply a circle, so in fact we exhibited $\mathbb{RP}^3$ as an $\mathbb{S}^1$–bundle over $\mathbb{S}^2$.

Now let us look back at the neighborhood of a singular point of $\mathbb{T}^4/\sigma$. It is a cone on $\mathbb{RP}^3$, and we can think of it as being built by attaching a disk to each circle-fiber of $\mathbb{RP}^3$, and then identifying all their centers in order to obtain the vertex of the cone, the singular point. When we desingularize, we replace this cone-neighborhood in $\mathbb{T}^4/\sigma$ with a copy of $\mathbb{D}T^*_{\mathbb{S}^2}$. This can be viewed simply as *not* identifying the centers of those disks attached to the fibers of $\mathbb{RP}^3$, but keeping them disjoint. The space of the circle-fibers of $\mathbb{RP}^3$ is the base $\mathbb{S}^2$ of the fibration. Thus the space of the attached disks is $\mathbb{S}^2$ as well, and thus their centers (now distinct) will draw a new 2–sphere, which replaced the singular point.

We can thus think of our desingularization as simply replacing each of the sixteen singular points of $\mathbb{T}^4/\sigma$ by a sphere with self-intersection -2.

32. We take "simple" to include "simple to describe". Smooth manifolds with simpler intersection forms already exist (*e.g.*, exotic $\#m\,\mathbb{S}^2 \times \mathbb{S}^2$'s, see page 553), and exotic $\mathbb{S}^4$'s could always appear.

33. Remember that the **cone** C_f of a map $f\colon A \to B$ is the function $C_f\colon C_A \to C_B$ defined by first extending $f\colon A \to B$ to $f \times id\colon A \times [0,1] \to B \times [0,1]$, then collapsing $A \times 1$ to a point and $B \times 1$ to another, with the the resulting function $C_f\colon C_A \to C_B$ sending vertex to vertex.

34. Remember that the **Hopf map** is defined to send a point $x \in \mathbb{S}^3 \subset \mathbb{C}^2$ to the point from $\mathbb{S}^2 = \mathbb{CP}^1$ that represents the complex line spanned by x inside $\mathbb{C}^2$. Topologically, the Hopf bundle $\mathbb{S}^3 \to \mathbb{S}^2$ is a circle-bundle of Euler class $+1$. Two distinct fibers will be two circles in $\mathbb{S}^3$ linked once (a so-called Hopf link, see figure 8.16 on page 318). The Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$ represents the generator of $\pi_3\mathbb{S}^2 = \mathbb{Z}$.

Holomorphic construction

A complex geometer would construct the Kummer $K3$ in a way that visibly exhibits its complex structure. Specifically, she would start with $\mathbb{T}^4$ being a complex torus—for example the simplest such, the product of two copies of $\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$. Such a $\mathbb{T}^4$ comes equipped with complex coordinates (w_1, w_2), and the involution σ can be described as $\sigma(w_1, w_2) = (-w_1, -w_2)$ (which is obviously holomorphic).

As before, the action of σ has sixteen fixed points, but, before taking the quotient, the complex geometer will blow-up[35] $\mathbb{T}^4$ at these sixteen points. This has the result of replacing each fixed point of σ with a sphere of self-intersection -1 (a neighborhood of which looks like a neighborhood of $\overline{\mathbb{CP}}^1$ inside $\overline{\mathbb{CP}}^2$). The map σ can be extended across this blown-up 4–torus: since she replaced the fixed points of σ by spheres, she can extend σ across the new spheres simply as the identity, thus letting the whole spheres be fixed by the resulting σ.

Only now will the complex geometer take the quotient by σ of the blown-up 4–torus. The result is the $K3$ surface. The spheres of self-intersection -1 created when blowing-up the torus will project to the quotient $K3$ as themselves (they were fixed by σ), but their neighborhoods are doubly-covered through the action of σ; thus these spheres inside $K3$ have now self-intersection -2.

Many K3's. This is the place to note that a complex geometer will in fact see a *multitude* of $K3$ surfaces. Indeed, "$K3$" is not the name of *one* complex surface, but the name of a class of surfaces.[36] Any non-singular simply-connected complex surface with $c_1 = 0$ is a **$K3$ surface.**

For example, in the construction above, if we start with a different complex structure on $\mathbb{T}^4$ (from factoring $\mathbb{C}^2$ by a different lattice), then we will end up with a different $K3$ surface. All $K3$'s that result from such a construction are called **Kummer surfaces.** However, $K3$ surfaces can be built in many other ways. One example is the hypersurface of $\mathbb{CP}^3$ given by the homogeneous equation

$$z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0$$

(or any other smooth surface of degree 4). Another is the $E(2)$ elliptic surface that we will describe in chapter 8 (page 301).

This whole multitude of complex $K3$ surfaces, through the blinded eyes of the topologist, are just *one* smooth 4–manifold: any two $K3$'s are complex-deformations of each other, and thus are diffeomorphic. Hence, in this book we will carelessly be saying "*the* $K3$ surface".

35. For a discussion of blow-ups, see ahead section 7.1 (page 286).

36. For instance, the moduli space of all $K3$ surfaces has dimension 20.

K3 as an elliptic fibration

The $K3$ surface can be structured as a singular fibration over $\mathbb{S}^2$, with generic fiber a torus. A (singular) fibration by tori of a complex surface is called an **elliptic fibration** (because a torus in complex geometry is called an *elliptic curve*). A complex surface that admits an elliptic fibration is called an **elliptic surface.** The Kummer $K3$ is such an elliptic surface. Other examples of elliptic surfaces, as well as a different elliptic fibration on the $K3$ manifold, will be discussed later.[37]

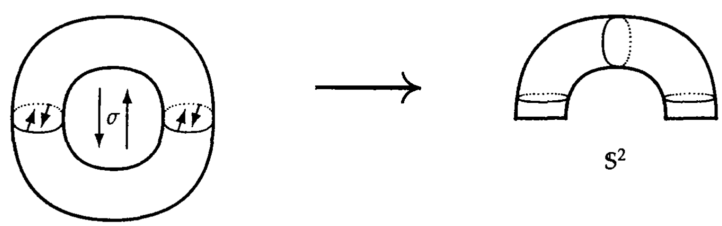
In any case, describing the elliptic fibration of $K3$ will help us better visualize this manifold. To exhibit it, we start with the projection

$$\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$$

of $\mathbb{T}^4$ onto its first two factors. After taking the quotient by the action of σ, this projection descends to a map

$$\mathbb{T}^4/\sigma \longrightarrow \mathbb{T}^2/\sigma\,.$$

Its target $\mathbb{T}^2/\sigma$ is a non-singular sphere $\mathbb{S}^2$, as suggested in figure 3.12 (it seems like it has four singular points at the corners, but these are merely metric-singular, and can be smoothed over).



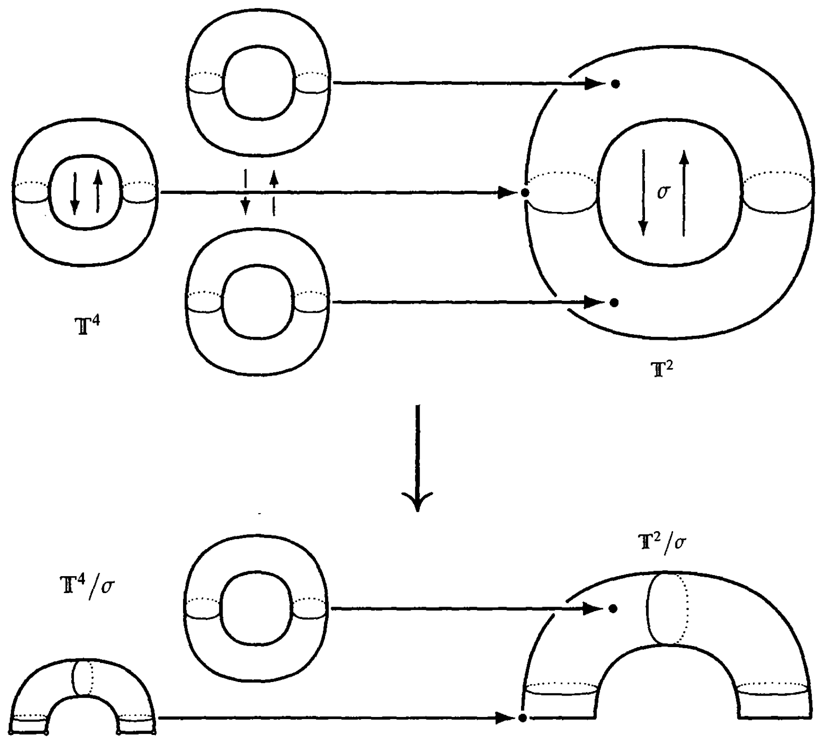
3.12. Obtaining the base sphere: $\mathbb{T}^2/\sigma = \mathbb{S}^2$

Aside from the corner-points of the base-sphere $\mathbb{T}^2/\sigma$, each of its other points comes from two distinct points (p,q) and $(\overline{p},\overline{q})$ of $\mathbb{T}^2$ identified by σ. Thus, the fiber of the map $\mathbb{T}^4/\sigma \to \mathbb{T}^2/\sigma$ over a generic point appears from σ's identifying two distinct tori $p \times q \times \mathbb{S}^1 \times \mathbb{S}^1$ and $\overline{p} \times \overline{q} \times \mathbb{S}^1 \times \mathbb{S}^1$ from $\mathbb{T}^4$. The resulting fiber will itself be a torus. This is the generic fiber of $\mathbb{T}^4/\sigma \to \mathbb{T}^2/\sigma$. See also figure 3.13 on the following page.

On the other hand, each of the four corner-points of the sphere $\mathbb{T}^2/\sigma$ comes from a single fixed point (p_0, q_0) of σ on $\mathbb{T}^2$. Thus, the fiber of $\mathbb{T}^4/\sigma \to \mathbb{T}^2/\sigma$ over such a corner appears from σ's sending a torus $p_0 \times q_0 \times \mathbb{S}^1 \times \mathbb{S}^1$ to itself. The quotient of this torus is again a cornered-sphere (just as before, in figure 3.12), but now its corners coincide with the sixteen global fixed points of σ on $\mathbb{T}^4$. In other words, each such sphere-fiber contains four

37. See chapter 8 (starting on page 301), which is devoted to these creatures.

of the sixteen singular points of the quotient $\mathbb{T}^4/\sigma$, points where the latter fails to be a manifold. See again figure 3.13.



3.13. The map $\mathbb{T}^4/\sigma \to \mathbb{T}^2/\sigma$ and its fibers

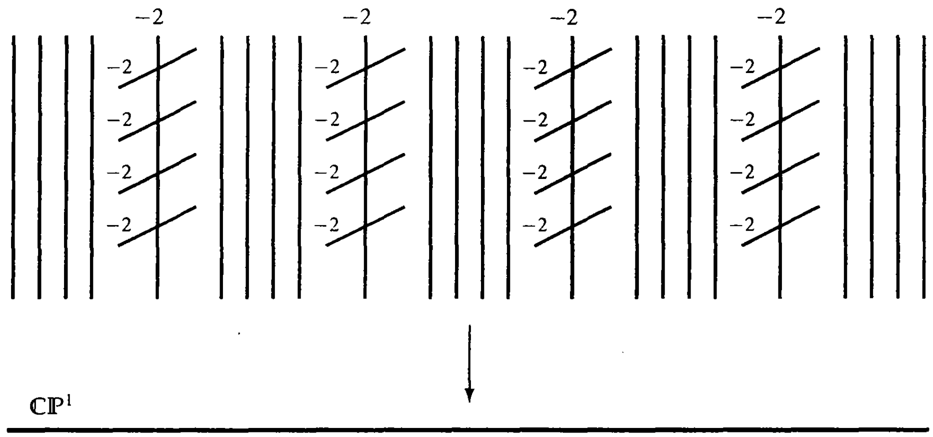
This might be a good moment to notice that $\mathbb{T}^4/\sigma$ is simply-connected. It fibrates over $\mathbb{S}^2$, which is simply-connected, and any loop in a generic torus fiber can be moved along to one of the singular sphere-fibers and contracted there. The desingularization of $\mathbb{T}^4/\sigma$ into $K3$ does not create any new loops, and therefore the $K3$ surface is, as claimed, simply-connected.

As explained before, we cut neighborhoods of the singular points out of $\mathbb{T}^4/\sigma$ and glue a copy of $\mathbb{D}T^*_{\mathbb{S}^2}$ in their stead, thus replacing each singular point by a sphere; the result is the $K3$ surface. The projection $\mathbb{T}^4/\sigma \to \mathbb{T}^2/\sigma$ survives the desingularization as a map

$$K3 \longrightarrow \mathbb{S}^2 .$$

Indeed, since we only replaced sixteen points by sixteen spheres, we can send each of these spheres wherever the removed point used to go in $\mathbb{S}^2$.

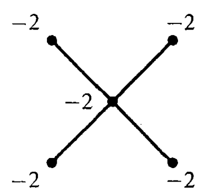
The generic fiber of $K3 \to \mathbb{S}^2$ is still a torus. However, there are now also four singular fibers, each made of five transversely-intersecting spheres: the old singular sphere-fiber of $\mathbb{T}^4/\sigma$, together with its four desingularizing spheres. A symbolic picture of this fibration is figure 3.14.



3.14. $K3$ as the Kummer elliptic fibration

Observe that the main sphere of the singular fiber must have self-intersection -2. *This can be can argued as follows: Denote by* S *the main sphere of a singular fiber and by* S_1, S_2, S_3, S_4 *the desingularizing spheres. Recall how the main sphere* S *appeared from factoring by* σ*: doubly-covered by a torus. Imagine a moving generic torus-fiber* F *of* $K3$ *approaching our singular fiber: it will wrap around the main sphere twice, covering it. Also, the approaching fiber will extend to cover the desingularizing spheres once, and so in homology we have* $F = 2S + S_1 + S_2 + S_3 + S_4$. *We know that* $F \cdot F = 0$ *(since it is a fiber), and that each* $S_k \cdot S_k = -2$*; then one can compute that we must also have* $S \cdot S = -2$.

Finally, note that a neighborhood of the singular fiber inside $K3$ can be obtained by plumbing five copies of $\mathbb{D}T^*_{S^2}$ following the diagram from figure 3.15.



3.15. Plumbing diagram for neighborhood of singular fiber

3.4. Notes

Note: Duals of complex bundles and orientations

The pretext for this note is to explain why the cotangent bundle $T^*_{S^2}$ (used earlier for building $K3$) has Euler class -2 rather than $+2$; that is to say, why $T^*_{S^2}$ and T_{S^2} have opposite orientations.

Let V be a real vector space, endowed with a complex structure. There are two ways to think of such a creature: (1) we can view V as a complex vector space, in other words, think of it as endowed with an action of the complex scalars $\mathbb{C} \times V \to V$ that makes V into a vector space over the field of complex numbers; or (2) we can view V as a real space endowed with an automorphism $J \colon V \to V$ with the property that $J \circ J = -id$. One should think of this J as a proxy for the multiplication by i. The two views are clearly equivalent, related by

$$J(v) = i \cdot v \,.$$

Nonetheless, they naturally lead to two different versions of a complex structure for the dual vector space.

The real version. Let us first discuss the case when we view V as a real vector space endowed with an anti-involution J. As a real vector space, the dual of V is

$$V^* = \operatorname{Hom}_{\mathbb{R}}(V; \mathbb{R}) \,.$$

A vector space and its dual are isomorphic, but there is no natural choice of isomorphism. To fix a choice of such an isomorphism, we endow V with an auxiliary inner-product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$. Then V and V^* are naturally isomorphic through

$$V \xrightarrow{\approx} V^* \colon \qquad v \longmapsto v^* = \langle \cdot, v \rangle_{\mathbb{R}} \,.$$

If V is endowed with a complex structure J, then it is quite natural to restrict the choice of inner-product to those that are compatible with J. This means that we only choose inner-products that are invariant under J: we require that

$$\langle Jv, Jw \rangle_{\mathbb{R}} = \langle v, w \rangle_{\mathbb{R}} \,.$$

An immediate consequence is that we have $\langle Jv, w \rangle_{\mathbb{R}} = -\langle v, Jw \rangle_{\mathbb{R}}$.

We now wish to endow the dual V^* with a complex structure of its own. In other words, we want to define a natural anti-involution $J^* \colon V^* \to V^*$ induced by J. Since an isomorphism $V \approx V^*$ was already chosen, it makes sense now to simply transport J from V to V^* through that isomorphism. Namely, we define the complex structure J^* of V^* by

$$J^*(v^*) = (Jv)^* \,.$$

More explicitly, if $f \in V^*$ is given by $f(x) = \langle x, v\rangle_{\mathbb{R}}$ for some $v \in V$, then $(J^*f)(x) = \langle x, Jv\rangle_{\mathbb{R}}$. However, this means that $(J^*f)(x) = -\langle Jx, v\rangle_{\mathbb{R}}$, and so we have

$$J^*f = -f(J\,\cdot)\,.$$

Notice that we ended up with a formula that does *not* depend on the choice of inner-product. Hence we have defined a natural complex structure J^* on the real vector space $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$.

The complex version. If, on the other hand, we think of the complex structure of V as an action of the complex scalars that makes V into a vector space $V_{\mathbb{C}}$ over the complex numbers, then a different notion of dual space comes to the fore. We must define the dual as

$$V_{\mathbb{C}}^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})\,.$$

This vector space comes from birth equipped with a complex structure, namely

$$(i \cdot f)(x) = i\,f(x)$$

for every $f \in V_{\mathbb{C}}^*$. To better grasp what this $V_{\mathbb{C}}^*$ looks like, we will endow $V_{\mathbb{C}}$ with an auxiliary inner-product. The appropriate notion of inner-product for complex vector spaces is that of **Hermitian** inner-products. This differs from the usual inner products by the facts that it is complex-valued, and it is complex-linear in its first variable, but complex anti-linear in the second. We have $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{C}} : V \times V \to \mathbb{C}$ with $\langle zv, w\rangle_{\mathbb{C}} = z\langle v, w\rangle_{\mathbb{C}}$, but $\langle v, zw\rangle_{\mathbb{C}} = \bar{z}\langle v, w\rangle_{\mathbb{C}}$ for every[1] $z \in \mathbb{C}$.

Any Hermitian inner product can then be used to define a complex-isomorphism of $V_{\mathbb{C}}^*$, though not with $V_{\mathbb{C}}$, but with its **conjugate** vector space $\overline{V}_{\mathbb{C}}$. The latter is defined as being the real vector space V endowed with an action of complex scalars that is conjugate to that of $V_{\mathbb{C}}$. That is to say, in $\overline{V}_{\mathbb{C}}$ we have $i \cdot v = -iv$. The complex-isomorphism with the dual is:

$$\overline{V}_{\mathbb{C}} \xrightarrow{\approx} V_{\mathbb{C}}^*: \qquad v \longmapsto v^* = \langle\,\cdot\,, v\rangle_{\mathbb{C}}\,.$$

Notice that in the definition of v^* we must put v as the second entry in $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{C}}$, so that v^* be a complex-linear function and thus indeed belong to $V_{\mathbb{C}}^*$.

If $f \in V_{\mathbb{C}}^*$ is given by $f(x) = \langle x, v\rangle_{\mathbb{C}}$ for some $v \in V$, then we have $(if)(x) = i\,f(x) = i\langle x, v\rangle_{\mathbb{C}} = \langle x, -iv\rangle_{\mathbb{C}}$. This means that we have

$$i \cdot v^* = (-iv)^*\,,$$

which shows that the complex-isomorphism above is indeed between the dual $V_{\mathbb{C}}^*$ and the *conjugate* vector space $\overline{V}_{\mathbb{C}}$.

Comparison. In review, if we view a complex vector space as (V, J), then its dual is (V^*, J^*) and the two are complex-isomorphic. If we view a complex vector space as $V_{\mathbb{C}}$, then its dual is $V_{\mathbb{C}}^*$, which is complex-isomorphic to $\overline{V}_{\mathbb{C}}$. To compare the two versions, it is enough to notice that $\overline{V}_{\mathbb{C}}$ translates simply as $(V, -J)$. Indeed, as *real* vector spaces (*i.e.*, ignoring the complex structures) V^* and $V_{\mathbb{C}}^*$ are

---

1. It is worth noticing that the concept of a real inner product compatible with a complex structure, and the concept of Hermitian inner product are equivalent: one can go from one to the other by using $\langle v, w\rangle_{\mathbb{C}} = \langle v, w\rangle_{\mathbb{R}} - i\,\langle iv, w\rangle_{\mathbb{R}}$ and $\langle v, w\rangle_{\mathbb{R}} = \mathrm{Re}\,\langle v, w\rangle_{\mathbb{C}}$.

naturally isomorphic. Specifically, the isomorphism $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) \approx \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ sends $f\colon V \to \mathbb{R}$ to the function $f_{\mathbb{C}}\colon V \to \mathbb{C}$ given by

$$f_{\mathbb{C}}(x) = \tfrac{1}{2}\big(f(x) - i f(Jx)\big) .$$

The duals (V^*, J^*) and $V^*_{\mathbb{C}}$ thus differ not as real vector spaces, but because their complex structures are conjugate. This could be checked directly against the isomorphism above, or, in the simplifying presence of an inner-product, we could simply write:

$$J^*(v^*) = (iv)^* \qquad \text{and} \qquad i \cdot v^* = (-iv)^* .$$

Usage. We should emphasize that, while the "complex" version of dual is certainly the most often used, nonetheless both these versions are important.

As a typical example, consider a complex manifold X, which is endowed with a tangent bundle T_X and a cotangent bundle T^*_X. Owing to the complex structure of X, the tangent bundle has a natural complex structure on its fibers. The complex structure on T^*_X is *always* taken to be dual to the one on T_X in its "complex" version: as complex bundles, we have $T^*_X \approx \overline{T_X}$. In general for vector bundles with complex structures, the dual is usually taken to be the "complex" dual.

The "real" version of dual is also used in complex geometry. Thinking now of the complex structure of T_X as $J\colon T_X \to T_X$, we let it induce its own dual complex structure J^* on T^*_X. We then extend J^* by linearity to the complexified vector space $T^*_X \otimes_{\mathbb{R}} \mathbb{C}$. The advantage of such an extension is that now J^* has eigenvalues $\pm i$, and thus splits the bundle $T^*_X \otimes \mathbb{C}$ into its $\pm i$–eigenbundles as

$$T^*_X \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1} ,$$

and hence separates complex-valued 1-forms on X into type $(1,0)$ and type $(0,1)$. This is simply a splitting into complex-linear and complex-anti-linear parts: indeed $J^*(\alpha) = -i\alpha$ if and only if $\alpha(Jx) = +i\alpha(x)$, and then $\alpha \in \Lambda^{1,0}$.

The advantage of using J lies in part with clarity of notation: for a complex-valued creature, J will denote the complex action on its arguments (living on X), while i denotes the complex action on its values (living in $\mathbb{C}$).

More on complex-valued forms. *Every complex-valued function $f\colon X \to \mathbb{C}$ has its differential $df \in \Gamma(T^*_X \otimes \mathbb{C})$ split into its $(1,0)$-part $\partial f \in \Gamma(\Lambda^{1,0})$ and its $(0,1)$-part $\bar\partial f \in \Gamma(\Lambda^{0,1})$. Hence, $\bar\partial f = 0$ means that f's derivative is complex-linear, $df(Jx) = i\,df$, and thus that f is* holomorphic.

By using local real coordinates $(x_1, y_1, \dots, x_m, y_m)$ on X such that $z_k = x_k + iy_k$ are local complex coordinates on X, we can define $dz_k = dx_k + i\,dy_k$ and $d\bar z_k = dx_k - i\,dy_k$, and write $\Lambda^{1,0} = \mathbb{C}\{dz_1, \dots, dz_m\}$ and $\Lambda^{0,1} = \mathbb{C}\{d\bar z_1, \dots, d\bar z_m\}$. Indeed, $J^*(d\bar z_k) = +i\,d\bar z_k$.

The split $\Lambda^1 \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}$ further leads to a splitting of all complex-valued forms into (p,q)-types, as in $\Lambda^k \otimes \mathbb{C} = \Lambda^{k,0} \oplus \Lambda^{k-1,1} \oplus \cdots \oplus \Lambda^{1,k-1} \oplus \Lambda^{0,k}$. Specifically, $\Lambda^{p,q}$ is made of all complex-valued forms that can be written using p of the dz_k's and q of the $d\bar z_k$'s. For example, $\Lambda^{2,0}$ contains all complex-bilinear 2-forms.

The exterior differential $d\colon \Gamma(\Lambda^k) \to \Gamma(\Lambda^{k+1})$ splits, after complexification, as $d = \partial + \bar\partial$ with $\partial\colon \Gamma(\Lambda^{p,q}) \to \Gamma(\Lambda^{p+1,q})$ and $\bar\partial\colon \Gamma(\Lambda^{p,q}) \to \Gamma(\Lambda^{p,q+1})$. Since $\bar\partial\bar\partial = 0$, this can be used to define cohomology groups $H^{p,q}(X) = \mathrm{Ker}\,\bar\partial / \mathrm{Im}\,\bar\partial$ (called **Dolbeault cohomology***), which offer a cohomology splitting $H^k(X;\mathbb{C}) = H^{k,0}(X) \oplus H^{k-1,1}(X) \oplus \cdots \oplus H^{1,k-1}(X) \oplus H^{0,k}(X)$, with $H^{p,q}(X) \approx \overline{H^{q,p}(X)}$; further, if X is Kähler, then the Hodge duality operator[2] $*$ will take*

2. The Hodge operator will be recalled in section 9.3 (page 350).

(p,q)-forms to $(m-q,\ m-p)$-forms, and lead into complex Hodge theory, to just drop some names. Any complex geometry book will explain these topics properly, for example **P. Griffiths** *and* **J. Harris***'s* ***Principles of algebraic geometry*** **[GH78, GH94]***; we ourselves will make use of (p,q)-forms for some technical points later on.*[3] *Part of this topic will be explained in more detail in the end-notes of chapter 9 (page 365).*

Orientations. Every vector space with a complex structure (defined either way) is naturally oriented by any basis like $\{e_1, ie_1, \ldots, e_k, ie_k\}$ (or $\{e_1, Je_1, \ldots, e_k, Je_k\}$). Thus its dual vector space, getting a complex structure itself, will be naturally oriented as well. However, the choice of duality matters: if our vector space V is odd-dimensional (over $\mathbb{C}$), then the two versions of dual complex structure lead to *opposite* orientations of V's dual. Specifically, the real-isomorphism $V \approx V^*_{\mathbb{C}}$ reverses orientations, while $V \approx (V^*, J^*)$ preserves them.

For complex manifolds and their tangent/cotangent bundles, as we mentioned above, one uses the "complex" version of duality. Therefore, for a complex curve C (for example, S^2) we have that the tangent bundle T_C and the cotangent bundle T^*_C, while isomorphic as real bundles, are naturally oriented by opposite orientations. In particular, the tangent bundle T_{S^2} is the plane bundle of Euler class $+2$, while the cotangent bundle $T^*_{S^2}$ is the plane bundle with Euler class -2.

For a complex surface M (for example, $K3$), the tangent and cotangent bundles do not have opposite orientations. Nonetheless, their complex structures are conjugate, and this leads to phenomena like $c_1(T^*_M) = -c_1(T_M)$.

Note: Positive E_8, negative E_8

In some texts, the E_8-form is sometimes described by the matrix

$$E_8 \approx \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{bmatrix}.$$

Correspondingly, the negative-E_8-form is sometimes written

$$-E_8 \approx \begin{bmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & 1 \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & \\ & & & & 1 & & & -2 \end{bmatrix}.$$

These alternative matrices are in fact equivalent with the ones presented earlier, because one can always find an isomorphism between the two versions: simply change the sign of "every other" element of the basis. Then the self-intersections

3. In section 6.2 (page 278), the end-notes of chapter 9 (connections and holomorphic bundles, page 365) and the end-notes of chapter 10 (Seiberg–Witten on Kähler and symplectic, page 457).

are preserved, but, if done properly, the intersections between distinct elements will all change signs. Peek back at the E_8 diagram for inspiration.

Complex geometers always prefer to have $+1$'s off the diagonal (thinking in terms of complex submanifolds, which always intersect positively), and so they will write $-E_8$ in the version displayed above.

More than this, certain texts prefer to switch the names of the E_8- and negative-E_8-matrices. Since what we denote here by $-E_8$ appears quite more often than E_8, calling it E_8 does save some writing.

Pick your own favorites.

Bibliography

For strong and general results about representing homology classes by submanifolds, see **R. Thom**'s celebrated paper ***Quelques propriétés globales des variétés différentiables*** [**Tho54**] (the results were first announced in [**Tho53a**]).

For a definition of intersections directly in terms of cycles (not necessarily submanifolds), see **P. Griffiths** and **J. Harris**'s ***Principles of algebraic geometry*** [GH78, **GH94**, sec 0.4]; there one can also find an intersection-based view of Poincaré duality. For the rigorous algebraic topology development of various products and pairings of co/homology, see **A. Dold**'s ***Lectures on algebraic topology*** [**Dol80, Dol95**]; the differential-forms approach is best culled from **R. Bott** and **L. Tu**'s ***Differential forms in algebraic topology*** [**BT82**].

Intersection forms can be defined in all dimensions $4k$, and their signature is an important invariant. For example, it is the main surgery obstruction in those dimensions, see for example **W. Browder**'s ***Surgery on simply-connected manifolds*** [**Bro72**].

That all 4–manifolds of zero signature bound was proved in **V. Rokhlin**'s ***New results in the theory of four-dimensional manifolds*** [**Rok52**], alongside his celebrated Rokhlin's theorem that we will discuss in the next chapter. A French translation of the paper can be read as [**Rok86**] from the volume ***À la recherche de la topologie perdue*** [GM86a]. A geometric proof of this result can be read from **R. Kirby**'s ***The topology of 4–manifolds*** [Kir89, ch VIII], and a slightly more complete outline will be presented on page 167 ahead. A different-flavored proof is contained in **R. Stong**'s ***Notes on cobordism theory*** [**Sto68**].

The E_8–manifold was defined, alongside the fake 4–balls Δ, in **M. Freedman**'s ***The topology of four-dimensional manifolds*** [**Fre82**]; see also, of course, **M. Freedman** and **F. Quinn**'s ***Topology of 4–manifolds*** [**FQ90**].

For the $K3$ surface, the algebro-geometric point-of-view is discussed at length in **W. Barth, C. Peters** and **A. Van de Ven**'s ***Compact complex surfaces*** [BPVdV84] (or the second edition [**BHPVdV04**], with **K. Hulek**). Also, inevitably, in **P. Griffiths** and **J. Harris**'s ***Principles of algebraic geometry*** [**GH78, GH94**]. For a topological point-of-view on $K3$, see **R. Gompf** and **A. Stipsicz**'s ***4–Manifolds and Kirby calculus*** [GS99], or **R. Kirby**'s ***The topology of 4–manifolds*** [**Kir89**]. We will come back to the $K3$ surface ourselves in chapter 8 (starting on page 301), where we will discuss it alongside its elliptic-surface brethren.