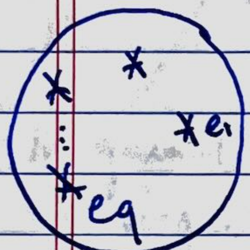
Day 4 Lecture 3

V Case study: Rational elliptic surfaces
(joint with Eduard Looijenga)

Let $M = Bl_{\{p_1, \dots, p_9\}}(\mathbb{CP}^2)$ ($n \le 8$ anticanonical bundle is ample)



$e_i = \mathbb{CP}^1$ exceptional divisor

$\downarrow \pi$ Blow down

$\mathbb{CP}^2$

$$Q_M = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} \qquad (H_2(M;\mathbb{Z}), \cdot) = H_M = (1) \oplus 9(-1)$$

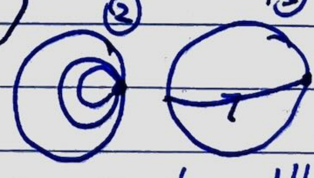
Freedman–Quinn: $\mathrm{Mod}(M) \xrightarrow{\cong} \mathrm{Aut}(H_M) \cong O(1,9)(\mathbb{Z})$

$\mathrm{Mod}(M) \curvearrowright H_M$

$O(1,9)(\mathbb{Z}) \curvearrowright \left(\mathbb{Z}^{10}, \begin{pmatrix} 1 & & \\ & -1 & \\ & & \ddots \\ & & & -1 \end{pmatrix}\right)$

Question: Given $A \in O(1,9)(\mathbb{Z})$, find a representative $f_A \in \mathrm{Diff}^+(M)$ or $\mathrm{Homeo}^+(M)$ s.t. $(f_A)_* = A$.

$O(1,9)(\mathbb{Z}) \subseteq O(1,9)(\mathbb{R}) = \mathrm{Isom}(\mathbb{H}^9)$ ② ③

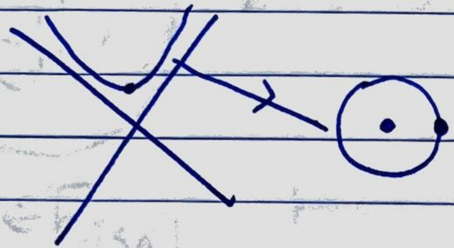


3 cases: ① A is finite order

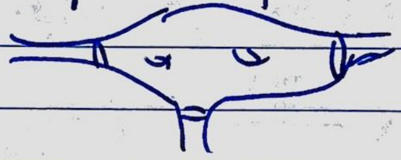
② A is parabolic type (i.e. A fixes a unique point on $\partial\mathbb{H}^9$)

③ A is hyperbolic type (i.e. A translates along a unique geodesic in $\mathbb{H}^9$)

Fact from hyperbolic geometry: $\exists !$ (up to conjugation in $O(1,9)(\mathbb{Z})$)
maximal parabolic subgroup $\Gamma < O(1,9)(\mathbb{Z})$ $\quad \Gamma = \mathrm{Stab}(v)$
$v \in (1) \oplus 9(-1)$ s.t. $v \cdot v = 0$



In general hyperbolic manifold with finite volume, you have finitely many ends



$$1 \to \underset{\cong \mathbb{Z}^8}{v^{\perp}/\mathbb{Z}v} \xrightarrow{\text{shear}} \mathrm{Stab}(v) \longmapsto O(v^{\perp}/v) \to 1$$

$v^{\perp}/v = \mathbb{Z}^8 = (-8)$
$v^{\perp} = (\cancel{1},8)$

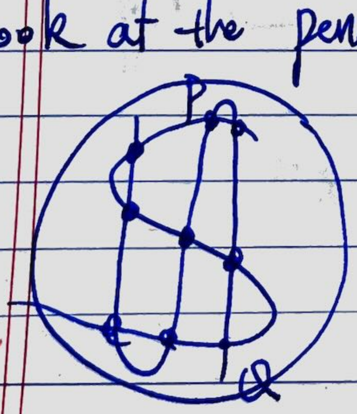
finite group, Weyl group

Problem Today: Given $A \in \mathrm{Stab}(v)$, construct $f \in \mathrm{Diff}^+(M)$.

$\boxed{\text{Classic Construction}}$

Take two (generic) smooth cubic curves $\underset{\cong T^2}{\{P=0\}}$ and $\underset{\cong T^2}{\{Q=0\}}$ in $\mathbb{CP}^2$.

Look at the pencil $E_{[s,t]} = \{sP + tQ = 0\}$ of cubic curves.



Bezout $\to$ 9 intersection points $p_1 \ldots p_9$

$E_{[s,t]}$ contains $\{p_1, \ldots, p_9\}$.

Moreover $\mathbb{C}P^2 = \bigcup_{[s:t]\in\mathbb{P}^1} E_{[s,t]}$

 $\xmapsto{\pi} \mathbb{C}P^1$

$x \in E_{[s,t]} \longmapsto [s,t]$ or $[x:y:z] \mapsto [Q(x,y,z) : P(x,y,z)]$

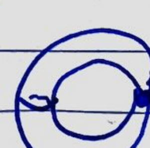
Not well-defined on $\{P=0\}\cap\{Q=0\}$, we blow up $\{p_1 \cdots p_9\}$

$\Rightarrow \quad Bl_{(p_1\cdots p_9)}(\mathbb{C}P^2) \xmapsto{\pi} \mathbb{C}P^1$

For generic P, Q (intersect transversally)

$\pi^{-1}([s,t]) = E_{[s,t]}$ is smooth $\cong T^2$

except at 12 values of $[s,t]$

At each 12 singular fibers $E_{[s,t]} \cong$  rational nodal curve

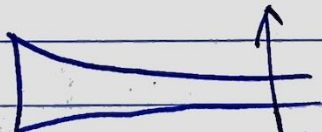
↑

Fishtail fiber

"elliptic fibration"

Where is "12" coming from?

① $\mathbb{P}^1 \longmapsto \overline{\mathcal{M}_1 = \mathbb{H}^2 / SL_2\mathbb{Z}}$ Deligne-Mumford compactification.

deg 12 points 

orbifold point is i & ω degree 2, 3.

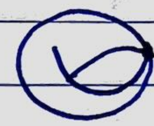
② resolvant deg for cubic curves is deg 12.

Day 4 Lecture 2.

Local monodromy
$$\begin{array}{c} Y \\ \downarrow \pi \\ \Delta \end{array}$$
complex dim 2

Δ disc in $\mathbb{C}$

$z \neq 0$ $\pi^{-1}(z) \cong T^2$ smooth

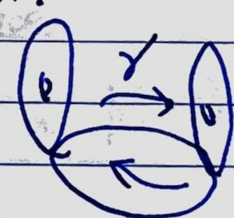
$\pi^{-1}(0)$ = singular 

$$\begin{array}{ccc} T^2 & \longrightarrow & Y - \pi^{-1}(0) \\ & & \downarrow \\ & & \Delta - \{0\} \end{array}$$

smooth fiber bundle over $\Delta - \{0\}$ $\rightsquigarrow$ Monodromy

$\pi_1(\Delta - \{0\}) \mapsto SL_2\mathbb{Z}$

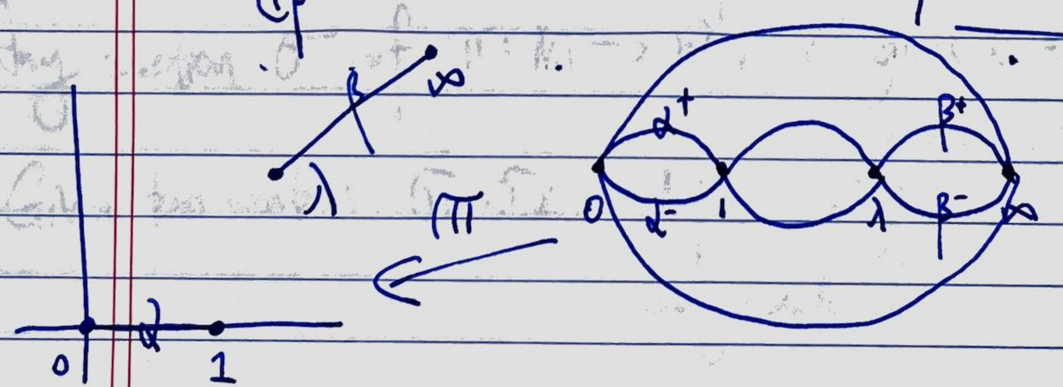
$\pi_1(\Delta - \{0\}) = \mathbb{Z}\langle\gamma\rangle$, $SL_2\mathbb{Z} = \mathrm{Mod}(T^2)$



Claim: Local monodromy is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
(Dehn twist along the vanishing cycle)

$\gamma \rightsquigarrow f_\gamma \in \mathrm{Diff}(T^2)$
the mapping class only depends on $\gamma \in \pi_1(\Delta - \{0\})$

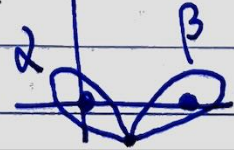
$y^2 = x(x-1)(x-\lambda) \subseteq \mathbb{C}^2 : E_\lambda$
$$\begin{array}{c} \downarrow \pi \\ \mathbb{CP}^1 \end{array}$$
$\pi(x,y) = x$



Exercise: Smooth family of complex tori

$$\begin{array}{ccc} T^2 & \longrightarrow & M \ni E_\lambda \\ & & \downarrow \\ & & \mathbb{P}^1 - \{0,1,\infty\} \ni \lambda \end{array}$$

$\overset{\text{Monodromy}}{\rightsquigarrow}$ $\pi_1(\mathbb{P}^1 - \{0,1,\infty\}) \xrightarrow{\rho} \mathrm{Mod}(T^2) = SL_2\mathbb{Z}$



What's $\rho(\alpha)$, $\rho(\beta)$?

Theorem (Moishezon) Given $A_1, A_2, \ldots A_{12} \in SL_2\mathbb{Z}$, each conjugate to $\begin{pmatrix}1&1\\0&1\end{pmatrix}$. $\prod_{i=1}^{12} A_i = I$.

$\Rightarrow$ can get $\left(A_1 \cdots A_{12}\right) = \left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-1&1\end{pmatrix}\cdots\right)$.

$T_a T_b T_a T_b \cdots T_a T_b$ $\square$

Claim: Each e_i gives a holomorphic section of

$$M \xrightarrow{\pi} \mathbb{P}^1, \quad \sigma: \mathbb{P}^1 \to M$$

$\sigma(\mathbb{P}^1)$ is a 2-sphere and intersect each fiber once.

$[s:t] \longmapsto E_{[s:t]} \cap e_i$ (a single point in $M = Bl_9(\mathbb{P}^2)$)

Back to Given $v \in H_M$ s.t $v^2 = 0$

Goal: Find f s.t $f_* \in \mathrm{Stab}(v)$.

Any section σ of $\pi: M \to \mathbb{P}^1$, it gives a group structure on each fiber.

Given two sections σ_1, σ_0, define a diffeomorphism by translating along each fiber $T_{\sigma_1 - \sigma_0}$.

$E_{[s:t]} \circlearrowleft T \quad T(z) = z + \sigma_1([s:t])$

Now the isotropic vector is $\pi^{-1}(pt)$ $pt \in \mathbb{P}^1$.

$T_{\sigma_1 - \sigma_0 *}([\sigma_0]) = [\sigma_1] \Rightarrow (T_{\sigma_1 - \sigma_0})_* \neq id$

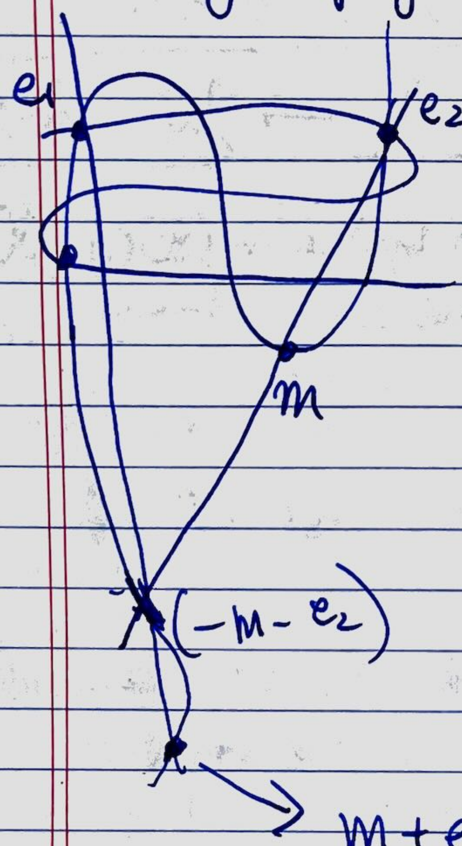
$(T_{\sigma_1-\sigma_0})_*([v]) = [v]$ (preserving each fiber as a set as homology class)

$\Rightarrow (T_{\sigma_1-\sigma_0})_* \in Stab(v)$

claim: $(T_{\sigma_1-\sigma_0})_* \in V^{\perp}/\mathbb{Z}v$

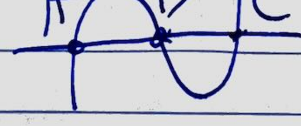
The fiber class $[v] = 3[H] - \sum_{i=1}^{9}[e_i]$

(by computing intersection #)



How to compute

$m+e_2-e_1$?



$A+B+C=0$

σ_0 fixes, σ_1 reflection $\rightsquigarrow$ $Ref_{\sigma_1,\sigma_0} \in O(V^{\perp}/\mathbb{Z}v)$

<u>affine reflection</u> ($\exists$ a conic through 5 points also gives a section of π)