Continuing ...

② Smooth vs. Topological

__Theorem__ (Ruberman, 90's)

Let $M = a\,\mathbb{P}^2 \# b\,\overline{\mathbb{P}^2}$ where $a = 2n$, $b = 10n+1$

$n > 1$. (e.g. $4\mathbb{P}^2 \# 21\overline{\mathbb{P}^2}$), then

$\ker(\pi_0(\mathrm{Diff}^+(M)) \to \pi_0(\mathrm{Homeo}^+(M))) \neq 0$.

In fact, ∞-generated!

---

③ Symplectic vs. smooth

(M, ω) symplectic 4-manifold

__Theorem.__ (Seidel, 1997): For certain special M,

$S \subseteq M$ Lagrangian 2-sphere, (-2).

Let T_S = Dehn twist. Then even though

$T_S^2 = 1 \in \pi_0(\mathrm{Diff}(M))$, T_S^2 is ∞-order in

$\pi_0(\mathrm{Symp}(M, \omega))$.

We don't know $\mathrm{Diff}_{cs}(\mathbb{R}^4)$!

C. What's left to do for $\mathrm{Mod}(M^4)$, $\pi_1(M) = 0$. (Farb - Looijenga)

<u>Problem 1</u> (The Realization Problem)

$\forall A \in O(H_M)$, write down explicitly some $F \in \mathrm{Homeo}^+(M)$ such that $F_* = A$.

<u>Note</u>: $\nexists$ smooth representatives always! e.g. $\mathrm{Diff}(K3) \leq \mathrm{Homeo}^+(K3)$ index 2, so there's a nonsmooth homeomorphism.

<u>INPUT</u>: Action on $H_2(M, \mathbb{Z})$

<u>OUTPUT</u>: Action on $M \circlearrowright f$

<u>Note</u>: Freedman proved surjectivity

<u>Problem 2</u>: Thurston-type normal forms

<u>Problem 3</u>: Section problems for which subgroups $G \leq \mathrm{Mod}(M)$ admit sections σ

$$\mathrm{Diff}(M) \xrightarrow[\pi]{} \mathrm{Mod}(M) \geq G$$

Geometric (Diff. and Alg.)

Problem 4 (Preserving structure)

Characterize which $A \in O(H_M)$ have a representative $F \in \mathrm{Diff}(M)$ with $F_* = A$ s.t. F preserves:

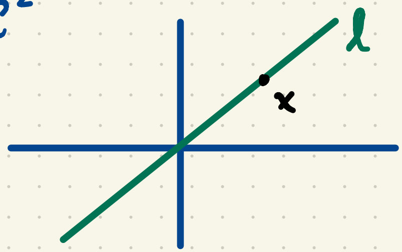
- some complex structure
- some special metrics (Ricci-Flat, Einstein, ...)
- some foliations (maybe w/ singularities)

More later.

IV. Case Study: $\mathrm{Mod}\left(Bl_{\{p_1,\dots,p_r\}}(\mathbb{P}^2)\right)$

A. (Complex) Blowups

$\mathbb{C}^2$



"blow up of $\mathbb{C}^2$ at 0"

$$B := Bl_0(\mathbb{C}^2) := \{(x,\ell) : x \in \ell\} \subseteq \mathbb{C}^2 \times \mathbb{P}^1$$

$$\downarrow \pi$$

$$\mathbb{C}^2$$

space of lines

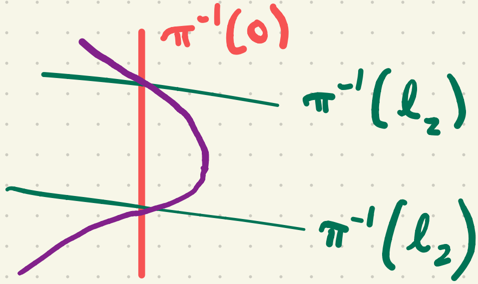
$$\pi(x,\ell) := x$$

The map π is the **blow down**.

- B is a 2-dimensional complex manifold
- $\pi^{-1}(x) = \begin{cases} x & x \neq 0 \\ \mathbb{CP}^1 & x = 0 \end{cases}$

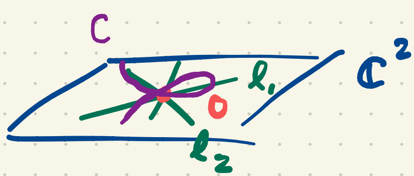
$$\mathbb{CP}^1 = \{(0, \ell)\}$$

- $\pi^{-1}(0) =: e$ is called the "exceptional divisor."



$Bl_0(\mathbb{C}^2) \to \mathbb{C}^2 - \{0\}$ is a biholomorphism.

See Scorpan for good pictures!



C gets "desingularized" by the blow up.

There a more general blow ups.

Now let $M =$ any complex surface

$p \in M$ any point.

$U = \text{nbhd}(p)$ parametrized by $\mathbb{C}^2$.

Let
$$\begin{array}{c} U' = \{(x, l) \in U \times \mathbb{P}^1 : x \in l\} \\ \pi \downarrow \qquad \pi(x,l) := x. \\ U \end{array}$$

$\pi : U' \setminus \{p \times \mathbb{P}^1\} \to U \setminus \{p\}$ biholomorphic diffeomorphism.

Let the "<u>blow up of M at p</u>" be

$$Bl_p(M) := (M \setminus U) \underset{\partial U = \partial U'}{\cup} U' \times (p \times \mathbb{P}^1)$$

- $Bl_p(M)$ is a $\mathbb{C}$-manifold
- M smooth proj. variety $\Rightarrow Bl_p(M)$ is
- $\pi^{-1}(p) \cong \mathbb{P}^1$ "exceptional divisor"

As a complex manifold,

$Bl_p(M)$ doesn't depend on choice of U.

(see Griffiths - Harris)

<u>Proposition</u>. Let M be a complex surface, $p \in M$, $M' := Bl_p(M)$. Then as smooth manifolds:

1. $M' \underset{\text{diffeo}}{\cong} M \# \overline{\mathbb{CP}^2}$

2. $Q_{M'} = Q_M \oplus \underbrace{(-1)}_{\text{generated by exceptional divisor}}$

ie. if $e = \pi^{-1}(p)$ = exceptional divisor

$e^2 = -1$ (see hand-out)

<u>PROOF</u>: You! $\square$

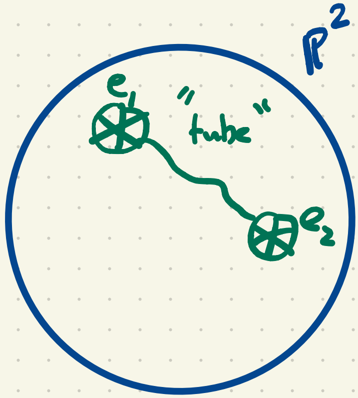
(cf. the tautological line bundle over $\mathbb{CP}^1$)

$$\begin{array}{ccc} \mathbb{C} & \to & T \\ & & \downarrow \\ & & \mathbb{CP}^1 \end{array}$$

Example. $M = Bl_{\{p_1, p_2\}}(\mathbb{P}^2)$

$e_1^2 = e_2^2 = -1.$

$e_1 \cong \mathbb{P}^1 \cong S^2$



Consider $[e_1 - e_2] \in H_2(M, \mathbb{Z}) = \mathbb{Z}^3$

then $(e_1 - e_2)^2 = -1 + (-1) = -2$

Claim: we can represent $[e_1] - [e_2] = [S^2]$.