III. Mapping Class Groups of 4-Manifolds

A. $\mathrm{Mod}(M^3)$

- Johannson (1979): M^3 closed, irreducible, atoroidal
  $\Longrightarrow \mathrm{Mod}(M^3)$ is finite
- M^3 Seifert fibered $S^1 \to M \neq T^3$
  $\Longrightarrow$

$$1 \to \mathbb{Z} \to \mathrm{Mod}(M^3) \to \mathrm{Mod}(\Sigma) \to 1 \qquad \begin{array}{c} S^1 \to M \\ \downarrow \\ \Sigma \\ \uparrow \\ \text{2 dim orbifold} \end{array}$$

- $\mathrm{Mod}(\overbrace{M_1 \# \cdots \# M_r}^{M}) \to \mathrm{Out}(\pi_1(M))$
  can be interesting,
  say $M_i = S^2 \times S^1$, $\leftrightarrow \mathrm{Out}(F_r)$
  (Chen - Tshishiku)

B. The Kinds of Things Known about $\mathrm{Mod}(M^4)$ (up until August 6, 2022)

⓪ First Examples :

a) M complex surfaces

$\mathrm{Aut}(M)$ usually finite group

biholo (true if M = projective variety of general type)

- $\exists$ ∞ - order complex automorphisms of K3 surfaces (McMullen, Filip)

b) $M = \mathbb{P}^2$

$$\sigma : \mathbb{P}^2 \longrightarrow \mathbb{P}^2, \quad [x:y:z] \longmapsto [-x:y:z]$$

$\sigma \sim I$

isotopic

Proof: $PGL_3(\mathbb{C}) \circlearrowright \mathbb{P}^2$

connected! $\sigma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \subseteq \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $\square$

Take a path $\sigma \rightsquigarrow I$

· Let $\tau : \mathbb{P}^2 \to \mathbb{P}^2$

$[x:y:z] \mapsto [\bar{x}:\bar{y}:\bar{z}]$ $\mathbb{C}$-conjugation on coordinates

<u>Exercise</u>: $\tau_* = -\mathrm{Id} \circlearrowright H_2(\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}$

$\Longrightarrow$ τ is to isotopic to I.

<u>Exercise</u>: Any diffeo. $f : \mathbb{P}^2 \circlearrowright$ preserves orientation.

<u>Note</u>: $\tau \in \mathrm{Diff}(\mathbb{P}^2)$, $\tau^2 = \mathrm{Id}$ orient. preserving

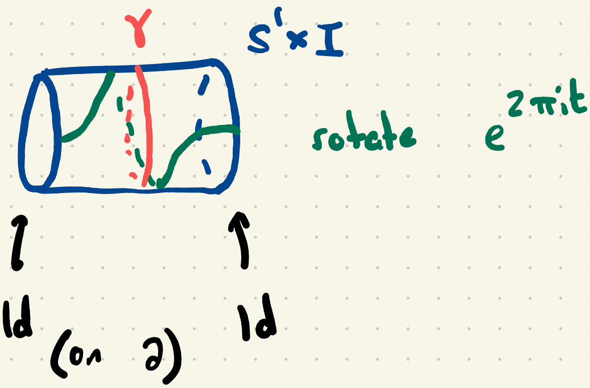
c) Dehn twists about (-2) 2-spheres

$i : S^2 \hookrightarrow M$

$v := i_*([S^2])$, $v^2 = -2$

<u>Recall</u>: Dehn twists about simple closed curves

$\gamma \subset \Sigma_g$:



Now let $S^2 \subseteq M$ be an embedded -2

2-sphere

$$\xRightarrow{\text{exercise}} \qquad \begin{array}{c} D^2 \longrightarrow \mathrm{Nbhd}_M(S^2) \\ \downarrow \\ S^2 \end{array}$$

is isomorphic as a D^2-bundle to the

unit disk bundle

* **Note**: this is actually $T^*S^2_{\leq 1}$

$$\begin{array}{c} D^2 \longrightarrow T^{*}S^2_{\leq 1} := \left\{ (u,v) \in S^2 \times \mathbb{R}^3 : \begin{array}{c} u \perp v \\ \|v\| \leq 1 \end{array} \right\} \\ \downarrow \\ S^2 \end{array}$$

Now construct $h' \in \mathrm{Diff}^+(TS^2_{\leq 1})$

identity on ∂, extend to M^4

Let h' be the time π map

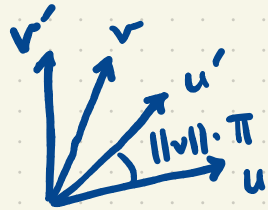
of the geodesic flow on S^2

$$\varphi_t : TS^2 \times \mathbb{R} \longrightarrow TS^2 \quad \text{where}$$

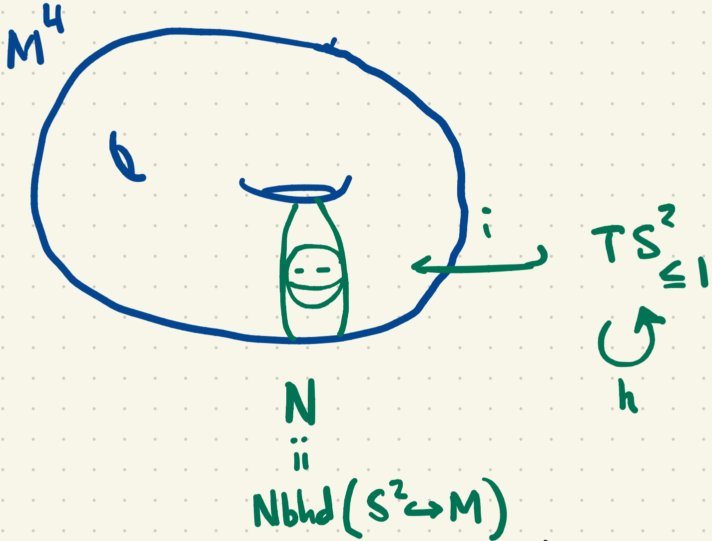
$(v_u, t) \mapsto$ parallel transport along the unique geodesic through u with tangent v_u.

So $h'(u,v) := \varphi(v_u, \pi) =: (u', v')$

$h(u,v) = h'(-u', -v')$



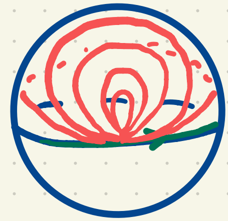
Now stick it in M:



$$T_{S^2}(M) := \begin{cases} m, & m \notin N \\ i \circ h \circ i^{-1}, & m \in N \end{cases}$$

Dehn twist about sphere

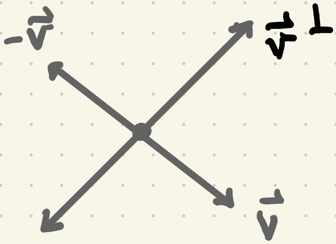
- $T_{S^2}^2 \sim I$



<u>Proof</u>: See exercises. $\square$

- T_{S^2} induces reflection on $H_2(M,\mathbb{Z})$

$\vec{v} = [S^2]$



$H_2(M,\mathbb{Z}) \subseteq H_2(M,\mathbb{R})$

- Can make T_{S^2} symplectic.

① <u>Theorem.</u> (Freedmann 1982, Quinn 1986)

Let M^4 be closed, oriented, $\pi_1(M^4) = 0$.

Then the natural homomorphism

$$\mathrm{Mod}(M) \longrightarrow O(H_M) := \mathrm{Aut}(\underbrace{H_2(M,\mathbb{Z})}_{\mathbb{Z}^d}, Q_M)$$
$$[f] \longmapsto f_*$$

is an isomorphism.

<u>Borel.</u> $G_{\mathbb{R}}$ semisimple real Lie group, e.g.

$O(p,q)(\mathbb{R})$, has an "arithmetic group" $G_{\mathbb{Z}} < G_{\mathbb{R}}$, is cofinite volume lattice.

<u>Corollary.</u> $\mathrm{Mod}(M)$ is arithmetic in $O(H_M \otimes \mathbb{R})$

<u>Examples.</u>

1. $Q_{\mathbb{P}^2} = (1) \Longrightarrow \mathrm{Mod}(\mathbb{CP}^2) \cong \mathrm{Aut}(\mathbb{Z}, \cdot)$
$$\cong \mathbb{Z}/2\mathbb{Z}$$

2. $\mathrm{Mod}(S^2 \times S^2)$
$$\cong$$
$$O\left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$
$$\cong$$
$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

<u>Exercise.</u> Find diffeo. reps for all 4 elts

3. $\mathrm{Mod}(a\mathbb{P}^2 \# b\overline{\mathbb{P}^2}) \cong O(\mathbb{Z}^{a+b}, (1)^{\oplus a} \oplus (-1)^{\oplus b})$

$$= O(a,b)(\mathbb{Z})$$

4. $\mathrm{Mod}(\text{the K3 manifold})$

$$\| $$

$$O(2E_8(-1) \oplus 3U) \subseteq O(3,19)(\mathbb{R})$$