E. Example: K3 Surfaces (Farb-Looijenga-"Nielsen...")

Definition. Let M be a closed complex surface. M is a K3 surface if:

(i) $\pi_1(M) = 0$

(ii) M admits a nowhere vanishing holomorphic 2-form (in local coordinates (z_1, z_2) e.g. $dz_1 \wedge dz_2$)

Huh?

Enriques-Kodaira Trichotomy (4-chotomy) of closed complex surfaces

like: $g=0$ 1. Rational (and ruled): $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$

$g=1$ 2. Complex tori $\mathbb{C}^2/\Lambda \cong T^4$, K3 surfaces!

$g \geq 2$ 3. General type: (e.g. $\mathbb{C}H^2/\Gamma$)

and blow ups of them.

Ex. Let $M_d :=$ a smooth degree d hypersurface $\subseteq \mathbb{P}^3$

$$= \{ [x_0 : \cdots : x_3] : F(x_0, \dots, x_3) = 0 \text{ for smooth } F \in \mathbb{C}[x_0, \dots, x_3]_{(d)} \}$$

homogeneous

Fact. $M_1 = \mathbb{P}^2$

$M_2 = \mathbb{P}^1 \times \mathbb{P}^1$

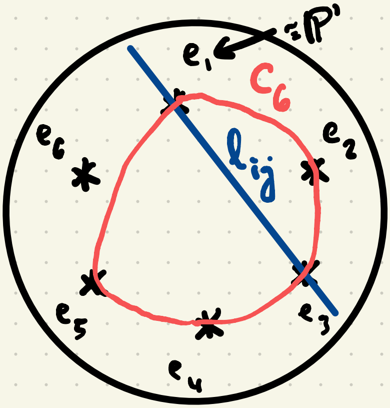
$M_3 = Bl_{\{p_1, \dots, p_6\}}(\mathbb{P}^2)$

$M_4 = K3$ surfaces

$M_{d \geq 5} =$ general type

Sidebar. smooth cubic surface $S \subseteq \mathbb{P}^3$ has 27 lines

- $S \underset{\text{diffeo}}{\cong} Bl_{\{p_1, \dots, p_6\}}(\mathbb{P}^2)$ $\quad \{p_i\}$ in general pos.
  - $p_i \neq p_j$
  - no 3 collinear



- not all 6 on a conic

Where are the 27 lines?

6 exceptional divisors
$15 = \binom{6}{2}$ strict transforms $\hat{\ell}_{ij}$ of ℓ_{ij}
6 strict transforms of conics $C_{i=1,\dots,6}$

27 lines!

Topology of K3 surfaces

1. M complex $\Longrightarrow$ M orientable $\Longrightarrow$
   $H_0(M,\mathbb{Z}) \cong H_4(M,\mathbb{Z}) \cong \mathbb{Z}$ $\underset{PD}{\Longrightarrow}$ Q_M unimodular
   $\pi_1(M) = 0 \Rightarrow H_1(M,\mathbb{Z}) \cong H_3(M,\mathbb{Z}) = 0$.

2. Hodge theory $\Longrightarrow$ a. Q_M even
   b. Q_M indefinite
   c. Q_M has σ-type $(3,19)$

Proposition. M a ~~K3~~ closed $\mathbb{C}$-surface $\Rightarrow$ M has a unique holomorphic 2-form up to scaling.

PROOF: Locally

$$\varpi_1 = f_1(z_1, z_2)\, dz_1 \wedge dz_2 \quad \leftarrow \text{any holo. 2-form}$$

$$\varpi_2 = f_2(z_1, z_2)\, dz_1 \wedge dz_2 \quad \leftarrow \text{nowhere } = 0 \text{ 2-form}$$

Consider the function

$$M \to \mathbb{C}$$
$$(z_1, z_2) \mapsto \frac{f_1(z_1, z_2)}{f_2(z_1, z_2)}$$

Bounded, well-defined, holomorphic $\xRightarrow[\text{max principle}]{}$ constant. $\square$

Seen through Hodge theory:

$$H^2(M, \mathbb{C}) \cong \underbrace{H^{2,0}(M)}_{\text{holom.}} \oplus \underbrace{H^{1,1}(M)}_{\text{type } (1,1)} \oplus \underbrace{H^{0,2}(M)}_{\text{anti-holom.}}$$

Classification of even, unimodular, indefinite quadratic forms $\Rightarrow$

$$Q_M = a \underbrace{E_8(-1)}_{\sigma = (0,8)} + b \underbrace{U}_{\sigma = (1,1)}, \qquad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so for K3s : $Q_M = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$

since $a(0,8) + b(1,1) = (3,19)$

Examples of K3 surfaces

1. Smooth quartic surfaces in $\mathbb{P}^3$

   e.g. $\{[x_0 : \cdots : x_3] : \sum x_i^4 = 0\}$ Fermat cubic

2. 2-sheeted branched covers of $\mathbb{P}^2$, branched over a smooth sextic curve C.

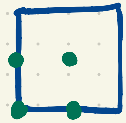
$$\begin{array}{l} M = \text{K3 surface!} \\ \downarrow 2:1 \\ \mathbb{P}^2 \supset C \quad \text{branch locus} \end{array}$$

3. Kummer surfaces : $A = \mathbb{C}^2/\Lambda$ complex torus, an abelian group under addition mod Λ

16 2-torsion points :

e.g. $A = \mathbb{C}^2/\mathbb{Z}[i]^2 = \mathbb{C}/\mathbb{Z}[i] \times \mathbb{C}/\mathbb{Z}[i]$

$$A[2] = \left\{ \begin{array}{c} \text{2-torsion pts} \\ \text{in } A \end{array} \right\}$$



1D example

$$= \left\{ (\epsilon_1, \dots, \epsilon_4) : \epsilon_i \in \{0, \tfrac{1}{2}\} \right\}$$

Let $\hat{M} := Bl_{A[2]} A$

- $i : A \to A : (z_1, z_2) \mapsto (-z_1, -z_2)$

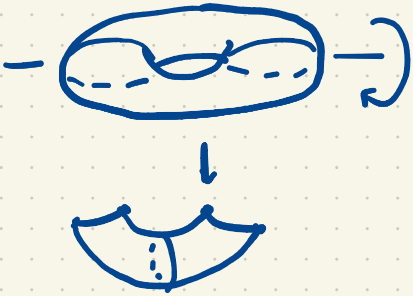
$\langle i \rangle$ acts on A with Fix Set $= A[2]$

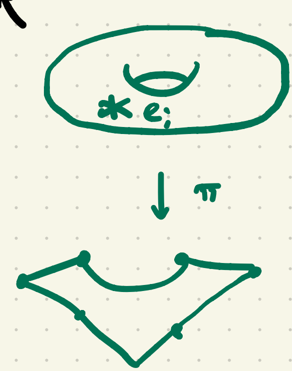
- i induces an involution of $\hat{M}$.

$M := \mathrm{Kum}(A) = \hat{M}/\langle i \rangle$ is a ~~complex~~ manifold

a K3 surface

Analogy of Latte example in complex dimension 1





$\pi(e_i) \cdot \pi(e_i) = -2.$

4. $M = X_2 \cap X_3$, $X_j \subseteq \mathbb{P}^4$ smooth hypersurface of degree j, is a K3 surface!

<u>Theorem (Kodaira, 1964)</u>: All K3 surfaces are diffeomorphic!

<u>Proof:</u> Not explicit. $\square$